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Theory of Hall Effect in a Two-Dimensional Electron System

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Hall conductivity σ_{XY} is studied in various approximations. Characteristics of σ_{XY} are obtained for the case of both short- and long-ranged scatterers in the self-consistent Born approximation, which is the simplest one free from the difficulty of divergence. In case of short-ranged scatterers, a relation is shown to hold between σ_{XY} and σ_{XX} within this approximation, if one uses a relaxation time under magnetic fields. Under strong magnetic fields, effects of higher Born scattering become important in low-lying Landau levels. They depend on the sign of scatterers and strongly on their concentrations. Effects of simultaneous scattering from many scatterers are calculated to the lowest order.

§ 1. Introduction

When a strong magnetic field is applied perpendicularly to a two-dimensional electron system such as inversion layers on semi-conductor surfaces, the energy spectrum becomes discrete because of the complete quantization of the orbital motion. Such a singular system provides an ideal tool for studying the quantum transport phenomena. In a series of previous papers,[1-4] which are referred to as I, II, III and IV in what follows, the transverse conductivity σ_{XX} was studied systematically. The present paper is concerned with another important quantity—the Hall conductivity σ_{XY}.

In order to see characteristics of σ_{XY}, we first employ the self-consistent Born approximation (SCBA) which is the simplest one free from the difficulty of divergence caused by the singular nature of our system.[1] One takes into account effects of scattering from an impurity in the lowest Born approximation, while the collision broadening is included in a self-consistent manner.

According to a simple phenomenological argument, the Hall conductivity σ_{XY} becomes

$$\begin{aligned}\sigma_{XY}&=-\frac{nec}{H}\frac{\omega_c^2\tau^2}{1+\omega_c^2\tau^2}\\&=-\frac{nec}{H}+\frac{1}{\omega_c\tau}\frac{ne^2\tau}{m}\frac{1}{1+\omega_c^2\tau^2}\,,\end{aligned}\tag{1.1}$$

where n is the total number of electrons in a unit area, ω_c is the cyclotron frequency given by $\omega_c=eH/mc$, and τ is the relaxation time. The above equation can be written as

$$\sigma_{XY}=-\frac{nec}{H}+\Delta\sigma_{XY}\,,\tag{1.2}$$

with

$$\Delta\sigma_{XY}=\frac{1}{\omega_c\tau}\sigma_{XX}\,.\tag{1.3}$$

In the SCBA such kind of relationship between $\Delta\sigma_{XY}$ and σ_{XX} holds in case of short-ranged scatterers if a relaxation time under the magnetic field is used in eq. (1.3). In contrast to the case of σ_{XX}, however, contributions from higher order approximations are relatively important and can be crucial especially under extremely strong magnetic fields. Those higher order corrections are also investigated.

In § 2, starting from the center migration theory of Kubo *et al.*[5,6] and using the technique of Green's function, we obtain necessary equations of $\Delta\sigma_{XY}$. In § 3, σ_{XY} is explicitly calculated in the SCBA under strong magnetic fields and characteristics in case of short- and long-ranged scatterers are obtained. Assuming scatterers with a short-ranged potential, we calculate the oscillatory σ_{XY} under magnetic fields of arbitrary strength and show that eq. (1.3) holds. In the first part of § 4, we investigate effects of higher Born scattering and show that they are important in low-lying Landau levels especially when the concentration of scatterers is not so large. Effects of simultaneous scattering from more than two scatterers are also investigated and are shown to be relatively important in low-lying Landau levels. Some discussions on the obtained results are given in § 5.

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§ 2. Hall Conductivity σ_{XY}

We consider a two-dimensional system described by the Hamiltonian

$$H=\frac{1}{2m}\left(\boldsymbol{p}+\frac{e}{c}\boldsymbol{A}\right)^2+\sum_{\mu}\sum_{i}v^{(\mu)}(\boldsymbol{r}-\boldsymbol{R}_i,Z_i)\ , \tag{2.1}$$

where bold-faced letters represent two-dimensional vectors in the xy-plane, $\boldsymbol{A}=(-Hy/2, Hx/2)$, and $v^{(\mu)}(\boldsymbol{r}-\boldsymbol{R}_i, Z_i)$ is effective two-dimensional potential of a scatterer located at $(\boldsymbol{R}_i, Z_i)$. The potential is assumed to be cylindrically symmetric for simplicity. It can be written in a mixed representation[1,5]

$$\begin{aligned}H=&\sum_{NX}E_N a^{+}_{NX}a_{NX}+\sum_{\mu}\sum_{i}\sum_{NX}\sum_{N'X'}\sum_{m}2\pi l^2\varphi^{*}_{N+mX}(\boldsymbol{R}_i)\\&\times\varphi_{N'+mX'}(\boldsymbol{R}_i)(Nm|v^{(\mu)}(Z)|N'm)a^{+}_{NX}a_{N'X'}\ ,\end{aligned} \tag{2.2}$$

where $E_N=(N+1/2)\hbar\omega_c$, $l^2=c\hbar/eH$, a^{+}_{NX} is the creation operator of an electron in the N-th Landau level having a center coordinate X, $\phi_{NX}(\boldsymbol{r})$ is its wave function, and $(Nm|v^{(\mu)}(Z)|N'm)$ is a matrix element of a potential between states with an angular momentum m around the position of a scatterer.

According to the center migration theory of Kubo *et al.*[5,6] one has

$$\Delta\sigma_{XY}=\frac{e^2\hbar}{i\pi L^2}\int f(E)\,\mathrm{d}E\left\langle \mathrm{Tr}\,\dot{X}\left(\frac{\partial}{\partial E}\,\mathrm{Re}\frac{1}{E-H+i0}\right)\dot{Y}\ \mathrm{Im}\frac{1}{E-H+i0}-(\dot{X}\leftrightarrow\dot{Y})\right\rangle\ , \tag{2.3}$$

where L^2 is the area of the system, $f(E)$ is the Fermi distribution function, a trace should be taken over one-electron states, $\langle\cdots\rangle$ means an average over all configurations of scatterers, and

$$\begin{aligned}\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix}&=\frac{l}{\hbar}\sum_{\mu}\sum_{i}\begin{pmatrix}l\dfrac{\partial}{\partial y}\\ -l\dfrac{\partial}{\partial x}\end{pmatrix}v^{(\mu)}(\boldsymbol{r}-\boldsymbol{R}_i,Z_i)\\&=-\frac{l}{\hbar}\sum_{\mu}\sum_{i}\sum_{NX}\sum_{N'X'}\sum_{m}\sum_{\pm}\begin{pmatrix}\left(Nm\left|l\dfrac{\partial v^{(\mu)}(Z)}{\partial y}\right|N'm\pm1\right)\\ -\left(Nm\left|l\dfrac{\partial v^{(\mu)}(Z)}{\partial x}\right|N'm\pm1\right)\end{pmatrix}\\&\quad\times2\pi l^2\varphi^{*}_{N+mX}(\boldsymbol{R}_i)\varphi_{N'+m\pm1X'}(\boldsymbol{R}_i)a^{+}_{NX}a_{N'X'}\ .\end{aligned} \tag{2.4}$$

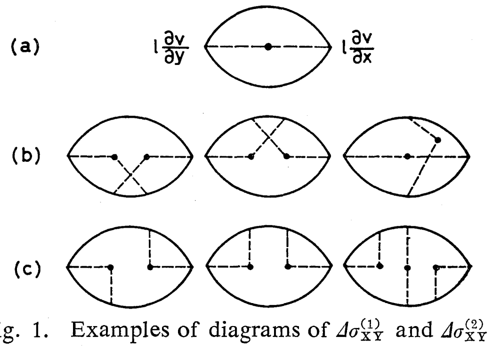

Fig. 1. Examples of diagrams of $\Delta\sigma^{(1)}_{XY}$ and $\Delta\sigma^{(2)}_{XY}$. (a) $\Delta\sigma^{(1)}_{XY}$ in the SCBA, which becomes identically zero, (b) $\Delta\sigma^{(1)}_{XY}$ in the lowest double-site approximation (see § 4), and (c) examples of $\Delta\sigma^{(2)}_{XY}$ in the SCBA.

We divide $\Delta\sigma_{XY}$ into two parts: $\Delta\sigma^{(1)}_{XY}$ and $\Delta\sigma^{(2)}_{XY}$. The former $\Delta\sigma^{(1)}_{XY}$ represents contributions from those diagrams which can not be cut into two parts by cutting two internal electron lines, and the latter $\Delta\sigma^{(2)}_{XY}$ contributions from other diagrams. Examples are shown in Fig. 1. As will be shown in the following, $\Delta\sigma^{(1)}_{XY}$ and $\Delta\sigma^{(2)}_{XY}$ are different in nature.

Introduce a quantity $\xi^{x}_{N+1,N}(E',E)$ or $\xi^{y}_{N+1,N}(E',E)$, which we call ξ-part. Each ξ-part is

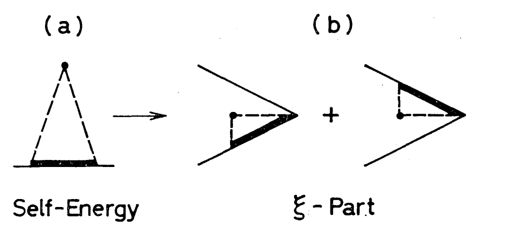

Fig. 2. Self-energy in the SCBA and corresponding ξ-part.

obtained graphically from self-energy diagrams by replacing one of matrix elements of the potential $v^{(\mu)}(\boldsymbol{r}, Z)$ by the corresponding $\partial v^{(\mu)}(\boldsymbol{r}, Z)/\partial x$ or $\partial v^{(\mu)}(\boldsymbol{r}, Z)\partial y$ as is shown in Fig. 2. From graphical consideration one gets

$$\Delta\sigma_{XY}^{(2)}=\frac{e^2}{8\pi^2\hbar}\int f(E)\,\mathrm{d}E\sum_{N_1}\sum_{\pm}\lim_{E'\to E}\mathrm{Im}_{(E)}\,\mathrm{Re}_{(E')}\frac{\partial}{\partial E'}[\xi^y_{N_1\pm1N_1}(E,E')$$
$$\times G_{N_1}(E)G_{N_1\pm1}(E')\Xi^x_{N_1\pm1N_1}(E',E)-(x\leftrightarrow y)]\ , \tag{2.5}$$

with

$$\Xi^x_{N_1\pm1N_1}(E',E)=\xi^x_{N_1\pm1N_1}(E',E)+\sum_{N_2}\gamma_{N_1\pm1N_1,N_2\pm1N_2}(E',E)G_{N_2\pm1}(E')G_{N_2}(E)\Xi^x_{N_2\pm1N_2}(E',E)\ , \tag{2.6}$$

where $\gamma_{N_1\pm1N_1,\,N_2\pm1N_2}$ is the proper vertex part defined in I and $G_N(E)$ is average Green's function

$$G_N(E)=\left\langle\!\left(0\ \middle|\ a_{NX}\frac{1}{E-H}a^+_{NX}\ \middle|\ 0\right)\!\right\rangle . \tag{2.7}$$

Making use of

$$\left(Nm\pm1\ \middle|\ l\frac{\partial v^{(\mu)}(Z)}{\partial y}\ \middle|\ N'm\right)=\pm i\left(Nm\pm1\ \middle|\ l\frac{\partial v^{(\mu)}(Z)}{\partial x}\ \middle|\ N'm\right)\ , \tag{2.8}$$

one can show that

$$\xi^y_{N\pm1N}(E',E)=\mp i\xi^x_{N\pm1N}(E',E)\ . \tag{2.9}$$

Further, as is easily seen, one has

$$\left.\begin{aligned}&\xi^x_{N\pm1N}(E',E)=\bar{\xi}^x_{NN\pm1}(E,E')\ ,\\&\xi^x_{N\pm1N}(E,E)=\xi^y_{N\pm1N}(E,E)=0\ ,\end{aligned}\right\} \tag{2.10}$$

where $\bar{\xi}$ can be obtained from ξ by taking the complex conjugate of matrix elements of potentials appearing in ξ. By the use of above relations, eq. (2.5) becomes

$$\Delta\sigma_{XY}^{(2)}=\frac{e^2}{4\pi^2\hbar}\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E\sum_{N_1}\sum_{\pm}(\pm i)\xi^x_{N_1N_1\pm1}(E+i0,E-i0)$$
$$\times G_{N_1}(E+i0)G_{N_1\pm1}(E-i0)\Xi^x_{N_1\pm1N_1}(E-i0,E+i0)\ . \tag{2.11}$$

Therefore, $\Delta\sigma_{XY}^{(2)}$ is determined only from quantities at the Fermi energy at zero temperature. Next let us consider the case under strong magnetic fields and assume that the Fermi level lies in the N-th Landau level. In this case, one has to retain terms with $N_1=N$ and $N+1$ in eq. (2.11), and further one has $\Xi^x_{N\pm1N}=\xi^x_{N\pm1N}$ and $G_{N\pm1}=\mp(\hbar\omega_c)^{-1}$. Therefore, eq. (2.11) is reduced to

$$\Delta\sigma_{XY}^{(2)}=\frac{e^2}{2\pi^2\hbar}\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E\frac{1}{\hbar\omega_c}\sum_{\pm}\xi^x_{N\pm1N}(E-i0,E+i0)\bar{\xi}^x_{N\pm1N}(E-i0,E+i0)\,\mathrm{Im}\,G_N(E+i0)\ . \tag{2.12}$$

§ 3. Self-Consistent Born Approximation (SCBA)

3.1 *Case of strong magnetic fields*

In the SCBA, $\Delta\sigma_{XY}^{(1)}$ is given by the diagram shown in Fig. 1(a) and becomes

$$\Delta\sigma_{XY}^{(1)}=\frac{e^2}{\pi^2\hbar}\int f(E)\,\mathrm{d}E\,2\pi l^2\sum_{\mu}\int\mathrm{d}Z\,N_i^{(\mu)}(Z)\sum_{N_1}\sum_{N_2}\sum_{m}\sum_{\pm}\frac{1}{2i}\left[\left(N_1m\ \middle|\ l\frac{\partial v^{(\mu)}(Z)}{\partial y}\ \middle|\ N_2m\pm1\right)\right.$$
$$\left.\times\left(N_2m\pm1\ \middle|\ l\frac{\partial v^{(\mu)}(Z)}{\partial x}\ \middle|\ N_1m\right)-(x\leftrightarrow y)\right]\mathrm{Im}\,G_{N_1}(E+i0)\frac{\partial}{\partial E}\mathrm{Re}\,G_{N_2}(E+i0)\ , \tag{3.1}$$

where $N_i^{(\mu)}(Z)$ is the concentration of scatterers in a unit volume. With the aid of eqs. (2.9) of I and (2.7) of III, one gets

$$\sum_m \sum_\pm \left(N_1 m \left| l \frac{\partial v^{(\mu)}(Z)}{\partial y} \right| N_2 m \pm 1 \right) \left(N_2 m \pm 1 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| N_1 m \right)$$
$$= \int \frac{d^2 r}{2\pi l^2} \int \frac{d^2 r'}{2\pi l^2}\, l \frac{\partial v^{(\mu)}(\boldsymbol{r}, Z)}{\partial y}\, l \frac{\partial v^{(\mu)}(\boldsymbol{r}', Z)}{\partial x'} J_{N_2 N_2}(\boldsymbol{r}-\boldsymbol{r}') J_{N_1 N_1}(\boldsymbol{r}-\boldsymbol{r}') , \tag{3.2}$$

which becomes identically zero because of the cylindrical symmetry of the potential. Therefore, $\Delta\sigma_{\mathrm{XY}}^{(1)}$ vanishes in the SCBA.

Next let us consider $\Delta\sigma_{\mathrm{XY}}^{(2)}$. As the ξ-part one should include those diagrams shown in Fig. 2(b).

$$\xi^x_{N\pm 1 N}(E', E) = 2\pi l^2 \sum_\mu \int dZ\, N_i^{(\mu)}(Z) \sum_{N'} \sum_m \left\{ \left(N\pm 1 m \mp 1 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| N'm \right) (N'm | v^{(\mu)}(Z) | Nm) \right.$$
$$\left. \times G_{N'}(E) + (N\pm 1 m | v^{(\mu)}(Z) | N'm) \left(N'm \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| Nm \pm 1 \right) G_{N'}(E') \right\} . \tag{3.3}$$

Again with the aid of (2.9) of I and (2.7) of III, one has

$$\sum_m (N\pm 1 m | v^{(\mu)}(Z) | N'm) \left(N'm \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| Nm \pm 1 \right)$$
$$= -\sum_m \left(N\pm 1 m \mp 1 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| N'm \right) (N'm | v^{(\mu)}(Z) | Nm)$$
$$= -\int \frac{d^2 r}{2\pi l^2} \int \frac{d^2 r'}{2\pi l^2}\, l \frac{\partial v^{(\mu)}(\boldsymbol{r}, Z)}{\partial x} v^{(\mu)}(\boldsymbol{r}', Z) J_{N N\pm 1}(\boldsymbol{r}-\boldsymbol{r}') J_{N' N'}(\boldsymbol{r}-\boldsymbol{r}') \exp[\mp i\theta(\boldsymbol{r}, \boldsymbol{r}')] . \tag{3.4}$$

Therefore, one sees that the ξ-part (3.3) satisfies eq. (2.10). Under strong magnetic fields, it becomes

$$\xi^x_{N\pm 1 N}(E - i0, E + i0) = \frac{1}{4} (\Gamma_N^{\pm})^2 [-2i\, \mathrm{Im}\, G_N(E+i0)] , \tag{3.5}$$

where

$$(\Gamma_N^{\pm})^2 = 4\cdot 2\pi l^2 \sum_\mu \int dZ\, N_i^{(\mu)}(Z) \sum_m (N\mp 1 m | v^{(\mu)}(Z) | Nm) \left(Nm \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| Nm \pm 1 \right) . \tag{3.6}$$

Substitution of eq. (3.5) into eq. (2.12) yields

$$\Delta\sigma_{\mathrm{XY}}^{(2)} = \frac{e^2}{\pi^2 \hbar} \int \left(-\frac{\partial f}{\partial E} \right) dE \frac{\Gamma_N}{\hbar\omega_c} \left(\frac{\Gamma_N^{xy}}{\Gamma_N} \right)^4 \left[1 - \left(\frac{E - E_N}{\Gamma_N} \right)^2 \right]^{3/2} , \tag{3.7}$$

with

$$(\Gamma_N^{xy})^4 = \sum_\pm |\Gamma_N^{\pm}|^4 , \tag{3.8}$$

$$\Gamma_N^2 = 4\cdot 2\pi l^2 \sum_\mu \int dZ\, N_i^{(\mu)}(Z) \sum_m |(Nm | v^{(\mu)}(Z) | Nm)|^2 , \tag{3.9}$$

where use has been made of eq. (3.5) of I.

When scatterers are of sufficiently short range, one replaces the potential by a δ-potential,

$$v^{(\mu)}(\boldsymbol{r}, Z) = V^{(\mu)}(Z) \delta^{(2)}(\boldsymbol{r}) . \tag{3.10}$$

Then

$$\left(N0 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| N' \pm 1 \right) = \mp \frac{V^{(\mu)}(Z)}{2\pi l^2} \sqrt{\frac{N' + 1/2 \pm 1/2}{2}} , \tag{3.11}$$

$$(N0 | v^{(\mu)}(Z) | N'0) = \frac{V^{(\mu)}(Z)}{2\pi l^2} , \tag{3.12}$$

and other matrix elements are zero. One gets

$$\Delta\sigma_{XY}^{(2)}=\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E\,\frac{e^2}{\pi^2\hbar}\frac{\Gamma}{\hbar\omega_c}\left(N+\frac{1}{2}\right)\left[1-\left(\frac{E-E_N}{\Gamma}\right)^2\right]^{3/2}, \tag{3.13}$$

and

$$(\Gamma_N{}^{xy})^4=\left(N+\frac{1}{2}\right)\Gamma^4\,, \tag{3.14}$$

where Γ^2 can be expressed in terms of the relaxation time τ_f obtained in the Born approximation by assuming the same scatterers in the absence of magnetic fields.

$$\Gamma^2=4\sum_\mu\int\mathrm{d}Z N_i^{(\mu)}(Z)\frac{|V^{(\mu)}(Z)|^2}{2\pi l^2}=\frac{2}{\pi}\hbar\omega_c\frac{\hbar}{\tau_f}\,. \tag{3.15}$$

When scatterers are of sufficiently slowly-varying type, eq. (3.6) becomes

$$\begin{aligned}(\Gamma_N{}^{\pm})^2=&4\cdot 2\pi l^2\sum_\mu\int\mathrm{d}ZN_i^{(\mu)}(Z)\int\frac{\mathrm{d}^2\boldsymbol{r}}{2\pi l^2}\int\frac{\mathrm{d}^2\boldsymbol{r}'}{2\pi l^2}\,l\,\frac{\partial v^{(\mu)}(\boldsymbol{r},Z)}{\partial x}v^{(\mu)}(\boldsymbol{r}+\boldsymbol{R})\\ &\times J_{NN\pm1}(\boldsymbol{R})J_{NN}(\boldsymbol{R})\exp\left(\mp i\theta(\boldsymbol{R})\right]\\ \simeq&-4\cdot 2\pi l^2\sum_\mu\int\mathrm{d}ZN_i^{(\mu)}(Z)\int\frac{\mathrm{d}^2\boldsymbol{r}}{2\pi l^2}\left[\,l\,\frac{\partial v^{(\mu)}(\boldsymbol{r},Z)}{\partial x}\right]^2\sqrt{\frac{N+1/2\pm1/2}{2}}\,,\end{aligned} \tag{3.16}$$

where $\theta(\boldsymbol{R})=\tan^{-1}R_y/R_x$. It can be written as

$$(\Gamma_N{}^{\pm})^2=-2\sqrt{\frac{N+1/2\pm1/2}{2}}\langle(l\boldsymbol{\nabla}V(\boldsymbol{r}))^2\rangle\,, \tag{3.17}$$

where $V(\boldsymbol{r})$ is the local potential energy. Therefore, one has

$$(\Gamma_N{}^{xy})^4=4\left(N+\frac{1}{2}\right)\langle(l\boldsymbol{\nabla}V(\boldsymbol{r}))^2\rangle^2\,, \tag{3.18}$$

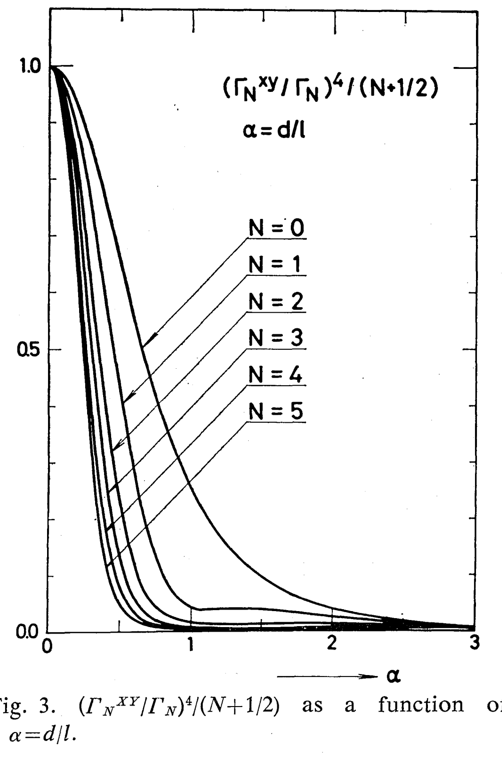

Fig. 3. $(\Gamma_N{}^{XY}/\Gamma_N)^4/(N+1/2)$ as a function of $\alpha=d/l$.

and $\Delta\sigma_{XY}$ decreases in proportion to $(l/d)^4$, where d is of the order of the range of scatterers.

In order to see such range dependence explicitly, we assume scatterers with a Gaussian potential

$$v^{(\mu)}(\boldsymbol{r},Z)=\frac{V^{(\mu)}(Z)}{\pi d^2}\exp\left(-\frac{\boldsymbol{r}^2}{d^2}\right). \tag{3.19}$$

After a little manipulation, one gets

$$(\Gamma_N{}^{\pm})^2=\Gamma^2h_N{}^{\pm}(\alpha)\,,$$

with

$$h_N{}^{\pm}(\alpha)=\pm\frac{1}{\sqrt{2N+1\pm1}}\times\int_0^\infty\mathrm{d}x\,xL_N{}^0(\alpha^2x)L^1_{N-1/2\pm1/2}(\alpha^2x)\,\mathrm{e}^{-(1+\alpha^2)x}\,, \tag{3.20}$$

where $\alpha=d/l$ and $L_N{}^m(x)$ is associated Laguerre's polynomial. Corresponding expression for $\Gamma_N{}^2$ is given by eqs. (2.24) and (2.26) in ref. 7. For the ground Landau level, for example, one gets

$$\left(\frac{\Gamma_0{}^{xy}}{\Gamma_0}\right)^4=\frac{1}{2(1+\alpha^2)^2}\,. \tag{3.21}$$

In Fig. 3, $(\Gamma_N{}^{xy}/\Gamma_N)^4/(N+1/2)$ is plotted against the range α for several Landau levels. Therefore,

$\Delta\sigma_{XY}^{(2)}$ decreases very rapidly with increasing range if Γ_N is kept constant.

3.2 *Short-ranged scatterers under magnetic fields of arbitrary strength*

In this section we specialize ourselves to the case of short-ranged scatterers. By the use of eqs. (3.11) and (3.12), eq. (3.3) becomes

$$\xi^x_{N\pm1\,N}(E-i0,\,E+i0)=\pm 2i\sqrt{\frac{N+1/2\pm1/2}{2}}\,\mathrm{Im}\,\Sigma(E+i0)\ , \tag{3.22}$$

where

$$\Sigma(E)=\frac{1}{4}\Gamma^2\sum_N G_N(E)\ . \tag{3.23}$$

Since the vertex part $\gamma_{N_1\pm1N_1,\,N_2\pm1N_2}$ vanishes, the Hall conductivity becomes

$$\begin{aligned}\Delta\sigma_{XY}&=\frac{e^2}{2\pi^2\hbar}\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E\,2(\mathrm{Im}\,\Sigma(E+i0))^2\sum_N(N+1)\,\mathrm{Im}\,[G_N(E+i0)G_{N+1}(E-i0)]\\&=\frac{e^2}{2\pi^2\hbar}\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E\,\frac{-2\,\mathrm{Im}\,\Sigma(E+i0)}{\hbar\omega_c}\sum_N(N+1)(\hbar\omega_c)^2\,\mathrm{Im}\,G_N(E+i0)\,\mathrm{Im}\,G_{N+1}(E+i0)\ .\end{aligned} \tag{3.24}$$

On the other hand, the transverse conductivity σ_{XX} is written as

$$\sigma_{XX}=\frac{e^2}{2\pi^2\hbar}\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E\sum_N(N+1)(\hbar\omega_c)^2\,\mathrm{Im}\,G_N(E+i0)\,\mathrm{Im}\,G_{N+1}(E+i0)\ . \tag{3.25}$$

One notices that at zero temperature

$$\Delta\sigma_{XY}=\frac{-2\,\mathrm{Im}\,\Sigma(E_F+i0)}{\hbar\omega_c}\sigma_{XX}\ , \tag{3.26}$$

where E_F is the Fermi energy. If one puts $\mathrm{Im}\,\Sigma(E_F+i0)=-\hbar/2\tau$, eq. (3.26) is the same as the phenomenological formula (1.3).*

Let us consider the case that the Fermi level lies in high Landau levels at zero temperature. The discussion in IV applies directly to the present σ_{XY}. When the magnetic field is sufficiently strong ($\omega_c\tau_f\gg1$), one gets.

$$\Delta\sigma_{XY}=\frac{e^2}{\pi^2\hbar}\left(N+\frac{1}{2}\right)\frac{\Gamma}{\hbar\omega_c}\times\left[1+\left(\frac{\pi^2}{8}-1\right)\left(\frac{\Gamma}{\hbar\omega_c}\right)^2+\cdots\right], \tag{3.27}$$

or

$$\frac{\Delta\sigma_{XY}}{\sigma_{XX}}=\frac{\Gamma}{\hbar\omega_c}\left[1+\frac{\pi^2}{24}\left(\frac{\Gamma}{\hbar\omega_c}\right)^2+\cdots\right], \tag{3.28}$$

at each peak associated with the N-th Landau level. When the magnetic field is rather weak ($\omega_c\tau_f\lesssim1$), on the other hand, one has

$$\sigma_{XY}=-\frac{ne^2\tau_f}{m}\frac{\omega_c\tau_f}{1+\omega_c^2\tau_f^2}\times\left[1-\frac{2(1+3\omega_c^2\tau_f^2)}{\omega_c^2\tau_f^2(1+\omega_c^2\tau_f^2)}\cos\frac{2\pi X'}{\hbar\omega_c}\times\exp\left(-\frac{\pi}{\omega_c\tau_f}\right)\right], \tag{3.29}$$

where X' is defined in IV.

* As is easily seen, eq. (3.26) is applicable to three-dimensional systems. Further it is easy to show that eq. (3.26) still holds for $\Delta\sigma_{XY}^{(2)}$ even if effects of higher Born scattering are included within the single-site approximation (SSA).

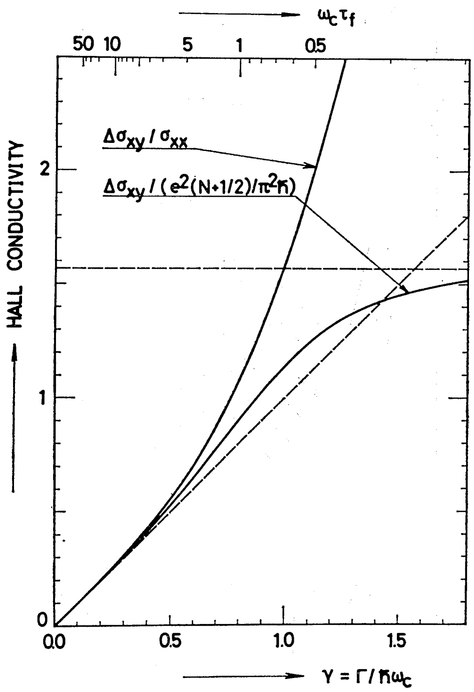

Fig. 4. $\Delta\sigma_{XY}$ and $\Delta\sigma_{XY}/\sigma_{XX}$ as a function of $\gamma=\Gamma/\hbar\omega_c$ at $X'=N\hbar\omega_c$. See also Fig. 2 in IV.

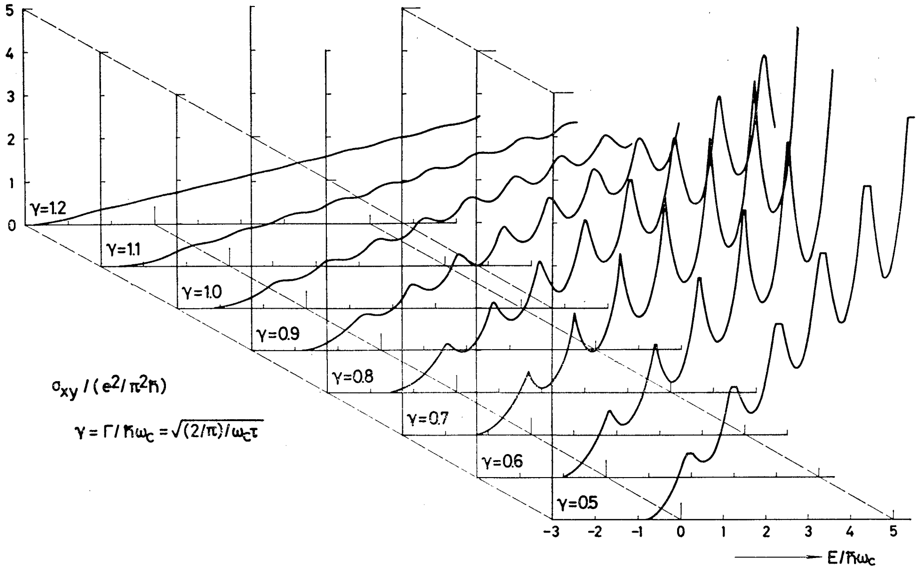

Fig. 5. Examples of oscillatory σ_{XY} for several $\gamma=\Gamma/\hbar\omega_c$.

In Fig. 4, $\Delta\sigma_{\text{XY}}$ and $\Delta\sigma_{\text{XY}}/\sigma_{\text{XX}}$ at $X'=N\hbar\omega_c$ for general $\gamma=\Gamma/\hbar\omega_c$, which are numerically calculated, are shown as a function of γ. The oscillatory conductivity σ_{XY} calculated numerically in a similar manner to IV is given in Fig. 5.

§ 4. Higher Approximations under Strong Magnetic Fields

First, we investigate effects of higher order scattering from a single scatterer employing the single-site approximation (SSA) as in II. When the magnetic field is sufficiently strong, one should include diagrams corresponding to those shown in Fig. 2 of II and one gets

$$\begin{aligned}\Delta\sigma_{\text{XY}}^{(1)}=&\frac{e^2}{\pi^2\hbar}\int f(E)\mathrm{d}E\,2\pi l^2\sum_{\mu}\int\mathrm{d}Z\,N_i^{(\mu)}(Z)\sum_{m}\sum_{\pm}\frac{1}{2i}\Bigg[\left(Nm\pm1\left|l\frac{\partial v^{(\mu)}(Z)}{\partial y}\right|Nm\right)\\&\times\left(Nm\left|l\frac{\partial v^{(\mu)}(Z)}{\partial x}\right|Nm\pm1\right)-(x\leftrightarrow y)\Bigg]\\&\times\mathrm{Im}\frac{1}{X_N-v_{Nm}^{(\mu)}(Z)}\frac{\partial}{\partial E}\mathrm{Re}\frac{1}{X_N-v_{Nm\pm1}^{(\mu)}(Z)},\end{aligned}\tag{4.1}$$

where $X_N(E)=G_N(E)^{-1}$, $v_{Nm}^{(\mu)}(Z)=(Nm|v^{(\mu)}(Z)|Nm)$, and one should choose N in such a way that E_N is closest to E. Making use of eq. (2.8) and integrating by parts, one has

$$\begin{aligned}\Delta\sigma_{\text{XY}}^{(1)}=&\frac{e^2}{\pi^2\hbar}\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E\,2\pi l^2\sum_{\mu}\int\mathrm{d}Z\,N_i^{(\mu)}(Z)\sum_{m}\left|\left(Nm\left|l\frac{\partial v^{(\mu)}(Z)}{\partial x}\right|Nm+1\right)\right|^2\\&\times\Bigg\{[v_{Nm}^{(\mu)}(Z)-v_{Nm+1}^{(\mu)}(Z)]^{-2}\,\mathrm{Im}\Bigg[\ln\frac{X_N}{X_N-v_{Nm}^{(\mu)}(Z)}-\frac{v_{Nm}^{(\mu)}(Z)}{X_N-v_{Nm}^{(\mu)}(Z)}\\&-\ln\frac{X_N}{X_N-v_{Nm+1}^{(\mu)}(Z)}+\frac{v_{Nm+1}^{(\mu)}(Z)}{X_N-v_{Nm+1}^{(\mu)}(Z)}\Bigg]\\&+[v_{Nm}^{(\mu)}(Z)-v_{Nm+1}^{(\mu)}(Z)]^{-1}\Bigg[v_{Nm}^{(\mu)}(Z)\,\mathrm{Im}\frac{1}{X_N-v_{Nm}^{(\mu)}(Z)}\\&\times\mathrm{Re}\frac{1}{X_N-v_{Nm+1}^{(\mu)}(Z)}+v_{Nm+1}^{(\mu)}(Z)\,\mathrm{Re}\frac{1}{X_N-v_{Nm}^{(\mu)}(Z)}\,\mathrm{Im}\frac{1}{X_N-v_{Nm+1}^{(\mu)}(Z)}\Bigg]\Bigg\},\end{aligned}\tag{4.2}$$

where such branch of logarithm should be chosen whose imaginary part lies between $-\pi$ and π. With the aid of eq. (2.9) of II and the relation

$$\left|\left(Nm\left|l\frac{\partial v^{(\mu)}(Z)}{\partial x}\right|Nm+1\right)\right|^2=\frac{1}{2}(N+m+1)[(Nm+1|v^{(\mu)}(Z)|Nm+1)-(Nm|v^{(\mu)}(Z)|Nm)]^2\,, \tag{4.3}$$

which is given in Appendix of II, the Hall conductivity becomes

$$\begin{aligned}\sigma_{XY}=&-\frac{nec}{H}+\Delta\sigma_{XY}^{(1)}\,,\\ =&-\frac{e^2}{\pi^2\hbar}\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E\left\{\frac{\pi}{2}\left[N+1-\frac{1}{\pi}\,\mathrm{Im}\ln X_N\right]\right.\\ &-2\pi l^2\sum_\mu\int\mathrm{d}Z\,N_i^{(\mu)}(Z)\sum_m\left|\left(Nm\left|l\frac{\partial v^{(\mu)}(Z)}{\partial x}\right|Nm+1\right)\right|^2\\ &\times[v_{Nm}^{(\mu)}(Z)-v_{Nm+1}^{(\mu)}(Z)]^{-1}\left[v_{Nm}^{(\mu)}(Z)\,\mathrm{Im}\frac{1}{X_N-v_{Nm}^{(\mu)}(Z)}\,\mathrm{Re}\frac{1}{X_N-v_{Nm+1}^{(\mu)}(Z)}\right.\\ &\left.\left.+v_{Nm+1}^{(\mu)}(Z)\,\mathrm{Re}\frac{1}{X_N-v_{Nm}^{(\mu)}(Z)}\,\mathrm{Im}\frac{1}{X_N-v_{Nm+1}^{(\mu)}(Z)}\right]\right\}\,.\end{aligned} \tag{4.4}$$

The Green's function G_N or X_N is determined by the self-consistency equation (2.8) of II. From the above equation, one can conclude that $\Delta\sigma_{XY}^{(1)}$ vanishes and σ_{XY} becomes $-nec/H=-e^2(N+1)/2\pi\hbar$, when the Fermi level lies in energy gaps between adjacent N-th and $N+1$-th Landau levels at zero temperature.

Let us consider the case that concentrations of scatterers are sufficiently small and impurity bands are separated from the N-th Landau level. At the energy where the spectrum has a gap between two impurity bands or between an impurity band and the main Landau level, X_N becomes a negative real number in case of attractive scatterers and a positive one in case of repulsive scatterers. When the Fermi level lies in those spectral gaps at zero temperature, therefore, the Hall conductivity becomes $-e^2N/2\pi\hbar$ in case of attractive scatterers and $-e^2(N+1)/2\pi\hbar$ in case of repulsive scatterers.* This means that electrons which fully occupy impurity bands do not contribute to the Hall current, while those which occupy the main Landau level give rise to the same Hall current as that obtained when all i.e. $1/2\pi l^2$ electrons of the Landau level move freely.

In case of high concentrations of scatterers, the absolute value of X_N becomes large compared with $(Nm|v^{(\mu)}(Z)|Nm)$ and further X_N approaches a solution in the SCBA

$$X_N=\frac{1}{2}[E-E_N+i\sqrt{\Gamma_N{}^2-(E-E_N)^2}]\,, \tag{4.5}$$

Therefore, one has

$$\Delta\sigma_{XY}^{(1)}=-\frac{e^2}{3\pi^2\hbar}\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E\left(\frac{\Gamma_N'}{\Gamma_N}\right)^3\left[1-\left(\frac{E-E_N}{\Gamma_N}\right)^2\right]^{3/2}\,, \tag{4.6}$$

with

$$\begin{aligned}(\Gamma_N')^3=2^3\cdot(2\pi l^2)\sum_\mu\int\mathrm{d}Z\,N_1^{(\mu)}(Z)\sum_m\left|\left(Nm\left|l\frac{\partial v^{(\mu)}(Z)}{\partial x}\right|Nm+1\right)\right|^2\\ \times[(Nm|v^{(\mu)}(Z)|Nm)-(Nm+1|v^{(\mu)}(Z)|Nm+1)]\,.\end{aligned} \tag{4.7}$$

Especially in case of short-ranged scatterers, eq. (4.7) becomes

$$(\Gamma_N')^3=2^3\cdot 2\pi l^2\sum_\mu\int\mathrm{d}Z\,N_i^{(\mu)}(Z)\left[\frac{V^{(\mu)}(Z)}{2\pi l^2}\right]^3\,. \tag{4.8}$$

As can be seen from the above equations, the effects of higher Born scattering does not vanish in the limit of $\Gamma/\hbar\omega_c\to 0$, and further they depend on the sign of scatterers: $\Delta\sigma_{XY}^{(1)}$ becomes positive (negative) when scatterers are attractive (repulsive). It should be noticed also that $\Delta\sigma_{XY}^{(1)}$ does

* A repulsive scatterer can have impurity bands in our system.

not increase with the Landau level N in contrast to $\Delta\sigma_{XY}^{(2)}$ and that it is independent of N in case of short-ranged scatterers. With the increase of the concentration of scatterers, $\Delta\sigma_{XY}^{(1)}$ becomes small, which is consistent with the result obtained in II that the SSA approaches the SCBA in case of high concentrations of scatterers.

As an illustrative example, we consider a model system in which a single kind of scatterers with the Gaussian potential (3.19) is distributed randomly. In a similar manner to in § 3 of II, one can calculate σ_{XY} of the ground Landau level. Examples of the results are shown in Fig. 6.

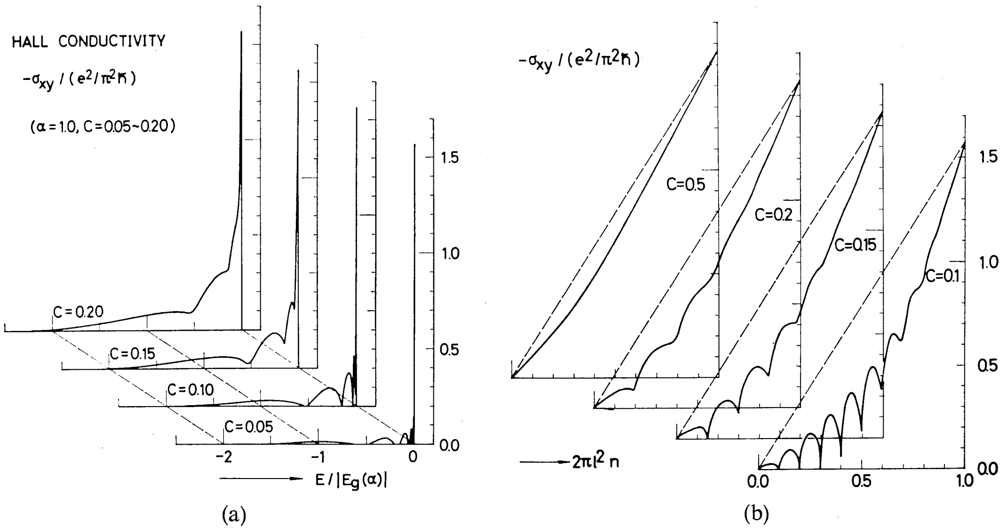

Fig. 6. Examples of σ_{XY} of the ground Landau level in case of scatterers with a Gaussian potential. $c=2\pi l^2 N_i$. See also Figs. 6~11. (a) σ_{XY} as a function of the energy. (b) σ_{XY} as a function of total number of electrons.

Next we investigate effects of simultaneous scattering from more than two scatterers to the lowest order, confining ourselves to the case of sufficiently high concentrations of short-ranged scatterers. One has to take into account those diagrams shown in Fig. 1(b) and gets

$$\Delta\sigma_{XY}^{(1)}=\frac{e^2}{\pi^2\hbar}\frac{\Gamma^4}{4^3}\int f(E)\,\mathrm{d}E\,h_N{}^{(2)}\left[\operatorname{Im}G_N{}^3\frac{\partial}{\partial E}\operatorname{Re}G_N-\operatorname{Im}G_N\frac{\partial}{\partial E}\operatorname{Re}G_N{}^3\right], \tag{4.9}$$

with

$$h_N{}^{(2)}=4\cdot\int\frac{\mathrm{d}^2\boldsymbol{R}}{2\pi l^2}\left\{\frac{N+1}{2}[J_{N+1N+1}(\boldsymbol{R})J_{NN}(\boldsymbol{R})+J_{N+1N}(\boldsymbol{R})^2]-\frac{N}{2}[J_{NN}(\boldsymbol{R})J_{N-1N-1}(\boldsymbol{R})+J_{NN-1}(\boldsymbol{R})^2]\right\}. \tag{4.10}$$

Partial integration of eq. (4.9) yields

$$\Delta\sigma_{XY}^{(1)}=\frac{e^2}{\pi^2\hbar}\frac{\Gamma^4}{4^3}\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E\,h_N{}^{(2)}\left[\frac{1}{2}\operatorname{Im}G_N{}^2|G_N|^2-\frac{1}{4}\operatorname{Im}G_N{}^4\right]. \tag{4.11}$$

If one uses G_N given by eq. (4.5) of the SCBA, it becomes

$$\Delta\sigma_{XY}^{(1)}=-\frac{e^2}{\pi^2\hbar}\frac{1}{2}h_N{}^{(2)}\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E\frac{E-E_N}{\Gamma}\left[1-\left(\frac{E-E_N}{\Gamma}\right)^2\right]^{3/2}, \tag{4.12}$$

the integrand of which becomes zero at $E=E_N$ and becomes large at tails of the spectrum. One has, for example, $h_0{}^{(2)}=1$, $h_1{}^{(2)}=3/4,\cdots$, and $h_N{}^{(2)}$ decreases with the increase of N. Therefore, many-site corrections are not important for large Landau level index N.

§ 5. Discussion and Conclusion

The relation (3.26) between $\Delta\sigma_{XY}$ and σ_{XX} in the SCBA can physically be understood as follows. When an electric field E_y is applied in the y-direction, a center of the cyclotron motion of electrons moves in the x-direction with a drift velocity $v_x = cE_y/H$, which gives rise to the Hall conductivity $-nec/H$. Effects of scattering can be regarded as a frictional force acting on each electron, the strength of which is given by $F_x = -mv_x/\tau = -eE_y/\omega_c\tau$, where τ is a relaxation time. The current in the x-direction due to this force becomes

$$\Delta j_x = \sigma_{XX} \frac{F_x}{(-e)} = \frac{1}{\omega_c\tau} \sigma_{XX} E_y = \Delta\sigma_{XY} E_y \,. \tag{5.1}$$

Therefore, one has eq. (3.26).

MOS inversion layers on Si (100) surface are a typical two-dimensional system. For usual electron concentrations ($N_{\rm inv} \gtrsim 10^{12}\,{\rm cm}^{-2}$), main scatterers of this system are considered to be of short range and their concentrations are considered to be relatively large. Therefore, the results in the SCBA together with eq. (4.6) are expected to apply to inversion layers. Measurements of both σ_{XX} and σ_{XY} will give useful informations on main scatterers and especially on the level broadening Γ. If one compares lineshapes obtained from eq. (3.6) of I and eqs. (3.13) and (4.6), one sees that the effective width of $\Delta\sigma_{XY}$ is smaller than that of σ_{XX}. Therefore, spin and valley splittings are expected to be seen in σ_{XY} at higher Landau levels than in σ_{XX}. Recently Igarashi, Wakabayashi and Kawaji measured these quantities.[8] The overall behavior of their results is satisfactorily in agreement with the present theory. Detailed comparison will be made in their paper.

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