Study of topological superconductors in 1D and 2D

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Introduction

The search for Majorana fermions goes back to relativistic quantum mechanics. The concept of particles which are their own antiparticles was firstly introduced by Ettore Majorana in a paper published in 1937. He suggested that the complex Dirac equation can be separated into a pair of real wave equations, each of which describes a real fermionic field, leading to the possibility of chargeless particle to be its own antiparticle [1].

Remarkably, it has been realized in the past two decades, that Majorana fermions may appear as quasi-particle excitations in some condensed matter systems. They may emerge as zero energy excitations in the middle of the gap of a superconductor [4].

This report firstly discuss Bogoliubov de Gennes theory to describe conventional superconductivity. Then we study the concept of topological superconductivity and its manifestation in low dimensional systems. Afterwards we will speak about the appearance of zero-mode energy excitations in the spectrum of a 1D superconducting wire (the well-known Kitaev chain) and a 2D lattice p-wave chiral superconductor.

1 s-wave superconductors

We start from the Hamiltonian for free electrons in a metal:

$$\mathcal{H} = \left(\frac{p^2}{2m} - \mu\right) I_{2\times 2},$$

where μ is the chemical potential defining the Fermi surface, m is an effective mass, $I_{2\times 2}$ is the identity matrix acting on vectors in the spin Hilbert space. The Hamiltonian of the system of free electrons in the second quantization looks as follows:

$$H = \sum_{\mathbf{p},\sigma} c^+_{\mathbf{p}\sigma} \left(\frac{p^2}{2m} - \mu\right) c_{\mathbf{p}\sigma} \equiv \sum_{\mathbf{p}\sigma} c^+_{\mathbf{p}\sigma} \varepsilon(p) c_{\mathbf{p}\sigma}, \tag{1}$$

where $c^+_{\mathbf{p}\sigma}$ creates an electron with momentum $\mathbf{p}$ and spin σ. The ground state of this Hamiltonian is obtained by filling all the levels below the Fermi energy.

Using the commutation relations for $c_{\mathbf{p}\sigma}$ and $c^+_{\mathbf{p}\sigma}$ the Hamiltonian (1) can be written in the form:

$$H = \frac{1}{2}\sum_{\mathbf{p}\sigma}\left[c^+_{\mathbf{p}\sigma}\varepsilon(p)c_{\mathbf{p}\sigma} - c_{-\mathbf{p}\sigma}\varepsilon(-p)c_{-\mathbf{p}\sigma}\right] + const. \tag{2}$$

Let us now introduce the spinor $\psi \equiv \begin{pmatrix} c_{\mathbf{p}\uparrow} & c_{\mathbf{p}\downarrow} & c^+_{-\mathbf{p}\uparrow} & c^+_{-\mathbf{p}\downarrow}\end{pmatrix}^T$. Using this definition the Hamiltonian (2) can be rewritten in, so called, Bogolubov-de-Gennes formalism [5]:

$$H = \sum_{\mathbf{p}}\psi^+_{\mathbf{p}}\mathcal{H}_{BdG}(\mathbf{p})\psi_{\mathbf{p}},$$

$$\mathcal{H}_{BdG}(\mathbf{p}) = \frac{1}{2}\begin{pmatrix} \varepsilon(p) & 0 & 0 & 0 \\ 0 & \varepsilon(p) & 0 & 0 \\ 0 & 0 & -\varepsilon(-p) & 0 \\ 0 & 0 & 0 & -\varepsilon(-p)\end{pmatrix}.$$

Rewriting the Hamiltonian in such a way, we get rid of the summation over spin, encoding it in our choice of basis. Now we have four energy eigenvalues instead of two. The important point is that only two out of four can be associated to independent quasi-particles states.

The next step is to use the formalism introduced above to calculate the spectrum of a conventional s-wave superconductor. Let us introduce the singlet pairing potential [6]:

$$H_\Delta = \sum_{\mathbf{p}}\Delta c^+_{\mathbf{p}\uparrow}c^+_{-p\downarrow} + \Delta^* c_{-p\downarrow}c_{\mathbf{p}\uparrow}, \tag{3}$$

here Δ is a complex parameter, whose amplitude, corresponds to the amplitude of the superconducting gap (up to a factor 2). The pairing potential describes the mechanism of creating a Cooper pair out of two electrons and its breaking into the constituents. The total superconducting Hamiltonian is obtained simply by adding the two terms (1) and (3):

$$H_{tot} = H + H_\Delta = \sum_{\mathbf{p}}\psi^+_{\mathbf{p}}\mathcal{H}_{BdG}\psi_{\mathbf{p}},$$

$$\mathcal{H}_{BdG} = \frac{1}{2}\begin{pmatrix} \varepsilon(p) & 0 & 0 & \Delta \\ 0 & \varepsilon(p) & -\Delta & 0 \\ 0 & -\Delta^* & -\varepsilon(-p) & 0 \\ \Delta^* & 0 & 0 & -\varepsilon(-p)\end{pmatrix}. \tag{4}$$

The Hamiltonian (4) can be decomposed as

$$\mathcal{H}_{BdG}(\mathbf{p},\Delta) = \varepsilon(p)\tau^z\otimes I_{2\times 2} - \mathrm{Re}(\Delta)\tau^y\otimes\sigma^y - \mathrm{Im}(\Delta)\tau^x\otimes\sigma^y, \tag{5}$$

where τ^α are the Pauli matrices defined in the particle-hole space and σ^α – in the spin Hilbert space. It is always convenient to have the Hamiltonian in the form (5), since

$$\mathcal{H}^2_{BdG} = (\varepsilon(p)^2 + |\Delta|^2)I_{2\times 2},$$

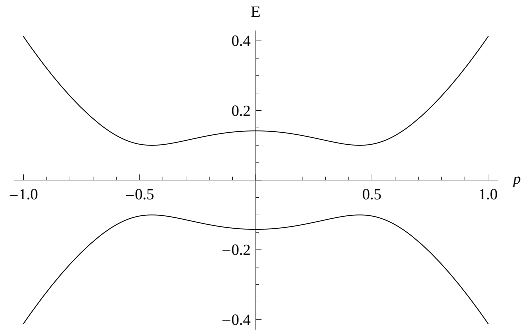


Figure 1: Plot of the dispersion relation for a s-wave superconductor $\mu = \Delta = 0.1$, $m = 1$.

so the energy spectrum is given by the expression $E_\pm = \pm\sqrt{\varepsilon(p)^2 + |\Delta|^2}$ and is shown in the Fig.1. It is always gapped for any value of the parameter μ, we will refer farther to such spectrum as topologically trivial.

To find a basis in which the Hamiltonian (4) is diagonal

$$H = \sum_{\mathbf{p},\sigma} E_\pm \gamma^+_{\mathbf{p}\sigma}\gamma_{\mathbf{p}\sigma}, \tag{6}$$

the quasi-particle operators $\gamma^+_{\mathbf{p}\sigma}$ and $\gamma_{\mathbf{p}\sigma}$ have to be introduced [5]. They are the linear combinations of creation and annihilation operators of the electrons:

$$\begin{aligned} \gamma_{\mathbf{p}\uparrow} &\equiv u_\mathbf{p} c_{\mathbf{p}\uparrow} - v_\mathbf{p} c^+_{-p\downarrow}, \\ \gamma_{\mathbf{p}\downarrow} &\equiv u_\mathbf{p} c_{\mathbf{p}\downarrow} + v_\mathbf{p} c^+_{-p\uparrow}. \end{aligned} \tag{7}$$

Since, $\gamma_{\mathbf{p}\uparrow}$ and $\gamma^+_{\mathbf{p}\uparrow}$ should satisfy the canonical fermion anticommutation relations, using (7) and the commutation relations for $c^+_{\mathbf{p}\sigma}$ and $c_{\mathbf{p}\sigma}$ we get normalisation conditions for the functions $u_\mathbf{p}$ and $v_\mathbf{p}$:

$$\begin{aligned} 1 &= \{\gamma^+_{\mathbf{p}\uparrow}, \gamma_{\mathbf{p}\uparrow}\} = |u_\mathbf{p}|^2 + |v_\mathbf{p}|^2, \\ 0 &= \{\gamma^+_{\mathbf{p}\uparrow}, \gamma_{-\mathbf{p}\downarrow}\} = u_\mathbf{p} v_{-\mathbf{p}} - v_\mathbf{p} u_{-\mathbf{p}}. \end{aligned}$$

Each quasi-particle operator $\gamma^+_{\mathbf{p}\uparrow}$ creates a state in the superposition of an electron with the quantum numbers $\{p, \uparrow\}$ and a hole with the opposite momentum and spin.

Situation studied above corresponds to the s-wave pairing, in another words, pairing between the electron wave functions with zero angular momentum, also this can be seen as a spin-singlet pairing. The simplest case of a spin-triplet pairing is a p-wave superconductor.

2 Topological p-wave superconductors

We are interested in the topologically nontrivial superconductors, the simplest model of such system is a BdG Hamiltonian of spin-polarized fermions [2].

2.1 The 1D Kitaev model

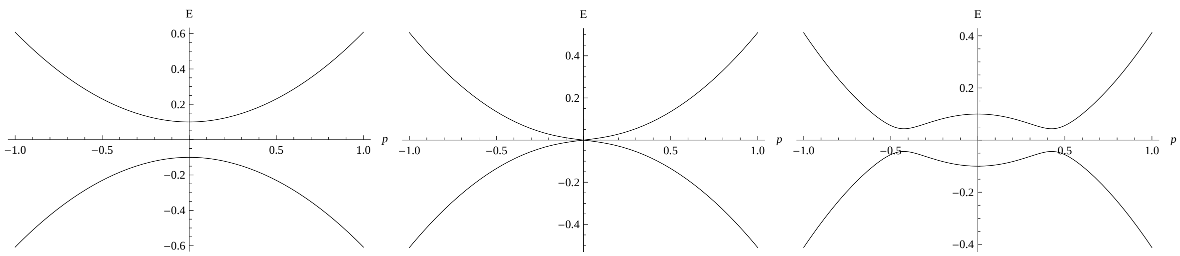
First, consider a 1D superconducting wire with a momentum dependent p-wave pairing potential given by the following Kitaev Hamiltonian:

$$H = \sum_{\mathbf{p}} c^{+}_{\mathbf{p}} \left(\frac{p^2}{2m} - \mu \right) c_{\mathbf{p}} + \frac{1}{2} \left(\Delta p c^{+}_{\mathbf{p}} c^{+}_{-\mathbf{p}} + \Delta^* p c_{-\mathbf{p}} c_{\mathbf{p}} \right).$$

Compared to the previous case, now the pairing potential is proportional to the odd power of the momentum. Using Nambu basis, which is of the form $\psi_{\mathbf{p}} = (c_{\mathbf{p}} \quad c^{+}_{-\mathbf{p}})^T$, the Hamiltonian can be rewritten as:

$$H = \sum_{\mathbf{p}} \frac{1}{2} \psi^{+}_{\mathbf{p}} \begin{pmatrix} \frac{p^2}{2m} - \mu & \Delta p \\ \Delta^* p & -\frac{p^2}{2m} + \mu \end{pmatrix} \psi_{\mathbf{p}} = \sum_{\mathbf{p}} \psi^{+}_{\mathbf{p}} \mathcal{H}_{BdG} \psi_{\mathbf{p}}. \tag{8}$$

Two energy bands of this Hamiltonian are given by the equation $E_{\pm} = \sqrt{\varepsilon(p)^2 + |\Delta|^2 p^2}$.



$\mu < 0$ $\mu = 0$ $\mu > 0$

Figure 2: Plot of the dispersion relation for p-wave superconductor $\Delta = 0.1$, $m = 1$.

The spectrum is gapped along $\mu \neq 0$. As can be seen from the Fig.2, the critical point $\mu = 0$ separates two phases. Considering the limit $m \to \infty$ and taking Δ to be real, the Hamiltonian reduces to the 1D Dirac Hamiltonian: $\mathcal{H}_{BdG} = |\Delta| p \sigma^x - \mu \sigma^z$. We are looking for zero-energy solution $|\psi\rangle$ such that $\mathcal{H}_{BdG}|\psi\rangle = 0|\psi\rangle$. Assuming that $\mu \to \mu(x)$ with $\mu(-\infty) < 0$ and $\mu(\infty) > 0$, we are looking for a ground state solution parametrized in the form

$$|\psi(x)\rangle = \exp\left(-\frac{1}{|\Delta|} \int_0^x \mu(x')dx' \right) |\phi_0\rangle.$$

Acting on it with $\mathcal{H}_{BdG}$, we get a secular equation for a spinor, solving it we obtain $|\phi_0\rangle$ and finally

$$|\psi(x)\rangle = \frac{1}{\sqrt{2}} \exp\left(-\frac{1}{|\Delta|} \int_0^x \mu(x')dx' \right) \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

The corresponding operator, which destroys this zero-mode looks as following:

$$\gamma_0 = \frac{1}{\mathcal{N}} \exp\left(-\frac{1}{|\Delta|} \int_0^x \mu(x')dx' \right) (c(x) - ic^+(x)),$$

where $c(x)$ is the annihilation fermion operator at the point x, and $\mathcal{N}$ is a normalization constant. It is easy to check, that multiplying γ_0 by a global phase $\bar{\gamma}_0 = e^{i\frac{\pi}{4}}\gamma_0$, we get that $\bar{\gamma}_0{}^+ = \bar{\gamma}_0$, thus $\bar{\gamma}_0$ is a Majorana fermion bound-state.

Now consider the Hamiltonian of a 1D p-wave spinless electronic chain on a lattice:

$$H = \sum_j \left[-\mu c_j^+ c_j - t(c_j^+ c_{j+1} + c_{j+1}^+ c_j) + |\Delta|(c_{j+1}^+ c_j^+ + c_j c_{j+1})\right], \tag{9}$$

which is the sum of the kinetic term and the pairing term. The amplitude t denotes the hopping between the nearest neighbours. Making the lattice Fourier transform, and going to Nambu basis the Hamiltonian takes the form

$$H = \frac{1}{2}\sum_p \psi_{\mathbf{p}}^+ \begin{pmatrix} -2t\cos p - \mu & 2i|\Delta|\sin p \\ -2i|\Delta|\sin p & 2t\cos p + \mu \end{pmatrix} \psi_{\mathbf{p}}.$$

The energy spectrum is $E_\pm(p) = \sqrt{(2t\cos p + \mu)^2 + 4|\Delta|^2\sin^2 p}$. Expanding around $p \sim 0$ it reduces to the continuum form derived from the Eq.(8).

To see the physics from a different perspective, one can describe the same system using the Majorana fermions representation firstly proposed by Kitaev [3]. Let us split the fermion operators c_j^+ and c_j as following:

$$\begin{aligned} c_j &= \frac{1}{2}(\gamma_{2j-1} + i\gamma_{2j}), \\ c_j^+ &= \frac{1}{2}(\gamma_{2j-1} - i\gamma_{2j}), \end{aligned}$$

where γ_{2j-1} and γ_{2j} are known as the Majorana operators. For c_j^+ and c_j to maintain the fermion anticommutation relations, γ_{2j-1} and γ_{2j} should satisfy:

$$\{\gamma_j, \gamma_{j'}\} = \delta_{jj'}.$$

Using the Majorana representation, the Hamiltonian (9) becomes

$$H = \frac{i}{2}\sum_{j=1}^{2N} \left[-\mu\gamma_{2j-1}\gamma_{2j} + (t + |\Delta|)\gamma_{2j}\gamma_{2j+1} + (-t + |\Delta|)\gamma_{2j-1}\gamma_{2j+2}\right]. \tag{10}$$

One can distinguish two specific cases.

1. When $|\Delta| = t = 0$, $\mu < 0$, the Hamiltonian in Eq.(10) reduces to

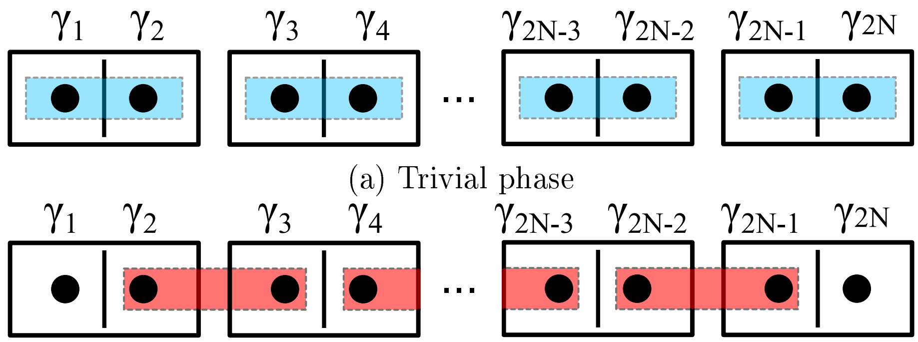
$$H = -\mu\frac{i}{2}\sum_j(\gamma_{2j-1}\gamma_{2j}).$$

The Majorana operators from the same site j are coupled, but the Majorana operators from the different sites are decoupled (see Fig.3a), this corresponds to the trivial phase.

2. When $|\Delta| = t > 0$, $\mu = 0$, in this case

$$H = it\sum_j(\gamma_{2j}\gamma_{2j+1}).$$

Now the Majorana operators from different sites are coupled (see Fig.3b). With the open boundary conditions, the Majorana operators γ_1 and γ_{2N} are unpaired, meaning that the two Majorana modes localized on the ends of the chain.



(a) Trivial phase

(b) Topological phase

Figure 3: Schematic illustration of the two types of pairing

The situation described above is just a 1D simple toy model where the Majorana modes emerge. In order to realize it experimentally, we need to incorporate back spin degrees of freedom. By combining a semiconducting wire with a strong spin-coupling and in proximity of a s-wave superconductor, it has been shown that this physics can be realized experimentally [8].

2.2 2D chiral p-wave superconductors

Moving in 2D, the lattice Hamiltonian of a chiral p-wave superconductor can be introduced by analogy to the 1D case:

$$\begin{aligned} H = \sum_{m,n}\Big[&-t\left(c^+_{m+1,n}c_{m,n} + h.c.\right) - t\left(c^+_{m,n+1}c_{m,n} + h.c\right) - (\mu - 4t)\, c^+_{m,n}c_{m,n} \\ &+ \left(\Delta c^+_{m+1,n}c^+_{m,n} + \Delta^* c_{m,n}c_{m+1,n}\right) + \left(i\Delta c^+_{m,n+1}c^+_{m,n} - i\Delta^* c_{m,n}c_{m,n+1}\right)\Big], \end{aligned} \tag{11}$$

where $c_{m,n}$ annihilate a fermion of the lattice site $\{m,n\}$.

Rewriting this Hamiltonian in the BdG form and making Fourier transformation, the Eq.(11) takes the form

$$H = \frac{1}{2}\sum_p \psi^+_{\mathbf{p}} \begin{pmatrix} \varepsilon(\mathbf{p}) & 2i\Delta(\sin p_x + i\sin p_y) \\ -2i\Delta^*(\sin p_x - i\sin p_y) & -\varepsilon(\mathbf{p}) \end{pmatrix} \psi_{\mathbf{p}}, \tag{12}$$

where $\varepsilon(\mathbf{p}) = -2t(\cos p_x + \cos p_y) - (\mu - 4t)$. For convenience a chemical potential has been shifted by a constant $4t$. Taking the limit $\mathbf{p} \to 0$ and denoting $m \equiv \frac{1}{2t}$, the model simplifies

$$H = \frac{1}{2}\sum_{\mathbf{p}} \psi_{\mathbf{p}}^{+} \begin{pmatrix} \frac{p^2}{2m} - \mu & 2i\Delta(p_x + ip_y) \\ -2i\Delta^*(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} \psi_{\mathbf{p}}. \tag{13}$$

Notice that the pairing potential is of the chiral form $p_x + ip_y$. The spectrum of the Hamiltonian is given by the formula

$$E_{\pm} = \pm\sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + 4|\Delta|^2 p^2},$$

which is gapped everywhere as long as $\mu \neq 0$.

To study the structure of energy bands of the exact Hamiltonian, let us for simplicity set $t = \frac{1}{2}$ and define $\Delta = |\Delta|e^{i\theta}$. We make the gauge transformation $c_{\mathbf{p}} \to e^{i\frac{\theta}{2}}c_{\mathbf{p}}$ to absorb the superconducting phase. Then the Hamiltonian can be written as

$$\mathcal{H}_{BdG}(\mathbf{p}) = (2 - \mu - \cos p_x - \cos p_y)\tau^z - 2|\Delta|\sin p_x \tau^y - 2|\Delta|\sin p_y \tau^x. \tag{14}$$

The energy spectrum for such Hamiltonian can be easily deduced:

$$E_{\pm} = \pm\sqrt{M(\mathbf{p})^2 + 4|\Delta|^2(\sin^2 p_x + \sin^2 p_y)},$$

where $M(\mathbf{p}) = 2 - \mu - \cos p_x - \cos p_y$. This spectrum is depicted in the Fig.4.

Changing the chemical potential, two different phases occur (see Fig. 4):

- $\mu < 0$ and $\mu > 4$ correspond to two trivial phases;
- $0 < \mu < 2$ and $2 < \mu < 4$ are the topological superconductor phases with the opposite chiralities.

To study the phase transition at $\mu = 0$, we expand the Hamiltonian around $(p_x, p_y) = (0, 0)$, which corresponds to the Eq.(13) and consider a domain wall between the two regions with $\mu < 0$ and $\mu > 0$. This implies the condition on $\mu \to \mu(x)$, such that $\mu(-\infty) = -\mu_0$ and $\mu(+\infty) = \mu_0$. We also keep translational invariance along y direction, meaning that p_y is a good quantum number to work with.

The Hamiltonian now becomes quasi-1D:

$$\mathcal{H}_{BdG} = \frac{1}{2}\begin{pmatrix} -\mu(x) & 2i|\Delta|(-i\frac{d}{dx} + ip_y) \\ -2i|\Delta|(-i\frac{d}{dx} - ip_y) & \mu(x) \end{pmatrix}, \tag{15}$$

It can be solved for each value of p_y by taking the eigenfunction of the form:

$$|\psi_{p_y}(x)\rangle = e^{ip_y y}\exp\left(-\frac{1}{2|\Delta|}\int_0^x \mu(x')dx'\right)|\phi_0\rangle.$$

First, we are searching for a ground state solution $\mathcal{H}_{BdG}|\psi_0(x)\rangle = 0$, by analogy to the one dimensional case, we get a secular equation for $|\phi_0\rangle$:

$$\begin{pmatrix} -\mu(x) & -\mu(x) \\ \mu(x) & \mu(x) \end{pmatrix} |\phi_0\rangle = 0.$$

The solution of which is a scalar spinor $|\phi_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix}^T$.

To find the solution for a finite p_y, notice that the term proportional to p_y in the Hamiltonian (15) is $-2|\Delta|p_y\tau^x$. Since $\tau^x|\phi_0\rangle = -|\phi_0\rangle$, then $|\psi_{p_y}\rangle$ is the eigenstate of $\mathcal{H}_{BdG}$ with the energy eigenvalue $E(p_y) = 2|\Delta|p_y$, meaning that there is a dispersive Majorana state propagating along the interface between the two phases with $\mu < 0$ and $\mu > 0$. So we cannot go from one region of a domain wall to another without crossing the bound states, which means that $\mu > 0$ and $\mu < 0$ are indeed two distinct topological phases separated by the propagating zero-mode solution.

Using the notion of a topological invariant, which can distinguish a trivial superconductor state from a topological one, it can be shown that $\mu < 0$ corresponds to the trivial superconductor, while $\mu > 0$ to the topological one.

3 Bound states on vortexes in a 2D chiral p-wave superconductors

Now we would like to study the system with periodic boundary conditions, for this let us consider a disk of a radius R with the $\mu < 0$ phase inside the disk and $\mu > 0$ – outside. Let us recall the explicit expression of the quasi-particle operator:

$$\gamma_{\mathbf{p}}^{+} = e^{\frac{i\theta}{2}} u(\mathbf{p}) c_{\mathbf{p}}^{+} + e^{-\frac{i\theta}{2}} v(\mathbf{p}) c_{\mathbf{p}},$$

where $u(\mathbf{p})$ and $v(\mathbf{p})$ do not depend on the phase of an order parameter.

It is easy to see, that if $\theta \to 2\pi$, $\gamma_{\mathbf{p}}^{+} \to -\gamma_{\mathbf{p}}^{+}$, giving rise to the antiperiodic boundary conditions on the quasi-particle wave function. This implies, that the states bounded to the domain wall between $\mu < 0$ and $\mu > 0$ should also obey the antiperiodic boundary conditions, meaning that the lowest allowed angular momentum is no more 0, but $\frac{1}{2}$. This leads therefore to an energy shift of the spectrum. It thus seems that it is impossible to have the Majorana zero-modes in this geometry.

Nevertheless, there is a way to get a Majorana zero-mode on the boundary of the disk by shifting the boundary conditions from the antiperiodic back to the periodic ones. The idea is to change boundaries using the Aharonov-Bohm effect [7], in another words, to insert a magnetic flux into the disk for the bound-states to pick up a phase. The magnetic flux applied to the superconductor will create a vortex [5]. This is interesting to modelize and to look for zero-state energy solution of the corresponding Hamiltonian.

In the presence of a vortex, the order parameter becomes $\Delta(r, \theta) = |\Delta(r)|e^{i\theta}$, taking the phase to be equal to the polar angle. Taking the gauge $\psi(r) \to \psi(r)e^{\frac{i\theta}{2}}$ and going to polar coordinates: $\mathbf{p} = p(\cos\theta, \sin\theta)$, the quasi-1D Hamiltonian (15) takes the following form:

$$\mathcal{H}_{BdG} = \frac{1}{2}\begin{pmatrix} -\mu & 2|\Delta(r)|e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right) \\ -2|\Delta(r)|e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right) & \mu \end{pmatrix}. \tag{16}$$

One can check that taking ψ_0 of the form:

$$\psi_0(r,\theta) = \frac{i}{\sqrt{r}}\exp\left[-\frac{1}{2}\int_0^r \frac{\mu(r')}{|\Delta(r')|}\right]\begin{pmatrix} -e^{\frac{i\theta}{2}} \\ e^{-\frac{i\theta}{2}} \end{pmatrix} \equiv ig(r)\begin{pmatrix} -e^{\frac{i\theta}{2}} \\ e^{-\frac{i\theta}{2}} \end{pmatrix},$$

where $g(r)$ is localized at the vortex core, satisfies $\mathcal{H}_{BdG}\psi_0(r,\theta) = 0$. Thus, there is a single Majorana state, which is trapped inside of the vortex. Therefore, we have shown that vortices in 2D chiral topological superconductors support localized Majorana bound states.

Conclusion

The search for Majorana zero-modes is an active field of study in condensed matter physics. As we have seen, it is related to the topology of the energy bands of the material. Here we discussed the way to obtain the Majorana zero-energy states energy using the properties of the topological superconductivity. We have seen that there are particular 1D and 2D models, which predict the emergence of zero-mode energy solutions in their spectra.

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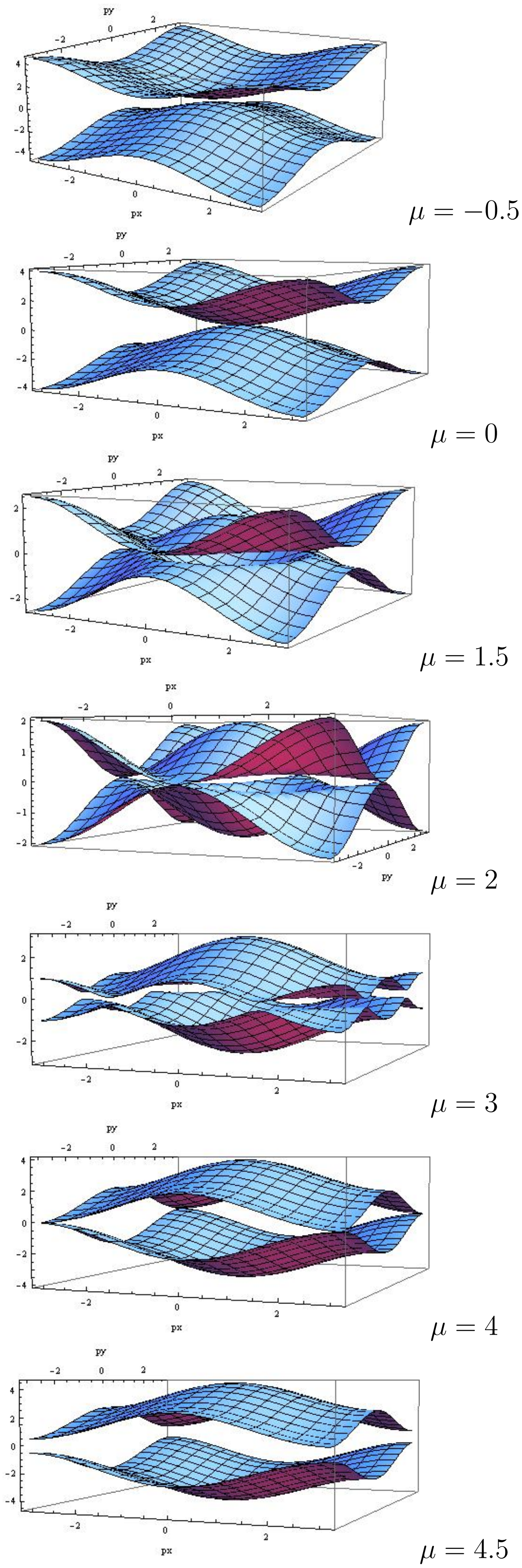

Figure 4: Energy spectrum of a 2D p-wave superconductor obtained by diagonalization the tight-binding Hamiltonian in Eq.(11)