The averaged current is just

$$\bar{I} = \frac{I_c}{Q\omega_p} \frac{2e}{\hbar} V = \frac{\hbar C}{2eI_c} \frac{I_c}{RC} \frac{2e}{\hbar} V = \frac{V}{R}. \tag{11.49}$$

Note that this result holds for any damping. It is evidently important to carefully specify whether a constant current or a constant voltage is imposed.

11.3 Bogoliubov-de Gennes Hamiltonian

It is often necessary to describe inhomogeneous systems, Josephson junctions are typical examples. So far, the only theory we know that is able to treat inhomogeneity is the Ginzburg-Landau theory, which has the disadvantage that the quasiparticles are not explicitly included. It is in this sense not a microscopic theory. We will now discuss a microscopic description that allows us to treat inhomogeneous systems. The essential idea is to make the BCS mean-field Hamiltonian spatially dependent. This leads to the *Bogoliubov-de Gennes Hamiltonian*. It is useful to revert to a first-quantized description. To this end, we introduce the condensate state $|\psi_{\mathrm{BCS}}\rangle$ as the ground state of the BCS Hamiltonian

$$H_{\mathrm{BCS}} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}}\, c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \Delta^*_{\mathbf{k}}\, c_{-\mathbf{k},\downarrow} c_{\mathbf{k}\uparrow} - \sum_{\mathbf{k}} \Delta_{\mathbf{k}}\, c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k},\downarrow} + \text{const.} \tag{11.50}$$

$|\psi_{\mathrm{BCS}}\rangle$ agrees with the BCS ground state defined in Sec. 9.2 in the limit $T \to 0$ (recall that $\Delta_{\mathbf{k}}$ and thus H_{BCS} is temperature-dependent). We have

$$H_{\mathrm{BCS}}\, |\psi_{\mathrm{BCS}}\rangle = E_{\mathrm{BCS}}\, |\psi_{\mathrm{BCS}}\rangle\,, \tag{11.51}$$

where E_{BCS} is the temperature-dependent energy of the condensate.

Since H_{BCS} is bilinear, it is sufficient to consider single-particle excitations. Many-particle excitations are simply product states, or more precisely Slater determinants, of single-particle excitations. We first define a two-component spinor

$$|\Psi_{\mathbf{k}}\rangle \equiv \begin{pmatrix} |\Psi_{\mathbf{k}1}\rangle \\ |\Psi_{\mathbf{k}2}\rangle \end{pmatrix} := \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k},\downarrow} \end{pmatrix} |\psi_{\mathrm{BCS}}\rangle\,. \tag{11.52}$$

It is easy to show that

$$[H_{\mathrm{BCS}}, c^\dagger_{\mathbf{k}\uparrow}] = \xi_{\mathbf{k}}\, c^\dagger_{\mathbf{k}\uparrow} - \Delta^*_{\mathbf{k}}\, c_{-\mathbf{k},\downarrow}, \tag{11.53}$$

$$[H_{\mathrm{BCS}}, c_{-\mathbf{k},\downarrow}] = -\xi_{\mathbf{k}}\, c_{-\mathbf{k},\downarrow} - \Delta_{\mathbf{k}}\, c^\dagger_{\mathbf{k}\uparrow}. \tag{11.54}$$

With these relations we obtain

$$\begin{aligned} H_{\mathrm{BCS}}\, |\Psi_{\mathbf{k}1}\rangle &= H_{\mathrm{BCS}}\, c^\dagger_{\mathbf{k}\uparrow}\, |\psi_{\mathrm{BCS}}\rangle = \Big(\xi_{\mathbf{k}}\, c^\dagger_{\mathbf{k}\uparrow} - \Delta^*_{\mathbf{k}}\, c_{-\mathbf{k},\downarrow} + c^\dagger_{\mathbf{k}\uparrow}\, H_{\mathrm{BCS}}\Big)\, |\psi_{\mathrm{BCS}}\rangle \\ &= (E_{\mathrm{BCS}} + \xi_{\mathbf{k}})\, |\Psi_{\mathbf{k}1}\rangle - \Delta^*_{\mathbf{k}}\, |\Psi_{\mathbf{k}2}\rangle \end{aligned} \tag{11.55}$$

and

$$\begin{aligned} H_{\mathrm{BCS}}\, |\Psi_{\mathbf{k}2}\rangle &= H_{\mathrm{BCS}}\, c_{-\mathbf{k},\downarrow}\, |\psi_{\mathrm{BCS}}\rangle = \Big(-\xi_{\mathbf{k}}\, c_{-\mathbf{k},\downarrow} - \Delta_{\mathbf{k}}\, c^\dagger_{\mathbf{k}\uparrow} + c_{-\mathbf{k},\downarrow}\, H_{\mathrm{BCS}}\Big)\, |\psi_{\mathrm{BCS}}\rangle \\ &= (E_{\mathrm{BCS}} - \xi_{\mathbf{k}})\, |\Psi_{\mathbf{k}2}\rangle - \Delta_{\mathbf{k}}\, |\Psi_{\mathbf{k}1}\rangle\,. \end{aligned} \tag{11.56}$$

Thus for the basis $\{|\Psi_{\mathbf{k}1}\rangle\,, |\Psi_{\mathbf{k}2}\rangle\}$ the Hamiltonian has the matrix form

$$\begin{pmatrix} E_{\mathrm{BCS}} + \xi_{\mathbf{k}} & -\Delta_{\mathbf{k}} \\ -\Delta^*_{\mathbf{k}} & E_{\mathrm{BCS}} - \xi_{\mathbf{k}} \end{pmatrix}. \tag{11.57}$$

This is the desired Hamiltonian in first-quantized form, except that we want to measure excitation energies relative to the condensate energy. Thus we write as the first-quantized Hamiltonian in $\mathbf{k}$ space

$$\mathcal{H}_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} \xi_{\mathbf{k}} & -\Delta_{\mathbf{k}} \\ -\Delta^*_{\mathbf{k}} & -\xi_{\mathbf{k}} \end{pmatrix}. \tag{11.58}$$

This is the Bogoliubov-de Gennes Hamiltonian for non-magnetic superconductors. Its eigenvalues are

$$\pm\sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2} = \pm E_{\mathbf{k}} \tag{11.59}$$

with corresponding eigenstates

$$u_{\mathbf{k}} |\Psi_{\mathbf{k}1}\rangle - v_{\mathbf{k}}^* |\Psi_{\mathbf{k}2}\rangle = \left(u_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger - v_{\mathbf{k}}^* c_{-\mathbf{k},\downarrow}\right) |\psi_{\text{BCS}}\rangle = \gamma_{\mathbf{k}\uparrow}^\dagger |\psi_{\text{BCS}}\rangle \tag{11.60}$$

and

$$v_{\mathbf{k}} |\Psi_{\mathbf{k}1}\rangle + u_{\mathbf{k}}^* |\Psi_{\mathbf{k}2}\rangle = \left(v_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger + u_{\mathbf{k}}^* c_{-\mathbf{k},\downarrow}\right) |\psi_{\text{BCS}}\rangle = \gamma_{-\mathbf{k},\downarrow} |\psi_{\text{BCS}}\rangle \tag{11.61}$$

with $u_{\mathbf{k}}, v_{\mathbf{k}}$ defined as above. (A lengthy but straightforward calculation has been omitted.) We can now understand why the second eigenvalue comes out negative: The corresponding eigenstate contains a quasiparticle *annihilation* operator, not a *creation* operator. Hence, $\mathcal{H}_{\text{BdG}}(\mathbf{k})$ reproduces the excitation energies we already know.

The next step is to Fourier-transform the Hamiltonian to obtain its real-space representation, which we write as

$$\mathcal{H}_{\text{BdG}}(\mathbf{r}) := \frac{1}{N}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\, \mathcal{H}_{\text{BdG}}(\mathbf{k}) = \begin{pmatrix} H_0(\mathbf{r}) & -\Delta(\mathbf{r}) \\ -\Delta^*(\mathbf{r}) & -H_0(\mathbf{r}) \end{pmatrix}, \tag{11.62}$$

where we expect

$$H_0(\mathbf{r}) = -\frac{\hbar^2}{2m}\nabla^2 - \mu + V(\mathbf{r}) \tag{11.63}$$

as the free-electron Hamiltonian. But in this form it becomes easy to include spatially inhomogeneous situations: Both $V(\mathbf{r})$ and $\Delta(\mathbf{r})$ can be chosen spatially dependent (and not simply lattice-periodic). The corresponding Schrödinger equation

$$\mathcal{H}_{\text{BdG}}(\mathbf{r})\,\Psi(\mathbf{r}) = E\,\Psi(\mathbf{r}) \tag{11.64}$$

with

$$\Psi(\mathbf{r}) = \begin{pmatrix} \Psi_1(\mathbf{r}) \\ \Psi_2(\mathbf{r}) \end{pmatrix} \tag{11.65}$$

is called the *Bogoliubov-de Gennes equation*. Note that in this context the gap $\Delta(\mathbf{r})$ is usually defined with the opposite sign, which is just a phase change, so that the explicit minus signs in the off-diagonal components of $\mathcal{H}_{\text{BdG}}$ are removed. Furthermore, in Bogoliubov-de Gennes theory, the gap function is typically not evaluated selfconsistantly from the averages $\langle c_{-\mathbf{k},\downarrow} c_{\mathbf{k}\uparrow}\rangle$. Rather, $\Delta(\mathbf{r})$ is treated as a given function characterizing the tendency of superconducting pairing.

11.4 Andreev reflection

As an application of the Bogoliubov-de Gennes approach, we study what happens to an electron that impinges on a normal-superconducting interface from the normal side. We model this situation by the Bogoliubov-de Gennes Hamiltonian

$$\mathcal{H}_{\text{BdG}} = \begin{pmatrix} -\frac{\hbar^2}{2m}\nabla^2 - \mu & \Delta_0\Theta(x) \\ \Delta_0\Theta(x) & \frac{\hbar^2}{2m}\nabla^2 + \mu \end{pmatrix} \tag{11.66}$$

(note the changed sign of $\Delta(\mathbf{r})$) so that

$$\mathcal{H}_{\text{BdG}}\,\Psi(\mathbf{r}) = E\,\Psi(\mathbf{r}). \tag{11.67}$$

In the normal region, $x < 0$, the two components $\Psi_1(\mathbf{r}), \Psi_2(\mathbf{r})$ are just superpositions of plane waves with wave vectors $\mathbf{k}_1, \mathbf{k}_2$ that must satisfy

$$k_1^2 = 2m(\mu + E) = k_F^2 + 2mE, \tag{11.68}$$

$$k_2^2 = 2m(\mu - E) = k_F^2 - 2mE, \tag{11.69}$$