

# ENG2009 – Modelling of Engineering Systems

## Tutorial 7

### ODE – Euler and Runge Kutta 2<sup>nd</sup> and 4<sup>th</sup> order

#### Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. (Thermal system)

Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8), \theta(0) = 1200K$$

Find the temperature at  $t = 480$  sec using Euler's method.

Assume a step size of  $h = 240$  sec



#### Solution

Step 2:

For

$$i = 1, t_1 = 240, \theta_1 = 106.09$$

Therefore

$$\begin{aligned} \theta_2 &= \theta_1 + f(t_1, \theta_1)h \\ &= 106.09 + f(240, 106.09)240 \\ &= 106.09 + (-2.2067 \times 10^{-12}(106.09^4 - 81 \times 10^8))240 \\ &= 106.09 + (0.017595)240 \\ &= 110.32K \end{aligned}$$

where  $\theta_2$  is the approximate temperature at

$$t = t_2 = t_1 + h = 240 + 240 = 480$$

Therefore

$$\theta(480) \approx \theta_2 = 110.32K$$

#### Solution

Step 1:

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

where

$$f(t, \theta) = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

Therefore

$$\begin{aligned} \theta_{i+1} &= \theta_i + f(t_i, \theta_i)h \\ \theta_1 &= \theta_0 + f(t_0, \theta_0)h \\ &= 1200 + f(0, 1200)240 \\ &= 1200 + (-2.2067 \times 10^{-12}(1200^4 - 81 \times 10^8))240 \\ &= 1200 + (-4.5579)240 \\ &= 106.09K \end{aligned}$$

where  $\theta_1$  is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240$$

Therefore

$$\theta(240) \approx \theta_1 = 106.09K$$

#### Solution continued...

Note that the exact solution of the ordinary differential equation is given by the solution of a non-linear eq. as

$$\begin{aligned} 0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.00333\theta) \\ = -0.22067 \times 10^{-3}t - 2.9282 \end{aligned}$$

The solution to this nonlinear equation at  $t = 480$  sec is:

$$\theta_{exact}(480) = 647.57K$$

vs numerical solution:

$$\theta_{euler}(480) \approx \theta_2 = 110.32K$$

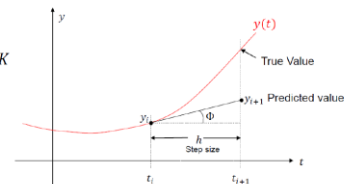


Figure 2. General graphical interpretation of Euler's method

### Exact vs Numerical Solutions

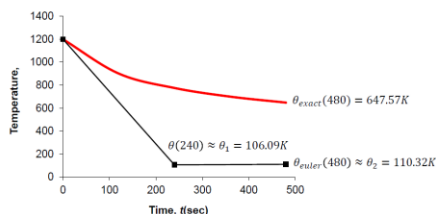


Figure 3. Comparing exact and Euler's method

### Effect of step size, $h$

Table 1. Temperature at 480 seconds as a function of step size,  $h$

Step, $h$	No of sim. data	$\theta(480)$	$E_t$ (error)	$ e_t \%$
480	1	-987.81	1635.4	252.54
240	2	110.32	537.26	82.964
120	3	546.77	100.80	15.566
60	4	614.97	32.607	5.0352
30	5	632.77	14.806	2.2864

exact solution:  $\theta(480) = 647.57K$

Example calculation:

$$E_t = 647.57 - 110.32 = 537.26$$

$$\begin{aligned} |e_t|\% &= \left| \frac{E_t}{\text{exact solution}} \times 100 \right| \\ &= \left| \frac{537.26}{647.57} \times 100 \right| = 82.964 \end{aligned}$$

## Comparison with exact results

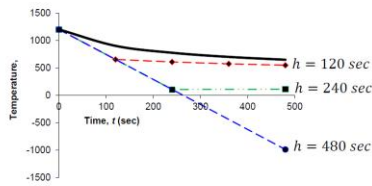


Figure 4. Comparison of Euler's method with exact solution for different step sizes

The smaller the step size  $h$ ,  
the closer to the numerical solution  
to the exact solution

## Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K.

Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8), \theta(0) = 1200K$$

Find the temperature at  $t = 480$  sec using Heun's method.

Assume step size of  $h = 240$  sec.



## Comparison with exact results

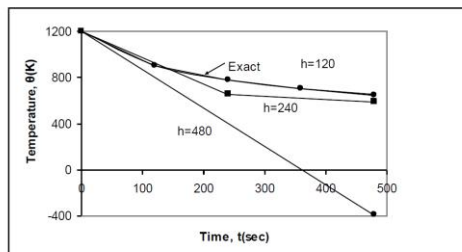


Figure 2. Heun's method results for different step sizes

## Euler vs Runge-Kutta 2<sup>nd</sup> Order Methods

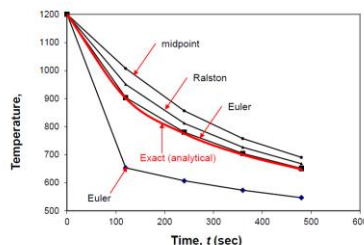


Figure 4. Comparison of Euler and Runge Kutta 2<sup>nd</sup> order methods with exact results. ( $h = 120$  sec)

## Effects of step size $h$ on Euler's Method

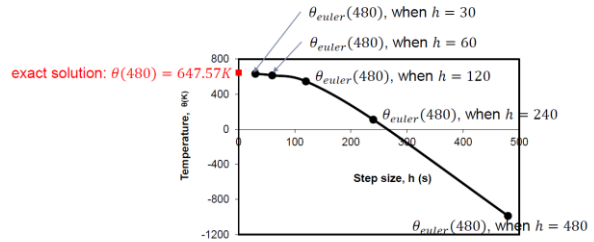


Figure 5. Effect of step size in Euler's method.

Temp  $\theta$  at 480 sec i.e.  $\theta(480)$  for various step size  $h$

## Solution

Given

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

therefore

$$f(t, \theta) = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

Heun's method:

$$\theta_{i+1} = \theta_i + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

## Euler vs Runge-Kutta 2<sup>nd</sup> Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, $h$	$ \epsilon_t  \%$			
	Euler	Heun	Midpoint	Ralston
480	252.54	160.82	86.612	30.544
240	82.964	9.7756	50.851	6.5537
120	15.566	0.58313	6.5823	3.1092
60	5.0352	0.36145	1.1239	0.72299
30	2.2864	0.097625	0.22353	0.15940

$$\theta_{exact}(480) = 647.57K \quad (\text{exact})$$

## Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K.

Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8), \theta(0) = 1200K$$

Find the temperature at  $t = 480$  sec using Runge-Kutta 4<sup>th</sup> order method.

Assume step size of  $h = 240$  sec.



## Solution

Given

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

therefore

$$f(t, \theta) = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

Runge-Kutta 4<sup>th</sup> order method:

$$\begin{aligned} y_{i+1} &= y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\ k_1 &= f(x_i, y_i) \\ k_2 &= f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right) \\ k_3 &= f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h\right) \\ k_4 &= f(x_i + h, y_i + k_3h) \end{aligned}$$

## Comparison with exact results

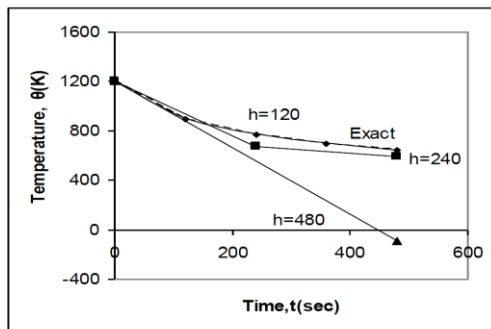


Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

## Exact solution

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$\begin{aligned} &0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.0033333\theta) \\ &= -0.22067 \times 10^{-3}t - 2.9282 \end{aligned}$$

The solution to this nonlinear equation at  $t = 480$  sec is

$$\theta_{\text{exact}}(480) = 647.57K$$

Compared with Runge-Kutta 4<sup>th</sup> order method:

$$\theta_{RK4}(480) = 594.91K$$

## Effects of $h$ on Runge-Kutta 4<sup>th</sup> Order Method

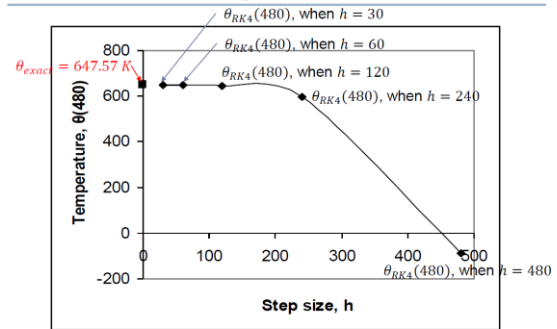


Figure 2. Effect of step size in Runge-Kutta 4th order method

## Comparison of Euler and Runge-Kutta Methods

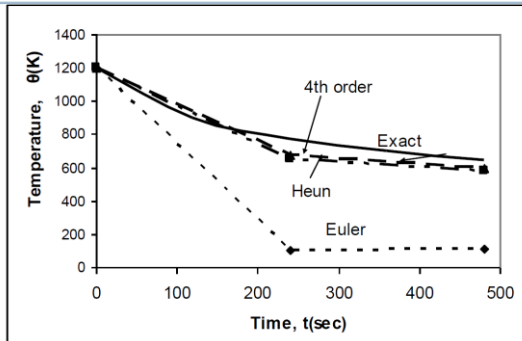


Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order ( $h = 240$  sec)

---

## Exercises

---

### Q1) Civil engineering example:

A polluted lake has an initial concentration of a bacteria of  $10^7$  parts/m<sup>3</sup>, while the acceptable level is only  $5 \times 10^6$  parts/m<sup>3</sup>. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration  $C$  of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt} + 0.06C = 0, \quad C(0) = 10^7$$

Using a step size of 3.5 weeks, find the concentration of the pollutant after 7 weeks. Use the following methods

- (a) Euler's methods
- (b) Heun's method
- (c) Runge-Kutta 4<sup>th</sup> order methods