ENG2009 – Modelling of Engineering Systems

Tutorial 6

<u>Simultaneous Linear Equations 1:</u>

<u>Naïve Gauss elimination:</u>

Example :

Example 1

The upward velocity of a rocket is given at three different times

Table 1 Velocity vs. time data.

Time t (sec)	Velocity v (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 , \qquad 5 \le t \le 12.$$

Find the velocity at $t = 6$ seconds .

Answer:

Example 1 Cont.

Rewrite as

$$t^2 a_1 + t a_2 + a_3 = v(t), \quad 5 \le t \le 12$$

Results in a matrix template of the form:

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Using data from Table 1, the matrix becomes:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

1.Forward Elimination
2.Back Substitution

Number of Steps of Forward Elimination

From

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

- 3 equations, 3 unknown
- Solve unknown: a_1, a_2, a_3
- Number of steps of forward elimination is

$$(n-1) = (3-1) = 2$$

Forward Elimination: Step 1

Idea: eliminate 64 from Equation 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64, i.e. $\frac{64}{25} = 2.56$

$$[25 \quad 5 \quad 1 \quad \vdots \quad 106.8] \times 2.56 = [64 \quad 12.8 \quad 2.56 \quad \vdots \quad 273.408]$$

Subtract the result from Equation 2

$$\begin{array}{r} [64 \quad 8 \quad 1 \quad \vdots \quad 177.2] \\ -[64 \quad 12.8 \quad 2.56 \quad \vdots \quad 273.408] \\ \hline [0 \quad -4.8 \quad -1.56 \quad \vdots \quad -96.208] \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

Idea: eliminate 144 from Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, i.e. $\frac{144}{25} = 5.7$

$$[25 \quad 5 \quad 1 \quad \vdots \quad 106.8] \times 5.76 = [144 \quad 28.8 \quad 5.76 \quad \vdots \quad 615.168]$$

Subtract the result from Equation 3

$$\begin{array}{r} [144 \quad 12 \quad 1 \quad \vdots \quad 279.2] \\ -[144 \quad 28.8 \quad 5.76 \quad \vdots \quad 615.168] \\ \hline [0 \quad -16.8 \quad -4.76 \quad \vdots \quad -335.968] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Forward Elimination: Step 2

Idea: eliminate -16.8 from Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Divide Equation 2 by -4.8 and multiply it by -16.8, i.e. $\frac{-16.8}{-4.8} = 3.5$.

$$[0 \quad -4.8 \quad -1.56 \quad \vdots \quad -96.208] \times 3.5 = [0 \quad -16.8 \quad -5.46 \quad \vdots \quad -336.728]$$

Subtract the result from Equation 3

$$\begin{array}{r} [0 \quad -16.8 \quad -4.76 \quad \vdots \quad 335.968] \\ -[0 \quad -16.8 \quad -5.46 \quad \vdots \quad -336.728] \\ \hline [0 \quad 0 \quad 0.7 \quad \vdots \quad 0.76] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix}$$

Back Substitution

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.2 \\ 0 & 0 & 0.7 & \vdots & 0.7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_3

$0.7a_3 = 0.76$

Therefore

$$a_3 = \frac{0.76}{0.7} = 1.08571$$

Back Substitution (cont.)

Last slide: $a_3 = 1.08571$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_2

$-4.8a_2 - 1.56a_3 = -96.208$

Therefore

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8} = \frac{-96.208 + 1.56 \times 1.08571}{-4.8} = 19.6905$$

Back Substitution (cont.)

Last slide: $a_3 = 1.08571$, $a_2 = 19.6905$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.2 \\ 0.76 \end{bmatrix}$$

Solving for a_1

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25} = \frac{106.8 - 5 \times 19.6905 - 1.08571}{25} = 0.290472$$

Example 1 solution

Original problem formulation:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

The solution vector is $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$

The polynomial that passes through the three data points is then:

$$\begin{aligned} v(t) &= a_1 t^2 + a_2 t + a_3 \\ &= 0.290472t^2 + 19.6905t + 1.08571, \qquad 5 \le t \le 12 \end{aligned}$$

Therefore at 6 sec:

$$\begin{aligned} v(6) &= 0.290472(6)^2 + 19.6905(6) + 1.08571 \\ &= 129.686 \text{ m/s.} \end{aligned}$$

Gaussian elimination: Partial pivoting example

Example:

Recall: Example

The upward velocity of a rocket is given at three different times

Table 1 Velocity vs. time data.

Time t (sec)	Velocity v (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 , \qquad 5 \le t \le 12.$$

Find the velocity at $t = 6$ seconds,

using Gaussian elimination with partial pivoting.

Answer:

Recall: Example

Rewrite as

$$t^2 a_1 + t a_2 + a_3 = v(t), \quad 5 \le t \le 12$$

Results in a matrix template of the form:

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Using data from Table 1, the matrix becomes:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

1. Forward Elimination (switch row if required)
2. Back Substitution

Recall: Number of Steps of Forward Elimination

From

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

- 3 equations, 3 unknown
- Solve unknown: a_1, a_2, a_3
- Number of steps of forward elimination is
  $$(n-1) = (3-1) = 2$$

Forward Elimination: Step 1

From

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Examine absolute values of first column, first row and below.

$$|25|, |64|, |144|$$

- Largest absolute value is 144 and exists in row 3.
- Switch row 1 and row 3.

Therefore

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

Idea: eliminate 64 from Equation 2

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide Equation 1 by 144 and multiply it by 64, i.e. $\frac{64}{144} = 0.4444$

$$[144 \quad 12 \quad 1 \quad \vdots \quad 279.2] \times 0.4444 = [63.99 \quad 5.333 \quad 0.4444 \quad \vdots \quad 124.1]$$

Subtract the result from Equation 2

$$\begin{array}{r} [64 \quad 8 \quad 1 \quad \vdots \quad 177.2] \\ -[63.99 \quad 5.333 \quad 0.4444 \quad \vdots \quad 124.1] \\ \hline [0 \quad 2.667 \quad 0.5556 \quad \vdots \quad 53.10] \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

Idea: eliminate 25 from Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide Equation 1 by 144 and multiply it by 25, i.e. $\frac{25}{144} = 0.1736$

$$[144 \quad 12 \quad 1 \quad \vdots \quad 279.2] \times 0.1736 = [25.00 \quad 2.083 \quad 0.1736 \quad \vdots \quad 48.47]$$

Subtract the result from Equation 3

$$\begin{array}{r} [25 \quad 5 \quad 1 \quad \vdots \quad 106.8] \\ -[25 \quad 2.083 \quad 0.1736 \quad \vdots \quad 48.47] \\ \hline [0 \quad 2.917 \quad 0.8264 \quad \vdots \quad 58.33] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix}$$

Forward Elimination: Step 2

From last step

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix}$$

Examine absolute values of second column, second row and below:

$$|2.667|, |2.917|$$

- Largest absolute value is 2.917 and exists in row 3.
- Switch row 2 and row 3.

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Forward Elimination: Step 2 (cont.)

Idea: eliminate 2.667 from Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Divide Equation 2 by 2.917 and multiply it by 2.667, i.e. $\frac{2.667}{2.917} = 0.9143$.

$$[0 \quad 2.917 \quad 0.8264 \quad \vdots \quad 58.33] \times 0.9143 = [0 \quad 2.667 \quad 0.7556 \quad \vdots \quad 53.33]$$

Subtract the result from Equation 3

$$\begin{array}{r} [0 \quad 2.667 \quad 0.5556 \quad \vdots \quad 53.10] \\ -[0 \quad 2.667 \quad 0.7556 \quad \vdots \quad 53.33] \\ \hline [0 \quad 0 \quad -0.2 \quad \vdots \quad -0.23] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix}$$

Back Substitution

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_3

$$-0.2a_3 = -0.23$$

Therefore

$$a_3 = \frac{-0.23}{-0.2} = 1.15$$

Back Substitution (cont.)

Last slide: $a_3 = 1.15$

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_2

$$2.917a_2 + 0.8264a_3 = 58.33$$

$$a_2 = \frac{58.33 - 0.8264a_3}{2.917} = \frac{58.33 - 0.8264 \times 1.15}{2.917} = 19.67$$

Back Substitution (cont.)

Last slide: $a_3 = 1.15$, $a_2 = 19.67$

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_1

$$144a_1 + 12a_2 + a_3 = 279.2$$

$$a_1 = \frac{279.2 - 12a_2 - a_3}{144} = \frac{279.2 - 12 \times 19.67 - 1.15}{144} = 0.2917$$

Gaussian Elim. with Partial Pivoting Solution

Original problem

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solution (similar to Naïve Gaussian elimination)

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2917 \\ 19.67 \\ 1.15 \end{bmatrix}$$

The polynomial that passes through the three data points is then:

$$v(t) = a_1t^2 + a_2t + a_3 = 0.2917t^2 + 19.67t + 1.15, \qquad 5 \le t \le 12$$

Therefore at 6 sec:

$$v(6) = 0.2917(6)^2 + 19.67(6) + 1.15 = 129.6712 \text{ m/s}$$ (similar to Naïve Gaussian elimination)

LU decomposition

Example:

Using LU Decomposition to solve SLEs

Example:
Solve the following set of linear equations using LU Decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$[A] \quad [X] = [C]$$

Using the procedure for finding the $[L]$ and $[U]$ matrices

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Answer:

Example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$[A] \quad [X] = [C]$$

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Set $[L][Z] = [C]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solve for $[Z]$

$$z_1 = 106.8$$
$$2.56z_1 + z_2 = 177.2$$
$$5.76z_1 + 3.5z_2 + z_3 = 279.2$$

Example

Solve for $[Z]$

$$z_1 = 106.8$$
$$2.56z_1 + z_2 = 177.2$$
$$5.76z_1 + 3.5z_2 + z_3 = 279.2$$

Complete the forward substitution to solve for $[Z]$

$$\begin{aligned} z_1 &= 106.8 \\ z_2 &= 177.2 - 2.56z_1 \\ &= 177.2 - 2.56(106.8) \\ &= -96.208 \\ z_3 &= 279.2 - 5.76z_1 - 3.5z_2 \\ &= 279.2 - 5.76(106.8) - 3.5(-96.208) \\ &= 0.76 \end{aligned}$$

Therefore

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Example

Set $[U][X] = [Z]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solve for $[X]$

The 3 equations become

$$25x_1 + 5x_2 + x_3 = 106.8$$
$$-4.8x_2 - 1.56x_3 = -96.208$$
$$0.7x_3 = 0.76$$

Example

From the 3rd equation

$$\begin{aligned} 0.7x_3 &= 0.76 \\ x_3 &= \frac{0.76}{0.7} \\ x_3 &= 1.0857 \end{aligned}$$

Substituting in x_3 and using the second equation

$$-4.8x_2 - 1.56x_3 = -96.208$$

$$\begin{aligned} x_2 &= \frac{-96.21 + 1.56x_3}{-4.8} \\ &= \frac{-96.21 + 1.56(1.0857)}{-4.8} \\ &= 19.691 \end{aligned}$$

Example

Substituting in x_3 and x_2 using the first equation

$$25x_1 + 5x_2 + x_3 = 106.8$$

$$\begin{aligned} x_1 &= \frac{106.8 - 5x_2 - x_3}{25} \\ &= \frac{106.8 - 5(19.691) - 1.0857}{25} \\ &= 0.29048 \end{aligned}$$

Hence the Solution Vector is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.29048 \\ 19.691 \\ 1.0857 \end{bmatrix}$$

Exercises

Q 1) (Kaw & Kalu)

A trunnion of diameter $12.363''$ has to be cooled from a room temperature of $80°F$ before it is shrink fitted into a steel hub (Figure 1).

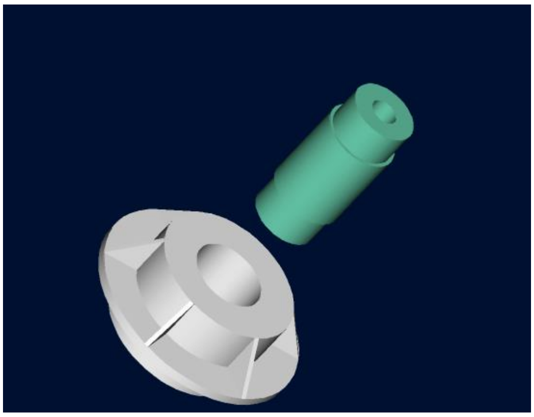

Figure 1 Trunnion to be slid through the hub after contracting.

The equation that gives the diametric contraction ΔD of the trunnion in a dry-ice/alcohol mixture (boiling temperature is $-108°F$) is given by

$$\Delta D = 12.363 \int_{80}^{-108} \alpha(T) dT$$

The equation for the thermal expansion coefficient, $\alpha = a_1 + a_2 T + a_3 T^2$, is obtained using regression analysis where the constants of the model are found by solving the following simultaneous linear equations.

$$\begin{bmatrix} 24 & -2860 & 7.26 \times 10^5 \\ -2860 & 7.26 \times 10^5 & -1.86472 \times 10^8 \\ 7.26 \times 10^5 & -1.86472 \times 10^8 & 5.24357 \times 10^{10} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057 \times 10^{-4} \\ -1.04162 \times 10^{-2} \\ 2.56799 \end{bmatrix}$$

Find the values of a_1, a_2, and a_3 using

a) naïve Gaussian elimination,
b) partial-pivoting Gaussian elimination,
c) LU decomposition

Solutions:

Q 1) (a) Solve using Naïve Gaussian elimination

Since there are three equations, there will be two steps of forward elimination of unknowns.

Original:

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}=\begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

First step

Divide Row 1 by 24 and then multiply it by −2860, that is, multiply Row 1 by $-2860/24 = -119.1667$.

$$\text{Row 1} \times (-119.1667) = [-2860 \quad 3.4081\times10^{5} \quad -8.6515\times10^{7}][-0.012596]$$

Subtract the result from Row 2 to get

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ 0 & 3.8518\times10^{5} & -9.9957\times10^{7} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.2436\times10^{10} \end{bmatrix}\begin{bmatrix} 1.057\times10^{-4} \\ 2.1797\times10^{-3} \\ 2.56799 \end{bmatrix}$$

Divide Row 1 by 24 and then multiply it by 7.26×10^{5}, that is, multiply Row 1 by $7.26\times10^{5}/24 = 30250$.

$$\text{Row 1} \times (30250) = [7.26\times10^{5} \quad -8.6515\times10^{7} \quad 2.1962\times10^{10}][3.1974]$$

Subtract the result from Row 3 to get

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ 0 & 3.8518\times10^{5} & -9.9957\times10^{7} \\ 0 & -9.9957\times10^{7} & 3.04742\times10^{10} \end{bmatrix}\begin{bmatrix} 1.057\times10^{-4} \\ 2.1797\times10^{-3} \\ -0.629435 \end{bmatrix}$$

Second step

We now divide Row 2 by 3.8518×10^{5} and then multiply it by -9.9957×10^{7}, that is, multiply Row 2 by $-9.9957\times10^{7}/3.8518\times10^{5} = -2.5950\times10^{2}$.

$$\text{Row 2} \times (-2.5950\times10^{2}) = [0 \quad -9.9957\times10^{7} \quad 2.5939\times10^{10}][-0.5656]$$

Subtract the result from Row 3 to get

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ 0 & 3.8518\times10^{5} & -9.9957\times10^{7} \\ 0 & 0 & 4.5348\times10^{9} \end{bmatrix}\begin{bmatrix} 1.057\times10^{-4} \\ 2.1797\times10^{-3} \\ -0.0638 \end{bmatrix}$$

Back Substitution

From the third equation,

$$4.5348 \times 10^{9} a_3 = -0.0638$$

$$\begin{aligned} a_3 &= \frac{-0.0638}{4.5348 \times 10^{9}} \\ &= -1.4066 \times 10^{-11} \end{aligned}$$

Substituting the value of a_3 in the second equation,

$$3.8518 \times 10^{5} a_2 + (-9.9957 \times 10^{7}) a_3 = 2.1797 \times 10^{-3}$$

$$\begin{aligned} a_2 &= \frac{2.1797 \times 10^{-3} - (-9.9957 \times 10^{7}) a_3}{3.8518 \times 10^{5}} \\ &= \frac{2.1797 \times 10^{-3} - (-9.9957 \times 10^{7}) \times (-1.4066 \times 10^{-11})}{3.8518 \times 10^{5}} \\ &= 2.0087 \times 10^{-9} \end{aligned}$$

Substituting the values of a_2 and a_3 in the first equation,

$$24 a_1 + (-2860) a_2 + 7.26 \times 10^{5} a_3 = 1.057 \times 10^{-4}$$

$$\begin{aligned} a_1 &= \frac{1.057 \times 10^{-4} - (-2860) a_2 - 7.26 \times 10^{5} a_3}{24} \\ &= \frac{1.057 \times 10^{-4} - (-2860) \times (2.0086 \times 10^{-9}) - 7.26 \times 10^{5} \times (-1.4066 \times 10^{-11})}{24} \\ &= 5.0690 \times 10^{-6} \end{aligned}$$

Hence the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5.0690 \times 10^{-6} \\ 2.0087 \times 10^{-9} \\ -1.4066 \times 10^{-11} \end{bmatrix}$$

Q 1) (b) Solve using partial pivoting Gaussian elimination

Since there are three equations, there will be two steps of forward elimination of unknowns.

Similar procedure to Naïve Gaussian elimination except for checking the step to examine absolute values of column of interest and swap rows (largest value on top).

Original:

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

Examine absolute values of first column, first row and below.

$|24|, |-2860|, |7.26\times10^{5}|$

Largest absolute value is $7.26\times10^{5}$and exists in row 3.

Switch row 1 and row 3 to yield

$$\begin{bmatrix} 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 24 & -2860 & 7.26\times10^{5} \end{bmatrix} \begin{bmatrix} 2.56799 \\ -1.04162\times10^{-2} \\ 1.057\times10^{-4} \end{bmatrix}$$

First step

Divide Row 1 by $7.26\times10^{5}$and then multiply it by −2860, that is, multiply Row 1 by $-2860/7.26\times10^{5} = -0.0039$.

$$\text{Row 1}\times(-0.0039) = [-2860 \quad 7.3459\times10^{5} \quad -2.0656\times10^{8}][-0.0101]$$

Subtract the result from Row 2 to get

$$\begin{bmatrix} 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \\ 0 & -8.5866\times10^{3} & 2.00928\times10^{7} \\ 24 & -2860 & 7.26\times10^{5} \end{bmatrix} \begin{bmatrix} 2.56799 \\ -2.9988\times10^{-4} \\ 1.057\times10^{-4} \end{bmatrix}$$

Divide Row 1 by 7.26×10^{5} and then multiply it by 24, that is, multiply Row 1 by $24/7.26\times10^{5} = 3.3058\times10^{-5}$.

$$\text{Row 1}\times(3.3058\times10^{-5}) = [24 \quad -6.1644\times10^{3} \quad 1.7334\times10^{6}][8.4892\times10^{-5}]$$

Subtract the result from Row 3 to get

$$\begin{bmatrix} 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \\ 0 & -8.5866\times10^{3} & 2.00928\times10^{7} \\ 0 & 3.3043\times10^{3} & -1.0074 \end{bmatrix} \begin{bmatrix} 2.56799 \\ -2.9988\times10^{-4} \\ 2.0808\times10^{-5} \end{bmatrix}$$

Second step

From

$$\begin{bmatrix} 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \\ 0 & -8.5866\times10^{3} & 2.00928\times10^{7} \\ 0 & 3.3043\times10^{3} & -1.0074 \end{bmatrix}\begin{bmatrix} 2.56799 \\ -2.9988\times10^{-4} \\ 2.0808\times10^{-5} \end{bmatrix}$$

Examine absolute values of second column, second row and below.

$|-8.5866\times10^{3}|,|3.3043\times10^{3}|$

Largest absolute value is 8.5866×10^{3} and exists in the existing row 2.

No need to swap rows.

We now divide Row 2 by -8.5866×10^{3} and then multiply it by 3.3043×10^{3}, that is, multiply Row 2 by $3.3043\times10^{3}/-8.5866\times10^{3}=-0.3848$.

$$\text{Row } 2\times(-0.3848)=[0 \quad 3.3044\times10^{3} \quad -7.7322\times10^{6}][1.1540\times10^{-4}]$$

Subtract the result from Row 3 to get

$$\begin{bmatrix} 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \\ 0 & -8.5866\times10^{3} & 2.00928\times10^{7} \\ 0 & 0 & 6.7248\times10^{6} \end{bmatrix}\begin{bmatrix} 2.56799 \\ -2.9988\times10^{-4} \\ -9.4592\times10^{-5} \end{bmatrix}$$

Back Substitution

From the third equation,

$$6.7248\times10^{6}a_3=-9.4592\times10^{-5}$$

$$a_3=\frac{-9.4592\times10^{-5}}{6.7248\times10^{6}}$$
$$=-1.4066\times10^{-11}$$

Substituting the value of a_3 in the second equation,

$$-8.5866\times10^{3}a_2+2.00928\times10^{7}a_3=-2.9988\times10^{-4}$$

$$a_2=\frac{-2.9988\times10^{-4}-2.00928\times10^{7}a_3}{-8.5866\times10^{3}}$$
$$=2.0087\times10^{-9}$$

Substituting the values of a_2 and a_3 in the first equation,

$$7.26\times10^{5}a_1+(-1.86472\times10^{8})a_2+5.24357\times10^{10}a_3=2.56799$$

$$a_1 = \frac{2.56799 - (-1.86472 \times 10^8)a_2 - 5.24357 \times 10^{10} a_3}{7.26 \times 10^5}$$
$$= 5.0690 \times 10^{-6}$$

Hence the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5.0690 \times 10^{-6} \\ 2.0087 \times 10^{-10} \\ -1.4066 \times 10^{-11} \end{bmatrix}$$

Q 1) (c) Solve using LU decomposition

Original:

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^5 \\ -2860 & 7.26\times10^5 & -1.86472\times10^8 \\ 7.26\times10^5 & -1.86472\times10^8 & 5.24357\times10^{10} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

First step: decompose matrix A into $[A] = [L][U]$

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- Find $[U]$
- Find $[L]$

$[U]$ is the same as the coefficient matrix at the end of the forward elimination step.
$[L]$ is obtained using the *multipliers* that were used in the forward elimination process

Finding U: Using the same Naïve Forward Elimination Procedure of Gauss Elimination:

Original:

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^5 \\ -2860 & 7.26\times10^5 & -1.86472\times10^8 \\ 7.26\times10^5 & -1.86472\times10^8 & 5.24357\times10^{10} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

First step

Divide Row 1 by 24 and then multiply it by −2860, that is, multiply Row 1 by $-2860/24 = -119.1667$.

$$\text{Row 1} \times (-119.1667) = [-2860 \quad 3.4081\times10^5 \quad -8.6515\times10^7][-0.012596]$$

Subtract the result from Row 2 to get

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^5 \\ 0 & 3.8518\times10^5 & -9.9957\times10^7 \\ 7.26\times10^5 & -1.86472\times10^8 & 5.2436\times10^{10} \end{bmatrix}\begin{bmatrix} 1.057\times10^{-4} \\ 2.1797\times10^{-3} \\ 2.56799 \end{bmatrix}$$

Divide Row 1 by 24 and then multiply it by 7.26×10^5, that is, multiply Row 1 by $7.26\times10^5/24 = 30250$.

$$\text{Row 1} \times (30250) = [7.26\times10^5 \quad -8.6515\times10^7 \quad 2.1962\times10^{10}][3.1974]$$

Subtract the result from Row 3 to get

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^5 \\ 0 & 3.8518\times10^5 & -9.9957\times10^7 \\ 0 & -9.9957\times10^7 & 3.04742\times10^{10} \end{bmatrix}\begin{bmatrix} 1.057\times10^{-4} \\ 2.1797\times10^{-3} \\ -0.629435 \end{bmatrix}$$

Second step

We now divide Row 2 by 3.8518×10^5 and then multiply it by -9.9957×10^7, that is, multiply Row 2 by $-9.9957 \times 10^7/3.8518 \times 10^5 = -2.5950 \times 10^2$.

$$\text{Row } 2 \times (-2.5950 \times 10^2) = [0 \quad -9.9957 \times 10^7 \quad 2.5939 \times 10^{10}][-0.5656]$$

Subtract the result from Row 3 to get

$$\begin{bmatrix} 24 & -2860 & 7.26 \times 10^5 \\ 0 & 3.8518 \times 10^5 & -9.9957 \times 10^7 \\ 0 & 0 & 4.5348 \times 10^9 \end{bmatrix} \begin{bmatrix} 1.057 \times 10^{-4} \\ 2.1797 \times 10^{-3} \\ -0.0638 \end{bmatrix}$$

Therefore

$$U = \begin{bmatrix} 24 & -2860 & 7.26 \times 10^5 \\ 0 & 3.8518 \times 10^5 & -9.9957 \times 10^7 \\ 0 & 0 & 4.5348 \times 10^9 \end{bmatrix}$$

Finding L: obtained using the *multipliers* that were used in the forward elimination process

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

From the original

$$A = \begin{bmatrix} 24 & -2860 & 7.26 \times 10^5 \\ -2860 & 7.26 \times 10^5 & -1.86472 \times 10^8 \\ 7.26 \times 10^5 & -1.86472 \times 10^8 & 5.24357 \times 10^{10} \end{bmatrix}$$

From the first step of forward elimination:
Divide Row 1 by 24 and then multiply it by −2860, that is, multiply Row 1 by $-2860/24 = -119.1667$

Therefore $l_{21} = -119.1667$

Divide Row 1 by 24 and then multiply it by 7.26×10^5, that is, multiply Row 1 by $7.26 \times 10^5/24 = 30250$.

Therefore $l_{31} = 30250$

From the first step of forward elimination:

$$\begin{bmatrix} 24 & -2860 & 7.26 \times 10^5 \\ 0 & 3.8518 \times 10^5 & -9.9957 \times 10^7 \\ 0 & -9.9957 \times 10^7 & 3.04742 \times 10^{10} \end{bmatrix} \begin{bmatrix} 1.057 \times 10^{-4} \\ 2.1797 \times 10^{-3} \\ -0.629435 \end{bmatrix}$$

We now divide Row 2 by 3.8518×10^5 and then multiply it by -9.9957×10^7, that is, multiply Row 2 by $-9.9957 \times 10^7/3.8518 \times 10^5 = -2.5950 \times 10^2$.

Therefore $l_{32} = -2.5950 \times 10^2$.

Therefore

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -119.1667 & 1 & 0 \\ 30250 & -2.5950 \times 10^2 & 1 \end{bmatrix}$$

Check $LU - A = 0$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ -119.1667 & 1 & 0 \\ 30250 & -2.5950\times10^{2} & 1 \end{bmatrix}\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ 0 & 3.8518\times10^{5} & -9.9957\times10^{7} \\ 0 & 0 & 4.5348\times10^{9} \end{bmatrix}\right) - \begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Second step

From $[A][X] = [C]$ where

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

And $[A] = [L][U]$

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -119.1667 & 1 & 0 \\ 30250 & -2.5950\times10^{2} & 1 \end{bmatrix}\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ 0 & 3.8518\times10^{5} & -9.9957\times10^{7} \\ 0 & 0 & 4.5348\times10^{9} \end{bmatrix}$$

Set $[L][Z] = [C]$

$$\begin{bmatrix} 1 & 0 & 0 \\ -119.1667 & 1 & 0 \\ 30250 & -2.5950\times10^{2} & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

Solve for $[Z]$

$$\begin{aligned} z_1 &= 1.057\times10^{-4} \\ -119.1667z_1 + z_2 &= -1.04162\times10^{-2} \\ 30250z_1 - 2.5950\times10^{3}z_2 + z_3 &= 2.56799 \end{aligned}$$

to yield

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1.0570\times10^{-4} \\ 0.0022 \\ -0.0638 \end{bmatrix}$$

Then set $[U][X] = [Z]$

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ 0 & 3.8518\times10^{5} & -9.9957\times10^{7} \\ 0 & 0 & 4.5348\times10^{9} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.0570\times10^{-4} \\ 0.0022 \\ -0.0638 \end{bmatrix}$$

Solve for $[X]$

$$
\begin{aligned}
24x_1 - 2860x_2 + 7.26 \times 10^5 x_3 &= 1.0570 \times 10^{-4} \\
3.8518 \times 10^5 x_2 - 9.9957 \times 10^7 x_3 &= 0.0022 \\
4.5348 \times 10^9 x_3 &= -0.0638
\end{aligned}
$$

to yield

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5.0690 \times 10^{-6} \\ 2.0087 \times 10^{-10} \\ -1.4066 \times 10^{-11} \end{bmatrix}
$$

Here $[X] = [a]$ therefore

$$
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5.0690 \times 10^{-6} \\ 2.0087 \times 10^{-10} \\ -1.4066 \times 10^{-11} \end{bmatrix}
$$

Remark:

Note that the solutions for a_1, a_2, a_3 are the same for the three methods (Naïve Gaussian Elimination, Gaussian elimination with partial pivoting and LU decomposition).

$$
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{naive\ GE} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5.0690 \times 10^{-6} \\ 2.0087 \times 10^{-9} \\ -1.4066 \times 10^{-11} \end{bmatrix}
$$

$$
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{partialPivotingGE} = \begin{bmatrix} 5.0690 \times 10^{-6} \\ 2.0087 \times 10^{-9} \\ -1.4066 \times 10^{-11} \end{bmatrix}
$$

$$
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{LU} = \begin{bmatrix} 5.0690 \times 10^{-6} \\ 2.0087 \times 10^{-10} \\ -1.4066 \times 10^{-11} \end{bmatrix}
$$

Python code example for Lecture 11-13: Rocket example

```
import numpy as np
import pandas as pd
from numpy import array,zeros
import pprint
import scipy
import scipy.linalg

####################################
## rocket example from Lecture 11
####################################

v=[[106.8000], [177.1999], [279.1997]]

A=np.array([[25, 5, 1],
[64, 8, 1],
[144, 12, 1]],dtype='float')

C=v

#check rank of A to ensure its nonsingular
np.linalg.matrix_rank(A) # check for division by 0 problem

#Augment the A and C for row manipulations
J=np.array([[25, 5, 1,106.8],
[64, 8, 1,177.1999],
[144, 12, 1,279.1997]],dtype='float')

# fwd elim 1
R1=J[0]/A[0,0]*A[1,0]
R2=J[1]-R1
J[1]=R2

R1=J[0]/A[0,0]*A[2,0]
R3=J[2]-R1
J[2]=R3


# fwd elim 2
R2=J[1]/J[1,1]*J[2,1]
R3=J[2]-R2
J[2]=R3


# back sub
C3=J[2,3]/J[2,2]
C2=(J[1,3]-(J[1,2]*C3))/J[1,1]
C1=(J[0,3]-(J[0,2]*C3)-(J[0,1]*C2))/J[0,0]

sln=([[C1],[C2],[C3]])
print('solution \n',sln)


# exact sln: solve x from A*x = C
exactSln=np.linalg.inv(A)@C
print('exact sloution \n',exactSln)

#part (b)
tu=6
v6=(C1*tu*tu)+(C2*tu)+C3
print('velocity at 6 sec: \n',v6)


####################################
## rocket example from Lecture 12
####################################
# gaussian elimination with partial pivotings: Lecture 12 example

v=[[106.8000], [177.1999], [279.1997]]

A=np.array([[25, 5, 1],
[64, 8, 1],
[144, 12, 1]],dtype='float')
```

```
C=v

#check rank of A to ensure its nonsingular
np.linalg.matrix_rank(A) # check for division by 0 problem

#augment the A and C for row manipulations
J=np.array([[25, 5, 1,106.8],
[64, 8, 1,177.1999],
[144, 12, 1,279.1997]],dtype='float')

#partial pivoting; check abs value of 1st col and swap row 1 and 3
#J=[J[2],J[1],J[0]]
J=np.vstack((J[2],J[1],J[0]))

#fwd elim 1
#R1=[0,0,0,0]
R1=J[0]/J[0,0]*J[1,0]
R2=J[1]-R1
J[1]=R2

#R1=[0,0,0,0]
R1=J[0]/J[0,0]*J[2,0]
R3=J[2]-R1
J[2]=R3

# partial pivoting; check abs value of 1st col and swap row 2 and 3
#J=[J[0],J[2],J[1]]
J=np.vstack((J[0],J[2],J[1]))

#fwd elim 2
#R2=[0,0,0,0]
R2=J[1]/J[1,1]*J[2,1]
R3=J[2]-R2
J[2]=R3

# back sub
C3=J[2,3]/J[2,2]
C2=(J[1,3]-(J[1,2]*C3))/J[1,1]
C1=(J[0,3]-(J[0,2]*C3)-(J[0,1]*C2))/J[0,0]


sln=([[C1],[C2],[C3]])
print('solution \n',sln)


# exact sln: solve x from A*x = C
exactSln=np.linalg.inv(A)@C
print('exact sloution \n',exactSln)

#part (b)
tu=6
v6=(C1*tu*tu)+(C2*tu)+C3
print('velocity at 6 sec: \n',v6)


###################################
# rocket example from Lecture 13
###################################
#LU decomposition:  Lecture 13 example

v=[[106.8000], [177.1999], [279.1997]]

#matrix form
A=np.array([[25, 5, 1],
[64, 8, 1],
[144, 12, 1]],dtype='float')

C=v

#check that matrix nonsingular and can be solved
np.linalg.matrix_rank(A)

def LU_decomposition(A):
    n = A.shape[0]
    U = A.copy()
    L = np.eye(n, dtype=np.double)
    #Loop over rows
```

```
    for i in range(n):
        #Eliminate entries below i with row operations
        #on U and reverse the row operations to
        #manipulate L
        factor = U[i+1:, i] / U[i, i]
        L[i+1:, i] = factor
        U[i+1:] -= factor[:, np.newaxis] * U[i]
    return L, U

#actual LU decomposition
[L,U] = LU_decomposition(A)
print('L \n',L)
print('U \n',U)

#to check LU=A
(L@U)-A

#LZ=C, solve for Z
L
C
z1=C[0]/L[0,0]
z2=C[1]-(L[1,0]*z1)
z3=C[2]-(L[2,0]*z1)-(L[2,1]*z2)

z=[z1,z2,z3]
Z=np.transpose(z)
Z=np.vstack((z1,z2,z3))
print('sloution Z \n',Z)


#UX=Z, solve for X
U
Z
x3=Z[2]/U[2,2]
x2=(Z[1]-(U[1,2]*x3))/(U[1,1])
x1=(Z[0]-(U[0,1]*x2)-(U[0,2]*x3))/(U[0,0])

#final solution
sln=np.vstack((x1, x2, x3))
print('sloution \n',sln)

#solve x from A*x = C
exactSln=np.linalg.inv(A)@C
print('exact sloution \n',exactSln)

#part (b)
tu=6
v6=(sln[0]*tu*tu)+(sln[1]*tu)+sln[2]
print('velocity at 6 sec: \n',v6)
```

Outputs

```
solution
 [[0.2904738095238099], [19.690473809523816], [1.0857857142856844]]
exact sloution
 [[ 0.29047381]
 [19.69047381]
 [ 1.08578571]]
velocity at 6 sec:
 129.68568571428574

solution
 [[0.2904738095238094], [19.690473809523812], [1.0857857142856995]]
exact sloution
 [[ 0.29047381]
 [19.69047381]
 [ 1.08578571]]
velocity at 6 sec:
 129.6856857142857
```

```
L
 [[1.   0.   0.  ]
 [2.56 1.   0.  ]
 [5.76 3.5  1.  ]]
U
 [[25.    5.    1.  ]
 [ 0.   -4.8  -1.56]
 [ 0.    0.    0.7 ]]
slou tion Z
 [[106.8    ]
 [-96.2081 ]
 [  0.76005]]
sloution
 [[ 0.29047381]
 [19.69047381]
 [ 1.08578571]]
exact sloution
 [[ 0.29047381]
 [19.69047381]
 [ 1.08578571]]
velocity at 6 sec:
 [129.68568571]
```