ENG2009 – Modelling of Engineering Systems

Tutorial 8

Finite Difference and Shooting Method

Finite Difference

Example

Consider a thick pressure vessel (thickness t) that is being tested in the laboratory to check its ability to withstand pressure.

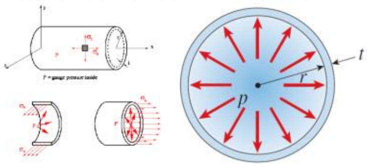
The inner radius $r = a$ and outer radius $r = b$.

The differential eq. for the radial displacement u of a point along the thickness is given by:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

Assume

- the inner radius $a = 5$ and the outer radius $b = 8$,
- the material of the pressure vessel is ASTM A36 steel with the yield strength of 36 ksi.



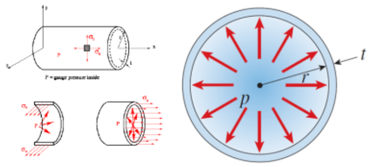
Example

Two strain gages that are bonded tangentially at the inner and the outer radius measure normal tangential strain which translates to radial displacements at a and b given by (boundary conditions):

$$u|_{r=a} = 0.003871 \text{ and } u|_{r=b} = 0.0030769$$

Task:

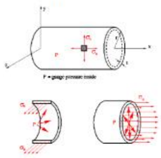
- Divide the radial thickness of the pressure vessel into 6 equidistant nodes,
- and find the radial displacement profile u along the thickness of the pressure vessel



Finite Difference Method

An example of a boundary value ordinary differential equation is (pressure vessel)

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$



with boundary conditions:

$$u(5) = 0.008731,$$
$$u(8) = 0.0030769$$

where u is the radial displacement, r is the radius

In general, the derivatives in such ordinary differential equation are substituted by finite divided differences approximations:

$$\frac{dy}{dx} \approx \frac{y_{i+1} - y_i}{\Delta x}$$
$$\frac{d^2y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}$$

Solution Cont

The example before uses the finite divided differences approximations

$$\frac{d^2u}{dr^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2}$$
$$\frac{du}{dr} \approx \frac{u_{i+1} - u_i}{\Delta r}$$

Which is first order accurate.

A better approximate is:

$$\frac{d^2u}{dr^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2}$$
$$\frac{du}{dr} \approx \frac{u_{i+1} - u_{i-1}}{2\Delta r}$$

Which is second order accurate.

Solution Cont

From

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0$$

And using the second order accurate approximation yield

$$\frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta r)^2}+\frac{1}{r_i}\frac{u_{i+1}-u_{i-1}}{2(\Delta r)}-\frac{u_i}{r_i{}^2}=0$$

which yield

$$\left(-\frac{1}{2r_i(\Delta r)}+\frac{1}{(\Delta r)^2}\right)u_{i-1}+\left(-\frac{2}{(\Delta r)^2}-\frac{1}{r_i{}^2}\right)u_i+\left(\frac{1}{(\Delta r)^2}+\frac{1}{2r_i\Delta r}\right)u_{i+1}=0 \text{ (eq**)}$$

vs

$$\frac{1}{(\Delta r)^2}u_{i-1}+\left(-\frac{2}{(\Delta r)^2}-\frac{1}{r_i\Delta r}-\frac{1}{r_i{}^2}\right)u_i+\left(\frac{1}{(\Delta r)^2}+\frac{1}{r_i\Delta r}\right)u_{i+1}=0 \quad \text{(eq*)}$$

Repeat the example with the second order accurate approximation…

Solution

- **Step 1 at node $i=0$,**

$$r_0=a=5,$$

$$u_0=u|_{r=a}=0.0038731 \text{ (boundary condition at } a\text{)}$$

- **Step 2 at node $i=1$,**

$$r_1=r_0+\Delta r=5+0.6=5.6$$

Therefore using $\left(-\frac{1}{2r_i(\Delta r)}+\frac{1}{(\Delta r)^2}\right)u_{i-1}+\left(-\frac{2}{(\Delta r)^2}-\frac{1}{r_i{}^2}\right)u_i+\left(\frac{1}{(\Delta r)^2}+\frac{1}{2r_i\Delta r}\right)u_{i+1}=0$ (eq**)

$$\left(-\frac{1}{2(5.6)(0.6)}+\frac{1}{(0.6)^2}\right)u_0+\left(-\frac{2}{(0.6)^2}-\frac{1}{(5.6)^2}\right)u_1+\left(\frac{1}{0.6^2}+\frac{1}{2(5.6)(0.6)}\right)u_2=0$$

$$2.6297u_0-5.5874u_1+2.9266u_2=0$$

- **Step 3 at node $i=2$,**

$$r_2=r_1+\Delta r=5.6+0.6=6.2$$

Therefore using $\left(-\frac{1}{2r_i(\Delta r)}+\frac{1}{(\Delta r)^2}\right)u_{i-1}+\left(-\frac{2}{(\Delta r)^2}-\frac{1}{r_i{}^2}\right)u_i+\left(\frac{1}{(\Delta r)^2}+\frac{1}{2r_i\Delta r}\right)u_{i+1}=0$ (eq**)

$$\left(-\frac{1}{2(6.2)(0.6)}+\frac{1}{0.6^2}\right)u_1+\left(-\frac{2}{0.6^2}-\frac{1}{6.2^2}\right)u_2+\left(\frac{1}{0.6^2}+\frac{1}{2(6.2)(0.6)}\right)u_3=0$$

$$2.6434u_1-5.5816u_2+2.9122u_3=0$$

Solution

- **Step 4 at node $i=3$,**

$$r_3=r_2+\Delta r=6.2+0.6=6.8$$

Therefore using $\left(-\frac{1}{2r_i(\Delta r)}+\frac{1}{(\Delta r)^2}\right)u_{i-1}+\left(-\frac{2}{(\Delta r)^2}-\frac{1}{r_i{}^2}\right)u_i+\left(\frac{1}{(\Delta r)^2}+\frac{1}{2r_i\Delta r}\right)u_{i+1}=0$ (eq**)

$$\left(-\frac{1}{2(6.8)(0.6)}+\frac{1}{0.6^2}\right)u_2+\left(-\frac{2}{0.6^2}-\frac{1}{6.8^2}\right)u_3+\left(\frac{1}{0.6^2}+\frac{1}{2(6.8)(0.6)}\right)u_4=0$$

$$2.6552u_2-5.5772u_3+2.9003u_4=0$$

- **Step 5 at node $i=4$,**

$$r_4=r_3+\Delta r=6.8+0.6=7.4$$

Therefore using $\left(-\frac{1}{2r_i(\Delta r)}+\frac{1}{(\Delta r)^2}\right)u_{i-1}+\left(-\frac{2}{(\Delta r)^2}-\frac{1}{r_i{}^2}\right)u_i+\left(\frac{1}{(\Delta r)^2}+\frac{1}{2r_i\Delta r}\right)u_{i+1}=0$ (eq**)

$$\left(-\frac{1}{2(7.4)(0.6)}+\frac{1}{0.6^2}\right)u_3+\left(-\frac{2}{0.6^2}-\frac{1}{(7.4)^2}\right)u_4+\left(\frac{1}{0.6^2}+\frac{1}{2(7.4)(0.6)}\right)u_5=0$$

$$2.6651u_3-5.5738u_4+2.8903u_5=0$$

- **Step 6 at node $i=5$,**

$$r_5=r_4+\Delta r=7.4+0.6=8$$

$$u_5=u|_{r=b}=0.0030769 \text{ (boundary condition at } b\text{)}$$

Solving system of equations

Using the second order accurate approximation, yield in the matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2.6297 & -5.5874 & 2.9266 & 0 & 0 & 0 \\ 0 & 2.6434 & -5.5816 & 2.9122 & 0 & 0 \\ 0 & 0 & 2.6552 & -5.5772 & 2.9003 & 0 \\ 0 & 0 & 0 & 2.6651 & -5.5738 & 2.8903 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}=\begin{bmatrix} 0.0038731 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0030769 \end{bmatrix}$$

Solving simultaneous linear equations (see previous lectures e.g. $[A][X]=[C]$ solve for $[X]$), to yield

$u_0=0.0038731$
$u_1=0.0036115$
$u_2=0.0034159$
$u_3=0.0032689$
$u_4=0.0031586$
$u_5=0.0030769$

Exercises

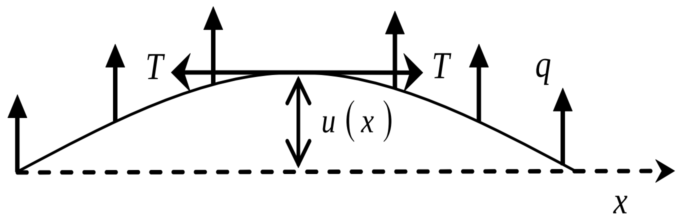
Q1) Given
$\frac{d^2y}{dx^2} = 6x - 0.5x^2, y(0) = 0, y(12) = 0$
the value of $\frac{d^2y}{dx^2}$ at $y(4)$ using the finite difference method and a step size of $h = 4$ can be approximated by

(A) $\frac{y(8)-y(0)}{8}$
(B) $\frac{y(8)-2y(4)+y(0)}{16}$
(C) $\frac{y(12)-2y(8)+y(4)}{16}$
(D) $\frac{y(4)-y(0)}{4}$

Q2) The transverse deflection u of a cable of length L that is fixed at both ends, is given as a solution to

$$\frac{d^2u}{dx^2} = \frac{Tu}{R} + \frac{qx(x-L)}{2R}$$

where
T = tension in cable
R = flexural stiffness
q = distributed transverse load



Given $L = 50, T = 2000, q = 75$, and $R = 75 \times 10^6$.
Using finite difference method modelling with second-order central divided difference accuracy and a step size of $h = 12.5$, the value of the deflection at the centre of the cable most likely is

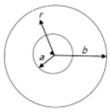
(A) 0.072737
(B) 0.080832
(C) 0.081380
(D) 0.084843

Shooting Method

Example

Consider the pressure vessel example, where

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0,$$



Where $a = 5$ and $b = 8$

with boundary conditions

$$u(5) = 0.0038731,$$
$$u(8) = 0.0030770$$

Let

$$\frac{du}{dr}=w$$

Then

$$\frac{dw}{dr}+\frac{1}{r}w-\frac{u}{r^2}=0$$

The idea is to use: the methods in initial value problems, to solve boundary value problem.

Solution

Two first order differential equations are given as

$$\frac{du}{dr}=w, u(5)=0.0038371$$
$$\frac{dw}{dr}=-\frac{w}{r}+\frac{u}{r^2}, w(5)=not\ known$$

Let us assume

$$w(5)=\frac{du}{dr}(5)\approx\frac{u(8)-u(5)}{8-5}=-0.00026538$$

To set up initial value problem

$$\frac{du}{dr}=w=f_1(r,u,w), u(5)=0.0038371$$
$$\frac{dw}{dr}=-\frac{w}{r}+\frac{u}{r^2}=f_2(r,u,w), w(5)=-0.00026538$$

Solution

Using Euler's method,

$$u_{i+1}=u_i+f_1(r_i,u_i,w_i)h$$
$$w_{i+1}=w_i+f_2(r_i,u_i,w_i)h$$

Let us consider 4 segments between the two boundaries, $r = 5$ and $r = 8$ then,

$$h=\frac{8-5}{4}=0.75$$

Solution

For

$$i=0, r_0=5, u_0=0.0038371, w_0=-0.00026538$$

$$\begin{aligned} u_1 &= u_0+f_1(r_0,u_0,w_0)h \\ &= 0.0038371+f_1(5,0.0038371,-0.00026538)(0.75) \\ &= 0.0038371+(-0.00026538)(0.75) \\ &= 0.0036741 \end{aligned}$$

$$\begin{aligned} w_1 &= w_0+f_2(r_0,u_0,w_0)h \\ &= -0.00026538+f_2(5,0.0038371,-0.00026538)(0.75) \\ &= -0.00026538+\left(-\frac{-0.00026538}{5}+\frac{0.0038371}{5^2}\right)(0.75) \\ &= -0.00010938 \end{aligned}$$

Solution

For

$$i=1, r_1=r_0+h=5+0.75=5.75,$$
$$u_1=0.0036741, w_1=-0.00010940$$

$$\begin{aligned} u_2 &= u_1+f_1(r_1,u_1,w_1)h \\ &= 0.0036741 \\ &= 0.0036741+(-0.00010938)(0.75) \\ &= 0.0035920 \end{aligned}$$

$$\begin{aligned} w_2 &= w_1+f_2(r_1,u_1,w_1)h \\ &= -0.00010938 \\ &= -0.00010938+(0.00013015)(0.75) \\ &= -0.000011769 \end{aligned}$$

Solution

For

$$i=2, r_2=r_1+h=5.75+0.75=6.5$$
$$u_2=0.0035920, w_2=-0.000011785$$

$$\begin{aligned} u_3 &= u_2+f_1(r_2,u_2,w_2)h \\ &= 0.0035920 \\ &= 0.0035920+(-0.000011769)(0.75) \\ &= 0.0035832 \end{aligned}$$

$$\begin{aligned} w_3 &= w_2+f_2(r_2,u_2,w_2)h \\ &= -0.000011769 \\ &= -0.000011769+(0.000086829)(0.75) \\ &= 0.000053352 \end{aligned}$$

Solution

For

$$i=3, r_3=r_2+h=6.50+0.75=7.25$$
$$u_3=0.0035832, w_3=0.000053332$$

$$\begin{aligned} u_4 &= u_3+f_1(r_3,u_3,w_3)h \\ &= 0.0035832+f_1(7.25,0.0035832,0.000053352)(0.75) \\ &= 0.0035832+(0.000053352)(0.75) \\ &= 0.0036232 \end{aligned}$$

$$\begin{aligned} w_4 &= w_3+f_2(r_3,u_3,w_3)h \\ &= -0.000011785 \\ &= 0.000053352+(0.000060811)(0.75) \\ &= 0.000098961 \end{aligned}$$

So at

$$r=r_4=r_3+h=7.25+0.75=8$$
$$u(8)\approx u_4=0.0036232$$

Actual boundary conditions
$u(5) = 0.0038731,$
$u(8) = 0.0030770$

Solution

Repeat the process…

Let us assume a new value for $\frac{du}{dr}(5)$

$$\begin{aligned} w(5) &= 2\frac{du}{dr}(5)\approx 2\frac{u(8)-u(5)}{8-5} \\ &= 2(-0.00026538) \\ &= -0.00053076 \end{aligned}$$

Before was $w(5)=\frac{du}{dr}(5)\approx\frac{u(8)-u(5)}{8-5}=-0.00026538$

Using $h = 0.75$ and Euler's method, we get

$$u(8)\approx u_4=0.0029665"$$

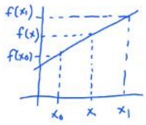
While the given value of this boundary condition is

$$u(8)\approx u_4=0.0030770$$

Actual boundary conditions
$u(5) = 0.0038731,$
$u(8) = 0.0030770$

Solution

Using linear interpolation



$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x) - f(x_0)}{x - x_0}$$

Therefore, rearranging to yield

$$f(x) = \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)(x - x_0) + f(x_0)$$

Recall:

$w(5) = \frac{du}{dr}(5) \approx \frac{u(8)-u(5)}{8-5} = -0.00026538$

To yield $u(8) \approx u_4 = 0.0029665$

and

$w(5) = 2\frac{du}{dr}(5) \approx 2\frac{u(8)-u(5)}{8-5} = 2(-0.00026538) = -0.00053076$

To yield $u(8) \approx u_4 = 0.0036232$

Actual boundary conditions
$u(5) = 0.0038731,$
$u(8) = 0.0030770$

Solution

Using linear interpolation on the obtained data for the two assumed values of

$$\frac{du}{dr}(5)$$

We get

$$\begin{aligned} w(5) = \frac{du}{dr}(5) = f(x) &= \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)(x - x_0) + f(x_0) \\ &\approx \frac{-0.00053076 - (-0.00026538)}{0.0036232 - 0.0029665}(0.0030770 - 0.0036232) + (-0.00026538) \\ &= -0.00048611 \end{aligned}$$

Recall:

$w(5) = \frac{du}{dr}(5) \approx \frac{u(8)-u(5)}{8-5} = -0.00026538$

To yield $u(8) \approx u_4 = 0.0029665$

and

$w(5) = 2\frac{du}{dr}(5) \approx 2\frac{u(8)-u(5)}{8-5} = 2(-0.00026538) = -0.00053076$

To yield $u(8) \approx u_4 = 0.0036232$

Actual boundary conditions
$u(5) = 0.0038731,$
$u(8) = 0.0030770$

Solution

Using $h = 0.75$ and repeating the Euler's method with

$$w(5) = -0.00048611$$
$$u(8) \approx u_4 = 0.0030769$$

Actual boundary conditions
$u(5) = 0.0038731,$
$u(8) = 0.0030770$

Using linear interpolation to refine the value of u_4 till one gets close to the actual value of $u(8)$ which gives you

$$\begin{aligned} u(5) &= u_1 = 0.0038731 \\ u(5.75) &\approx u_2 = 0.0035085 \\ u(6.50) &\approx u_3 = 0.0032858 \\ u(7.25) &\approx u_4 = 0.0031518 \\ u(8.00) &\approx u_5 = 0.0030770 \end{aligned}$$

Exercises

Q3) Given

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2,\ y(0) = 0,\ y(12) = 0$$

If one was using the shooting method with Euler's method with a step size of $h = 4$, and an assumed value of $\frac{dy}{dx}(0) = 20$, then the estimated value of $y(12)$ in the first iteration most likely is

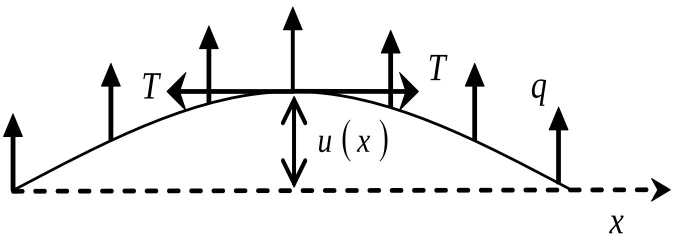
(A) 60.00
(B) 496.0
(C) 1088
(D) 1102

Q4) The transverse deflection, u of a cable of length, L, fixed at both ends, is given as a solution to

$$\frac{d^2u}{dx^2} = \frac{Tu}{R} + \frac{qx(x-L)}{2R}$$

where

T = tension in cable
R = flexural stiffness
q = distributed transverse load



Given are $L = 50$, $\mathrm{T} = 2000$, $\mathrm{q} = 75$ $\mathrm{R} = 75 \times 10^6$. The shooting method is used with Euler's method assuming a step size of $h = 12.5$. Initial slope guesses at $x = 0$ of $\frac{du}{dx} = 0.003$ and $\frac{du}{dx} = 0.004$ are used in order, and then refined for the next iteration using linear interpolation after the value of $u(L)$ is found. The deflection in inches at the centre of the cable found during the second iteration is most likely is:

(A) 0.075000
(B) 0.10000
(C) -0.061291
(D) 0.00048828