ENG2009 – Modelling of Engineering Systems
Tutorial 10

PDE 1 & 2 – Parabolic PDE using Explicit, Implicit and Crank-Nicolson methods

What is PDE

Ordinary Differential Equations have only one independent variable

$$3\frac{dy}{dt} + 5y^2 = 3e^{-t}, y(0) = 5$$

Partial Differential Equations have more than one independent variable

$$3\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$$

subject to certain conditions: where u is the dependent variable, and x and y are the independent variables.

Classification of 2nd Order Linear PDE's

Consider

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

where A, B and C are functions of x and y, and D is a function of x, y, u and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$

Can be:

- Elliptic - If $B^2 - 4AC < 0$
- Parabolic - If $B^2 - 4AC = 0$
- Hyperbolic - If $B^2 - 4AC > 0$

Physical Example of a Parabolic PDE

The internal temperature of a metal rod exposed to two different temperatures at each end can be found using the heat conduction equation.

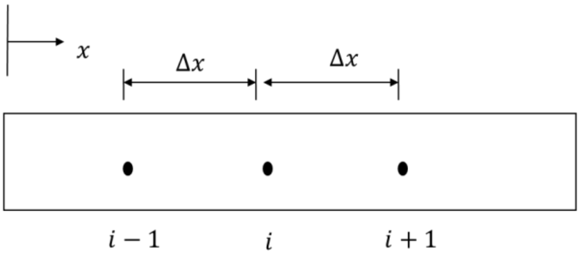
$$\alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

Discretizing the Parabolic PDE

For a rod of length L divided into $n + 1$ nodes $\Delta x = \frac{L}{n}$

The time is similarly broken into time steps of Δt

Hence T_i^j corresponds to the temperature at node i, that is,
$x = (i)(\Delta x)$ and time $t = (j)(\Delta t)$



Schematic diagram showing interior nodes

The Explicit Method

From $\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$

$$\alpha \frac{T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}}{(\Delta x)^2} = \frac{T_i^{j+1} - T_i^{j}}{\Delta t}$$

Solving for the temp at the time node $j+1$ gives

$$T_i^{j+1} = T_i^{j} + \alpha \frac{\Delta t}{(\Delta x)^2}\left(T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}\right)$$

choosing,

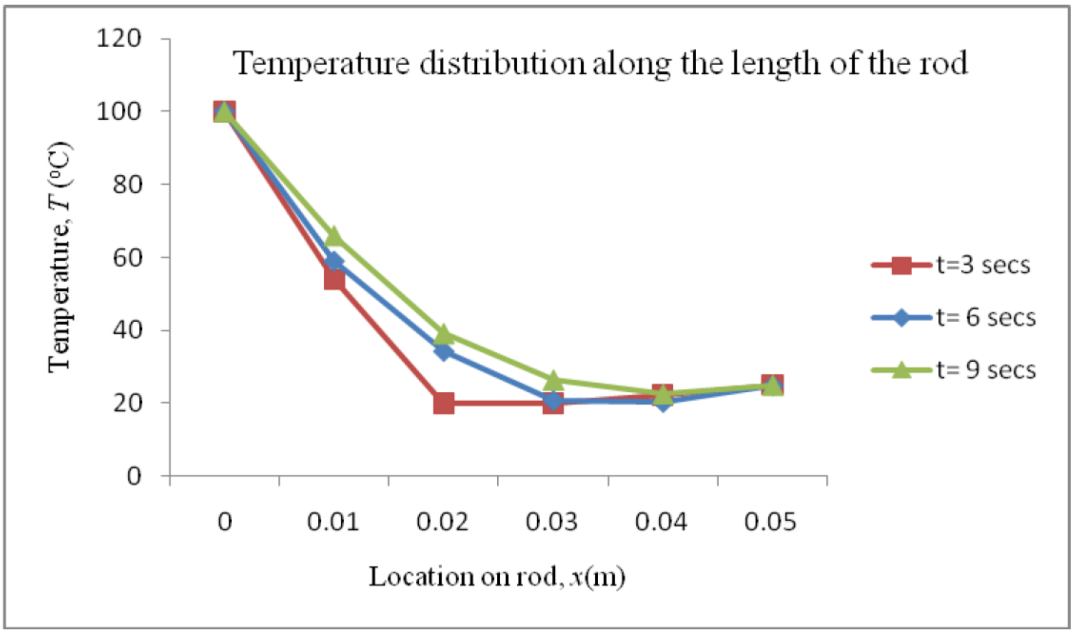
$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

we can write the equation as:

$$T_i^{j+1} = T_i^{j} + \lambda\left(T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}\right)$$

Last lecture: Example 1: Explicit Method

Temperature distribution along the length of the rod at different times



Summary: The Implicit Method

From

$$\alpha\frac{T_{i+1}^{j+1}-2T_i^{j+1}+T_{i-1}^{j+1}}{(\Delta x)^2}=\frac{T_i^{j+1}-T_i^j}{\Delta t}$$

Rearranging yields

$$\boxed{-\lambda T_{i-1}^{j+1}+(1+2\lambda)T_i^{j+1}-\lambda T_{i+1}^{j+1}=T_i^j}$$

given that,

$$\lambda=\alpha\frac{\Delta t}{(\Delta x)^2}$$

The rearranged equation can be written for every node during each time step. These equations can then be solved as a simultaneous system of linear equations to find the nodal temperatures at a particular time.

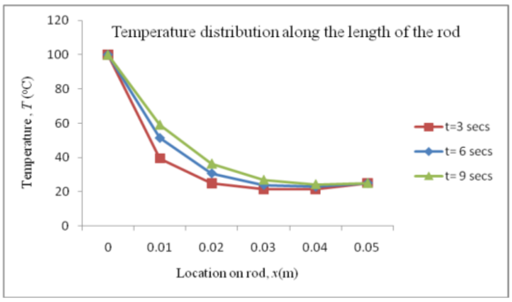
Internal Temperatures at 9 sec.

The table below allows you to compare the results from all three methods discussed in juxtaposition with the analytical solution.

Node	Explicit	Implicit	Crank-Nicolson	Analytical
T_1^3	65.953	59.043	62.604	62.510
T_2^3	39.132	36.292	37.613	37.084
T_3^3	27.266	26.809	26.562	25.844
T_4^3	22.872	24.243	24.042	23.610

Example 2: Implicit method

Temperature distribution along the length of the rod at different times



Exercises

Q1) In a general second order linear partial differential equation with two independent variables

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+D=0$$

where A, B, C are functions of x and y, and D is a function of $x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, then the partial differential equation is parabolic if

(A) $B^2-4AC<0$
(B) $B^2-4AC>0$
(C) $B^2-4AC=0$
(D) $B^2-4AC\neq 0$

<u>SOLUTION</u>

The correct answer is (c)

A general second order linear partial differential equation is parabolic if $B^2-4AC=0$

Q2) The region in which the following partial differential equation

$$x^3\frac{\partial^2 u}{\partial x^2}+27\frac{\partial^2 u}{\partial y^2}+3\frac{\partial^2 u}{\partial x\partial y}+5u=0$$

acts as parabolic equation is

(A) $x>\left(\frac{1}{12}\right)^{1/3}$

(B) $x<\left(\frac{1}{12}\right)^{1/3}$

(C) for all values of x

(D) $x=\left(\frac{1}{12}\right)^{1/3}$

SOLUTION

The correct answer is (D).

A general partial differential equation with two independent variables is of the form

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial xy}+C\frac{\partial^2 u}{\partial y^2}+D=0$$

where A, B, and C are functions of x and y and is a function of x, y, u and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$.

For this equation to be parabolic, $B^2-4AC=0$.

In the above question, $A=x^3, B=3, C=27$, giving

$$B^2-4AC=0$$

$$(3)^2-4(x^3)(27)=0$$

$$9-108x^3=0$$

$$108x^3=9$$

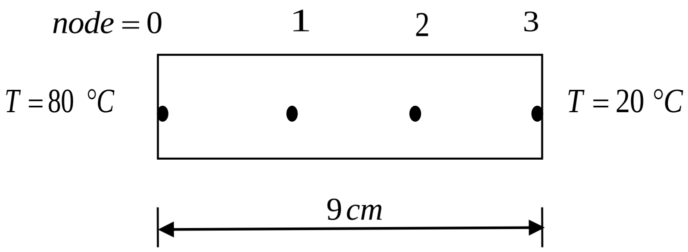
$$x^3=\frac{9}{108}$$

$$x^3=\frac{1}{12}$$

$$x=\left(\frac{1}{12}\right)^{1/3}$$

Q3) The partial differential equation of the temperature in a long thin rod is given by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$



If $\alpha = 0.8cm^2/s$, the initial temperature of rod is $40°C$, and the rod is divided into three equal segments, the temperature at node 1 (using $\Delta t = 0.1s$) by using an <u>explicit</u> solution at $t = 0.2sec$ is

(A) 40.7134 ^0C
(B) 40.6882 ^0C
(C) 40.7033 ^0C
(D) 40.6956 ^0C

Formula: Explicit method $T_i^{j+1} = T_i^j + \lambda(T_{i+1}^j - 2T_i^j + T_{i-1}^j)$, where $\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$

<u>SOLUTION</u>

The correct answer is (C).

Given

$$\alpha = 0.8cm^2/s$$
$$\Delta t = 0.1s$$
$$t = 0.2\sec s$$
$$L = 9\text{cms}$$

Number of divisions of the rod, $n = 3$

$$\Delta x = \frac{L}{n}$$
$$= \frac{9}{3}$$
$$= 3$$

Number of time steps$=\dfrac{t_{final} - t_{initial}}{\Delta t}$

$$= \frac{0.2 - 0}{0.1}$$
$$= 2$$

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

$$= 0.8\frac{0.1}{(3)^2}$$
$$= 0.0089$$

The boundary conditions

$$\left.\begin{aligned} T_0^j &= 80°C \\ T_3^j &= 20°C \end{aligned}\right\} \text{for all } j = 0,1 \qquad \text{(E3.1)}$$

The initial temperature of the rod is $40°C$, that is, all the temperatures of the nodes inside the rod are at $40°C$ when time, $t = 0\,\text{sec}$ except for the boundary nodes as given by Equation (E3.1). This could be represented as

$$T_i^0 = 20°C, \text{ for all } i = 1,2. \qquad \text{(E3.2)}$$

<u>Initial temperature at the nodes inside the rod (when *t*=0 sec)</u>

$$T_0^0 = 80°C \quad \text{from Equation (E3.1)}$$
$$\left.\begin{aligned} T_1^0 &= 40°C \\ T_2^0 &= 40°C \end{aligned}\right\} \text{from Equation (E3.2)}$$
$$T_3^0 = 20°C \quad \text{from Equation (E3.1)}$$

<u>Temperature at the nodes inside the rod when *t*=0.1 sec</u>

Setting $j = 0$ and $i = 0,1,2,3$ in Equation (7) (from Chapter 10.02) gives the temperature of the nodes inside the rod when time, $t = 0.1\,\text{sec}$.

$$T_0^1 = 80°C \quad \text{Boundary Condition (E3.1)}$$

$$\begin{aligned} T_1^1 &= T_1^0 + \lambda\left(T_2^0 - 2T_1^0 + T_0^0\right) \\ &= 40 + 0.0089(40 - 2(40) + 80) \\ &= 40 + 0.0089(40) \\ &= 40 + 0.3556 \\ &= 40.3556°C \end{aligned}$$

$$\begin{aligned} T_2^1 &= T_2^0 + \lambda\left(T_3^0 - 2T_2^0 + T_1^0\right) \\ &= 40 + 0.0089(20 - 2(40) + 40) \\ &= 40 + 0.0089(-20) \\ &= 40 - 0.1778 \\ &= 39.8222°C \end{aligned}$$

$$T_3^1 = 20°C \quad \text{Boundary Condition (E3.1)}$$

Temperature at the nodes inside the rod when t=0.2 sec

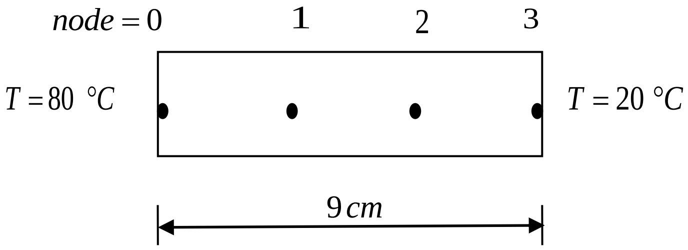
Setting $j=1$ and $i=0,1,2,3$ in Equation (6) (from Chapter 11.02) gives the temperature of the nodes inside the rod when time, $t=0.2\,\text{sec}$

$$T_0^2 = 80°C \quad \text{Boundary Condition (E3.1)}$$

$$\begin{aligned}
T_1^2 &= T_1^1 + \lambda\left(T_2^1 - 2T_1^1 + T_0^1\right) \\
&= 40.3556 + 0.0089\left(39.8222 - 2(40.3556) + 80\right) \\
&= 40.3556 + 0.0089\left(39.1110\right) \\
&= 40.3556 + 0.3477 \\
&= 40.7033°C
\end{aligned}$$

Q4) The partial differential equation of the temperature in a long thin rod is given by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$



If $\alpha = 0.8cm^2/s$, the initial temperature of rod is $40°C$, and the rod is divided into three equal segments, the temperature at node 1 (using $\Delta t = 0.1s$) by using an <u>implicit</u> solution for $t = 0.2sec$ is

(A) 40.7134 ^0C
(B) 40.6882 ^0C
(C) 40.7033 ^0C
(D) 40.6956 ^0C

Implicit method: $-\lambda T_{i-1}^{j+1} + (1+2\lambda)T_i^{j+1} - \lambda T_{i+1}^{j+1} = T_i^j$ where $\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$

<u>SOLUTION</u>

The correct answer is (B).

Given

$$\alpha = 0.8 cm^2/s$$
$$\Delta t = 0.1 s$$
$$t = 0.2 \sec s$$
$$L = 9 \text{cms}$$

Number of divisions of the rod, $n = 3$

$$\Delta x = \frac{L}{n}$$
$$= \frac{9}{3}$$
$$= 3$$

Number of time steps$= \frac{t_{final} - t_{initial}}{\Delta t}$

$$= \frac{0.2 - 0}{0.1}$$
$$= 2$$

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

$$= 0.8\frac{0.1}{(3)^2}$$
$$= 0.0089$$

The boundary conditions

$$\left.\begin{aligned} T_0^j &= 80°C \\ T_3^j &= 20°C \end{aligned}\right\} \text{for all } j = 0,1 \qquad \text{(E4.1)}$$

The initial temperature of the rod is $40°C$, that is, all the temperatures of the nodes inside the rod are at $40°C$ when time, $t = 0\,\text{sec}$ except for the boundary nodes as given by Equation (E3.1). This could be represented as

$$T_i^0 = 20°C, \text{ for all } i = 1,2\,. \qquad \text{(E4.2)}$$

<u>Initial temperature at the nodes inside the rod (when t=0 sec)</u>

$$T_0^0 = 80°C \quad \text{from Equation (E4.1)}$$

$$\left.\begin{aligned} T_1^0 &= 40°C \\ T_2^0 &= 40°C \end{aligned}\right\} \text{from Equation (E4.2)}$$

$$T_3^0 = 20°C \quad \text{from Equation (E4.1)}$$

<u>Temperature at the nodes inside the rod when t=0.1 sec</u>

$$\left.\begin{aligned} T_0^1 &= 80°C \\ T_3^1 &= 20°C \end{aligned}\right\} \text{Boundary Condition (E4.1)}$$

For all the interior nodes, putting $j = 0$ and $i = 1,2$ in Equation (11) (from Chapter 10.02) gives the following equations

<u>i=1</u>

$$-\lambda T_0^1 + (1+2\lambda)T_1^1 - \lambda T_2^1 = T_1^0$$
$$(-0.0089\times 80) + (1+2\times 0.0089)T_1^1 - (0.0089T_2^1) = 40$$
$$-0.7111 + 1.0178T_1^1 - 0.0089T_2^1 = 40$$
$$1.0178T_1^1 - 0.0089T_2^1 = 40.7111 \qquad \text{(E4.3)}$$

<u>i=2</u>

$$-\lambda T_1^1 + (1+2\lambda)T_2^1 - \lambda T_3^1 = T_2^0$$
$$-0.0089T_1^1 + 1.0178T_2^1 - (0.0089\times 20) = 40$$
$$-0.0089T_1^1 + 1.0178T_2^1 - 0.1778 = 40$$
$$-0.0089T_1^1 + 1.0178T_2^1 = 40.1778 \qquad \text{(E4.4)}$$

The simultaneous linear equations (E4.3) – (E4.4) can be written in matrix form as

$$\begin{bmatrix} 1.0178 & -0.0089 \\ -0.0089 & 1.0178 \end{bmatrix}\begin{bmatrix} T_1^1 \\ T_2^1 \end{bmatrix} = \begin{bmatrix} 40.7111 \\ 40.1778 \end{bmatrix}$$

The above coefficient matrix is tri-diagonal. Special algorithms such as Thomas' algorithm can be used to solve simultaneous linear equation with tri-diagonal coefficient matrices. The solution is given by

$$\begin{bmatrix} T_1^1 \\ T_2^1 \end{bmatrix} = \begin{bmatrix} 40.3478 \\ 39.8284 \end{bmatrix}$$

<u>Temperature at the nodes inside the rod when t=0.2 sec</u>

$$\left.\begin{aligned} T_0^2 &= 80°C \\ T_3^2 &= 20°C \end{aligned}\right\} \text{Boundary Condition} \quad (E4.1)$$

For all the interior nodes, putting $j = 1$ and $i = 1,2$ in Equation (11) (from Chapter 10.02) gives the following equations

<u>i=1</u>

$$-\lambda T_0^2 + (1+2\lambda)T_1^2 - \lambda T_2^2 = T_1^1$$

$$(-0.0089 \times 80) + (1 + 2 \times 0.0089)T_1^2 - 0.0089 T_2^2 = 40.3478$$

$$-0.7111 + 1.0178 T_1^2 - 0.0089 T_2^2 = 40.3478$$

$$1.0178 T_1^2 - 0.0089 T_2^2 = 41.0590 \qquad (E4.5)$$

<u>i=2</u>

$$-\lambda T_1^2 + (1+2\lambda)T_2^2 - \lambda T_3^2 = T_2^1$$

$$-0.0089 T_1^2 + 1.0178 T_2^2 - (0.0089 \times 20) = 39.8284$$

$$-0.0089 T_1^2 + 1.0178 T_2^2 - 0.1778 = 39.8284$$

$$-0.0089 T_1^2 + 1.0178 T_2^2 = 40.0061 \qquad (E4.6)$$

The simultaneous linear equations (E4.5) – (E4.6) can be written in matrix form as

$$\begin{bmatrix} 1.0178 & -0.0089 \\ -0.0089 & 1.0178 \end{bmatrix} \begin{bmatrix} T_1^2 \\ T_2^2 \end{bmatrix} = \begin{bmatrix} 41.0590 \\ 40.0061 \end{bmatrix}$$

The above coefficient matrix is tri-diagonal. Special algorithms such as Thomas' algorithm can be used to solve simultaneous linear equation with tri-diagonal coefficient matrices. The solution is given by

$$\begin{bmatrix} T_1^2 \\ T_2^2 \end{bmatrix} = \begin{bmatrix} 40.6882 \\ 39.6627 \end{bmatrix}$$

PDE 3 – Elliptic PDE using Direct and Gauss-Seidel methods

Recall: Classification of 2nd Order Linear PDE's

Given

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

Example

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

where

$$A = 1, B = 0, C = 1$$

which yield

$$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$$

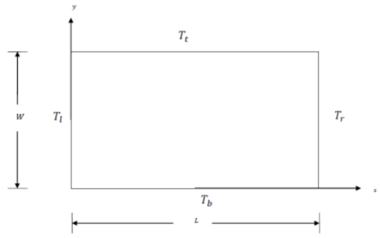
therefore the equation is elliptic.

Summary: Physical Example of an Elliptic PDE

Schematic diagram of a plate with specified temperature boundary conditions

The (Laplace) equation that governs the temperature:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$



Summary: Discretizing the Elliptic PDE

From

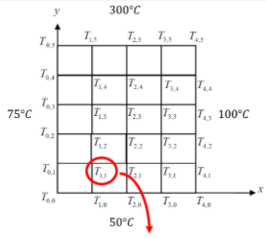
$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

Once the governing equation has been discretized there are several numerical methods that can be used to solve the problem.

We examined the:

- Direct Method
- Gauss-Seidel Method

Summary: Example 1: Direct Method



Example calculations:

The equation for the temperature at the node (1,1) is given by:

$i = 1$ and $j = 1$

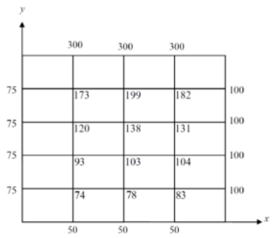
$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$
$$T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0} - 4T_{1,1} = 0$$
$$T_{2,1} + 75 + T_{1,2} + 50 - 4T_{1,1} = 0$$
$$-4T_{1,1} + T_{1,2} + T_{2,1} = -125$$

Summary: Example 1: Direct Method

We can develop similar equations for every interior node leaving us with an equal number of equations and unknowns.

Question: How many equations would this generate? Answer: 12

Solving yields:
$$\begin{bmatrix} T_{1,1} \\ T_{1,2} \\ T_{1,3} \\ T_{1,4} \\ T_{2,1} \\ T_{2,2} \\ T_{2,3} \\ T_{2,4} \\ T_{3,1} \\ T_{3,2} \\ T_{3,3} \\ T_{3,4} \end{bmatrix} = \begin{bmatrix} 73.8924 \\ 93.0252 \\ 119.907 \\ 173.355 \\ 77.5443 \\ 103.302 \\ 138.248 \\ 198.512 \\ 82.9833 \\ 104.389 \\ 131.271 \\ 182.446 \end{bmatrix} {}^\circ C \rightarrow$$



Summary: The Gauss-Seidel Method

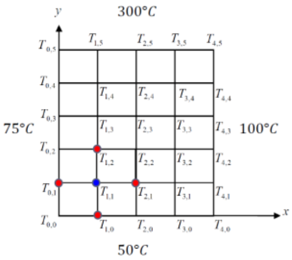
- Recall the discretized equation
$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$
- This can be rewritten as
$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$
- For the Gauss-Seidel Method, this equation is solved iteratively for all interior nodes until a pre-specified tolerance is met.

Example 2: Gauss-Seidel Method

- **Now we can begin to solve for the temperature at each interior node using**

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

- **Assume all internal nodes to have an initial temperature of <u>zero</u>.**



Iteration #1 (example calculations)

<u>$i = 1$ and $j = 1$</u>

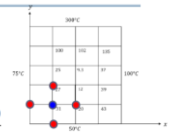
$$\begin{aligned} T_{1,1} &= \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4} \\ &= \frac{0 + 75 + 0 + 50}{4} \\ &= 31.2500°C \end{aligned}$$

Example 2: Gauss-Seidel Method

Iteration #2 (example calculations)

<u>$i = 1$ and $j = 1$</u>

$$\begin{aligned} T_{1,1} &= \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4} \\ &= \frac{20.3125 + 75 + 26.5625 + 50}{4} \\ &= 42.9688°C \end{aligned}$$



Absolute relative error

$$\begin{aligned} |\varepsilon_a|_{1,1} &= \left| \frac{T_{1,1}^{present} - T_{1,1}^{previous}}{T_{1,1}^{present}} \right| \times 100 \\ &= \left| \frac{42.9688 - 31.2500}{42.9688} \right| \times 100 \\ &= 27.27\% \end{aligned}$$

Exercises

Q5) In a general second order linear partial differential equation with two independent variables,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

where A, B, C are functions of x and y, and D is a function of $x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, then the PDE is elliptic if

(A) $B^2 - 4AC < 0$
(B) $B^2 - 4AC > 0$
(C) $B^2 - 4AC = 0$
(D) $B^2 - 4AC \neq 0$

SOLUTION

The correct answer is (A).

In a general second order linear partial differential equation with two independent variables,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

where A, B, C are functions of x and y, and D is a function of x, y, $\frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$.

If $B^2 - 4AC < 0$, the second order linear partial differential equation is elliptic.

Q6) The region in which the following equation

$$x^3\frac{\partial^2 u}{\partial x^2}+27\frac{\partial^2 u}{\partial y^2}+3\frac{\partial^2 u}{\partial x\partial y}+5u=0$$

acts as an elliptic equation is

(A) $x>\left(\frac{1}{12}\right)^{1/3}$

(B) $x<\left(\frac{1}{12}\right)^{1/3}$

(C) for all values of x

(D) $x=\left(\frac{1}{12}\right)^{1/3}$

SOLUTION

The correct answer is (A).

A general partial differential equation with two independent variables is of the from

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial xy}+C\frac{\partial^2 u}{\partial y^2}+D=0$$

where A, B, and C are functions of x and y and is a function of x, y, u and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$.

For this equation to be elliptic,

$$B^2-4AC<0.$$

In the above question,

$$A=x^3, B=3, C=27, D=5u\ ,$$

giving

$$B^2-4AC<0$$

$$(3)^2-4(x^3)(27)<0$$

$$9-108x^3<0$$

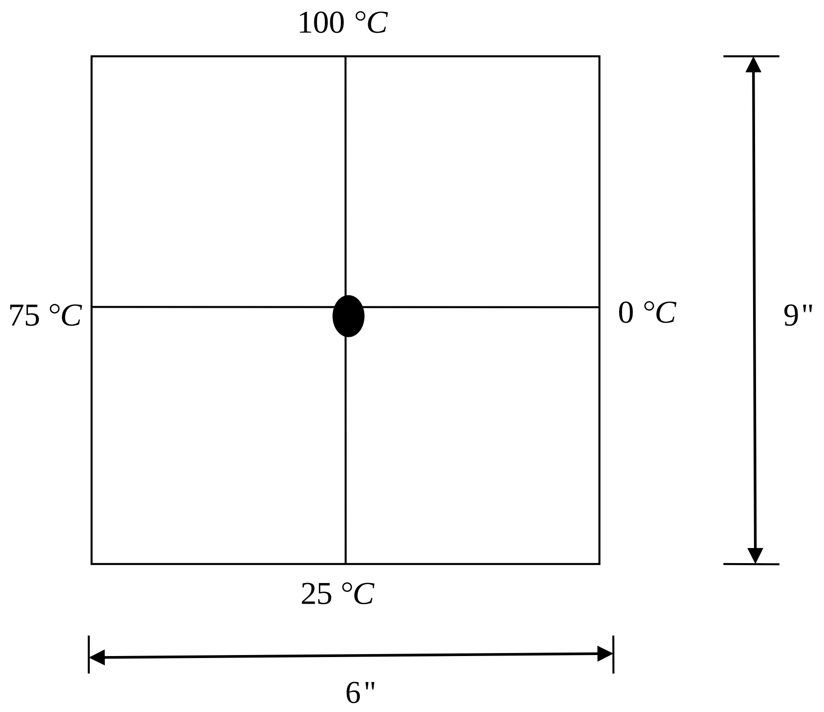
$$108x^3>9$$

$$x^3>\frac{9}{108}$$

$$x^3>\frac{1}{12}$$

$$x>\left(\frac{1}{12}\right)^{1/3}$$

Q7) Find the temperature at the interior node given in the following figure using the direct method



(A) $45.19°C$
(B) $48.64°C$
(C) $50.00°C$
(D) $56.79°C$

Formula: Direct method.
From

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Substituting the approximations into the equation yields:

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

if,

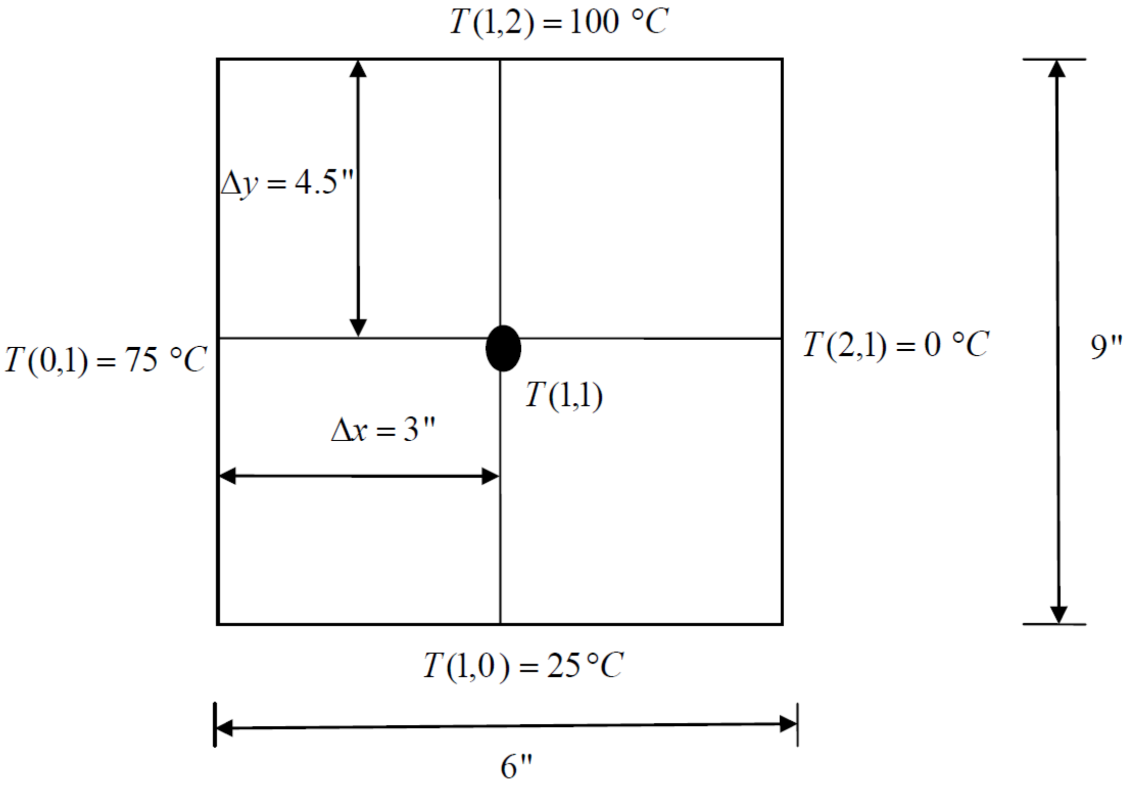
$$\Delta x = \Delta y$$

the equation can be rewritten as

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

SOLUTION

The correct answer is (A)



From

$$\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{(\Delta x)^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{(\Delta y)^2}=0$$

putting $i=1$ and $j=1$, we have

$$\frac{T_{2,1}-2T_{1,1}+T_{0,1}}{(\Delta x)^2}+\frac{T_{1,2}-2T_{1,1}+T_{1,0}}{(\Delta y)^2}=0$$

$$\frac{0-2T_{1,1}+75}{(3)^2}+\frac{100-2T_{1,1}+25}{(4.5)^2}=0$$

$$\frac{75-2T_{1,1}}{9}+\frac{125-2T_{1,1}}{20.25}=0$$

$$\frac{75}{9}+\frac{125}{20.25}-2T_{1,1}\left(\frac{1}{9}+\frac{1}{20.25}\right)=0$$

$$8.333+6.173-2T_{1,1}(0.1605)=0$$

$$14.51-2T_{1,1}(0.1605)=0$$

$$2T_{1,1}(0.1605)=14.51$$

$$T_{1,1}=45.19\ °C$$