ENG2009 – Modelling of Engineering Systems

Tutorial 2

Linear systems and Laplace Transform

Linear Systems:

1) What is system order?

Answer: (lecture slides)
Order of a system:

- is the *highest power* of the derivative in the *differential equation*,
- or the *highest power of "s"* in the *denominator* of the *transfer function* (ratio of output/input of a system)

System order:

Example : What is the order of the following systems

$$\frac{dy(t)}{dt} + y(t) = u(t)$$

Answer: 1st order

Exercises

Q1) What is the order of the following systems. Explain your reasoning.

(a) $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = u(t)$
(b) $3\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = u(t)$
(c) $G(s) = \frac{1}{2s+1}$
(d) $G(s) = \frac{1}{2s^2+2s+1}$
(e) $G(s) = \frac{1}{4s^3+2s^2+2s+1}$

Answer:

(a) 2nd order
(b) 3rd order
(c) 1st order
(d) 2nd order
(e) 3rd order

Linear vs nonlinear systems

Example: Is the following is linear or not linear: $f(x) = 5\cos(x)$. Explain your reasoning.
Answer: nonlinear due to the cos term.
The term does not satisfy the principle of superposition (check lecture notes).

Exercises

Q2) Which of the following is linear and which is not linear:

(a) $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 10y(t) = u(t)$
(b) $m\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = u(t)$, where m, b, k are constants
(c) $m(t)\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = u(t)$, where $m(t)$ is time varying and b, k are constants
(d) $\frac{d^2x}{dt^2} + (1 - \cos(2t))x = 0$
(e) $\frac{d^2x}{dt^2} + (x^2 - 1)\frac{dx}{dt} + x = 0$
(f) $\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + x^3 = \sin(\omega t)$

Answer:

(a) LTI Linear differential eq
(b) LTI Linear differential eq
(c) LTV. Linear structure, parameter $m(t)$ is time varying
(d) Nonlinear - due to time varing parameter $(1 - \cos(2t))$
(e) Nonlinear – The equation contain powers x^2 .
(f) Nonlinear – The equation contain powers x^3

Linearisation

Example: (Lecture notes – pendulum example)

Linearise $\sin(\theta)$ around $\theta = 0$ (small angle)

Answer:

From taylor series expansion

$$f(x) \approx f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$$

Or

$$f(x) - f(x_0) \approx \dot{f}(x_0)(x - x_0)$$

Therefore, for $\theta_0 = 0$

$f(x) = \sin(\theta)$

$f(0) = \sin(0) = 0$

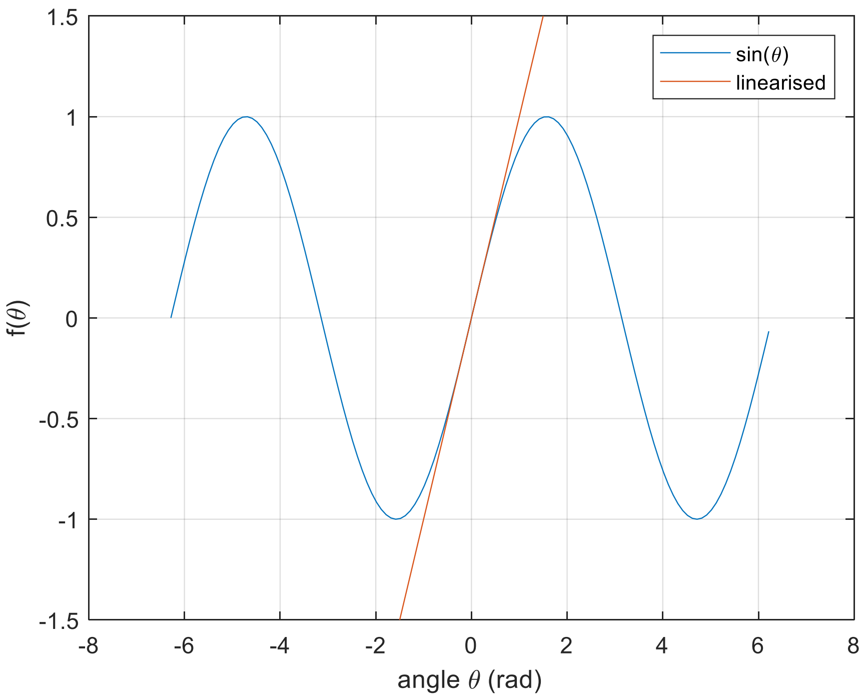
$\dot{f}(x_0) = \left.\frac{d(\sin(\theta))}{dt}\right|_{\theta=0} = \cos(0) = 1$

Therefore

$f(x) \approx f(\theta_0) + \left.\frac{df}{dx}\right|_{\theta=0} (\theta - \theta_0) = 0 + (1)(\theta - 0) = \theta$

Therefore linerisation of $\sin(\theta)$ around small angle $\theta = 0$ is

$$\sin(\theta) \approx \theta$$



Exercises

Q3) Linearise the following:

(i) $f(x) = 5\cos(x)$, about $x = \pi/2$

(ii) $f(x) = 2\,ln(x)$, about $x = 1$

Answer:

(i) Solution

$f(x) \approx f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$

Therefore, for $x_0 = \frac{\pi}{2}$

$f(x) = 5\cos(x)$

$f(\pi/2) = 5\cos(\pi/2) = 0$

$\dot{f}(x_0) = 5\left.\frac{d(\cos(x))}{dt}\right|_{x=\pi/2} = -5\sin(x)|_{x=\pi/2} = -5$

Therefore

$f(x) \approx f\left(\frac{\pi}{2}\right) + \left.\frac{df}{dx}\right|_{x=\frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) = 0 + (-5)\left(x - \frac{\pi}{2}\right) = -5\left(x - \frac{\pi}{2}\right)$

Therefore linerisation of $5\cos(x)$ around $x = \frac{\pi}{2}$ is

$$5\cos(x) \approx -5\left(x - \frac{\pi}{2}\right)$$

(ii) Solution

$f(x) \approx f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$

Therefore, for $x_0 = 1$

$f(x) = 2\ln(x)$

$f(1) = 2\ln(1) = 0$

$\dot{f}(x_0) = 2\left.\frac{d(\ln(x))}{dt}\right|_{x=1} = 2\left.\frac{1}{x}\right|_{x=1} = 2$

Therefore

$f(x) \approx f(1) + \left.\frac{df}{dx}\right|_{x=1} (x - x_0) = 0 + (2)(x - 1) = 2(x - 1)$

Therefore linerisation of $2\ln(x)$ around $x = 1$ is

$$2\ln(x) \approx 2(x - 1)$$

Laplace Transform:

Find the Laplace transform for $f(t) = 1$. (See lecture notes).

Answer:

For a signal $f(t)$, the Laplace transform is defined as:

$$\mathcal{L}(f(t)) = \int_0^{\infty} f(t)e^{-st}\,dt$$

where $s = \sigma + \omega j$ (complex number)

For example if $f(t) = 1$ then

$$F(s) := \mathcal{L}(1) = \int_0^{\infty} 1e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_0^{\infty} = \frac{1}{s}$$

Exercise

Q4) Find the Laplace transform of the following functions:

$f(t) = Ae^{-at}$, where A and a are constants

Answer:

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt = \int_0^{\infty} Ae^{-at}e^{-st}\,dt = A\int_0^{\infty} e^{-(s+a)t}\,dt$$

$$= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a}$$

Q5) Using the Laplace transform Table and the Laplace transform theorems table (attached at the back of this paper), derive the Laplace transforms for the following time functions:

a) $e^{-at}\sin(\omega t)$
b) $e^{-at}\cos(\omega t)$
c) t^3

Answers

a. Using the frequency shift theorem and the Laplace transform of sin ωt, $F(s) = \frac{\omega}{(s+a)^2+\omega^2}$.

b. Using the frequency shift theorem and the Laplace transform of cos ωt, $F(s) = \frac{(s+a)}{(s+a)^2+\omega^2}$.

c. Using the integration theorem, and successively integrating u(t) three times, $\int dt = t$; $\int t\,dt = \frac{t^2}{2}$;

$\int \frac{t^2}{2}dt = \frac{t^3}{6}$, the Laplace transform of $t^3u(t)$, $F(s) = \frac{6}{s^4}$.

Final value theorem:

Example: (Final Value Theorem)

Find the final value of the system corresponding to the following system, where the input $u(t)$ is an unit step i.e. $u(t) = 1$.

$$Y(s) = U(s)\frac{6}{s+2}$$

Hint: use Final Value Theorem

Solution:

System model represented in Laplace as

$$Y(s) = U(s)\frac{6}{s+2}$$

Step 1: Simple check shows that this system is stable as the denominator has a root at left half of s-plane, i.e. $s = -2$. Therefore Final Value Theorem is valid.

Step 2: The input $u(t)$ is an unit step i.e. $u(t) = 1$. From Laplace table

$$\mathcal{L}(u(t)) = \mathcal{L}(1) = \frac{1}{s}$$

Therefore,

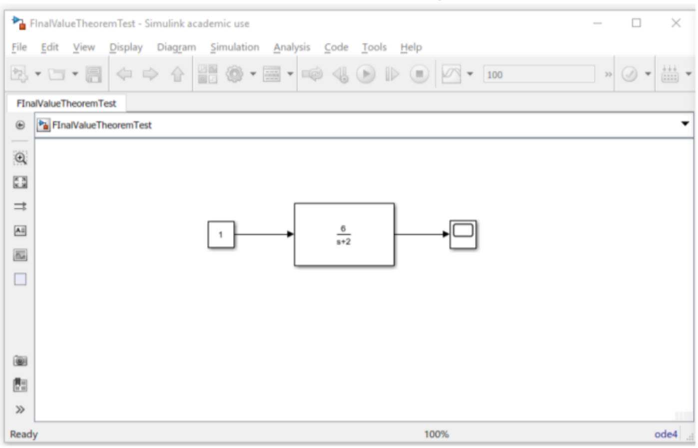
$$Y(s) = \frac{1}{s}\left(\frac{6}{s+2}\right)$$

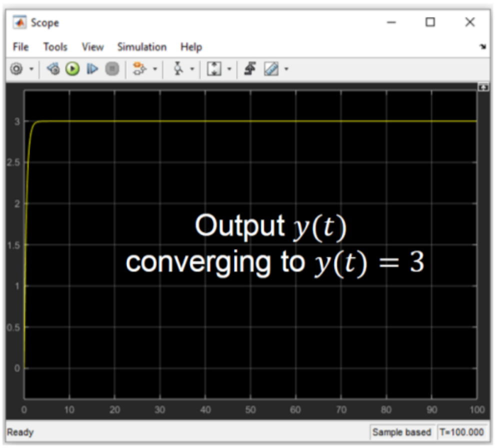
Step 3: Therefore, the final value of $y(t)$ is:

$$\lim_{t\to\infty}(y(t)) = \lim_{s\to 0}(sY(s)) = s\frac{1}{s}\left(\frac{6}{s+2}\right)\Big|_{s=0} = \frac{6}{2} = 3$$

Thus, after the transients have decayed to zero, $y(t)$ will settle to a constant value of 3.

Check with simulation results:





Exercises

Q6) Without solving the differential equation, find the steady state values for the following systems (if available), using Final Value Theorem. Assume that the initial conditions are zero, and the input for each systems are unit input $u(t) = 1$.

(a) Mechanical system (mass-spring-damper):

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F_e(t)$$

where the mass $m = 2$, damping factor $b = 5$, spring contant $k = 1$, and $F_e(t)$ is the input, and displacement $x(t)$ is the output

(b) Electrical system (simple RLC circuit)

$$\frac{3}{4}\frac{dv_o(t)}{dt} + \frac{1}{2}v_o(t) = \frac{1}{2}\frac{di_i(t)}{dt} + i_i(t)$$

where $v_o(t)$ is the output and $i_i(t)$ is the input.

Answer:

(a)

given $m\ddot{x} + b\dot{x} + kx = Fe$

$m = 2,\ b = 5,\ k = 1$

$\Rightarrow \quad 2\ddot{x} + 5\dot{x} + x = Fe$

assume zero ic, $x(0) = 0$, $\dot{x}(0) = 0$

unit step $Fe(t) = 1$

Laplace : $2s^2X(s) + 5sX(s) + X(s) = F_e(s)$

$X(s)(2s^2 + 5s + 1) = Fe(s)$

TF $\quad \dfrac{X(s)}{Fe(s)} = \dfrac{1}{(2s^2 + 5s + 1)}$

poles $\Rightarrow$ roots of denominator : $(2s^2 + 5s + 1) = 0$

using $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

poles @ -2.2808, -0.2192, LHP $\therefore$ stable system

FVT valid

$X(s) = \dfrac{1}{(2s^2 + 5s + 1)} Fe(s)$

$fe(t) = 1 \xrightarrow[\mathcal{L}]{} Fe(s) = \dfrac{1}{s}$

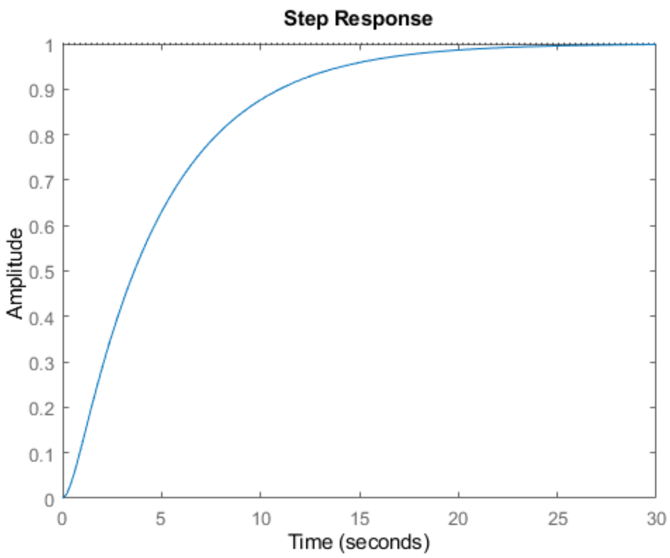
$$\Rightarrow \quad X(s) = \frac{1}{(2s^2+5s+1)}\left(\frac{1}{s}\right)$$

FVT $\lim_{t \to \infty} (y(t)) = \lim_{s \to 0} (s\,Y(s))$

$$= \cancel{s}\,\frac{1}{(2\cancel{s}^2+5\cancel{s}+1)}\left(\frac{1}{\cancel{s}}\right)\Bigg|_{s=0} = \underline{1} \ast \quad \text{Final value.}$$

check with sim

Simulation result



(b)

given $\frac{3}{4}\dot{v}_o(t) + \frac{1}{2}v_o(t) = \frac{1}{2}\dot{i}_i(t) + i_i(t)$

initial cond $v_o(0) = 0$, $\dot{v}_o(0) = 0$, $\dot{i}_i(0) = 0$, $i_i(0) = 0$

input $i_i(t) = 1$ (unit step)

Laplace: $\frac{3}{4}sV_o(s) + \frac{1}{2}V_o(s) = \frac{1}{2}sI_i(s) + I_i(s)$

$$V_o(s)\left(\frac{3}{4}s + \frac{1}{2}\right) = I_i(s)\left(\frac{1}{2}s + 1\right)$$

Transfer func

$$\frac{V_o(s)}{I_i(s)} = \frac{\left(\frac{1}{2}s+1\right)}{\left(\frac{3}{4}s+\frac{1}{2}\right)}$$

Poles: roots of denominator

roots of $\left(\frac{3}{4}s+\frac{1}{2}\right)=0$

$$s = -\frac{1}{\cancel{2}}\left(\frac{\overset{2}{\cancel{4}}}{3}\right) = -\frac{2}{3} = -0.6667$$

Stable pole (LHP) $\therefore$ FVT applies

$$V_o(s) = \frac{\left(\frac{1}{2}s+1\right)}{\left(\frac{3}{4}s+\frac{1}{2}\right)} I_i(s) \quad , \quad \text{input } i_i(t) = 1$$

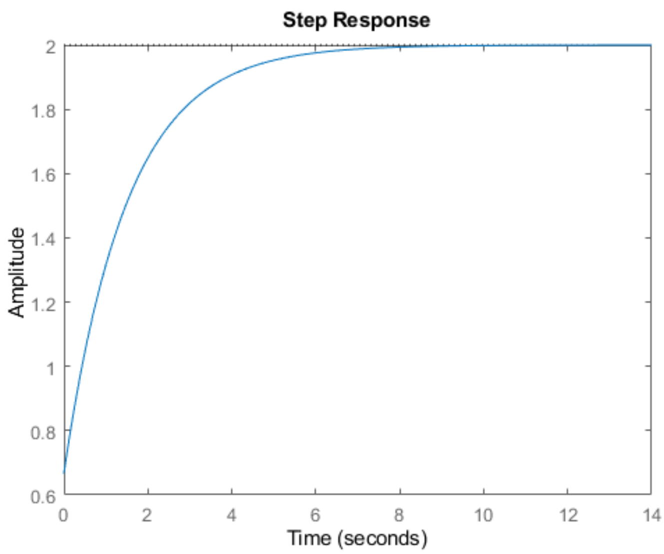
$$I_i(s) = \frac{1}{s}$$

FVT

$$\lim_{t\to\infty}\left(v_o(t)\right) = \lim_{s\to 0}\left(sV_o(s)\right) = \cancel{s}\,\frac{\left(\frac{1}{2}\cancel{s}+1\right)}{\left(\frac{3}{4}\cancel{s}+\frac{1}{2}\right)}\left(\frac{1}{\cancel{s}}\right)\Bigg|_{s=0} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

Final value
To check with sim

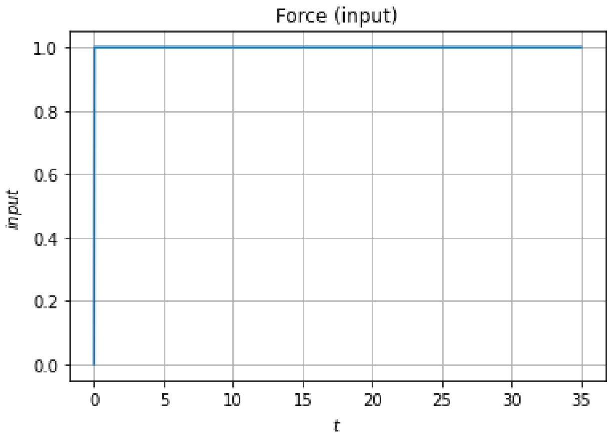
Simulation results

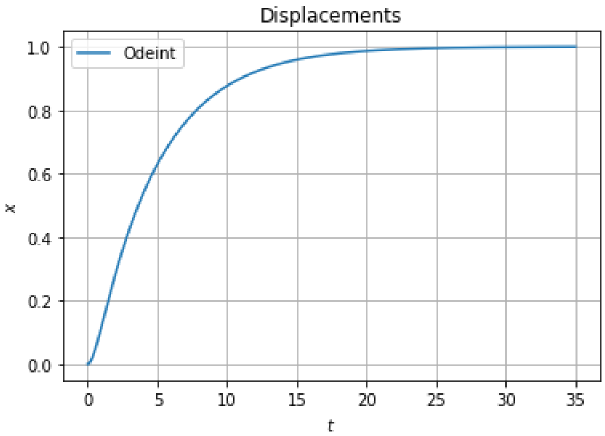


Python code for Q6(a) mechanical system

```
#import numpy, integrate and pyplot
import numpy as np
from scipy import integrate
from matplotlib import pyplot as plt

#sim data
m=2 #mass of the oscillator
b=5 #damper of the oscillator
k=1 #stiffness of the oscillator

#create 1000 uniformly distributed points in the interval [0,35]
t = np.linspace(0,35,1000)

#lambda returning the force
force = lambda t:t>=0.01 #unit step

#lambda with the RHS of the eq., it evaluates the force and returns a list
f = lambda X,t: [X[1],1/m*(force(t)-k*X[0]-b*X[1])]

#solve equation using odeint
XOdeint = integrate.odeint(f,[0,0],t)

plt.plot(t,force(t))
plt.title('Force (input)')
plt.xlabel('$t$')
plt.ylabel('$input$')
plt.grid()
plt.show()

#plot displacements from both solutions (first column of each solution)
plt.plot(t,XOdeint[:,0])
plt.legend(['Odeint'])
plt.title('Displacements')
plt.xlabel('$t$')
plt.ylabel('$x$')
plt.grid()
plt.show()

#plot velocities from both solutions (first column of each solution)
plt.plot(t,XOdeint[:,1])
plt.legend(['Odeint'])
plt.title('Velocitiy')
plt.xlabel('$t$')
plt.ylabel('$\dot{x}$')
plt.grid()
plt.show()

#Save data
np.savetxt('t.txt',t)
np.savetxt('x.txt',XOdeint[:,0])
```

Simulation results





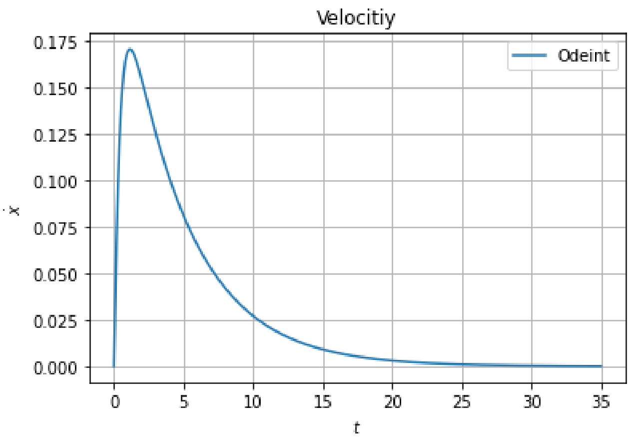


Table of Laplace Transforms

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$\frac{1}{s}$	$1(t)$
3	$\frac{1}{s^2}$	t
4	$\frac{2!}{s^3}$	t^2
5	$\frac{3!}{s^4}$	t^3
6	$\frac{m!}{s^{m+1}}$	t^m
7	$\frac{1}{(s+a)}$	e^{-at}
8	$\frac{1}{(s+a)^2}$	te^{-at}
9	$\frac{1}{(s+a)^3}$	$\frac{1}{2!}t^2e^{-at}$
10	$\frac{1}{(s+a)^m}$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\frac{a}{s(s+a)}$	$1-e^{-at}$
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at-1+e^{-at})$
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\frac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-at}-ae^{-at}$
17	$\frac{a}{(s^2+a^2)}$	$\sin at$
18	$\frac{s}{(s^2+a^2)}$	$\cos at$
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\frac{a^2+b^2}{s\left[(s+a)^2+b^2\right]}$	$1-e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right)$

Properties of Laplace Transforms

Number	Laplace Transform	Time Function	Comment
—	$F(s)$	$f(t)$	Transform pair
1	$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Superposition
2	$F(s)e^{-s\lambda}$	$f(t-\lambda)$	Time delay ($\lambda \geq 0$)
3	$\frac{1}{\lvert a\rvert} F\left(\frac{s}{a}\right)$	$f(at)$	Time scaling
4	$F(s+a)$	$e^{-at} f(t)$	Shift in frequency
5	$s^m F(s) - s^{m-1} f(0)$ $-s^{m-2} \dot{f}(0) - \cdots - f^{(m-1)}(0)$	$f^{(m)}(t)$	Differentiation
6	$\frac{1}{s} F(s)$	$\int_0^t f(\zeta)\, d\zeta$	Integration
7	$F_1(s)F_2(s)$	$f_1(t) * f_2(t)$	Convolution
8	$\lim_{s\to\infty} sF(s)$	$f(0^+)$	Initial Value Theorem
9	$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$	Final Value Theorem
10	$\frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F_1(\zeta)F_2(s-\zeta)\,d\zeta$	$f_1(t)f_2(t)$	Time product
11	$\frac{1}{2\pi} \int_{-j\infty}^{+j\infty} Y(-j\omega)U(j\omega)\, d\omega$	$\int_0^\infty y(t)u(t)\, dt$	Parseval's Theorem
12	$-\frac{d}{ds} F(s)$	$tf(t)$	Multiplication by time