ENG2009 – Modelling of Engineering Systems

Tutorial 3

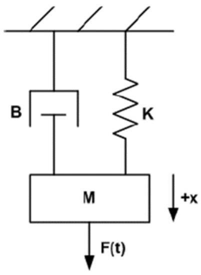
Laplace Transform and Transfer Function

Laplace Transform

Properties of Laplace Transform (2^{nd} order derivatives)

Example: The differential equation of the mass-spring-damper systems is (see Lecture 2):

$$\ddot{x}(t) = \frac{1}{M}(-B\dot{x}(t) - Kx(t) + u(t))$$



Physical Model

Find the Laplace transform of the system, assuming zero initial conditions ($x(0) = 0,\ \dot{x}(0) = 0$)

Solution:

Step 1 and 2: Laplace transform and apply initial condition: $x(0) = 0$ and $\dot{x}(0) = 0$,

Assuming zero initial conditions ($x(0) = 0,\ \dot{x}(0) = 0$):

- Let the applied force as input, i.e. $f(t) = u(t)$, therefore

$$\mathcal{L}(x(t)) = X(s)$$

$$\mathcal{L}(u(t)) = U(s)$$

- Using property $\mathcal{L}\left(\frac{df(t)}{dt}\right) = sF(s) - f(0)$, therefore

$$\mathcal{L}\left(\frac{dx(t)}{dt}\right) = \mathcal{L}(\dot{x}(t)) = sX(s) - x(0) = sX(s)$$

- Using property$\mathcal{L}\left(\frac{d^2f(t)}{dt^2}\right) = s^2F(s) - sf(0) - \dot{f}(0)$, therefore

$$\mathcal{L}\left(\frac{d^2x(t)}{dt^2}\right) = \mathcal{L}(\ddot{x}(t)) = s^2X(s) - sx(0) - \dot{x}(0) = s^2X(s)$$

Step 3: solve for $X(s)$

Therefore

$$\ddot{x}(t) = \frac{1}{M}(-B\dot{x}(t) - Kx(t) + u(t))$$

becomes

$$s^2X(s) = \frac{1}{M}\big(-BsX(s) - kX(s) + U(s)\big)$$

Rearrange to yield

$$X(s)(Ms^2 + Bs + K) = U(s)$$

■

Properties of Laplace Transform (Final value theorem)

Example: (Final Value Theorem)

Find the final value of the system corresponding to the following system, where the input $u(t)$ is an unit step i.e. $u(t) = 1$.

$$Y(s) = U(s)\frac{6}{s+2}$$

Hint: use Final Value Theorem

Solution:

System model represented in Laplace as

$$Y(s) = U(s)\frac{6}{s+2}$$

Step 1: Simple check shows that this system is stable as the denominator has a root at left half of s-plane, i.e. $s = -2$. Therefore Final Value Theorem is valid.

Step 2: The input $u(t)$ is an unit step i.e. $u(t) = 1$. From Laplace table

$$\mathcal{L}\big(u(t)\big) = \mathcal{L}(1) = \frac{1}{s}$$

Therefore,

$$Y(s) = \frac{1}{s}\left(\frac{6}{s+2}\right)$$

Step 3: Therefore, the final value of $y(t)$ is:

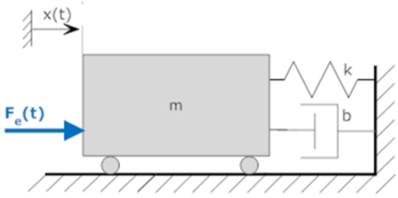
$$\lim_{t\to\infty}(y(t)) = \lim_{s\to 0}\big(sY(s)\big) = s\frac{1}{s}\left(\frac{6}{s+2}\right)\Big|_{s=0} = \frac{6}{2} = 3$$

Thus, after the transients have decayed to zero, $y(t)$ will settle to a constant value of 3.

■

Solving Differential equation

Example: Consider the mechanical systems below, where input=force $f(t) = u(t)$, output=position of mass $x(t)$ and $m = 1,\ b = 2,\ k = 10$:



The differential equation is given by:

$$\ddot{x}(t) + 2\dot{x}(t) + 10x(t) = u(t)$$

Assume that the initial conditions are: $x(0) = 0,\ \dot{x}(0) = 1, u(t) = 0$.

Find the position $x(t)$ over time t.

Solution:

The differential equation:

$$\ddot{x}(t) + 2\dot{x}(t) + 10x(t) = u(t)$$

Given $x(0) = 0,\ \dot{x}(t) = 1, u(t) = 0$.

Step 1:

- $\mathcal{L}\big(x(t)\big) = X(s)$
- $\mathcal{L}\big(\dot{x}(t)\big) = sX(s) - x(0)$: 1st derivative property
- $\mathcal{L}\big(\ddot{x}(t)\big) = s^2X(s) - sx(0) - \dot{x}(0)$: 2nd derivative property
- $\mathcal{L}\big(u(t)\big) = 0$

Laplace of $\ddot{x}(t) + 2\dot{x}(t) + 10x(t) = 0$:

$$\big(s^2X(s) - sx(0) - \dot{x}(0)\big) + 2\big(sX(s) - x(0)\big) + 10\big(X(s)\big) = 0$$

Step 2: apply initial condition: $x(0) = 0$ and $\dot{x}(0) = 1$,

$$\big(s^2X(s) - sx(0) - \dot{x}(0)\big) + 2\big(sX(s) - x(0)\big) + 10\big(X(s)\big) = 0$$

$$(s^2X(s) - s.0 - 1) + 2(sX(s) - 0) + 10\big(X(s)\big) = 0$$

$$s^2X(s) - 1 + 2sX(s) + 10X(s) = 0$$

Step 3: solve for $X(s)$

$$s^2X(s) - 1 + 2sX(s) + 10X(s) = 0$$

$$s^2X(s) + 2sX(s) + 10X(s) = 1$$

$$X(s)(s^2 + 2s + 10) = 1$$

$$X(s) = \frac{1}{s^2+2s+10}$$

Step 4: get the results from Laplace Transform Table

By completing the square we can rewrite the denominator

$$X(s) = \frac{1}{s^2 + 2s + 10}$$

$$= \frac{1}{(s+1)^2 + 9} = \frac{1}{(s+1)^2 + 3^2}$$

From table of Laplace transform no. (20): $\frac{b}{(s+a)^2+b^2} \overset{\mathcal{L}}{\leftrightarrow} e^{-at}\sin(bt),\ a = 1,\ b = 3$

Therefore

$$\frac{1}{3}\frac{3}{(s+1)^2 + 3^2} \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{3}e^{-1t}\sin(3t)$$

Therefore the solution for the differential equation $\ddot{x}(t) + 2\dot{x}(t) + 10x(t) = u(t)$ is:

$$\boldsymbol{x(t) = \frac{1}{3}e^{-1t}\sin(3t)}$$

■

Exercises:

Q1) Solve the differential equation (using inverse Laplace transform)

$$\ddot{y}(t) + y(t) = 0$$

where $y(0) = 1, \dot{y}(0) = 2$

Q2) Use Laplace transform to solve

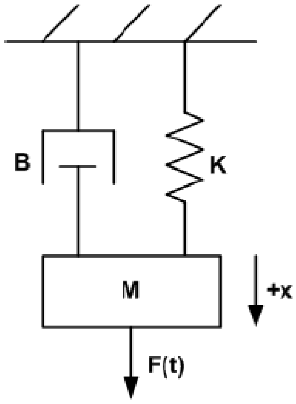
$$\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = e^{-3t}$$

where $y(0) = 1$ and $\dot{y}(0) = 1.$

Hint: use *partial fraction*

<u>Transfer Function:</u>

Example : Obtain the differential equation of the mass-spring-damper system below (Lecture 2) and evaluate its transfer function. (position $x(t)$ is the output and applied force $u(t)$ is the input):



Physical Model

Solution:

Step 1:

The differential equation of the mass-spring-damper systems in Lecture 2 is (position $x(t)$ is the output and applied force $u(t)$ is the input):

$$\ddot{x}(t) = \frac{1}{M}(-B\dot{x}(t) - Kx(t) + u(t))$$

Step 2:

In Laplace form, assuming *zero initial conditions* ($x(0) = 0,\ \dot{x}(0) = 0$):

$$s^2X(s) = \frac{1}{M}\big(-BsX(s) - kX(s) + U(s)\big)$$

Step 3:

Manipulate to yield the transfer function:

$$\frac{X(s)}{U(s)} = \frac{1}{Ms^2 + Bs + K}$$

■

Exercises:

Q3) Find the transfer functions for the following systems. Then analyse if the systems is stable.

a) $\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = 3u(t)$

b) $\dddot{y}(t) - \ddot{y}(t) + 3\dot{y}(t) + 5y(t) = 2\dot{u}(t) + 7u(t)$

Table of Laplace Transforms

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$\dfrac{1}{s}$	$1(t)$
3	$\dfrac{1}{s^2}$	t
4	$\dfrac{2!}{s^3}$	t^2
5	$\dfrac{3!}{s^4}$	t^3
6	$\dfrac{m!}{s^{m+1}}$	t^m
7	$\dfrac{1}{(s+a)}$	e^{-at}
8	$\dfrac{1}{(s+a)^2}$	te^{-at}
9	$\dfrac{1}{(s+a)^3}$	$\dfrac{1}{2!}t^2e^{-at}$
10	$\dfrac{1}{(s+a)^m}$	$\dfrac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\dfrac{a}{s(s+a)}$	$1-e^{-at}$
12	$\dfrac{a}{s^2(s+a)}$	$\dfrac{1}{a}(at-1+e^{-at})$
13	$\dfrac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$
14	$\dfrac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\dfrac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$
16	$\dfrac{(b-a)s}{(s+a)(s+b)}$	$be^{-at}-ae^{-at}$
17	$\dfrac{a}{(s^2+a^2)}$	$\sin at$
18	$\dfrac{s}{(s^2+a^2)}$	$\cos at$
19	$\dfrac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\dfrac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\dfrac{a^2+b^2}{s\left[(s+a)^2+b^2\right]}$	$1-e^{-at}\left(\cos bt+\dfrac{a}{b}\sin bt\right)$

Properties of Laplace Transforms

Number	Laplace Transform	Time Function	Comment
—	$F(s)$	$f(t)$	Transform pair
1	$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Superposition
2	$F(s)e^{-s\lambda}$	$f(t-\lambda)$	Time delay ($\lambda \geq 0$)
3	$\frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right)$	$f(at)$	Time scaling
4	$F(s+a)$	$e^{-at} f(t)$	Shift in frequency
5	$s^m F(s) - s^{m-1} f(0) - s^{m-2} \dot{f}(0) - \cdots - f^{(m-1)}(0)$	$f^{(m)}(t)$	Differentiation
6	$\frac{1}{s} F(s)$	$\int_0^t f(\zeta)\, d\zeta$	Integration
7	$F_1(s) F_2(s)$	$f_1(t) * f_2(t)$	Convolution
8	$\lim_{s\to\infty} sF(s)$	$f(0^+)$	Initial Value Theorem
9	$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$	Final Value Theorem
10	$\frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F_1(\zeta) F_2(s-\zeta) d\zeta$	$f_1(t) f_2(t)$	Time product
11	$\frac{1}{2\pi} \int_{-j\infty}^{+j\infty} Y(-j\omega) U(j\omega)\, d\omega$	$\int_0^\infty y(t) u(t)\, dt$	Parseval's Theorem
12	$-\frac{d}{ds} F(s)$	$t f(t)$	Multiplication by time