

ENG2009 – Modelling of Engineering Systems

Tutorial 8

Finite Difference and Shooting Method

Finite Difference

Example

Consider a thick pressure vessel (thickness t) that is being tested in the laboratory to check its ability to withstand pressure.

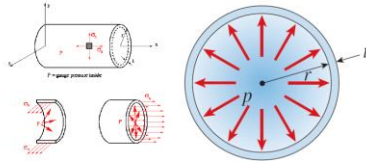
The inner radius $r = a$ and outer radius $r = b$.

The differential eq. for the radial displacement u of a point along the thickness is given by:

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = 0$$

Assume

- the inner radius $a = 5$ and the outer radius $b = 8$,
- the material of the pressure vessel is ASTM A36 steel with the yield strength of 36 ksi.



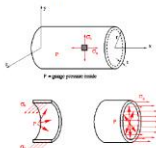
Finite Difference Method

An example of a boundary value ordinary differential equation is (pressure vessel)

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = 0$$

with boundary conditions:

$$u(5) = 0.008731, \\ u(8) = 0.0030769$$



where u is the radial displacement, r is the radius

In general, the derivatives in such ordinary differential equation are substituted by **finite divided differences approximations**:

$$\frac{dy}{dx} \approx \frac{y_{i+1} - y_i}{\Delta x} \\ \frac{d^2 y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}$$

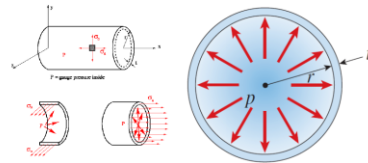
Example

Two strain gages that are bonded tangentially at the inner and the outer radius measure normal tangential strain which translates to radial displacements at a and b given by (boundary conditions):

$$u|_{r=a} = 0.003871 \text{ and } u|_{r=b} = 0.0030769$$

Task:

- Divide the radial thickness of the pressure vessel into 6 equidistant nodes,
- and find the radial displacement profile u along the thickness of the pressure vessel



Solution Cont

The example before uses the finite divided differences approximations

$$\frac{d^2 u}{dr^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} \\ \frac{du}{dr} \approx \frac{u_{i+1} - u_i}{\Delta r}$$

Which is first order accurate.

A better approximate is:

$$\frac{d^2 u}{dr^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} \\ \frac{du}{dr} \approx \frac{u_{i+1} - u_{i-1}}{2\Delta r}$$

Which is second order accurate.

Solution Cont

From

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = 0$$

And using the second order accurate approximation yield

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} + \frac{1}{r_i} \frac{u_{i+1} - u_{i-1}}{2(\Delta r)} - \frac{u_i}{r_i^2} = 0$$

which yield

$$\left(-\frac{1}{2r_i(\Delta r)} + \frac{1}{(\Delta r)^2}\right)u_{i-1} + \left(-\frac{2}{(\Delta r)^2} - \frac{1}{r_i^2}\right)u_i + \left(\frac{1}{(\Delta r)^2} + \frac{1}{2r_i\Delta r}\right)u_{i+1} = 0 \text{ (eq**)}$$

vs

$$\frac{1}{(\Delta r)^2}u_{i-1} + \left(-\frac{2}{(\Delta r)^2} - \frac{1}{r_i\Delta r} - \frac{1}{r_i^2}\right)u_i + \left(\frac{1}{(\Delta r)^2} + \frac{1}{r_i\Delta r}\right)u_{i+1} = 0 \text{ (eq*)}$$

Repeat the example with the second order accurate approximation...

Solution

- Step 4 at node $i = 3$,

$$r_3 = r_2 + \Delta r = 6.2 + 0.6 = 6.8$$

$$\text{Therefore using } \left(-\frac{1}{2r_i(\Delta r)} + \frac{1}{(\Delta r)^2}\right)u_{i-1} + \left(-\frac{2}{(\Delta r)^2} - \frac{1}{r_i^2}\right)u_i + \left(\frac{1}{(\Delta r)^2} + \frac{1}{2r_i\Delta r}\right)u_{i+1} = 0 \text{ (eq**)}$$

$$\left(-\frac{1}{2(6.8)(0.6)} + \frac{1}{0.6^2}\right)u_2 + \left(-\frac{2}{0.6^2} - \frac{1}{6.8^2}\right)u_3 + \left(\frac{1}{0.6^2} + \frac{1}{2(6.8)(0.6)}\right)u_4 = 0$$

$$2.6552u_2 - 5.5772u_3 + 2.9003u_4 = 0$$

- Step 5 at node $i = 4$,

$$r_4 = r_3 + \Delta r = 6.8 + 0.6 = 7.4$$

$$\text{Therefore using } \left(-\frac{1}{2r_i(\Delta r)} + \frac{1}{(\Delta r)^2}\right)u_{i-1} + \left(-\frac{2}{(\Delta r)^2} - \frac{1}{r_i^2}\right)u_i + \left(\frac{1}{(\Delta r)^2} + \frac{1}{2r_i\Delta r}\right)u_{i+1} = 0 \text{ (eq**)}$$

$$\left(-\frac{1}{2(7.4)(0.6)} + \frac{1}{0.6^2}\right)u_3 + \left(-\frac{2}{0.6^2} - \frac{1}{7.4^2}\right)u_4 + \left(\frac{1}{0.6^2} + \frac{1}{2(7.4)(0.6)}\right)u_5 = 0$$

$$2.6651u_3 - 5.5738u_4 + 2.8903u_5 = 0$$

- Step 6 at node $i = 5$,

$$r_5 = r_4 + \Delta r = 7.4 + 0.6 = 8$$

$$u_5 = u|_{r=b} = 0.0030769 \text{ (boundary condition at } b)$$

Solution

- Step 1 at node $i = 0$,

$$r_0 = a = 5,$$

$$u_0 = u|_{r=a} = 0.0038731 \text{ (boundary condition at } a)$$

- Step 2 at node $i = 1$,

$$r_1 = r_0 + \Delta r = 5 + 0.6 = 5.6$$

$$\text{Therefore using } \left(-\frac{1}{2r_i(\Delta r)} + \frac{1}{(\Delta r)^2}\right)u_{i-1} + \left(-\frac{2}{(\Delta r)^2} - \frac{1}{r_i^2}\right)u_i + \left(\frac{1}{(\Delta r)^2} + \frac{1}{2r_i\Delta r}\right)u_{i+1} = 0 \text{ (eq**)}$$

$$\left(-\frac{1}{2(5.6)(0.6)} + \frac{1}{0.6^2}\right)u_0 + \left(-\frac{2}{0.6^2} - \frac{1}{5.6^2}\right)u_1 + \left(\frac{1}{0.6^2} + \frac{1}{2(5.6)(0.6)}\right)u_2 = 0$$

$$2.6297u_0 - 5.5874u_1 + 2.9266u_2 = 0$$

- Step 3 at node $i = 2$,

$$r_2 = r_1 + \Delta r = 5.6 + 0.6 = 6.2$$

$$\text{Therefore using } \left(-\frac{1}{2r_i(\Delta r)} + \frac{1}{(\Delta r)^2}\right)u_{i-1} + \left(-\frac{2}{(\Delta r)^2} - \frac{1}{r_i^2}\right)u_i + \left(\frac{1}{(\Delta r)^2} + \frac{1}{2r_i\Delta r}\right)u_{i+1} = 0 \text{ (eq**)}$$

$$\left(-\frac{1}{2(6.2)(0.6)} + \frac{1}{0.6^2}\right)u_1 + \left(-\frac{2}{0.6^2} - \frac{1}{6.2^2}\right)u_2 + \left(\frac{1}{0.6^2} + \frac{1}{2(6.2)(0.6)}\right)u_3 = 0$$

$$2.6434u_1 - 5.5816u_2 + 2.9122u_3 = 0$$

Solving system of equations

Using the second order accurate approximation, yield in the matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2.6297 & -5.5874 & 2.9266 & 0 & 0 & 0 \\ 0 & 2.6434 & -5.5816 & 2.9122 & 0 & 0 \\ 0 & 0 & 2.6552 & -5.5772 & 2.9003 & 0 \\ 0 & 0 & 0 & 2.6651 & -5.5738 & 2.8903 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0.0038731 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0030769 \end{bmatrix}$$

Solving simultaneous linear equations (see previous lectures e.g. $[A][X] = [C]$ solve for $[X]$), to yield

$$\begin{aligned} u_0 &= 0.0038731 \\ u_1 &= 0.0036115 \\ u_2 &= 0.0034159 \\ u_3 &= 0.0032689 \\ u_4 &= 0.0031586 \\ u_5 &= 0.0030769 \end{aligned}$$

Exercises

Q1) Given

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2, y(0) = 0, y(12) = 0$$

the value of $\frac{d^2y}{dx^2}$ at $y(4)$ using the finite difference method and a step size of $h = 4$ can be approximated by

- (A) $\frac{y(8)-y(0)}{8}$
- (B) $\frac{y(8)-2y(4)+y(0)}{16}$
- (C) $\frac{y(12)-2y(8)+y(4)}{16}$
- (D) $\frac{y(4)-y(0)}{4}$

Q2) The transverse deflection u of a cable of length L that is fixed at both ends, is given as a solution to

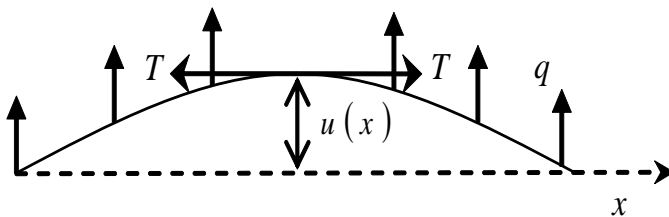
$$\frac{d^2u}{dx^2} = \frac{Tu}{R} + \frac{qx(x-L)}{2R}$$

where

T = tension in cable

R = flexural stiffness

q = distributed transverse load



Given $L = 50$, $T = 2000$, $q = 75$, and $R = 75 \times 10^6$.

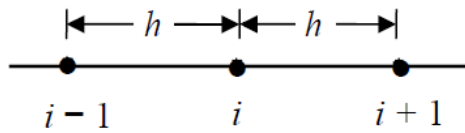
Using finite difference method modelling with second-order central divided difference accuracy and a step size of $h = 12.5$, the value of the deflection at the centre of the cable most likely is

- (A) 0.072737
- (B) 0.080832
- (C) 0.081380
- (D) 0.084843

Solutions:

Q1)

The correct answer is (B).



The finite difference approximation for $\frac{d^2y}{dx^2}$ at node i , that is $x = x_i$, is given by

$$\left. \frac{d^2y}{dx^2} \right|_i \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

where

$$y_{i+1} = y(8)$$

$$y_i = y(4)$$

$$y_{i-1} = y(0)$$

$$h = 4$$

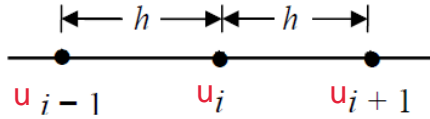
Thus

$$\begin{aligned} \frac{d^2y}{dx^2} &\approx \frac{y(8) - 2y(4) + y(0)}{(4)^2} \\ &\approx \frac{y(8) - 2y(4) + y(0)}{16} \end{aligned}$$

Solutions:

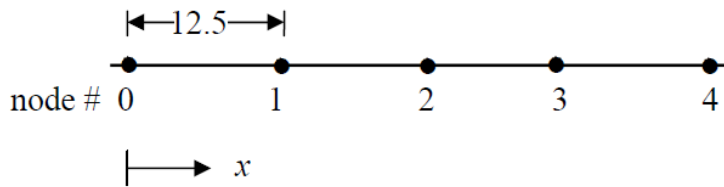
Q2)

The correct answer is (D).



At node i , using finite difference approximations,

$$\left. \frac{d^2 u}{dx^2} \right|_i \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = \frac{(2000)u_i}{(75 \times 10^6)} + \frac{75 \times x_i(x_i - 50)}{2(75 \times 10^6)}$$



For $i = 0$, $x_0 = 0$

$$u_0 = 0 \quad (1)$$

For $i = 1$, $x_1 = 12.5$

$$\frac{u_2 - 2u_1 + u_0}{h^2} = \frac{(2000)u_1}{(75 \times 10^6)} + \frac{75 \times x_1(x_1 - 50)}{2(75 \times 10^6)}$$

$$\frac{u_2 - 2u_1 + u_0}{12.5^2} = \frac{(2000)u_1}{(75 \times 10^6)} + \frac{75 \times 12.5(12.5 - 50)}{2(75 \times 10^6)}$$

$$\frac{u_2 - 2u_1 + u_0}{156.25} = 2.6667 \times 10^{-5} u_1 - 2.3438 \times 10^{-4}$$

$$u_2 - 2u_1 + u_0 = 4.1667 \times 10^{-3} u_1 - 3.6621 \times 10^{-2}$$

$$u_2 - 2.0042u_1 + u_0 = -3.6621 \times 10^{-2} \quad (2)$$

For $i = 2$, $x_2 = 25$

$$\frac{u_3 - 2u_2 + u_1}{h^2} = \frac{(2000)u_2}{(75 \times 10^6)} + \frac{75 \times x_2(x_2 - 50)}{2(75 \times 10^6)}$$

$$\frac{u_3 - 2u_2 + u_1}{12.5^2} = \frac{(2000)u_2}{(75 \times 10^6)} + \frac{75 \times 25(25 - 50)}{2(75 \times 10^6)}$$

$$\frac{u_3 - 2u_2 + u_1}{156.25} = 2.6667 \times 10^{-5} u_2 - 3.1250 \times 10^{-4}$$

$$u_3 - 2u_2 + u_1 = 4.1667 \times 10^{-3} u_2 - 4.8828 \times 10^{-2}$$

$$u_3 - 2.0042u_2 + u_1 = -4.8828 \times 10^{-2} \quad (3)$$

For $i = 3$, $x_3 = 37.5$

$$\begin{aligned}
 \frac{u_4 - 2u_3 + u_2}{h^2} &= \frac{(2000)u_3}{(75 \times 10^6)} + \frac{75 \times x_3(x_3 - 50)}{2(75 \times 10^6)} \\
 \frac{u_4 - 2u_3 + u_2}{12.5^2} &= \frac{(2000)u_3}{(75 \times 10^6)} + \frac{75 \times 37.5(37.5 - 50)}{2(75 \times 10^6)} \\
 \frac{u_4 - 2u_3 + u_2}{156.25} &= 2.6667 \times 10^{-5} u_3 - 2.3438 \times 10^{-4} \\
 u_4 - 2u_3 + u_2 &= 4.1667 \times 10^{-3} u_3 - 3.6621 \times 10^{-2} \\
 u_4 - 2.0042u_3 + u_2 &= -3.6621 \times 10^{-2}
 \end{aligned} \tag{4}$$

For $i = 4$, $x_4 = 50$

$$u_4 = 0'' \tag{5}$$

Rewrite Equations 1-5 in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2.0042 & 1 & 0 & 0 \\ 0 & 1 & -2.0042 & 1 & 0 \\ 0 & 0 & 1 & -2.0042 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -3.6621 \times 10^{-2} \\ -4.8828 \times 10^{-2} \\ -3.6621 \times 10^{-2} \\ 0 \end{bmatrix}$$

Solving this matrix by using Gauss elimination gives

$$u_0 = 0''$$

$$u_1 = 0.060605''$$

$$u_2 = 0.084843''$$

$$u_3 = 0.060605''$$

$$u_4 = 0''$$

Hence

$$u(25) \approx u_2 = 0.084843''$$

Shooting Method

Example

Consider the pressure vessel example, where

$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = 0,$$

with boundary conditions

$$u(5) = 0.0038731,$$

$$u(8) = 0.0030770$$

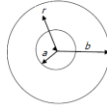
Let

$$\frac{du}{dr} = w$$

Then

$$\frac{dw}{dr} + \frac{1}{r}w - \frac{u}{r^2} = 0$$

The idea is to use the methods in initial value problems, to solve boundary value problems.



Where $a = 5$
and $b = 8$

Solution

Using Euler's method,

$$u_{i+1} = u_i + f_1(r_i, u_i, w_i)h$$

$$w_{i+1} = w_i + f_2(r_i, u_i, w_i)h$$

Let us consider 4 segments between the two boundaries, $r = 5$ and $r = 8$ then,

$$h = \frac{8-5}{4} = 0.75$$

Solution

For

$$i = 1, r_1 = r_0 + h = 5 + 0.75 = 5.75,$$

$$u_1 = 0.0036741, w_1 = -0.00010940$$

$$\begin{aligned} u_2 &= u_1 + f_1(r_1, u_1, w_1)h \\ &= 0.0036741 \\ &= 0.0036741 + (-0.00010938)(0.75) \\ &= 0.0035920 \end{aligned}$$

$$\begin{aligned} w_2 &= w_1 + f_2(r_1, u_1, w_1)h \\ &= -0.00010938 \\ &= -0.00010938 + (0.00013015)(0.75) \\ &= -0.000011769 \end{aligned}$$

Solution

For

$$i = 3, r_3 = r_2 + h = 6.50 + 0.75 = 7.25$$

$$u_3 = 0.0035832, w_3 = 0.000053332$$

$$\begin{aligned} u_4 &= u_3 + f_1(r_3, u_3, w_3)h \\ &= 0.0035832 + f_1(7.25, 0.0035832, 0.000053332)(0.75) \\ &= 0.0035832 + (0.000053332)(0.75) \\ &= 0.0036232 \end{aligned}$$

$$\begin{aligned} w_4 &= w_3 + f_2(r_3, u_3, w_3)h \\ &= -0.000011785 \\ &= 0.000053332 + (0.000060811)(0.75) \\ &= 0.000098961 \end{aligned}$$

So at

$$r = r_4 = r_3 + h = 7.25 + 0.75 = 8$$

$$u(8) \approx u_4 = 0.0036232$$

Actual boundary conditions
 $u(5) = 0.0038731,$
 $u(8) = 0.0030770$

Solution

Two first order differential equations are given as

$$\frac{du}{dr} = w, u(5) = 0.0038731$$

$$\frac{dw}{dr} = -\frac{w}{r} + \frac{u}{r^2}, w(5) = \text{not known}$$

Let us assume

$$w(5) = \frac{du}{dr}(5) \approx \frac{u(8) - u(5)}{8 - 5} = -0.00026538$$

To set up initial value problem

$$\frac{du}{dr} = w = f_1(r, u, w), u(5) = 0.0038731$$

$$\frac{dw}{dr} = -\frac{w}{r} + \frac{u}{r^2} = f_2(r, u, w), w(5) = -0.00026538$$

Solution

For

$$i = 0, r_0 = 5, u_0 = 0.0038731, w_0 = -0.00026538$$

$$\begin{aligned} u_1 &= u_0 + f_1(r_0, u_0, w_0)h \\ &= 0.0038731 + f_1(5, 0.0038731, -0.00026538)(0.75) \\ &= 0.0038731 + (-0.00026538)(0.75) \\ &= 0.0036741 \end{aligned}$$

$$\begin{aligned} w_1 &= w_0 + f_2(r_0, u_0, w_0)h \\ &= -0.00026538 + f_2(5, 0.0038731, -0.00026538)(0.75) \\ &= -0.00026538 + \left(\frac{-0.00026538}{5} + \frac{0.0038731}{5^2} \right)(0.75) \\ &= -0.00010938 \end{aligned}$$

Solution

For

$$i = 2, r_2 = r_1 + h = 5.75 + 0.75 = 6.5$$

$$u_2 = 0.0035920, w_2 = -0.000011785$$

$$\begin{aligned} u_3 &= u_2 + f_1(r_2, u_2, w_2)h \\ &= 0.0035920 \\ &= 0.0035920 + (-0.000011769)(0.75) \\ &= 0.0035832 \end{aligned}$$

$$\begin{aligned} w_3 &= w_2 + f_2(r_2, u_2, w_2)h \\ &= -0.000011769 \\ &= -0.000011769 + (0.000086829)(0.75) \\ &= 0.000053352 \end{aligned}$$

Solution

Repeat the process...

Let us assume a new value for $\frac{du}{dr}(5)$

$$\begin{aligned} w(5) &= 2 \frac{du}{dr}(5) \approx 2 \frac{u(8) - u(5)}{8 - 5} \\ &= 2(-0.00026538) \\ &= -0.00053076 \end{aligned}$$

$$\text{Before was } w(5) = \frac{du}{dr}(5) \approx \frac{u(8) - u(5)}{8 - 5} = -0.00026538$$

Using $h = 0.75$ and Euler's method, we get

$$u(8) \approx u_4 = 0.0029665''$$

While the given value of this boundary condition is

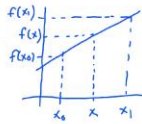
$$u(8) \approx u_4 = 0.0030770$$

Actual boundary conditions
 $u(5) = 0.0038731,$
 $u(8) = 0.0030770$

Solution

Using linear interpolation

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x) - f(x_0)}{x - x_0}$$



Therefore, rearranging to yield

$$f(x) = \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) (x - x_0) + f(x_0)$$

Recall:

$$w(5) = \frac{du}{dr}(5) \approx \frac{u(8) - u(5)}{8 - 5} = -0.00026538$$

To yield $u(8) \approx u_4 = 0.0029665$

and

$$w(5) = 2 \frac{du}{dr}(5) \approx 2 \frac{u(8) - u(5)}{8 - 5} = 2(-0.00026538) = -0.00053076$$

To yield $u(8) \approx u_4 = 0.0036232$

Actual boundary conditions
 $u(5) = 0.0038731$,
 $u(8) = 0.0030770$

Solution

Using $h = 0.75$ and repeating the Euler's method with

$$w(5) = -0.00048611$$

$$u(8) \approx u_4 = 0.0030769$$

Actual boundary conditions
 $u(5) = 0.0038731$,
 $u(8) = 0.0030770$

Using linear interpolation to refine the value of u_4

till one gets close to the actual value of $u(8)$ which gives you

$$\begin{aligned} u(5) &= u_1 = 0.0038731 \\ u(5.75) &\approx u_2 = 0.0035085 \\ u(6.50) &\approx u_3 = 0.0032858 \\ u(7.25) &\approx u_4 = 0.0031518 \\ u(8.00) &\approx u_5 = 0.0030770 \end{aligned}$$

Solution

Using linear interpolation on the obtained data for the two assumed values of

$$\frac{du}{dr}(5)$$

We get

$$\begin{aligned} w(5) &= \frac{du}{dr}(5) = f(x) = \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) (x - x_0) + f(x_0) \\ &\approx \frac{-0.00053076 - (-0.00026538)}{0.0030770 - 0.0029665} (0.0030770 - 0.0036232) + (-0.00026538) \\ &= -0.00048611 \end{aligned}$$

Recall:

$$w(5) = \frac{du}{dr}(5) \approx \frac{u(8) - u(5)}{8 - 5} = -0.00026538$$

To yield $u(8) \approx u_4 = 0.0029665$

and

$$w(5) = 2 \frac{du}{dr}(5) \approx 2 \frac{u(8) - u(5)}{8 - 5} = 2(-0.00026538) = -0.00053076$$

To yield $u(8) \approx u_4 = 0.0036232$

Actual boundary conditions
 $u(5) = 0.0038731$,
 $u(8) = 0.0030770$

Exercises

Q3) Given

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2, y(0) = 0, y(12) = 0$$

If one was using the shooting method with Euler's method with a step size of $h = 4$, and an assumed value of $\frac{dy}{dx}(0) = 20$, then the estimated value of $y(12)$ in the first iteration most likely is

- (A) 60.00
- (B) 496.0
- (C) 1088
- (D) 1102

Q4) The transverse deflection, u of a cable of length, L , fixed at both ends, is given as a solution to

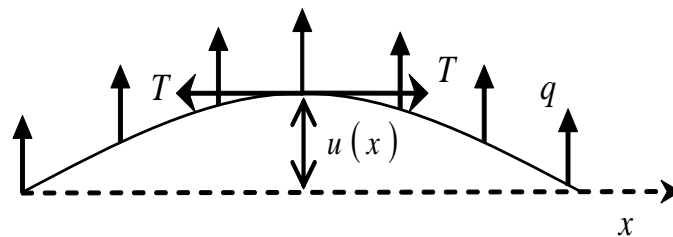
$$\frac{d^2u}{dx^2} = \frac{Tu}{R} + \frac{qx(x-L)}{2R}$$

where

T = tension in cable

R = flexural stiffness

q = distributed transverse load



Given are $L = 50$, $T = 2000$, $q = 75$, $R = 75 \times 10^6$. The shooting method is used with Euler's method assuming a step size of $h = 12.5$. Initial slope guesses at $x = 0$ of $\frac{du}{dx} = 0.003$ and $\frac{du}{dx} = 0.004$ are used in order, and then refined for the next iteration using linear interpolation after the value of $u(L)$ is found. The deflection in inches at the centre of the cable found during the second iteration is most likely is:

- (A) 0.075000
- (B) 0.10000
- (C) -0.061291
- (D) 0.00048828

Solutions:

Q3)

The correct answer is (B)

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2$$

Let

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = 6x - 0.5x^2$$

$$\frac{dy}{dx} = z = f_1(x, y, z), y(0) = 0$$

$$\frac{dz}{dx} = 6x - 0.5x^2 = f_2(x, y, z), \frac{dy}{dx}(0) = z(0) = 20$$

$$y_{i+1} = y_i + f_1(x_i, y_i, z_i)h$$

$$z_{i+1} = z_i + f_2(x_i, y_i, z_i)h$$

For $i = 0, x_0 = 0, y_0 = 0, z_0 = 20$

$$x_1 = x_0 + h$$

$$= 0 + 4$$

$$= 4$$

$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$

$$= 0 + f(0, 0, 20) \times 4$$

$$= 0 + (20) \times 4$$

$$= 80$$

$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$

$$= 20 + f_2(0, 0, 20) \times 4$$

$$= 20$$

For $i = 1, x_1 = 4, y_1 = 80, z_1 = 20$

$$x_2 = x_1 + h$$

$$= 4 + 4$$

$$= 8$$

$$y_2 = y_1 + f(x_1, y_1, z_1)h$$

$$= 80 + f(4, 80, 20) \times 4$$

$$= 80 + (20) \times 4$$

$$= 160$$

$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$

$$= 20 + f_2(4, 8, 20) \times 4$$

$$= 20 + (6 \times 4 - 0.5 \times 4^2) \times 4$$

$$= 84$$

For $i = 2, x_2 = 8, y_2 = 160, z_2 = 84$

$$x_3 = x_2 + h$$

$$= 8 + 4$$

$$= 12$$

$$y_3 = y_2 + f(x_2, y_2, z_2)h$$

$$= 160 + f(8, 160, 84) \times 4$$

$$= 160 + (84) \times 4$$

$$= 496$$

$$z_3 = z_2 + f_2(x_2, y_2, z_2)h$$

$$= 84 + f_2(8, 160, 84) \times 4$$

$$= 84 + (6 \times 8 - 0.5(8)^2) \times 4$$

$$= 148$$

$$y_3 \cong y(x_3)$$

$$= y(12)$$

$$= 496$$

Solutions:

Q4)

The correct answer is (C)

$$\frac{d^2u}{dx^2} = \frac{Tu}{R} + \frac{qx(x-L)}{2R}$$

Let

$$\frac{du}{dx} = z = f_1(x, u, z)$$

$$\begin{aligned} \frac{dz}{dx} &= \frac{Tu}{R} + \frac{q \times x(x-L)}{2R} \\ &= \frac{2000u}{75 \times 10^6} + \frac{75 \times x(x-L)}{2(75 \times 10^6)} \\ &= f_2(x, u, z) \end{aligned}$$

$$u_{i+1} = u_i + f_1(x_i, u_i, z_i)h$$

$$z_{i+1} = z_i + f_2(x_i, u_i, z_i)h$$

The first iteration

For $i = 0, x_0 = 0, u_0 = 0, z_0 = 0.003$

$$\begin{aligned} u_1 &= u_0 + f_1(x_0, u_0, z_0)12.5 \\ &= 0 + f_1(0, 0, 0.003) \times 12.5 \\ &= 0 + (0.003) \times 12.5 \\ &= 0.0375 \end{aligned}$$

$$\begin{aligned} z_1 &= z_0 + f_2(x_0, u_0, z_0)h \\ &= 0.003 + f_2(0, 0, 0.003) \times 12.5 \\ &= 0.003 \end{aligned}$$

For $i = 1, x_1 = 12.5, u_1 = 0.0375, z_1 = 0.003$

$$\begin{aligned} u_2 &= u_1 + f_1(x_1, u_1, z_1)12.5 \\ &= 0.0375 + f_1(12.5, 0.0375, 0.003) \times 12.5 \\ &= 0.0375 + (0.003) \times 12.5 \\ &= 0.075 \end{aligned}$$

$$\begin{aligned} z_2 &= z_1 + f_2(x_1, u_1, z_1)h \\ &= 0.003 + f_2(12.5, 0.0375, 0.003) \times 12.5 \\ &= 0.003 + \left(\frac{+2000 \times 0.0375}{75 \times 10^6} + \frac{75 \times 12.5(12.5 - 50)}{2(75 \times 10^6)} \right) \times 12.5 \\ &= \cancel{0.005917} \quad 8.2812\text{e-}05 \end{aligned}$$

For $i = 2, x_2 = 25, u_2 = 0.075, z_2 = 0.005917$

$$u_3 = u_2 + f_1(x_2, u_2, z_2)h$$

$$= 0.14896 - 0.0760$$

$$z_3 = z_2 + f_2(x_2, u_2, z_2)h$$

$$= 0.0097984 - 0.0038$$

For $i = 3, x_3 = 37.5, u_3 = 0.14896, z_3 = 0.0097984$

$$u_4 = u_3 + f_1(x_3, u_3, z_3)h$$

$$= 0.27145 - 0.0286$$

$$z_4 = z_3 + f_2(x_3, u_3, z_3)h$$

$$= 0.012678 - 0.0067$$

The second iteration

For $i = 0, x_0 = 0, u_0 = 0, z_0 = 0.004$

$$\begin{aligned} u_1 &= u_0 + f_1(x_0, u_0, z_0)h \\ &= 0 + f_1(0, 0, 0.004) \times 12.5 \\ &= 0 + (0.004) \times 12.5 \\ &= 0.05 \end{aligned}$$

$$\begin{aligned} z_1 &= z_0 + f_2(x_0, u_0, z_0)h \\ &= 0.004 + f_2(0, 0, 0.004) \times 12.5 \\ &= 0.004 \end{aligned}$$

For $i = 1, x_1 = 12.5, u_1 = 0.05, z_1 = 0.004$

$$\begin{aligned} u_2 &= u_1 + f_1(x_1, u_1, z_1)h \\ &= 0.05 + f_1(12.5, 0.05, 0.004) \times 12.5 \\ &= 0.05 + (0.004) \times 12.5 \\ &= 0.1 \end{aligned}$$

$$\begin{aligned} z_2 &= z_1 + f_2(x_1, u_1, z_1)h \\ &= 0.004 + f_2(12.5, 0.05, 0.004) \times 12.5 \\ &= 0.004 + \left(\frac{+2000 \times 0.05}{75 \times 10^6} + \frac{75 \times 12.5(12.5 - 50)}{2(75 \times 10^6)} \right) \times 12.5 \\ &= \cancel{0.006913} \quad 0.0011 \end{aligned}$$

For $i = 2, x_2 = 25, \cancel{u_2 = 0.1}, \cancel{z_2 = 0.006913}$

$$\begin{aligned} &= 0.18641 \quad 0.1136 \\ z_3 &= z_2 + f_2(x_2, u_2, z_2)h \end{aligned}$$

$$= 0.010786 \quad -0.0028$$

For $i = 3, x_3 = 37.5, u_3 = \cancel{0.18641}, z_3 = \cancel{0.010786}$

$$u_4 = u_3 + f_1(x_3, u_3, z_3)12.5$$

$$= 0.32124 \quad 0.0788$$

$$z_4 = z_3 + f_2(x_3, u_3, z_3)h$$

$$= 0.013653 \quad -0.0057$$

linear interpolation

$$\frac{du}{dr}(50) \cong \frac{0.003 - 0.004}{\cancel{0.271445} - \cancel{0.32124}}(0 - \cancel{0.32124}) + (0.004)$$

$$\frac{du}{dr}(50) \cong \cancel{-0.00245162} \quad 0.0024$$

For $i = 0, x_0 = 0, u_0 = 0, z_0 = \cancel{-0.0024516} \quad 0.0024$

$$u_1 = u_0 + f_1(x_0, u_0, z_0)12.5$$

$$= \cancel{-0.030645} \quad 0.0304$$

$$z_1 = z_0 + f_2(x_0, u_0, z_0)h$$

$$= \cancel{-0.0024516} \quad 0.0024$$

For $i = 1, x_1 = 12.5, u_1 = \cancel{0.030645}, z_1 = \cancel{-0.0024516}$

$$u_2 = u_1 + f_1(x_1, u_1, z_1)12.5$$

$$= -0.061291 \quad 0.0608$$

$$z_2 = z_1 + f_2(x_1, u_1, z_1)h$$

$$= 0.00048828 \quad -4.8828\text{e-}04$$

For $i = 2, x_2 = 25, u_2 = \cancel{-0.061291}, z_2 = \cancel{0.00048828}$

$$u_3 = u_2 + f_1(x_2, u_2, z_2)12.5$$

$$= -\cancel{0.055187} \quad 0.0547$$

$$z_3 = z_2 + f_2(x_2, u_2, z_2)h$$

$$= \cancel{0.004415} \quad -0.0044$$

For $i = 3, x_3 = 37.5, u_3 = \cancel{0.055187}, z_3 = \cancel{0.004415}$

$$u_4 = u_3 + f_1(x_3, u_3, z_3)12.5$$

$$= -2.4286 \times 10^{-16} \quad 0$$

thus the value of u at $x = 25$

$$\cancel{u(25) = -0.061291''}$$

$$u(25) = u_2 = 0.0608$$

$u_4 = 0$ (deflection at $x=50$), which is correct

$$z_4 = -0.0073$$

See python code example: Q4exampleSln_v2.py