ENG2009 – Modelling of Engineering Systems

Tutorial 5

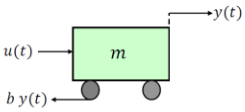
System Responses and Stability

Quick revision: System Response (unit step response) and Stability (Lecture 9-10)

First order system

First-Order Systems

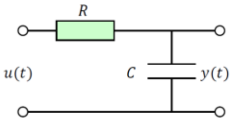
Example:



$u(t)$ - force $y(t)$ - velocity

$$m\frac{dy(t)}{dt} + by(t) = u(t)$$

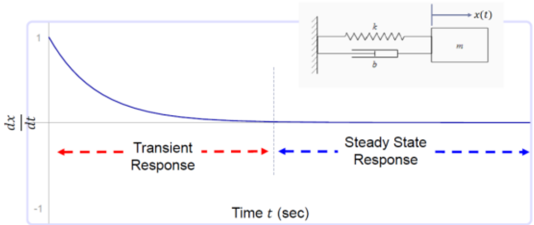
Other 1st order system



$$RC\frac{dy(t)}{dt} + y(t) = u(t)$$

1st order system: system response

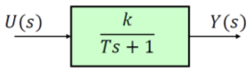
Example: Mass-damper animation



Spring constant $k = 0$
Input: $u(t) = 0$
Output: $y(t)$: velocity
Initial condition: $v(0) = \frac{dx}{dt} = 1$

First-Order Systems

- General form: $T\frac{dy(t)}{dt} + y(t) = \mathrm{k}u(t)$
- Transfer function:



k – constant (steady-state gain)

T - time constant

- Pole: $s = -\frac{1}{T}$

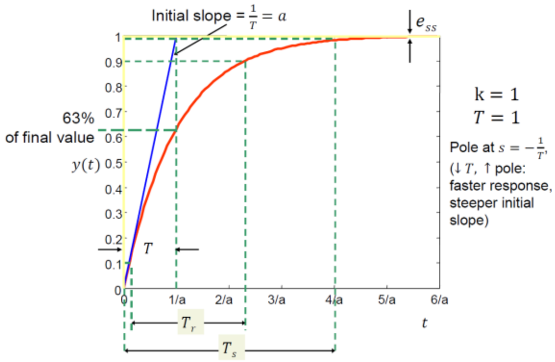
Step Response of 1st Order Systems

- General output analysis:
- General form: $T\frac{dy(t)}{dt} + y(t) = \mathrm{k}u(t) \rightarrow \mathcal{L} \rightarrow \frac{Y(s)}{U(s)} = \frac{\mathrm{k}}{(Ts+1)}$
- Pole: $s = -\frac{1}{T}$, ($\downarrow T$, $\uparrow$ pole: faster response)
- Unit-step, $U(s) = \frac{1}{s} \rightarrow$ $Y(s) = \frac{1}{s(Ts+1)}$
- Partial fraction expansion: $Y(s) = \frac{1}{s} - \frac{1}{s+1/T}$
- Solve: Inverse Laplace transform:

$$\mathbf{y(t) = 1 - e^{(-t/T)}}$$

$y_{ss}(t)$ $y_t(t)$

Step Response of 1st Order Systems

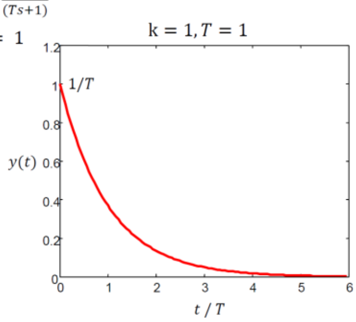


Impulse Response of 1st Order Systems

- General form: $\frac{Y(s)}{U(s)} = \frac{\mathrm{k}}{(Ts+1)}$
- Unit-impulse, $R(s) = 1$

$\rightarrow$ $Y(s) = \frac{1}{Ts+1}$

- Inverse Laplace transform: $\boldsymbol{y(t) = e^{-(t/T)} / T}$

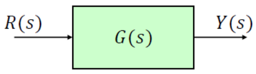


- *Impulse response:*

$y_{ss}(t) = 0$

Second order system

Second-Order Systems



- Transfer function: $G(s) = \dfrac{Y(s)}{R(s)} = \dfrac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

k = *gain of system (sometime called DC gain)*

ζ = *damping ratio* - controls the rate of rise or decay of oscillations in the system (unitless)

ω_n = *natural frequency* - frequency of oscillation of the system without damping (rad/s)

Real example: mass-spring-damper system

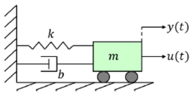
$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = k\omega_n^2 u(t) \text{ (general form)}$$

$$m\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = u(t)$$

$$\frac{d^2y(t)}{dt^2} + \underbrace{\left(\frac{b}{m}\right)}_{2\zeta\omega_n}\frac{dy(t)}{dt} + \underbrace{\left(\frac{k}{m}\right)}_{\omega_n^2} y(t) = \underbrace{\left(\frac{1}{m}\right)}_{k\omega_n^2} u(t)$$

Natural frequency, $\omega_n = \sqrt{k/m}$

Damping factor, $\zeta = \dfrac{b}{2\omega_n m}$



Therefore
- $\uparrow b \Rightarrow \uparrow \zeta$: (more damper, more damping factor, less oscillatory)
- $\uparrow k \Rightarrow \uparrow \omega_n, \downarrow \zeta$: (more spring, less damping factor, more oscillatory)
- $\uparrow m \Rightarrow \downarrow \omega_n, \downarrow \zeta$: (more weight, less damping factor, more oscillatory)

System output influenced by damping factor ζ?

Second-Order Systems

- Solutions of the characteristic equation (quadratic):

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Roots of characteristic equation (pole of TF)

$s_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$ $\quad s_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$

- We will consider five cases:
  1) $\zeta = 1$, *critically damped (stable system) - last lecture*
  2) $\zeta > 1$, *overdamped (stable system) - last lecture*
  3) $0 < \zeta < 1$, *underdamped (stable system)*
  4) $\zeta = 0$, *undamped (marginally stable system)*
  5) $\zeta < 0$, *exponential growth (unstable system)*

Step Response of 2nd Order Systems

Case (3) Underdamped, $0 < \zeta < 1$:

- Two complex poles:

$s_1 = -\zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2}$

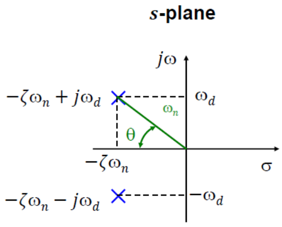
$s_2 = -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}$

Damped natural frequency

$\omega_d = \omega_n\sqrt{1 - \zeta^2}$

$\theta = \tan^{-1}\left(\dfrac{\sqrt{1 - \zeta^2}}{\zeta}\right)$

$\cos(\theta) = \zeta$



Step Response of 2nd Order Systems

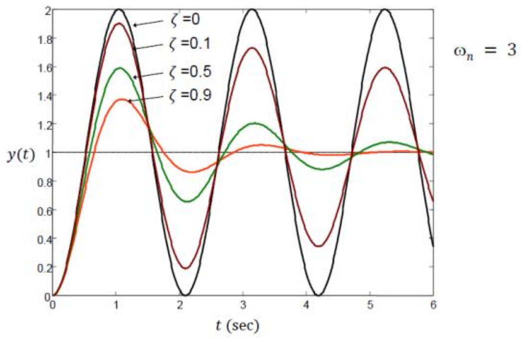
- Unit-step time response:

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\sin(\omega_d t + \theta)$$

$y_{ss}(t)$ $\quad y_t(t)$

$\theta = \tan^{-1}\left(\dfrac{\sqrt{1 - \zeta^2}}{\zeta}\right)$ $\qquad \omega_d = \omega_n\sqrt{1 - \zeta^2}$

Step Response of 2nd Order Systems



Stability of LTI systems

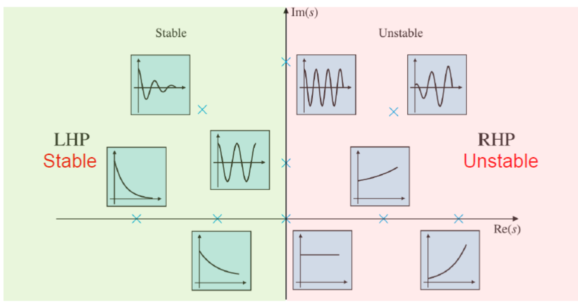
An LTI system is said to be *stable*
if
all the poles
have
negative real parts

(i.e. they are *all* in the left half of the s-plane).

Note:
poles: the roots of the transfer function denominator polynomial)

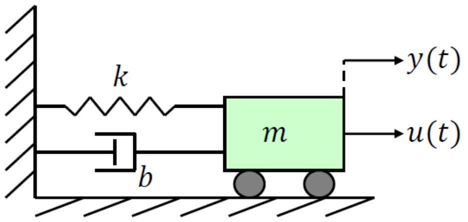
Recall: Stability of LTI systems

Effect of pole location on system response and stability



Example: Mass-spring-damper animation example

Given the differential equation: $m\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = u(t)$ for the mass-sprind-damper system below, where $m = 1,\ b = 0.2, k = 1$. Assume that the input is a unit impulse $u(t) = \delta$.



Obtain the following:

(a) Transfer function (assume zero initial conditions)
(b) The order of the system
(c) Poles and zeros
(d) Draw the s-plane
(e) Is the system stable?
(f) Without solving the diffrential equation, predict the response of the system.
(g) Use Python to create a simulation of the system. Assume that the input force $u(t)$ is a unit impulse

Solutions:

(a) Differential eq: $m\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = u(t)$.
Laplace Transfrom (assuming zero initial conditions):

$$ms^2Y(s) + bsY(s) + kY(s) = U(s)$$
$$(ms^2 + bs + k)Y(s) = U(s)$$

Transfer function:

$$\frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$

(b) 2nd order system (highest power of denominator)

(c) substituting $m = 1,\ b = 0.2, k = 1$ to the transfer function:

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 0.2s + 1}$$

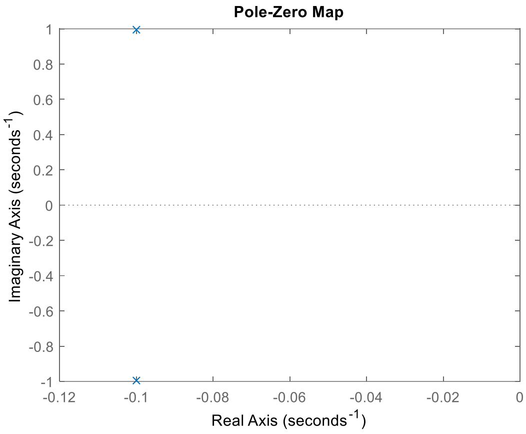
Zeros: roots of numerator: no zero

Poles: roots of denominator

$s^2 + 0.2s + 1 = 0$

$\rightarrow$ roots: $s = -0.1000\ +\ 0.9950i, s = -0.1000 - 0.9950i$ (two complex pair poles)

(d) s-plane



(e) Stable: all poles in LHP

(f) For secod order system, analyse damping ratio ζ:
Note that the general transfer function of a second order system is given by

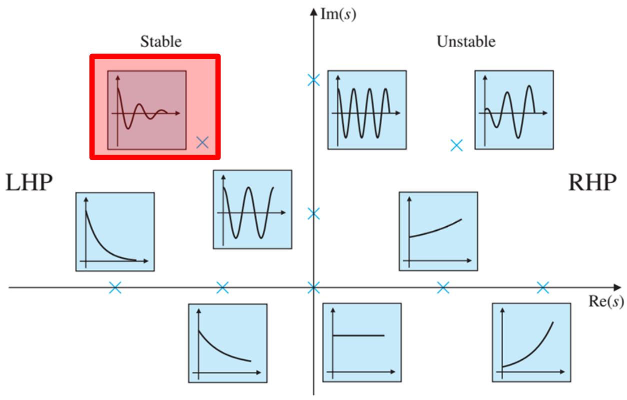
$$G(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s^2 + 0.2s + 1}$$

where k is the gain of the system, ζ is the damping ratio and ω is the natural frequency.

Therefore in this case $\omega_n = 1$ and $2\zeta\omega_n = 0.2$
Which yield damping ratio $\underline{\zeta = 0.1}$ which is underdamped system.

Checking poles: $s = -0.1000 + 0.9950i, s = -0.1000 - 0.9950i$. Two complex pair farly close to the imaginary axis: underdamped system, slightly oscillatory, but converge to zero (due to impulse input).

Example:



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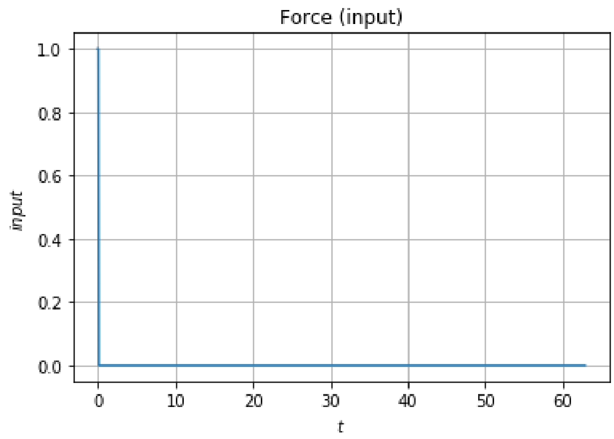
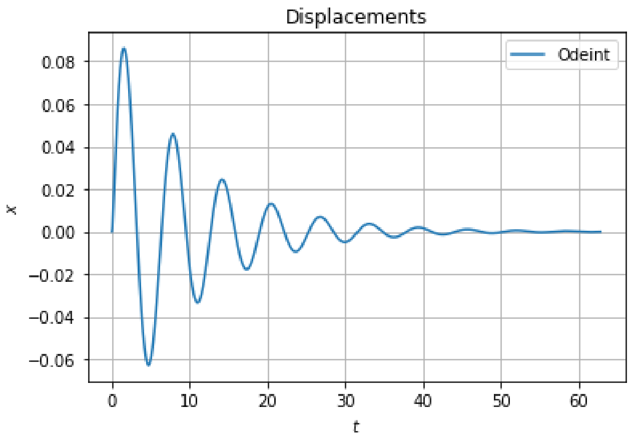
(g) Python code example:

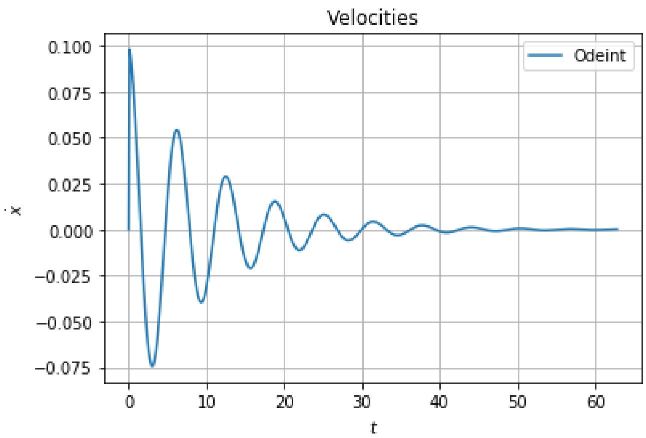
```
#import numpy, integrate and pyplot
import numpy as np
from scipy import integrate
from matplotlib import pyplot as plt

#sim data
m=1 #mass of the oscillator
b=0.2 #damper of the oscillator
k=1 #stiffness of the oscillator
omega = (k/m)**0.5 #angular frequency of the oscillator
T=2*np.pi/omega #period of the oscillator

#create 1000 unifirmply distributed points in the interval [0,10T]
t = np.linspace(0,10*T,1000)

#lambda returning the force
force = lambda t:t<=0.1

#lambda with the RHS of the equation, it evaluates the force and returns a list
f = lambda X,t: [X[1],1/m*(force(t)-k*X[0]-b*X[1])]

#solve equation using odeint, now the values of X for t=0 are given by a list
XOdeint = integrate.odeint(f,[0,0],t)

ti = np.linspace(0,10*T,1000)

#plot force
plt.plot(ti,force(ti))
plt.title('Force (input)')
plt.grid()
plt.xlabel('$t$')
plt.ylabel('$input$')
plt.show()

#plot displacements from both solutions (first column of each solution)
plt.plot(t,XOdeint[:,0])
plt.legend(['Odeint'])
plt.title('Displacement')
plt.grid()
plt.xlabel('$t$')
plt.ylabel('$x$')
plt.show()

#plot velocities from both solutions (first column of each solution)
plt.plot(t,XOdeint[:,1])
plt.legend(['Odeint'])
plt.title('Velocitie')
plt.grid()
plt.xlabel('$t$')
plt.ylabel('$\dot{x}$')
plt.show()

#Save data
np.savetxt('t.txt',t)
np.savetxt('x.txt',XOdeint[:,0])
```



Force (input)
input
t



Displacements
Odeint
x
t



Velocities
Odeint
$\dot{x}$
t

Exercises:

For each of the transfer functions shown below, find:

(a) Order of the system,
(b) The poles and zeros,
(c) plot the s-plane,
(d) state wheather the system stable or unstable,
(e) without solving for the inverse Laplace transform, state the nature of each response to a step input (state if the system is overdamped, underdamped, and so on).
(f) DIY: Use Python to create a simulation of the system. Assume that the input force $u(t)$ is a unit impulse

Q1) $G(s) = \frac{2s+1}{s^2+3s+2}$

Q2) $G(s) = \frac{2}{s+2}$

Q3) $G(s) = \frac{10(s+7)}{(s+10)(s+20)}$

Q4) $G(s) = \frac{20}{s^2+6s+144}$

Q5) $G(s) = \frac{s+2}{s^2+9}$

Solutions:

Q1) Solution

Transfer function: $G(s) = \frac{2s+1}{s^2+3s+2}$

(a) 2nd order system (highest power of denominator)

(b) **Zeros**: roots of numerator: $2s + 1 = 0$

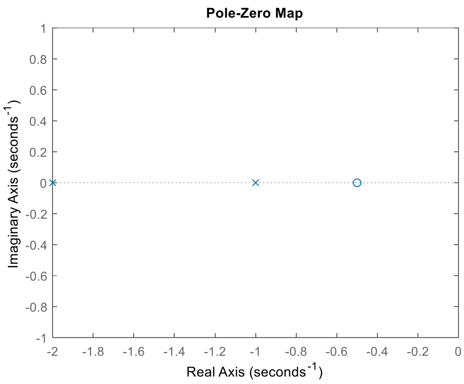
$\rightarrow$ roots: $s = -\frac{1}{2}$ (one real zero)

Poles: roots of denominator

$\mathrm{s}^2 + 3\mathrm{s} + 2 = (\mathrm{s}+2)(\mathrm{s}+1) = 0$

$\rightarrow$ roots: $s = -1, s = -2$ (two real poles)

(c) s-plane:



(d) Stable: all poles in LHP (poles at -1 and -2)

(e) For secod order system, analyse damping ratio ζ:

Note that the general transfer function of a second order system is given by

$$G(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{2s+1}{s^2+3s+2}$$

where k is the gain of the system, ζ is the damping ratio and ω is the natural frequency.

Therefore in this case $\omega_n = \sqrt{2}$ and $2\zeta\omega_n = 3$

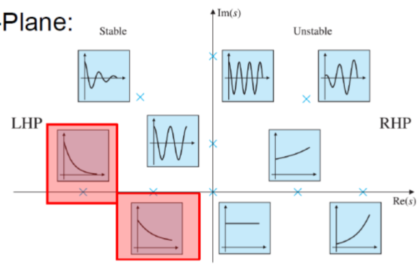
Which yield damping ratio $\underline{\zeta = 1.0607}$ which is <u>critical damped</u> system.

Checking the poles (roots of characteristic equation):

$s = -1, \Rightarrow$ (real pole) critical damped response

$s = -2, \ \Rightarrow$ (real pole) critical damped (more dominant than the first pole)

Example s-Plane:



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Q2) Solution

Transfer function: $G(s) = \frac{2}{s+2}$

(a) 1st order system (highest power of denominator)

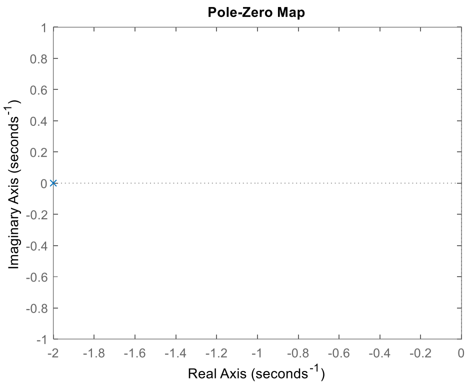
(b) **Zeros**: roots of numerator: no zero

Poles: roots of denominator

$(\mathrm{s}+2) = 0$

$\rightarrow \text{roots: } s = -2$ (one real poles)

(c) s-plane:



(d) Stable: all poles in LHP (poles at -2)

(e) Poles (roots of characteristic equation):

First order response (see Lecture 15), converge to steady state value (stable response/ non diverging).

Critical damp behaviour

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Q3) Solution

Transfer function: $G(s) = \frac{10(s+7)}{(s+10)(s+20)}$

(a) 2nd order system (highest power of denominator)

(b) **Zeros**: roots of numerator: $10(s+7) = 0$

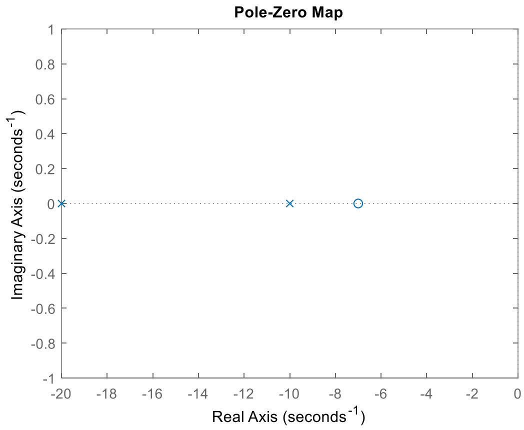
$\rightarrow$ roots: $s = -7$ (one real zero)

Poles: roots of denominator

$(s+10)(s+20) = 0$

$\rightarrow$ roots: $s = -10, s = -20$ (two real poles)

(c) s-plane:



(d) Stable: all poles in LHP (poles at -10 and -20)

(e) For secod order system, analyse damping ratio ζ:

Note that the general transfer function of a second order system is given by

$$G(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{10(s+7)}{(s+10)(s+20)} = \frac{10(s+7)}{s^2 + 30s + 200}$$

where k is the gain of the system, ζ is the damping ratio and ω is the natural frequency.

Therefore in this case $\omega_n = \sqrt{200}$ and $2\zeta\omega_n = 30$

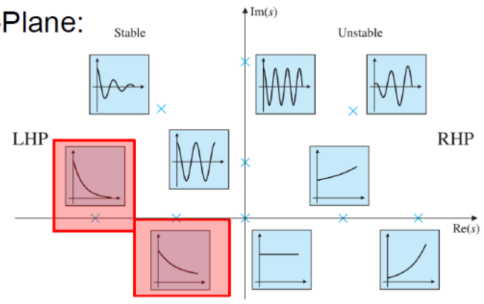
Which yield damping ratio $\underline{\zeta = 1.0607}$ which is <u>critical damped</u> system.

Checking poles (roots of characteristic equation):

$s = -10, \Rightarrow$ (real pole) critical damped response: far from imaginary axis

$s = -20, \ \Rightarrow$ (real pole) critical damped (more dominant than the first pole): far from imaginary axis

Example s-Plane:



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Q4) Solution

Transfer function: $G(s) = \frac{20}{s^2+6s+144}$

(a) 2nd order system (highest power of denominator)

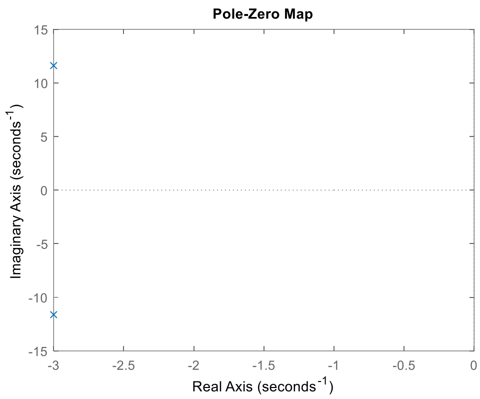
(b) **Zeros**: roots of numerator: no zero

Poles: roots of denominator (using roots of quadratic equation: $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$S^2 + 6s + 144 = 0$

$\rightarrow$ roots: $s = -3 + 3\sqrt{15}j$, $s = -3 - 3\sqrt{15}j$ (two complex pair poles)

(c) s-plane:



(d) Stable: all poles in LHP (two complex poles at $= -3 \pm 3\sqrt{15}j$)

(e) For secod order system, analyse damping ratio ζ:

Note that the general transfer function of a second order system is given by

$$G(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{20}{s^2 + 6s + 144}$$

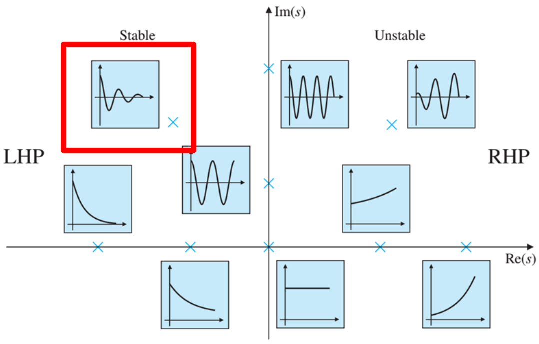
where k is the gain of the system, ζ is the damping ratio and ω is the natural frequency.

Therefore in this case $\omega_n = \sqrt{144} = 12$ and $2\zeta\omega_n = 6$

Which yield damping ratio $\underline{\zeta = 0.25}$ which is <u>under damped</u> system.

Checking poles (roots of characteristic equation):

$s = -3 \pm 3\sqrt{15}j, \Rightarrow$ (two complex poles) <u>underdamped</u> response



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Q5) Solution

Transfer function: $G(s) = \frac{s+2}{s^2+9}$

(a) 2nd order system (highest power of denominator)

(b) **Zeros**: roots of numerator: $s + 2 = 0$

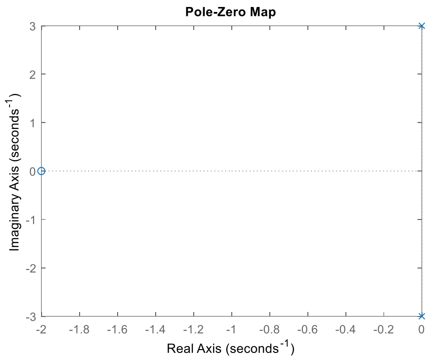
$\rightarrow$ roots: $s = -2$ (one real zero)

Poles: roots of denominator

$s^2 + 9 = 0$

$\rightarrow$ roots: $s = 3j, s = -3j$ (two complex poles at imaginary axis)

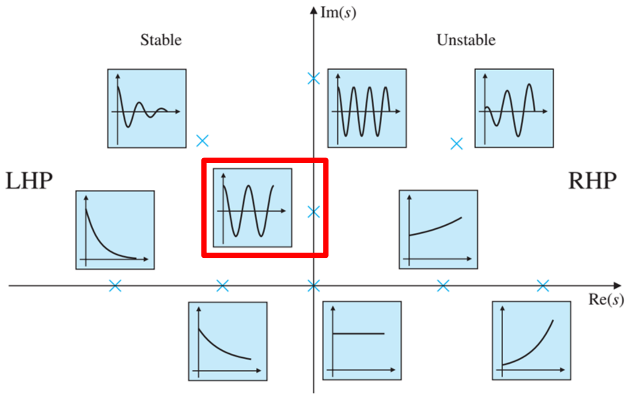
(c) s-plane:



(d) marginally stable: poles in imaginary axis (poles at $3j$ and $-3j$)

(e) Poles (roots of characteristic equation):

$s = 3j, s = -3j,$ $\Rightarrow$ (two complex pole at imaginary axis) <u>undamped</u>



For secod order system, analyse damping ratio ζ:

Note that the general transfer function of a second order system is given by

$$G(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s+2}{s^2+9}$$

where k is the gain of the system, ζ is the damping ratio and ω is the natural frequency.

Therefore in this case $\omega_n = \sqrt{9} = 3$ and $2\zeta\omega_n = 0$

Which yield damping ratio <u>$\zeta = 0$</u> which is <u>undamped</u> system.

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