Final Submission

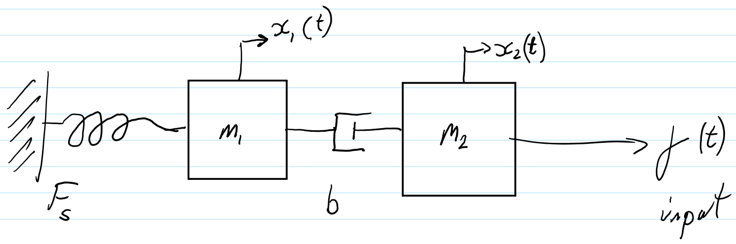
31 December 2023 18:41

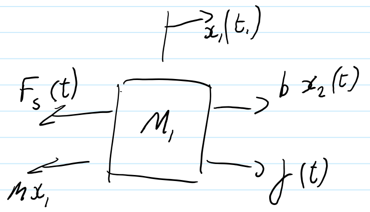
Q1. Real life Example: Suspension System on a vehicle i.e. Van, shock absorber = damper

spring = Suspension

mass = Car / tyre

Q2.





Assumptions: Case 1:

$x_1 \neq 0$

$x_2 = 0$

$f(t) = 0$

Case 2:

$x_1 = 0$

$x_2 \neq 0$

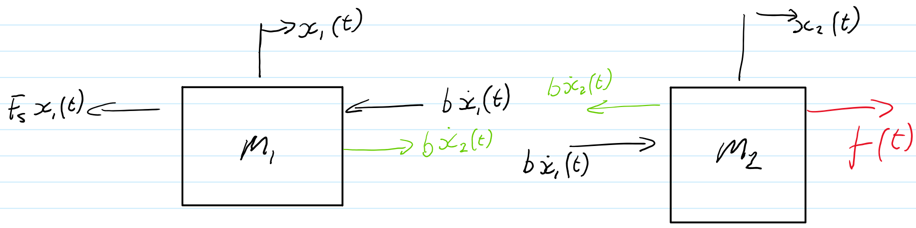
$f(t) = 0$

Case 3:

$x_1 = 0$

$x_2 = 0$

$f(t) \neq 0$



$$\Sigma F = M\,a$$

$$-b\dot{x}_1(t) + b\dot{x}_2(t) - F_s\,x_1(t) = m_1\ddot{x}_1(t)$$

$$m\ddot{x}_1(t) + b\dot{x}_1(t) + F_s\,x_1(t) - b\dot{x}_2(t) = 0$$

$$\frac{d^2x_1(t)}{dt^2} + \frac{dx_1(t)}{dt} + 2x_1^2(t) - \frac{dx_2(t)}{dt} = 0$$

$$\Sigma F = m\,a$$

$$f(t) + bx_1(t) - bx_2(t) = m_2\,a$$

$$f(t) + b\dot{x}_1(t) - b\dot{x}_2(t) = m_2\ddot{x}_2(t)$$

$$f(t) = m_2\ddot{x}_2(t) + b\dot{x}_2(t) - b\dot{x}_1(t)$$

$$\frac{d^2x_2(t)}{dt^2} + \frac{dx_2(t)}{dt} - \frac{dx_1(t)}{dt} = f(t)$$

Q3.

① $\frac{d^2x_1(t)}{dt^2} + \frac{dx_1(t)}{dt} + 2x_1^2(t) - \frac{dx_2(t)}{dt} = 0$

② $\frac{d^2x_2(t)}{dt^2} + \frac{dx_2(t)}{dt} - \frac{dx_1(t)}{dt} = f(t)$

$x_1(0) = 1$, $x_2(0) = 0$, $f(0) = 0$

let $x_1(t) = (\delta x_1(t) + 1)$

Small excursions of

let $x_1(t) = (\delta x_1(t) + 1)$

Small excursions of $x_1(t)$ about $x_1(0)=1$

let $x_2(t) = (\delta x_2(t) + 0)$

Small excursions of $x_2(t)$ about $x_2(0)=0$

let $f(t) = (\delta f(t) + 0)$

Small excursions of $f(t)$ about $f(0)=0$

Substituting:

$$① \quad \frac{d^2(\delta x_1(t)+1)}{dt^2} + \frac{d(\delta x_1(t)+1)}{dt} + 2(\delta x_1(t)+1)^2 - \frac{d(\delta x_2(t)+0)}{dt}$$

$$② \quad \frac{d^2(\delta x_2(t)+0)}{dt} + \frac{d(\delta x_2(t)+0)}{dt} - \frac{d(\delta x_1(t)+1)}{dt} = \delta f(t) + 0$$

Since 1 & 0 are constants: $\frac{d^2(0)}{dt^2} = 0$ & $\frac{d(1)}{dt} = 0$:

$$\frac{d(\delta x_1(t)+1)}{dt} \Rightarrow \frac{d(\delta x_1(t))}{dt}$$

&

$$\frac{d(\delta x_2(t)+0)}{dt} \Rightarrow \frac{d(\delta x_2(t))}{dt}$$

&

$$\frac{d^2(\delta x_1(t)+1)}{dt^2} \Rightarrow \frac{d^2(\delta x_1(t))}{dt^2}$$

&

$$\frac{d^2(\delta x_2(t)+0)}{dt^2} \Rightarrow \frac{d^2(\delta x_2(t))}{dt^2}$$

$$① \quad \frac{d^2(\delta x_1(t))}{dt^2} + \frac{d(\delta x_1(t))}{dt} + 2(\delta x_1(t)+1)^2 - \frac{d(\delta x_2(t))}{dt} = 0$$

$$② \quad \frac{d^2(\delta x_2(t))}{dt^2} + \frac{d(\delta x_2(t))}{dt} - \frac{d(\delta x_1(t))}{dt} = \delta f(t) \Rightarrow \text{As Required.}$$

Now to linearise: $(\delta x_1(t)+1)^2$

Using Taylor Series: $f(x) - f(x_0) \approx \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$

$$f((\delta x_1(t)+1)^2) - f(x_1^2(0)) = \left.\frac{d}{dx}(x_1^2(t))\right|_{x_1(0)=1} \delta x_1(t)$$

$$f((\delta x_1(t)+1)^2) - 1^2 = 2x_1(0) \quad \delta x_1(t)$$

$$f((\delta x_1(t)+1)^2) - 1 = 2\delta x_1(t)$$

$$f((\delta x_1(t)+1)^2) = 2\delta x_1(t) + 1$$

$$① \quad \frac{d^2(\delta x_1(t))}{dt^2} + \frac{d(\delta x_1(t))}{dt} + 2(1 + 2\delta x_1(t)) - \frac{d(\delta x_2(t))}{dt} = 0 \quad \text{As Required}$$

Q4.

$$L(f(t)) = F(s) = \int_{0^-}^{\infty} f(t) e^{-st} \, dt$$

① $\frac{d^2x_1(t)}{dt^2} + \frac{dx_1(t)}{dt} + 2(1+2x_1(t)) - \frac{dx_2(t)}{dt} = 0$

② $\frac{d^2x_2(t)}{dt^2} + \frac{dx_2(t)}{dt} - \frac{dx_1(t)}{dt} = f(t)$

Zero initial conditions:
$x_1(0)=0$
$\dot{x}_1(0)=0$
$x_2(0)=0$
$\dot{x}_2(0)=0$
$\boxed{f(0)=0}$

②: $\mathcal{L}(f(t)) = F(s) = \int_{0-}^{\infty} \left(\frac{d^2x_2(t)}{dt^2} + \frac{dx_2(t)}{dt} - \frac{dx_1(t)}{dt}\right) e^{-st}\, dt$

$= \int_{0-}^{\infty} \left(\ddot{x}_2(t) + \dot{x}_2(t) - \dot{x}_1(t)\right) e^{-st}\, dt$

$F(s) = [s^2X_2(s) - sx_2(0) - \dot{x}_2(0)] + [sX_2(s) - x_2(0)] - [sX_1(s) - x_1(0)]$

Apply initial (zero) conditions

$F(s) = [s^2X_2(s) - 0 - 0] + [sX_2 s - 0] - [sX_1(s) - 0]$

② $F(s) = s^2X_2(s) + sX_2(s) - sX_1(s)$ $\qquad F(s) + sX_1(s) = s^2X_2(s) + sX_2(s)$

① $\frac{d^2x_1(t)}{dt^2} + \frac{dx_1(t)}{dt} + 2(1+2x_1(t)) - \frac{dx_2(t)}{dt} = 0$

$\mathcal{L}(f(t)) = F(s): [s^2X_1(s) - sx_1(0) - \dot{x}_1(0)] + [sX_1(s) - x_1(0)] + \left[\frac{2}{s} + 4X_1(s)\right] - [sX_2(s) - x_2(0)]$

Applying initial (zero) conditions

① $F(s): s^2X_1(s) + sX_1(s) + \frac{2}{s} + 4X_1(s) - sX_2(s) = 0$

$F(s): X_1(s)(s^2+s+4) - sX_2(s) + \frac{2}{s} = 0$

Q5.

From Q4:

①: $X_1(s)(s^2+s+4) - sX_2(s) + \frac{2}{s} = 0$

②: $F(s) = s^2X_2(s) + sX_2(s) - sX_1(s)$

a) output: $x_2(t)$
input: $f(t)$

Make eq.① $= X_1(s)$

Substitute into ②

① $X_1(s)(s^2+s+4) = sX_2(s) - \frac{2}{s}$

$X_1(s) = \frac{sX_2(s) - 2/s}{s^2+s+4}$

② $F(s) = s^2X_2(s) + sX_2(s) - sX_1(s)$

$$F(s) = s^2X_2(s) + sX_2(s) - s\left(\frac{sX_2(s)}{s^2+s+4} - \frac{2/s}{s^2+s+4}\right)$$

$$= s^2X_2(s) + sX_2(s) - \frac{s^2X_2(s)}{s^2+s+4} + \frac{2}{s^2+s+4}$$

$$= X_2(s)\left(s^2+s - \frac{s^2}{s^2+s+4}\right) + \frac{2}{s^2+s+4}$$

$$F(s) = X_2(s)\left(\frac{(s^2+s)(s^2+s+4) - s^2}{s^2+s+4}\right) + \frac{2}{s^2+s+4}$$

Cannot get transfer function until $\frac{2}{s^2+s+4} = 0$

$$\therefore \frac{X_2(s)}{F(s)} = \frac{s^2+s+4}{(s^2+s)(s^2+s+4)-s^2}$$

$$= \frac{s^2+s+4}{s^4+s^3+4s^2+s^3+s^2+4s-s^2}$$

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{s^2+s+4}{s^4+2s^3+4s^2+4s}$$

b) 4th order system

because of s^4 in denominator

highest power of s in the denominator.

6. 2 zeros → in numerator

s^2+s+4 is the numerator polynomial.

$s^2+s+4=0$ ⇒ to find zeros

$$s = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$s = \frac{-1 \pm \sqrt{1-4\cdot 4}}{2}$$

$$= \frac{-1 \pm \sqrt{-15}}{2}$$

$$= \frac{-1}{2} \pm \frac{\sqrt{15}}{2}i$$

$$\boxed{\begin{array}{l} s = -0.5 + \frac{\sqrt{15}}{2}i \\ \& \\ s = -0.5 - \frac{\sqrt{15}}{2}i \end{array}}$$

4 poles → in Denominator

$s=0$
&
$s = -0.3522 \pm 1.7214i$ Given

Denominator Polynomial: $s^4+2s^3+4s^2+4s$

let denominator $=0$ to find poles

$$s^4+2s^3+4s^2+4s = 0$$

$$s(s^3+2s^2+4s+4) = 0$$

$s=0$ Given

$$s^3+2s^2+4s+4=0$$

$s = -0.3522 \pm 1.7214i$ Given

Last s value is between -2 and 0

Using cubic calculator:

last pole: $s = -1.2956$

Zeros:

$s_1 = -0.5 + \frac{\sqrt{15}}{2}i$

$s_2 = -0.5 - \frac{\sqrt{15}}{2}i$

poles:

$s_1 = -1.2956$

$s_2 = -0.3522 + 1.7214i$

$s_1 = -0.5 + \frac{\sqrt{15}}{2} i$

$s_2 = -0.5 - \frac{\sqrt{15}}{2} i$

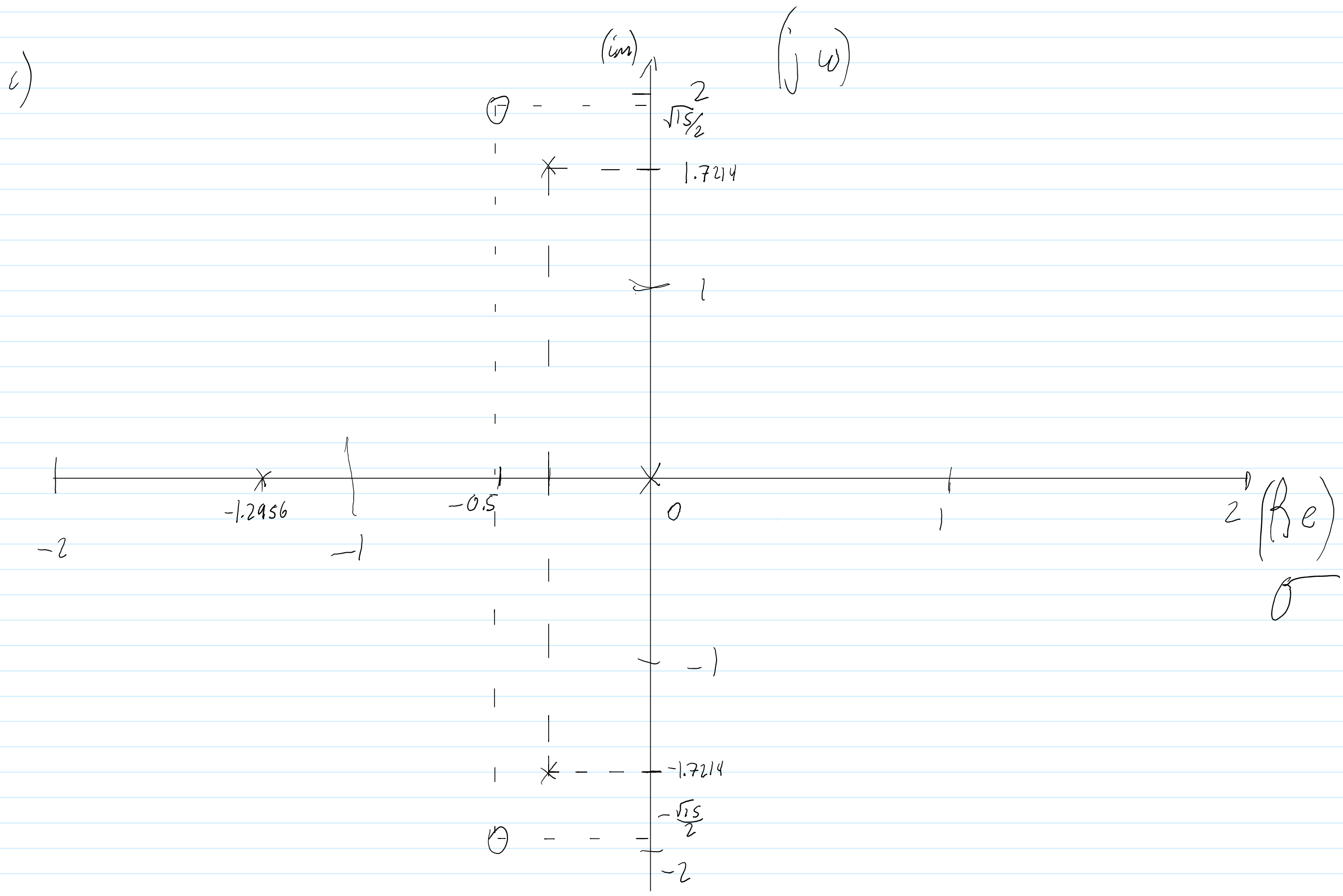
$s_1 = -1.2956$

Given $\begin{cases} s_2 = -0.3522 + 1.7214i \\ s_3 = -0.3522 - 1.7214i \\ s_4 = 0 \end{cases}$

b) Yes the system is marginally stable

As three poles are in the left-hand plane, -ve real values, and one value lies on the imaginary axis, $s_4 = 0$

c)



Q7.

Final value Theorem (FVT) can be used because the system has one pole at the origin and the rest in the LHP, i.e. Type 1 system.

Unit impulse

$U(s) = 1 \longrightarrow \boxed{\dfrac{s^2+s+4}{s^4+2s^3+4s^2+4s}} \longrightarrow Y(s)$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} s \cdot \frac{s^2+s+4}{s^4+2s^3+4s^2+4s}$$

$$= \lim_{s\to 0} s \cdot \frac{s^2+s+4}{s(s^3+2s^2+4s+4)}$$

$$= \lim_{s \to 0} s \cdot \frac{s^2 + s + 4}{s(s^3 + 2s^2 + 4s + 4)}$$

$$= \lim_{s \to 0} \frac{s^2 + s + 4}{s^3 + 2s^2 + 4s + 4}$$

$$= \frac{0^2 + 0 + 4}{0^3 + 2 \cdot 0^2 + 4 \cdot 0 + 4}$$

$$= \frac{4}{4} = \underline{1}$$

Final Value of the system is 1 for $x_2(t)$

This is ~~possible~~ as it is a type 1 system where FVT will equal a real value.