ENG2009 – Modelling of Engineering Systems

Tutorial 2

Higher order and Coupled ODE

Higher order and coupled ODE – Examples

Example 1

Rewrite the following differential equation as a set of first order differential equations.

$$3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x}, y(0) = 5, y'(0) = 7$$

Solution

The ordinary differential equation would be rewritten as follows. Assume

$$\frac{dy}{dx} = z,$$

Then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

Substituting this in the given second order ordinary differential equation gives

$$3\frac{dz}{dx} + 2z + 5y = e^{-x}$$

$$\frac{dz}{dx} = \frac{1}{3}(e^{-x} - 2z - 5y)$$

The set of two simultaneous first order ordinary differential equations complete with the initial conditions then is

$$\frac{dy}{dx} = z, y(0) = 5$$

$$\frac{dz}{dx} = \frac{1}{3}(e^{-x} - 2z - 5y), z(0) = 7.$$

Now one can apply any of the numerical methods used for solving first order ordinary differential equations.

Example 2

Given

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}, y(0) = 1, \frac{dy}{dt}(0) = 2, \text{ find by Euler's method}$$

a) $y(0.75)$
b) the absolute relative true error for part(a), if $y(0.75)|_{exact} = 1.668$
c) $\frac{dy}{dt}(0.75)$

Use a step size of $h = 0.25$.

Solution

First, the second order differential equation is written as two simultaneous first-order differential equations as follows. Assume

$$\frac{dy}{dt} = z$$

then

$$\frac{dz}{dt} + 2z + y = e^{-t}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y$$

So the two simultaneous first order differential equations are

$$\frac{dy}{dt} = z = f_1(t, y, z), y(0) = 1 \quad \text{(E2.1)}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y = f_2(t, y, z), z(0) = 2 \quad \text{(E2.2)}$$

Using Euler's method on Equations (E2.1) and (E2.2), we get

$$y_{i+1} = y_i + f_1(t_i, y_i, z_i)h \qquad \text{(E2.3)}$$
$$z_{i+1} = z_i + f_2(t_i, y_i, z_i)h \qquad \text{(E2.4)}$$

a) To find the value of $y(0.75)$ and since we are using a step size of 0.25 and starting at $t = 0$, we need to take three steps to find the value of $y(0.75)$.

For $i = 0, t_0 = 0, y_0 = 1, z_0 = 2$,

From Equation (E2.3)

$$y_1 = y_0 + f_1(t_0, y_0, z_0)h$$
$$= 1 + f_1(0,1,2)(0.25)$$
$$= 1 + 2(0.25)$$
$$= 1.5$$

y_1 is the approximate value of y at

$$t = t_1 = t_0 + h = 0 + 0.25 = 0.25$$
$$y_1 = y(0.25) \approx 1.5$$

From Equation (E2.4)

$$z_1 = z_0 + f_2(t_0, y_0, z_0)h$$
$$= 2 + f_2(0,1,2)(0.25)$$
$$= 2 + (e^{-0} - 2(2) - 1)(0.25)$$
$$= 1$$

$z_1$is the approximate value of z (same as $\frac{dy}{dt}$) at $t = 0.25$

$$z_1 = z(0.25) \approx 1$$

For $i = 1, t_1 = 0.25, y_1 = 1.5, z_1 = 1$,

From Equation (E2.3)

$$y_2 = y_1 + f_1(t_1, y_1, z_1)h$$
$$= 1.5 + f_1(0.25,1.5,1)(0.25)$$
$$= 1.5 + (1)(0.25)$$
$$= 1.75$$

y_2 is the approximate value of y at

$$t = t_2 = t_1 + h = 0.25 + 0.25 = 0.50$$
$$y_2 = y(0.5) \approx 1.75$$

From Equation (E2.4)

$$z_2 = z_1 + f_2(t_1, y_1, z_1)h$$
$$= 1 + f_2(0.25,1.5,1)(0.25)$$
$$= 1 + (e^{-0.25} - 2(1) - 1.5)(0.25)$$
$$= 1 + (-2.7211)(0.25)$$
$$= 0.31970$$

$z_2$is the approximate value of $z$at

$$t = t_2 = 0.5$$
$$z_2 = z(0.5) \approx 0.31970$$

For $i = 2, t_2 = 0.5, y_2 = 1.75, z_2 = 0.31970$,

From Equation (E2.3)

$$y_3 = y_2 + f_1(t_2, y_2, z_2)h$$
$$= 1.75 + f_1(0.50,1.75,0.31970)(0.25)$$
$$= 1.75 + (0.31970)(0.25)$$
$$= 1.8299$$

y_3 is the approximate value of y at

$$t = t_3 = t_2 + h = 0.5 + 0.25 = 0.75$$
$$y_3 = y(0.75) \approx 1.8299$$

From Equation (E2.4)

$$z_3 = z_2 + f_2(t_2, y_2, z_2)h$$
$$= 0.31972 + f_2(0.50,1.75,0.31970)(0.25)$$
$$= 0.31972 + (e^{-0.50} - 2(0.31970) - 1.75)(0.25)$$
$$= 0.31972 + (-1.7829)(0.25)$$

$$= -0.1260$$

z_3 is the approximate value of z at

$$t = t_3 = 0.75$$

$$z_3 = z(0.75) \approx -0.12601$$

$$y(0.75) \approx y_3 = 1.8299$$

b) The exact value of $y(0.75)$ is

$$y(0.75)|_{exact} = 1.668$$

The absolute relative true error in the result from part (a) is

$$|\epsilon_t| = \left|\frac{1.668 - 1.8299}{1.668}\right| \times 100$$

$$= 9.7062\%$$

c) $\frac{dy}{dx}(0.75) = z_3 \approx -0.12601$

Exercises

Q1) Given

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}, y(0) = 1, \frac{dy}{dt}(0) = 2,$$

find by Heun's method

a) $y(0.75)$
b) $\frac{dy}{dx}(0.75)$.

Use a step size of $h = 0.25$.