ENG2009 – Modelling of Engineering Systems

Tutorial 2

Higher order and Coupled ODE

Higher order and coupled ODE – Examples

Example 1

Rewrite the following differential equation as a set of first order differential equations.

$$3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x}, y(0) = 5, y'(0) = 7$$

Solution

The ordinary differential equation would be rewritten as follows. Assume

$$\frac{dy}{dx} = z,$$

Then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

Substituting this in the given second order ordinary differential equation gives

$$3\frac{dz}{dx} + 2z + 5y = e^{-x}$$
$$\frac{dz}{dx} = \frac{1}{3}(e^{-x} - 2z - 5y)$$

The set of two simultaneous first order ordinary differential equations complete with the initial conditions then is

$$\frac{dy}{dx} = z, y(0) = 5$$
$$\frac{dz}{dx} = \frac{1}{3}(e^{-x} - 2z - 5y), z(0) = 7.$$

Now one can apply any of the numerical methods used for solving first order ordinary differential equations.

Example 2

Given

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}, y(0) = 1, \frac{dy}{dt}(0) = 2, \text{ find by Euler's method}$$

a) $y(0.75)$
b) the absolute relative true error for part(a), if $y(0.75)|_{exact} = 1.668$
c) $\frac{dy}{dt}(0.75)$

Use a step size of $h = 0.25$.

Solution

First, the second order differential equation is written as two simultaneous first-order differential equations as follows. Assume

$$\frac{dy}{dt} = z$$

then

$$\frac{dz}{dt} + 2z + y = e^{-t}$$
$$\frac{dz}{dt} = e^{-t} - 2z - y$$

So the two simultaneous first order differential equations are

$$\frac{dy}{dt} = z = f_1(t, y, z), y(0) = 1 \quad \text{(E2.1)}$$
$$\frac{dz}{dt} = e^{-t} - 2z - y = f_2(t, y, z), z(0) = 2 \quad \text{(E2.2)}$$

Using Euler's method on Equations (E2.1) and (E2.2), we get

$$y_{i+1} = y_i + f_1(t_i, y_i, z_i)h \quad \text{(E2.3)}$$
$$z_{i+1} = z_i + f_2(t_i, y_i, z_i)h \quad \text{(E2.4)}$$

a) To find the value of $y(0.75)$ and since we are using a step size of 0.25 and starting at $t = 0$, we need to take three steps to find the value of $y(0.75)$.

For $i = 0, t_0 = 0, y_0 = 1, z_0 = 2$,

From Equation (E2.3)

$$y_1 = y_0 + f_1(t_0, y_0, z_0)h$$
$$= 1 + f_1(0,1,2)(0.25)$$
$$= 1 + 2(0.25)$$
$$= 1.5$$

y_1 is the approximate value of y at

$$t = t_1 = t_0 + h = 0 + 0.25 = 0.25$$
$$y_1 = y(0.25) \approx 1.5$$

From Equation (E2.4)

$$z_1 = z_0 + f_2(t_0, y_0, z_0)h$$
$$= 2 + f_2(0,1,2)(0.25)$$
$$= 2 + (e^{-0} - 2(2) - 1)(0.25)$$
$$= 1$$

$z_1$is the approximate value of z (same as $\frac{dy}{dt}$) at $t = 0.25$

$$z_1 = z(0.25) \approx 1$$

For $i = 1, t_1 = 0.25, y_1 = 1.5, z_1 = 1$,

From Equation (E2.3)

$$y_2 = y_1 + f_1(t_1, y_1, z_1)h$$
$$= 1.5 + f_1(0.25,1.5,1)(0.25)$$
$$= 1.5 + (1)(0.25)$$
$$= 1.75$$

y_2 is the approximate value of y at

$$t = t_2 = t_1 + h = 0.25 + 0.25 = 0.50$$
$$y_2 = y(0.5) \approx 1.75$$

From Equation (E2.4)

$$z_2 = z_1 + f_2(t_1, y_1, z_1)h$$
$$= 1 + f_2(0.25,1.5,1)(0.25)$$
$$= 1 + (e^{-0.25} - 2(1) - 1.5)(0.25)$$
$$= 1 + (-2.7211)(0.25)$$
$$= 0.31970$$

$z_2$is the approximate value of $z$at

$$t = t_2 = 0.5$$
$$z_2 = z(0.5) \approx 0.31970$$

For $i = 2, t_2 = 0.5, y_2 = 1.75, z_2 = 0.31970$,

From Equation (E2.3)

$$y_3 = y_2 + f_1(t_2, y_2, z_2)h$$
$$= 1.75 + f_1(0.50,1.75,0.31970)(0.25)$$
$$= 1.75 + (0.31970)(0.25)$$
$$= 1.8299$$

y_3 is the approximate value of y at

$$t = t_3 = t_2 + h = 0.5 + 0.25 = 0.75$$
$$y_3 = y(0.75) \approx 1.8299$$

From Equation (E2.4)

$$z_3 = z_2 + f_2(t_2, y_2, z_2)h$$
$$= 0.31972 + f_2(0.50,1.75,0.31970)(0.25)$$
$$= 0.31972 + (e^{-0.50} - 2(0.31970) - 1.75)(0.25)$$
$$= 0.31972 + (-1.7829)(0.25)$$

$= -0.1260$

z_3 is the approximate value of z at

$$t = t_3 = 0.75$$

$$z_3 = z(0.75) \approx -0.12601$$

$$y(0.75) \approx y_3 = 1.8299$$

b) The exact value of $y(0.75)$ is

$$y(0.75)|_{exact} = 1.668$$

The absolute relative true error in the result from part (a) is

$$|\in_t| = \left|\frac{1.668 - 1.8299}{1.668}\right| \times 100$$

= 9.7062%

c) $\frac{dy}{dx}(0.75) = z_3 \approx -0.12601$

Exercises

Q1) Given

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}, y(0) = 1, \frac{dy}{dt}(0) = 2,$$

find by Heun's method

a) $y(0.75)$
b) $\frac{dy}{dx}(0.75)$.

Use a step size of $h = 0.25$.

Solutions

First, the second order differential equation is rewritten as two simultaneous first-order differential equations as follows. Assume

$$\frac{dy}{dt} = z$$

then

$$\frac{dz}{dt} + 2z + y = e^{-t}$$
$$\frac{dz}{dt} = e^{-t} - 2z - y$$

So the two simultaneous first order differential equations are

$$\frac{dy}{dt} = z = f_1(t, y, z), y(0) = 1 \quad \text{(E3.1)}$$
$$\frac{dz}{dt} = e^{-t} - 2z - y = f_2(t, y, z), z(0) = 2 \quad \text{(E3.2)}$$

Using Heun's method on Equations (1) and (2), we get

$$y_{i+1} = y_i + \frac{1}{2}\left(k_1^y + k_2^y\right)h \quad \text{(E3.3)}$$
$$k_1^y = f_1\,(t_i, y_i, z_i) \quad \text{(E3.4a)}$$
$$k_2^y = f_1\left(t_i + h, y_i + hk_1^y, z_i + hk_1^z\right) \quad \text{(E 3.4b)}$$
$$z_{i+1} = z_i + \frac{1}{2}\,(k_1^z + k_2^z)h \quad \text{(E3.5)}$$
$$k_1^z = f_2\,(t_i, y_i, z_i) \quad \text{(E3.6a)}$$
$$k_2^z = f_2\left(t_i + h, y_i + hk_1^y, z_i + hk_1^z\right) \quad \text{(E3.6b)}$$

For $i = 0, t_o = 0, y_o = 1, z_o = 2$

From Equation (E3.4a)

$$k_1^y = f_1(t_o, y_o, z_o)$$
$$= f_1(0,1,2)$$
$$= 2$$

From Equation (E3.6a)

$$k_1^z = f_2(t_0, y_0, z_0)$$
$$= f_2(0,1,2)$$
$$= e^{-0} - 2(2) - 1$$
$$= -4$$

From Equation (E3.4b)

$$k_2^y = f_1\left(t_0 + h, y_0 + hk_1^y, z_0 + hk_1^z\right)$$
$$= f_1\left(0 + 0.25, 1 + (0.25)(2), 2 + (0.25)(-4)\right)$$
$$= f_1(0.25, 1.5, 1)$$
$$= 1$$

From Equation (E3.6b)

$$k_2^z = f_2\left(t_0 + h, y_0 + hk_1^y, z_0 + hk_1^z\right)$$
$$= f_2\left(0 + 0.25, 1 + (0.25)(2), 2 + (0.25)(-4)\right)$$
$$= f_2(0.25, 1.5, 1)$$
$$= e^{-0.25} - 2(1) - 1.5$$
$$= -2.7212$$

From Equation (E3.3)

$$y_1 = y_0 + \frac{1}{2}\left(k_1^y + k_2^y\right)h$$
$$= 1 + \frac{1}{2}(2 + 1)(0.25)$$
$$= 1.375$$

y_1 is the approximate value of y at

$$t = t_1 = t_0 + h = 0 + 0.25 = 0.25$$
$$y_1 = y(0.25) \cong 1.375$$

From Equation (E3.5)

$$z_1 = z_0 + \frac{1}{2}(k_1^z + k_2^z)h$$
$$= 2 + \frac{1}{2}(-4 + (-2.7212))(0.25)$$
$$= 1.1598$$

z_1 is the approximate value of z at

$$t = t_1 = 0.25$$
$$z_1 = z(0.25) \approx 1.1598$$

For $i = 1, t_1 = 0.25, y_1 = 1.375, z_1 = 1.1598$

From Equation (E3.4a)

$$k_1^y = f_1(t_1, y_1, z_1)$$
$$= f_1(0.25, 1.375, 1.1598)$$
$$= 1.1598$$

From Equation (E3.6a)

$$k_1^z = f_2(t_1, y_1, z_1)$$
$$= f_2(0.25, 1.375, 1.1598)$$
$$= e^{-0.25} - 2(1.1598) - 1.375$$
$$= -2.9158$$

From Equation (E3.4b)

$$k_2^y = f_1\left(t_1 + h, y_1 + hk_1^y, z_1 + hk_1^z\right)$$
$$= f_1\left(0.25 + 0.25, 1.375 + (0.25)(1.1598), 1.1598 + (0.25)(-2.9158)\right)$$
$$= f_1(0.50, 1.6649, 0.43087)$$
$$= 0.43087$$

From Equation (E3.6b)

$$k_2^z = f_2\left(t_1 + h, y_1 + hk_1^y, z_1 + hk_1^z\right)$$
$$= f_2\left(0.25 + 0.25, 1.375 + (0.25)(1.1598), 1.1598 + (0.25)(-2.9158)\right)$$
$$= f_2(0.50, 1.6649, 0.43087)$$
$$= e^{-0.50} - 2(0.43087) - 1.6649$$
$$= -1.9201$$

From Equation (E3.3)

$$y_2 = y_1 + \frac{1}{2}\left(k_1^y + k_2^y\right)h$$
$$= 1.375 + \frac{1}{2}(1.1598 + 0.43087)(0.25)$$
$$= 1.5738$$

y_2 is the approximate value of y at

$$t = t_2 = t_1 + h = 0.25 + 0.25 = 0.50$$
$$y_2 = y(0.50) \approx 1.5738$$

From Equation (E3.5)

$$z_2 = z_1 + \frac{1}{2}(k_1^z + k_2^z)h$$
$$= 1.1598 + \frac{1}{2}(-2.9158 + (-1.9201))(0.25)$$
$$= 0.55533$$

z_2 is the approximate value of z at

$$t = t_2 = 0.50$$
$$z_2 = z(0.50) \approx 0.55533$$

For $i = 2, t_2 = 0.50, y_2 = 1.57384, z_2 = 0.55533$

From Equation (E3.4a)

$$k_1^y = f_1(t_2, y_2, z_2)$$
$$= f_1(0.50, 1.5738, 0.55533)$$
$$= 0.55533$$

From Equation (E3.6a)

$$k_1^z = f_2(t_2, y_2, z_2)$$

$$= f_2(0.50, 1.5738, 0.55533)$$
$$= e^{-0.50} - 2(0.55533) - 1.5738$$
$$= -2.0779$$

From Equation (E3.4b)

$$k_2^y = f_2(t_2 + h, y_2 + hk_1^y, z_2 + hk_1^z)$$
$$= f_1(0.50 + 0.25, 1.5738 + (0.25)(0.55533), 0.55533 + (0.25)(-2.0779))$$
$$= f_1(0.75, 1.7126, 0.035836)$$
$$= 0.035836$$

From Equation (E3.6b)

$$k_2^z = f_2(t_2 + h, y_2 + hk_1^y, z_2 + hk_1^z)$$
$$= f_2(0.50 + 0.25, 1.5738 + (0.25)(0.55533), 0.55533 + (0.25)(-2.0779))$$
$$= f_2(0.75, 1.7126, 0.035836)$$
$$= e^{-0.75} - 2(0.035836) - 1.7126$$
$$= -1.3119$$

From Equation (E3.3)

$$y_3 = y_2 + \frac{1}{2}(k_1^y + k_2^y)h$$
$$= 1.5738 + \frac{1}{2}(0.55533 + 0.035836)(0.25)$$
$$= 1.6477$$

y_3 is the approximate value of y at

$$t = t_3 = t_2 + h = 0.50 + 0.25 = 0.75$$
$$y_3 = y(0.75) \approx 1.6477$$

b) From Equation (E3.5)

$$z_3 = z_2 + \frac{1}{2}(k_1^z + k_2^z)h$$
$$= 0.55533 + \frac{1}{2}(-2.0779 + (-1.3119))(0.25)$$
$$= 0.13158$$

z_3 is the approximate value of z at

$$t = t_3 = 0.75$$
$$z_3 = z(0.75) \cong 0.13158$$

The intermediate and the final results are shown in Table 1.

Table 1 Intermediate results of Heun's method.

i	0	1	2
t_i	0	0.25	0.50
y_i	1	1.3750	1.5738
z_i	2	1.1598	0.55533
k_1^y	2	1.1598	0.55533
k_1^z	-4	-2.9158	-2.0779
k_2^y	1	0.43087	0.035836
k_2^z	-2.7211	-1.9201	-1.3119
y_{i+1}	1.3750	1.5738	1.6477
z_{i+1}	1.1598	0.55533	0.13158

Recall Tutorial 5 - Python code example:

From differential eq:

$$m\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = u(t).$$

Rearrange to yield

$$\frac{d^2y(t)}{dt^2} = \frac{1}{m}\left(-b\frac{dy(t)}{dt} - ky(t) + u(t)\right)$$

Assume

$$\frac{dy}{dt} = z(t)$$

Therefore

$$\frac{dz}{dt} = \frac{1}{m}\left(-bz(t) - ky(t) + u(t)\right)$$

Therefore the python code:

```
#import numpy, integrate and pyplot
import numpy as np
from scipy import integrate
from matplotlib import pyplot as plt

#sim data
m=1 #mass of the oscillator
b=0.2 #damper of the oscillator
k=1 #stiffness of the oscillator
omega = (k/m)**0.5 #angular frequency of the oscillator
T=2*np.pi/omega #period of the oscillator

#create 1000 unifirmply distributed points in the interval [0,10T]
t = np.linspace(0,10*T,1000)

#lambda returning the force
force = lambda t:t<=0.1

#lambda with the RHS of the equation, it evaluates the force and returns a list
f = lambda X,t: [X[1],1/m*(force(t)-k*X[0]-b*X[1])]

#solve equation using odeint, now the values of X for t=0 are given by a list
XOdeint = integrate.odeint(f,[0,0],t)

ti = np.linspace(0,10*T,1000)

#plot force
plt.plot(ti,force(ti))
plt.title('Force (input)')
plt.grid()
plt.xlabel('$t$')
plt.ylabel('$input$')
plt.show()

#plot displacements from both solutions (first column of each solution)
plt.plot(t,XOdeint[:,0])
plt.legend(['Odeint'])
plt.title('Displacement')
plt.grid()
plt.xlabel('$t$')
plt.ylabel('$x$')
plt.show()
```

```
#plot velocities from both solutions (first column of each solution)
plt.plot(t,XOdeint[:,1])
plt.legend(['Odeint'])
plt.title('Velocitie')
plt.grid()
plt.xlabel('$t$')
plt.ylabel('$\dot{x}$')
plt.show()

#Save data
np.savetxt('t.txt',t)
np.savetxt('x.txt',XOdeint[:,0])
```