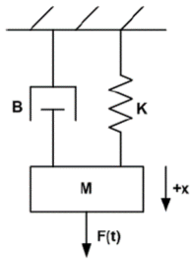
ENG2009 – Modelling of Engineering Systems

Tutorial 4

State space

State Space:

Example 1:
Find the state space for the following mass-spring-damper system:



Physical Model

Solution 1:
Step 1: The differential equation of the mass-spring-damper systems in Lecture 2 is (position $x(t)$ is the output and applied force $u(t)$ is the input):

$$\ddot{x}(t) = \frac{1}{M}(-B\dot{x}(t) - Kx(t) + u(t))$$

Step 2: Identify the states:
Let position as 1st states: $x_1(t) = x(t)$
Let velocity as 2nd states: $x_2(t) = \dot{x}(t)$

Step 3: Derivative of states:
$\dot{x}_1(t) = \dot{x}(t) = x_2(t)$
$\dot{x}_2(t) = \ddot{x}(t) = \frac{1}{M}(-B\dot{x}(t) - Kx(t) + u(t))$

Step 4: Rearrange the derivative of states

$$\begin{aligned} \dot{x}_1(t) &= \quad 0x_1(t) + 1x_2(t) + \ 0\,u(t) \\ \dot{x}_2(t) &= -\frac{K}{M}x_1(t) - \frac{B}{M}x_2(t) + \frac{1}{M}u(t) \end{aligned}$$

Step 5: Rewrite in matrix form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t)$$

where the output equation

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

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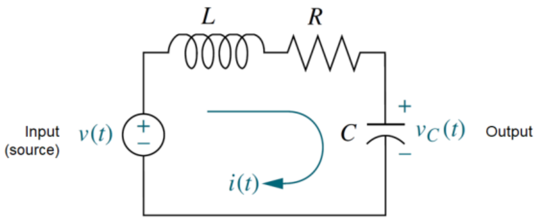
Example 2:
Find the state space equation relating the capacitor voltage $v_c(t)$ (output voltage), to the input voltage, $v(t)$.



Simple RLC circuit

Solution 2:
Step 1: using Kirchoff's voltage law, the differential equation (output: $v_c(t)$, input: $v(t)$):

$$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = v(t) \text{ or}$$

$$\frac{d^2v_c(t)}{dt^2} = \frac{1}{LC}\left(-RC\frac{dv_c(t)}{dt} - v_c(t) + v(t)\right)$$

Step 2: Identify the states and inputs:
Let 1st states: $x_1(t) = v_c(t)$
Let 2nd states: $x_2(t) = \dot{v}_c(t)$
Input: $u(t) = v(t)$

Step 3: Derivative of states:
$\dot{x}_1(t) = \dot{v}_c(t) = x_2(t)$
$\dot{x}_2(t) = \ddot{v}_c(t) = \frac{1}{LC}(-RCx_2(t) - x_1(t) + u(t))$

Step 4: Rearrange the derivative of states

$$\dot{x}_1(t) = \quad 0x_1(t) + 1x_2(t) + 0\,u(t)$$

$$\dot{x}_2(t) = -\frac{1}{LC}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{LC}u(t)$$

Step 5: Rewrite in matrix form:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1/LC & -R/L\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1/LC\end{bmatrix}u(t)$$

where the output equation

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

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Example 3:
For a DC motor, find the state space equation relating the input voltage $v(t) = e_{in}(t)$, and output shaft speed $\omega(t) = \dot{\theta}(t)$.

Solution 3:
Step 1: Two differential equations (input: $e_{in}(t)$, output: $\dot{\theta}(t)$):

$J\ddot{\theta} + B_r\,\dot{\theta} = \tau_e = K_t i$ (A)
$e_{in} = L\frac{di}{dt} + iR + K_e\dot{\theta}$ (B)

Step 2: Identify the states and input:
Let 1st states: $x_1(t) = \theta(t)$
Let 2nd states: $x_2(t) = \dot{\theta}(t)$
Let 3rd states: $x_3(t) = i(t)$
Input: $u(t) = e_{in}(t)$

Step 3: Derivative of states:
$\dot{x}_1(t) = \dot{\theta}(t) = x_2(t)$
$\dot{x}_2(t) = \ddot{\theta}(t) = \frac{1}{J}(-B_r x_2(t) + K_t x_3(t))$ (from eq (A))
$\dot{x}_3(t) = \frac{di}{dt} = \frac{1}{L}(-R x_3(t) - K_e x_2(t) + u(t))$ (from eq (B))

Step 4: Rearrange the derivative of states

$$\dot{x}_1(t) = 0x_1(t) + 1x_2(t) + 0x_3(t) + 0\,u(t)$$
$$\dot{x}_2(t) = 0x_1(t) - \frac{B_r}{J}x_2(t) + \frac{K_t}{J}x_3(t) + 0\,u(t)$$
$$\dot{x}_3(t) = 0x_1(t) - \frac{K_e}{L}x_2(t) - \frac{R}{L}x_3(t) + \frac{1}{L}u(t)$$

Step 5: Rewrite in matrix form:
$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & -B_r/J & K_t/J\\ 0 & -K_e/L & -R/L\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1/L\end{bmatrix}u(t)$$

where the output equation
$$y(t) = \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

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Example 4:
Find the eigenvalues for the following system and determine if the system stable?

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Answer:
Eigenvalues λ:

$$\det(A - \lambda I_n) = 0$$

Find λ:

$$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix}\right) = 0$$

$$(-\lambda)(-2-\lambda) - (-1)(1) = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)(\lambda + 1) = 0$$

Therefore eigenvalues (poles) at $\lambda = -1$ and $\lambda = -1$
⇒ Stable system

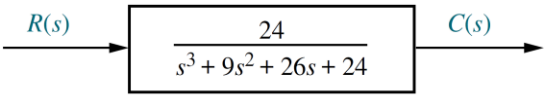
Exercises

Q1) Find the eigenvalues for the following system and determine if the system stable?

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Q2) Find the state-space representation for the transfer function shown below:



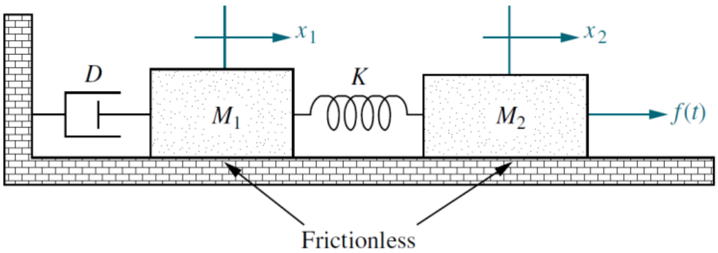
Q3) For the translational mechanical system shown below,

(a) Show that the differential equations are given by:

$$M_1\frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + Kx_1 - Kx_2 = 0$$

$$-Kx_1 + M_2\frac{d^2x_2}{dt^2} + Kx_2 = f(t)$$

(b) Find the state space equations (where the input is $f(t)$ and the output is x_2)



SOLUTIONS:

Q1) Solution:

Find the eigenvalues for the following system and determine if the system stable?

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Answer:

$$\det(A - \lambda I_n) = 0$$

Find λ:

$$\det\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}-\lambda & 1\\ -1 & -\lambda\end{bmatrix}\right) = 0$$

$$(-\lambda)(-\lambda) - (-1)(1) = 0$$

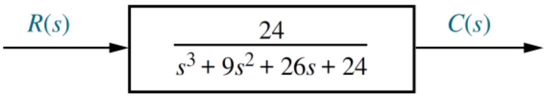
$$\lambda^2 + 1 = 0$$

Therefore eigenvalues (poles) at $\lambda = \pm j$

⇒ marginally stable system (neither positive nor negative poles)

Q2) Solution:

Find the state-space representation for the transfer function shown below:



Solution:

given $\frac{Y(s)}{U(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$

$$Y(s)(s^3 + 9s^2 + 26s + 24) = 24\,U(s)$$

$$\Rightarrow \quad \dddot{y} + 9\ddot{y} + 26\dot{y} + 24y = 24\,u(t)$$

Let
$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y} \end{aligned}\right\} \frac{d}{dt} \left\{\begin{aligned} \dot{x}_1 &= \dot{y} = x_2 \\ \dot{x}_2 &= \ddot{y} = x_3 \\ \dot{x}_3 &= \dddot{y} = -9\ddot{y} - 26\dot{y} - 24y + 24u \\ &= -9x_3 - 26x_2 - 24x_1 + 24u \end{aligned}\right.$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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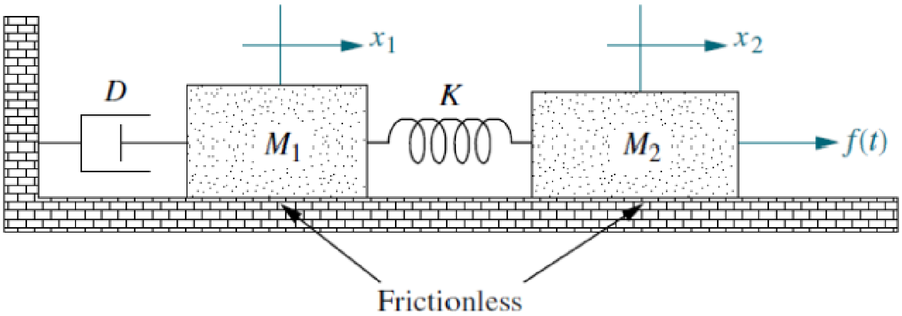
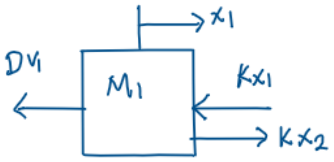
Q3) Solution:

3(a)



Mass M_1

FBD



$$\Sigma F = M_1 a_1$$

$$-DV_1 - Kx_1 + Kx_2 = M_1 a_1 \quad , \quad V_1 = \frac{dx_1}{dt} = \dot{x}_1$$

$$a_1 = \frac{d^2x_1}{dt^2} = \ddot{X}_2$$

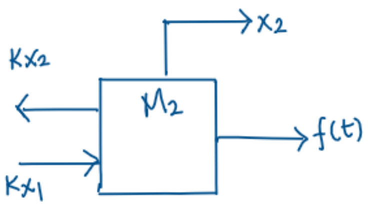
$$-D\dot{X}_1 - Kx_1 + KX_2 = M_1 \ddot{X}_1$$

rearrange

$$M_1\ddot{X}_1 + D\dot{X}_1 + KX_1 - KX_2 = 0 \quad \text{※} \quad (1)$$

Mass M_2

FBD



$$\Sigma F = M_2 a_2$$

$$f(t) + KX_1 - KX_2 = M_2 a_2 \quad , \quad a_2 = \frac{d^2X_2}{dt^2} = \ddot{X}_2$$

rearrange

$$M_2\ddot{x}_2 + Kx_2 - Kx_1 = f(t) \quad * \quad (2)$$

Overall

Therefore from (1) and (2)

$$M_1\ddot{x}_1 + D\dot{x}_1 + Kx_1 - Kx_2 = 0$$
$$M_2\ddot{x}_2 \quad - Kx_1 + Kx_2 = f(t) \quad *$$

3(b)

given
$$M_1\ddot{x}_1 + D\dot{x}_1 + Kx_1 - Kx_2 = 0 \qquad (1)$$
$$-Kx_1 + M_2\ddot{x}_2 + Kx_2 = f \qquad (2)$$

let
$$\left.\begin{aligned} z_1 &= x_1 \\ z_2 &= \dot{x}_1 \\ z_3 &= x_2 \\ z_4 &= \dot{x}_2 \end{aligned}\right\} \frac{d}{dt} \left\{ \begin{aligned} \dot{z}_1 &= \dot{x}_1 = z_2 \\ \dot{z}_2 &= \ddot{x}_1 = \frac{1}{M}\left(-D\dot{x}_1 - Kx_1 + Kx_2\right) \\ &= \frac{1}{M_1}\left(-Dz_2 - Kz_1 + Kz_3\right) \\ \dot{z}_3 &= \dot{x}_2 = z_4 \\ \dot{z}_4 &= \ddot{x}_2 = \frac{1}{M_2}\left(Kx_1 - Kx_2 + f\right) \\ &= \frac{1}{M_2}\left(Kz_1 - Kz_3 + f\right) \end{aligned}\right.$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & \end{bmatrix}}_{A} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix}}_{B} f$$

$$y = x_2 = z_3 = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

Python Example for Q2 solution (eigenvalues vs poles)

```
import numpy as np
from numpy import linalg as LA


A = np.array([[0, 1, 0], [0, 0, 1], [-24, -26, -9]])
print('A:\n',A)

B = np.array([[0], [0], [2]])
print('B:\n',B)

C = np.array([1, 0, 0])
print('C:\n',C)

w, v = LA.eig(A)
print('Eigenvalues:\n', w)
print('Eigenvectors:\n',v)

den=[1, 9, 26, 24]
poles=np.roots(den)
print('poles:\n',poles)
```

Output:

```
A:
 [[  0   1   0]
 [  0   0   1]
 [-24 -26  -9]]
B:
 [[0]
 [0]
 [2]]
C:
 [1 0 0]
Eigenvalues:
 [-2. -3. -4.]
Eigenvectors:
 [[ 0.21821789  0.10482848  0.06052275]
 [-0.43643578 -0.31448545 -0.24209101]
 [ 0.87287156  0.94345635  0.96836405]]
poles:
 [-4. -3. -2.]
```

Note that aigenvalues of the matrix A = poles of the system