ENG2009 – Modelling of Engineering Systems
Tutorial 10

PDE 1 & 2 – Parabolic PDE using Explicit and Implicit methods

What is PDE

Ordinary Differential Equations have only one independent variable

$$3\frac{dy}{dt} + 5y^2 = 3e^{-t}, y(0) = 5$$

Partial Differential Equations have more than one independent variable

$$3\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$$

subject to certain conditions: where u is the dependent variable, and x and y are the independent variables.

Classification of 2nd Order Linear PDE's

Consider

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

where A, B and C are functions of x and y, and D is a function of x, y, u and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$

Can be:

- Elliptic - If $B^2 - 4AC < 0$
- Parabolic - If $B^2 - 4AC = 0$
- Hyperbolic - If $B^2 - 4AC > 0$

Physical Example of a Parabolic PDE

The internal temperature of a metal rod exposed to two different temperatures at each end can be found using the heat conduction equation.

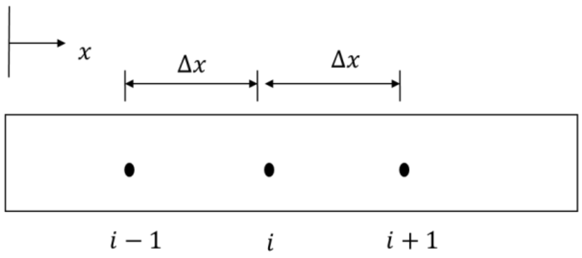
$$\alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

Discretizing the Parabolic PDE

For a rod of length L divided into $n + 1$ nodes $\Delta x = \frac{L}{n}$

The time is similarly broken into time steps of Δt

Hence T_i^j corresponds to the temperature at node i, that is,
$x = (i)(\Delta x)$ and time $t = (j)(\Delta t)$



Schematic diagram showing interior nodes

The Explicit Method

From $\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$

$$\alpha \frac{T_{i+1}^{j} - 2T_i^j + T_{i-1}^j}{(\Delta x)^2} = \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

Solving for the temp at the time node $j+1$ gives

$$T_i^{j+1} = T_i^j + \alpha \frac{\Delta t}{(\Delta x)^2}\left(T_{i+1}^j - 2T_i^j + T_{i-1}^j\right)$$

choosing,

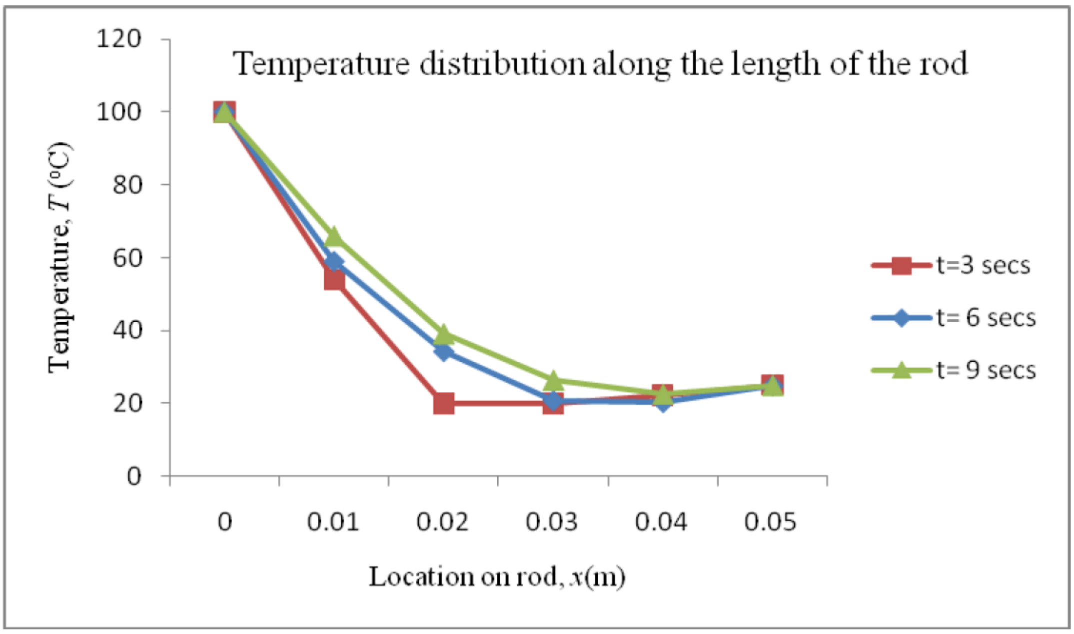
$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

we can write the equation as:

$$T_i^{j+1} = T_i^j + \lambda\left(T_{i+1}^j - 2T_i^j + T_{i-1}^j\right)$$

Last lecture: Example 1: Explicit Method

Temperature distribution along the length of the rod at different times



Summary: The Implicit Method

From

$$\alpha \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2} = \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

Rearranging yields

$$\boxed{-\lambda T_{i-1}^{j+1} + (1+2\lambda)T_i^{j+1} - \lambda T_{i+1}^{j+1} = T_i^j}$$

given that,

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

The rearranged equation can be written for every node during each time step. These equations can then be solved as a simultaneous system of linear equations to find the nodal temperatures at a particular time.

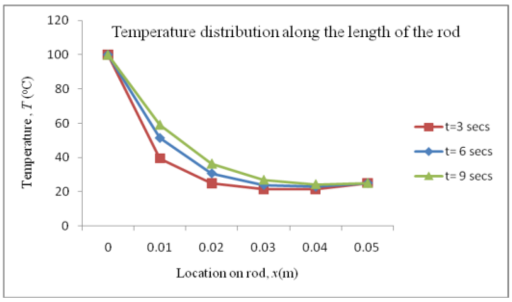
Internal Temperatures at 9 sec.

The table below allows you to compare the results from all three methods discussed in juxtaposition with the analytical solution.

Node	Explicit	Implicit	Crank-Nicolson	Analytical
T_1^3	65.953	59.043	62.604	62.510
T_2^3	39.132	36.292	37.613	37.084
T_3^3	27.266	26.809	26.562	25.844
T_4^3	22.872	24.243	24.042	23.610

Example 2: Implicit method

Temperature distribution along the length of the rod at different times



Exercises

Q1) In a general second order linear partial differential equation with two independent variables

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

where A, B, C are functions of x and y, and D is a function of $x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, then the partial differential equation is parabolic if

(A) $B^2 - 4AC < 0$
(B) $B^2 - 4AC > 0$
(C) $B^2 - 4AC = 0$
(D) $B^2 - 4AC \neq 0$

Q2) The region in which the following partial differential equation

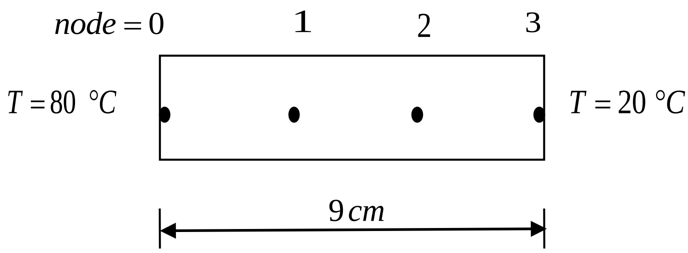
$$x^3\frac{\partial^2 u}{\partial x^2} + 27\frac{\partial^2 u}{\partial y^2} + 3\frac{\partial^2 u}{\partial x \partial y} + 5u = 0$$

acts as parabolic equation is

(A) $x > \left(\frac{1}{12}\right)^{1/3}$
(B) $x < \left(\frac{1}{12}\right)^{1/3}$
(C) for all values of x
(D) $x = \left(\frac{1}{12}\right)^{1/3}$

Q3) The partial differential equation of the temperature in a long thin rod is given by

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$$



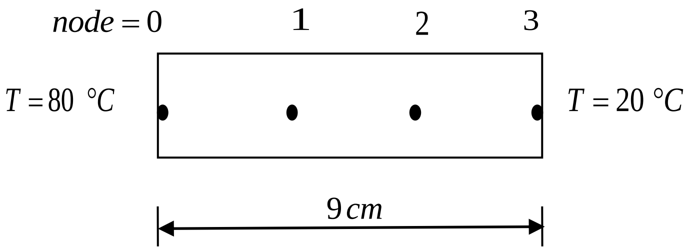
If $\alpha = 0.8 cm^2/s$, the initial temperature of rod is $40°C$, and the rod is divided into three equal segments, the temperature at node 1 (using $\Delta t = 0.1s$) by using an <u>explicit</u> solution at $t = 0.2sec$ is

(A) 40.7134 ^0C
(B) 40.6882 ^0C
(C) 40.7033 ^0C
(D) 40.6956 ^0C

Formula: Explicit method $T_i^{j+1} = T_i^j + \lambda\left(T_{i+1}^j - 2T_i^j + T_{i-1}^j\right)$, where $\lambda = \alpha\frac{\Delta t}{(\Delta x)^2}$

Q4) The partial differential equation of the temperature in a long thin rod is given by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$



If $\alpha = 0.8cm^2/s$, the initial temperature of rod is $40°C$, and the rod is divided into three equal segments, the temperature at node 1 (using $\Delta t = 0.1s$) by using an <u>implicit</u> solution for $t = 0.2sec$ is

(A) 40.7134 ^0C
(B) 40.6882 ^0C
(C) 40.7033 ^0C
(D) 40.6956 ^0C

Implicit method: $-\lambda T_{i-1}^{j+1} + (1 + 2\lambda)T_i^{j+1} - \lambda T_{i+1}^{j+1} = T_i^j$ where $\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$

PDE 3 – Elliptic PDE using Direct and Gauss-Seidel methods

Recall: Classification of 2nd Order Linear PDE's

Given

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

Example

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

where

$$A = 1, B = 0, C = 1$$

which yield

$$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$$

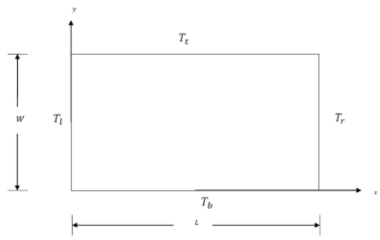
therefore the equation is elliptic.

Summary: Physical Example of an Elliptic PDE

Schematic diagram of a plate with specified temperature boundary conditions

The (Laplace) equation that governs the temperature:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$



Summary: Discretizing the Elliptic PDE

From

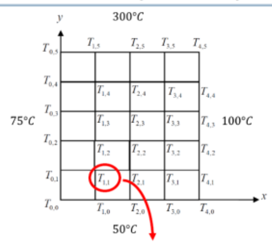
$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

Once the governing equation has been discretized there are several numerical methods that can be used to solve the problem.

We examined the:

- Direct Method
- Gauss-Seidel Method

Summary: Example 1: Direct Method



Example calculations:

The equation for the temperature at the node (1,1) is given by:

$i = 1$ and $j = 1$

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$
$$T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0} - 4T_{1,1} = 0$$
$$T_{2,1} + 75 + T_{1,2} + 50 - 4T_{1,1} = 0$$
$$-4T_{1,1} + T_{1,2} + T_{2,1} = -125$$

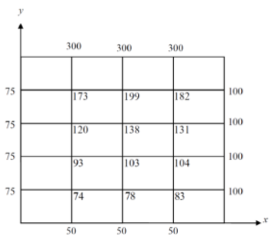
Summary: Example 1: Direct Method

We can develop similar equations for every interior node leaving us with an equal number of equations and unknowns.

Question: How many equations would this generate? Answer: 12

Solving yields:

$$\begin{bmatrix} T_{1,1} \\ T_{1,2} \\ T_{1,3} \\ T_{1,4} \\ T_{2,1} \\ T_{2,2} \\ T_{2,3} \\ T_{2,4} \\ T_{3,1} \\ T_{3,2} \\ T_{3,3} \\ T_{3,4} \end{bmatrix} = \begin{bmatrix} 73.8924 \\ 93.0252 \\ 119.907 \\ 173.355 \\ 77.5443 \\ 103.302 \\ 138.248 \\ 198.512 \\ 82.9833 \\ 104.389 \\ 131.271 \\ 182.446 \end{bmatrix} {}^\circ C$$



Summary: The Gauss-Seidel Method

- Recall the discretized equation

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

- This can be rewritten as

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

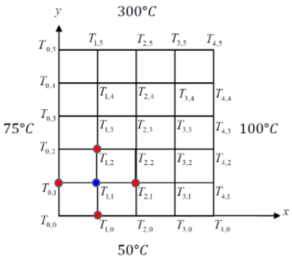
- For the Gauss-Seidel Method, this equation is solved iteratively for all interior nodes until a pre-specified tolerance is met.

Example 2: Gauss-Seidel Method

- **Now we can begin to solve for the temperature at each interior node using**

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

- **Assume all internal nodes to have an initial temperature of <u>zero</u>.**



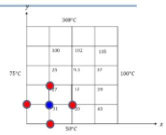
Iteration #1 (example calculations)

<u>$i = 1$ and $j = 1$</u>

$$T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4}$$
$$= \frac{0 + 75 + 0 + 50}{4}$$
$$= 31.2500°C$$

Example 2: Gauss-Seidel Method

Iteration #2 (example calculations)



<u>$i = 1$ and $j = 1$</u>

$$T_{1,1} = \frac{T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0}}{4}$$
$$= \frac{20.3125 + 75 + 26.5625 + 50}{4}$$
$$= 42.9688°C$$

Absolute relative error

$$|\varepsilon_a|_{1,1} = \left|\frac{T_{1,1}^{present} - T_{1,1}^{previous}}{T_{1,1}^{present}}\right| \times 100$$
$$= \left|\frac{42.9688 - 31.2500}{42.9688}\right| \times 100$$
$$= 27.27\%$$

Exercises

Q5) In a general second order linear partial differential equation with two independent variables,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

where A, B, C are functions of x and y, and D is a function of $x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, then the PDE is elliptic if

(A) $B^2 - 4AC < 0$
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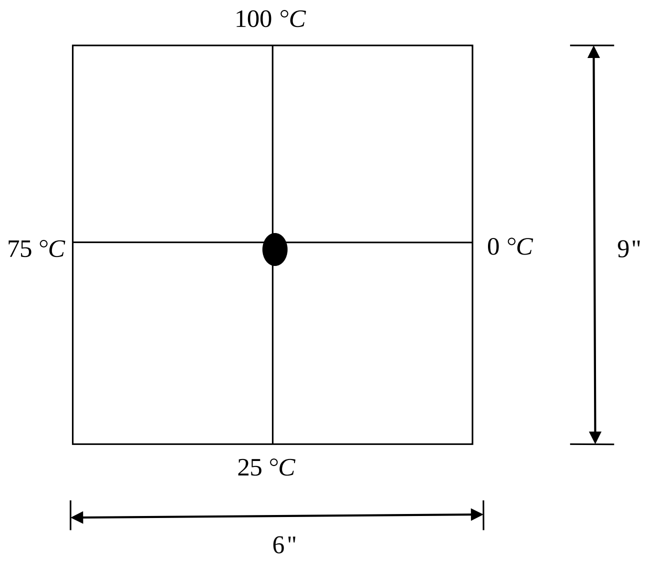
Q6) The region in which the following equation

$$x^3\frac{\partial^2 u}{\partial x^2} + 27\frac{\partial^2 u}{\partial y^2} + 3\frac{\partial^2 u}{\partial x \partial y} + 5u = 0$$

acts as an elliptic equation is

(A) $x > \left(\frac{1}{12}\right)^{1/3}$
(B) $x < \left(\frac{1}{12}\right)^{1/3}$
(C) for all values of x
(D) $x = \left(\frac{1}{12}\right)^{1/3}$

Q7) Find the temperature at the interior node given in the following figure using the direct method



(A) $45.19°C$
(B) $48.64°C$
(C) $50.00°C$
(D) $56.79°C$

Formula: Direct method.
From

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Substituting the approximations into the equation yields:

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

if,

$$\Delta x = \Delta y$$

the equation can be rewritten as

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$