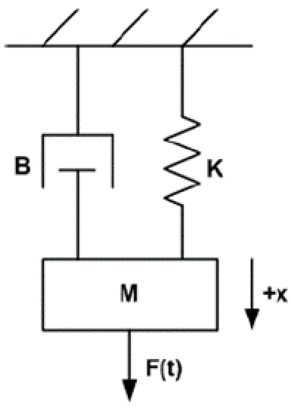
ENG2009 – Modelling of Engineering Systems

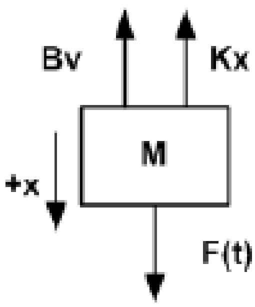
Tutorial 1

Mechanical Systems (Translational)

Example: Write the mathematical equation for the following mechanical system



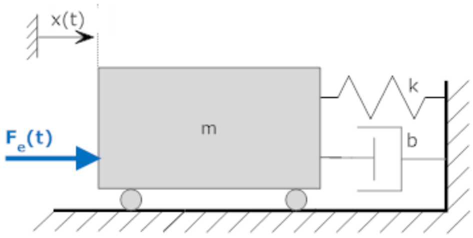
Answer:
Free body diagram:



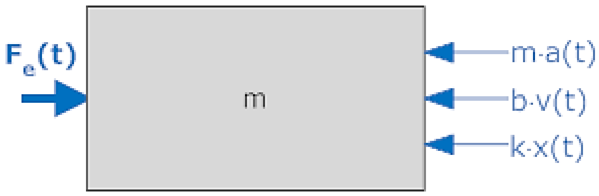
Math model:

$$\sum F_y = Ma$$
$$-Bv - Kx + F = Ma$$
$$a = \frac{1}{M}(-Bv - Kx + F)$$
Let $a = \ddot{x},\ v = \dot{x}$
$$\ddot{x} = \frac{1}{M}(-B\dot{x} - Kx + F)$$

Example: (Mass spring damper from Lecture 2)
Write the mathematical equation for the following mechanical system, where the input is the external force F_e and the outpus is position $x(t)$.
(Mass spring damper from Lecture 2)



Answer:
Free body diagram:



Math model:

$$\sum_{all} F = 0$$

$$F_e(t) - ma(t) - bv(t) - kx(t) = 0$$

or

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F_e(t)$$

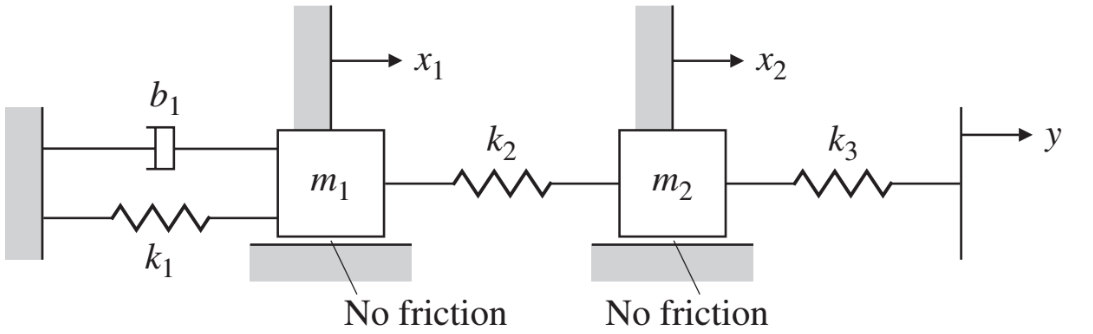
Or dot notation

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F_e(t)$$

Remark:
This equation is in our standard form (input-output notation):
Left hand side: system outputs (the unknown variables)
right hand side: system inputs (the known variables)

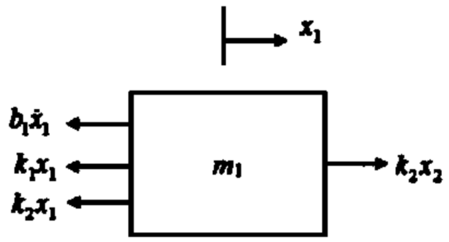
Excercise:

Q1) Write the differential equations for the mechanical systems shown below. Assume that there are nonzero initial conditions for both masses and there is no input.



Q1) Answer:

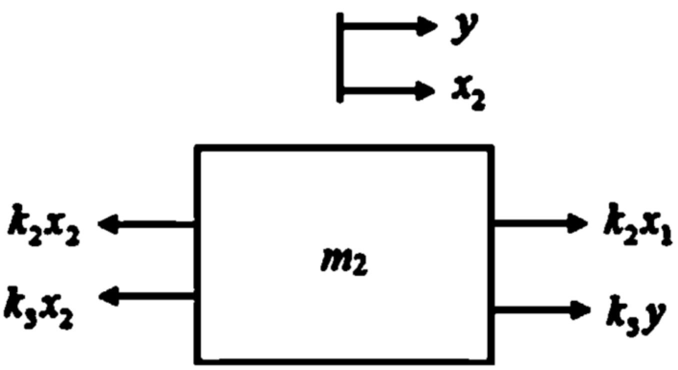
Draw the free body diagram of mass m_1.



Write the differential equation describing the system.

$$m_1\ddot{x}_1 = -b_1\dot{x}_1 - k_1x_1 - k_2x_1 + k_2x_2$$
$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0$$

Draw the free body diagram of mass m_2.



Write the differential equation describing the system.

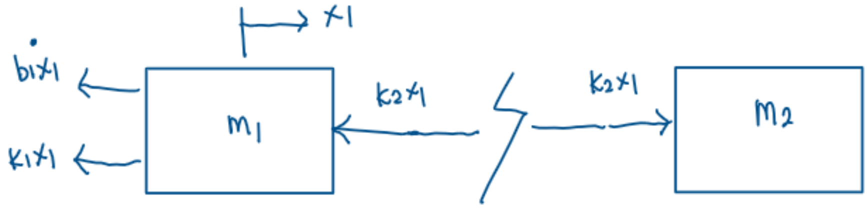
$$m_2\ddot{x}_2 = -k_2x_2 - k_3x_2 + k_2x_1 + k_3y$$
$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3(x_2 - y) = 0$$

Thus, the differential equations describing the system are,

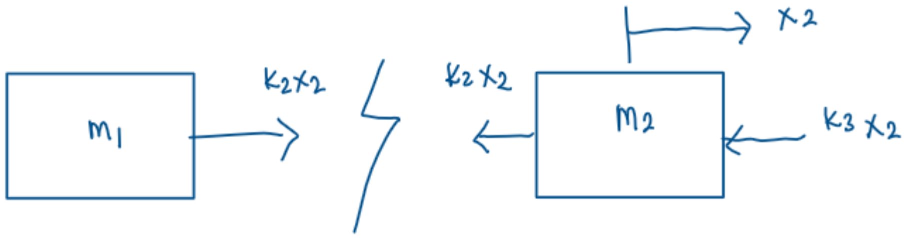
$$\boxed{\begin{aligned} &m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0 \\ &m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3(x_2 - y) = 0 \end{aligned}}.$$

case ① $x_1 \neq 0$, $x_2 = 0$, $y = 0$

FBD

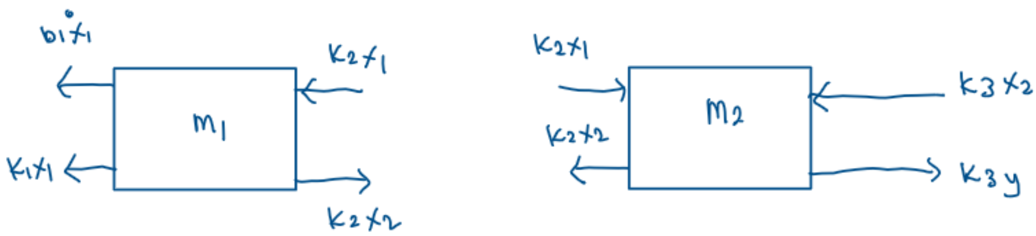


case ② $x_1 = 0$, $x_2 \neq 0$, $y = 0$



case ③ $x_1 = 0$, $x_2 = 0$, $y \neq 0$

<u>superposition</u>



Electrical systems

Notes:

Kirchhoff's voltage law: *the algebraic sum of all voltages taken around any closed path in a circuit is zero*.

$$\sum_j v_j = 0$$

where v_j denotes the voltage across the j_{th} element in the loop

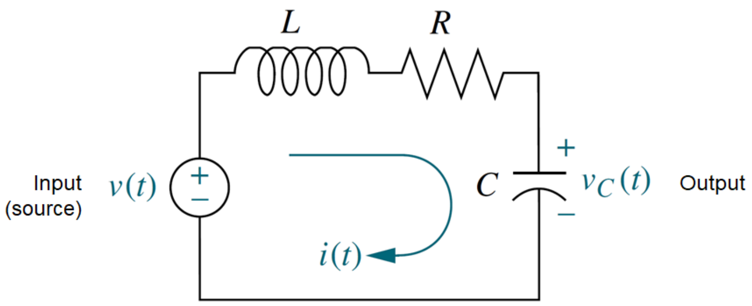
Kirchhoff's current law: *the algebraic sum of the currents at **any junction** is zero*.

$$\sum_j i_j = 0$$

where i_j denotes the current at the j_{th} node.

Example:

Find the differential equation relating the capacitor voltage $v_c(t)$ (output voltage), to the input voltage, $v(t)$



Answer:

- Summing the voltages around the loop, assuming zero initial conditions, yields the integral-differential equation:

$$v_{inductor} + v_{resistor} + v_{capacitor} = v_{in}$$

$$L\frac{di(t)}{dt} + i(t)R + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t)$$

- Changing variables from current to charge using

$$i = \frac{dq}{dt}$$

(see Table 1 in lecture notes) yields:

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t)$$

- Changing variables from voltage to charge using

$$q = Cv_c$$

(see Table 1 in lecture notes) yields:

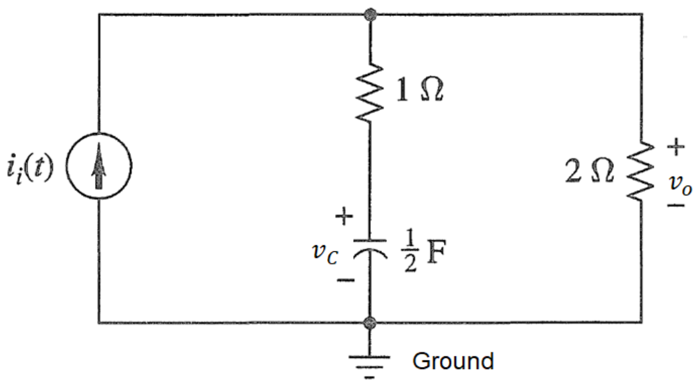
$$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = v(t)$$

Where $v(t)$ is the input voltage (source)

and $v_c(t)$ is the output voltage

Excercises:

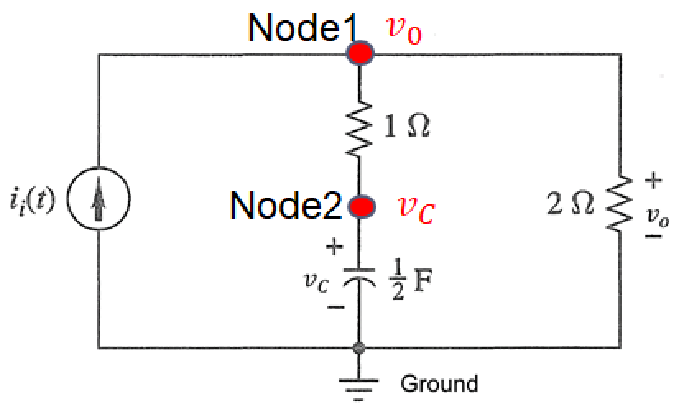
Q1) Find the input-output differential equation relating the output voltage $v_o(t)$, to the input current, $i_i(t)$ for the following circuit



Answer:

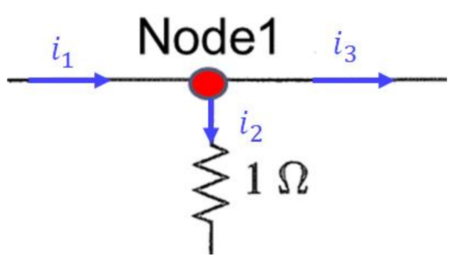
Strategy:

- Input: current $i_i(t)$
- Output: the 2 Ω resistor voltage $v_o(t)$
- The node voltage with respect to ground at the top of the node is v_o
- The node voltage with respect to ground at the top of the capacitor is v_c
- Two simultaneous equations
- Use Kirchoff's current law



Solution:

<u>Node 1</u>



- Current law
- $i_1 = i_i$
- $i_2 = \frac{v_R}{R} = \frac{v_0 - v_C}{1}$

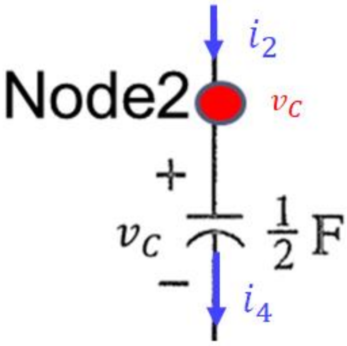
- $i_3 = \frac{v_o}{R} = \frac{v_o}{2}$

Therefore at Node 1

$i_1 = i_2 + i_3$

$i_i(t) = (v_o - v_C) + \frac{1}{2}v_o$ (A)

Node 2



- Current law
- $i_2 = \frac{v_R}{R} = \frac{v_0 - v_C}{1}$ (as before)
- $i_4 = C\frac{dv_c}{dt} = \frac{1}{2}\frac{dv_c}{dt}$

Therefore at Node 2

$i_2 = i_4$

$(v_o - v_c) = \frac{1}{2}\frac{dv_c}{dt}$ (B)

Overall equations, after rearranging both equation (A) and (B), to yield:

$$-i_i(t) + (v_o - v_C) + \frac{1}{2}v_o = 0 \quad \text{(A)}$$

$$\frac{1}{2}\frac{dv_c}{dt} + (v_C - v_o) = 0 \quad \text{(B)}$$

From (A),

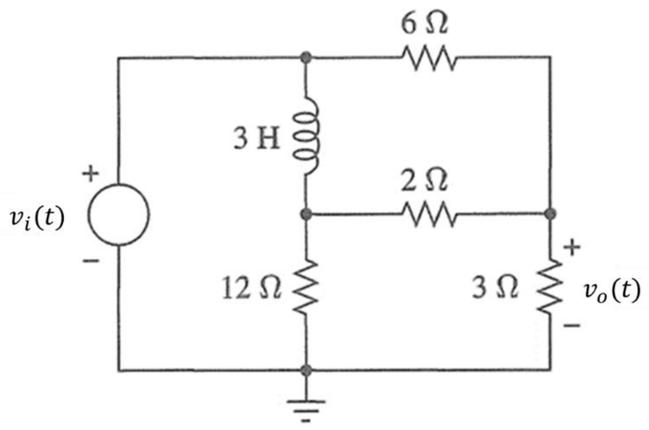
$$v_C = \frac{3}{2}v_o - i_i$$

Sub into (B) to yield

$$\frac{3}{4}\frac{dv_o}{dt} + \frac{1}{2}v_o = \frac{1}{2}\frac{di_i}{dt} + i_i$$

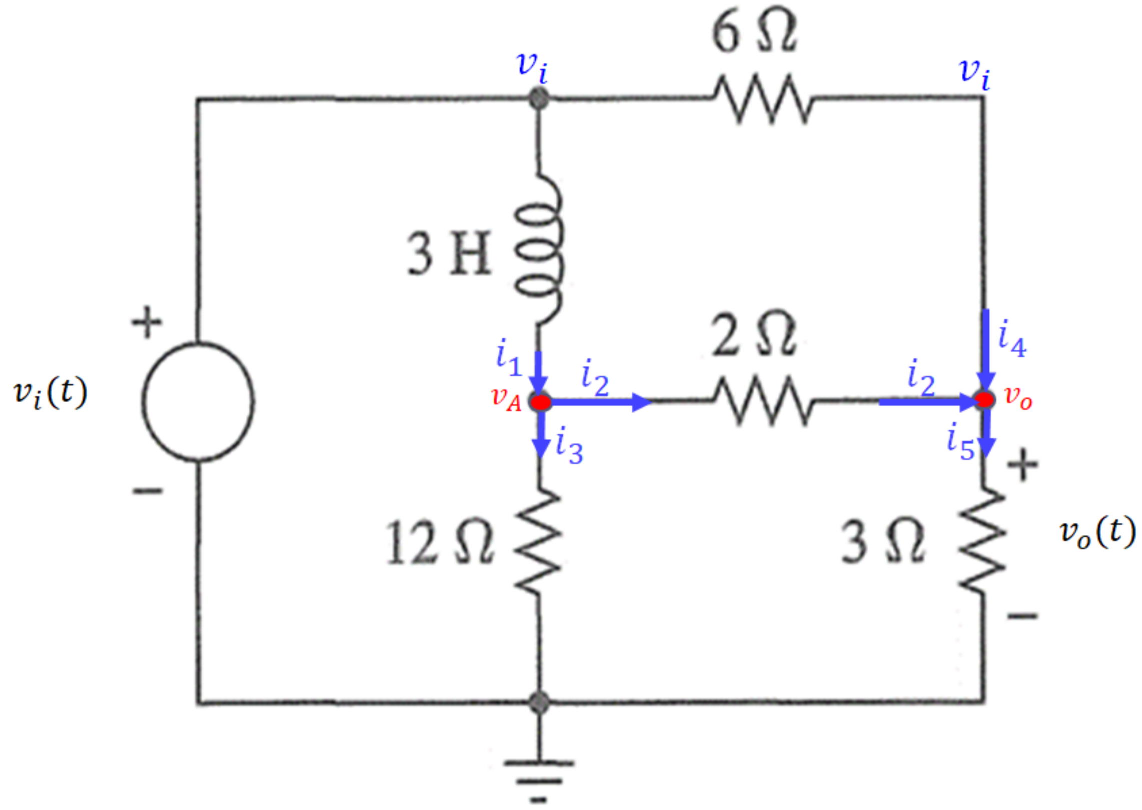
which is the differential equation relating the
output voltage $v_o(t)$,
to the input current $i_i(t)$

Q2) Find the input-output differential equation relating the output voltage $v_o(t)$, to the input voltage, $v_i(t)$ for the following circuit

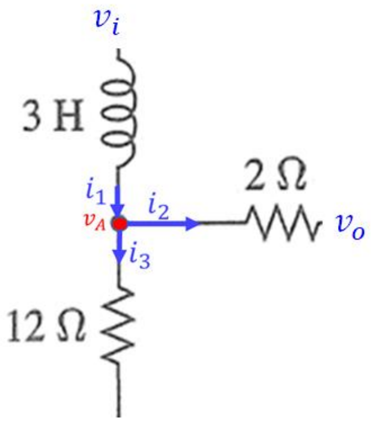


Answer:

Define node A and node *O*



Node A



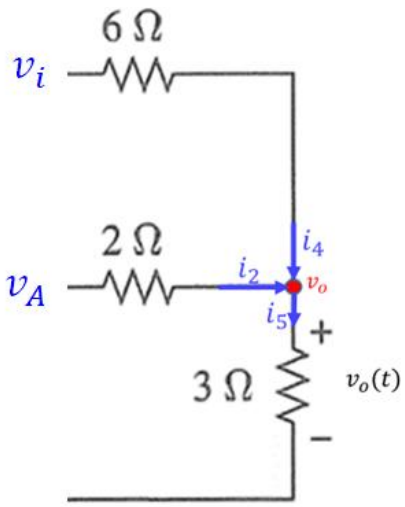
- Write current law at Node A
- $i_1 = \frac{1}{L}\int (v_i - v_A)\, d\tau = \frac{1}{3}\int (v_i - v_A)\, d\tau$
- $i_2 = \frac{v_A - v_o}{R} = \frac{v_A - v_o}{2}$
- $i_3 = \frac{v_A}{R} = \frac{v_A}{12}$

Therefore at Node A

$$i_1 = i_2 + i_3$$

$$\frac{1}{L}\int (v_i - v_A)\, d\tau = \frac{v_A - v_o}{2} + \frac{v_A}{12}$$

Node *O*



- Write current law at Node *O*
- $i_2 = \frac{v_A - v_o}{R} = \frac{v_A - v_o}{2}$ (as before)
- $i_4 = \frac{v_i - v_o}{R} = \frac{v_i - v_o}{6}$
- $i_5 = \frac{v_o}{R} = \frac{v_o}{3}$
-

Therefore at Node *O*

$$i_5 = i_2 + i_4$$

$$\frac{v_o}{3} = \frac{v_A - v_o}{2} + \frac{v_i - v_o}{6}$$

Overall

$\frac{1}{3}\int (v_i - v_A)\,d\tau = \frac{v_A - v_o}{2} + \frac{v_A}{12}$ (A)

$\frac{v_o}{3} = \frac{v_A - v_o}{2} + \frac{v_i - v_o}{6}$ (B)

Differentiating (A) and simplify both (A) and (B) to yield

$7\dot{v}_A + 4v_A - 6\dot{v}_o = 4v_i$ (A)

$-3v_A + 6v_o = v_i$ (B)

From (B)

$v_A = 2v_0 - \frac{1}{3}v_i$

Sub into (A) to yield

$\boldsymbol{\dot{v}_0 + v_o = \frac{7}{24}\,\dot{v}_i + \frac{2}{3}v_i}$

which is the differential equation relating the

output voltage $v_o(t)$,

to the **input voltage $v_i(t)$**

given $\frac{1}{3}\int (v_i - v_A)\,d\tau = \frac{v_A - v_o}{2} + \frac{v_A}{12}$ (A)

$$\frac{v_o}{3} = \frac{v_A - v_o}{2} + \frac{v_i - v_o}{6} \quad (B)$$

from (A), differentiate:

$$\frac{d}{d\tau}\left(\frac{\cancel{1}^{4}}{\cancel{3}}\int (v_i - v_A)\,d\tau \right) = \left(\frac{\cancel{12}^{6}}{\cancel{2}}(v_A - v_o) + v_A \right)\frac{d}{d\tau}$$

$$4(v_i - v_A) = 6(\dot{v}_A - \dot{v}_o) + \dot{v}_A$$

$$4v_i - 4v_A = 7\dot{v}_A - 6\dot{v}_o$$

rearange

$$7\dot{v}_A - 6\dot{v}_o + 4v_A = 4v_i \quad (\overset{*}{A})$$

from (B) rearange

$$2\frac{\cancel{6}}{\cancel{3}}v_o = \frac{\cancel{6}^{3}}{\cancel{2}}(v_A - v_o) + (v_i - v_o)$$

$$6v_o - 3v_A = v_i \quad (\overset{*}{B})$$

rearrange

$$v_A = \frac{1}{3}(6v_o - v_i)$$

$v_A = 2v_o - \frac{1}{3}v_i$ sub into $(\overset{*}{A})$

$$7\frac{d}{dt}\left(2v_o - \frac{1}{3}v_i\right) - 6\dot{v}_o + 4\left(2v_o - \frac{1}{3}v_i\right) = 4v_i$$

$$14\dot{v}_o - \frac{7}{3}\dot{v}_i - 6\dot{v}_o + 8v_o - \frac{4}{3}v_i = 4v_i$$

$$8\dot{v}_o - \frac{7}{3}\dot{v}_i + 8v_o = \left(4 + \frac{4}{3}\right)v_i$$

$$8\left(\dot{v}_o + v_o\right) = \frac{7}{3}\dot{v}_i + \left(\frac{12+4}{3}\right)v_i$$

$$\left(\dot{v}_o + v_o\right) = \frac{7}{24}\dot{v}_i + \left(\frac{\cancel{16}^{2}}{3}\right)\left(\frac{1}{\cancel{8}}\right)v_i$$

$$\boxed{\left(\dot{v}_o + v_o\right) = \frac{7}{24}\dot{v}_i + \frac{2}{3}v_i}$$

EXTRA: Python simulation of a mass-spring-damper system

```
#import numpy, integrate and pyplot
import numpy as np
from scipy import integrate
from matplotlib import pyplot as plt

#sim data
m=2 #mass of the oscillator
b=0.1 #damper of the oscillator
k=1 #stiffness of the oscillator
omega = (k/m)**0.5 #angular frequency of the oscillator
T=2*np.pi/omega #period of the oscillator

#create 1000 unifirmply distributed points in the interval [0,10T]
t = np.linspace(0,10*T,1000)

#lambda returning the force
force = lambda t: np.sin(1.0*omega*t)
Inputs=np.sin(1.0*omega*t)

#lambda with the RHS of the equation, it evaluates the force and
returns a list
f = lambda X,t: [X[1],1/m*(force(t)-k*X[0]-b*X[1])]

#solve equation using odeint, now the values of X for t=0 are given
by a list
XOdeint = integrate.odeint(f,[0,0],t)

plt.plot(t,Inputs)
plt.title('Force (input)')
plt.xlabel('$t$')
plt.ylabel('$input$')
plt.show()

#plot displacements from both solutions (first column of each
solution)
plt.plot(t,XOdeint[:,0])
plt.legend(['Odeint'])
plt.title('Displacements')
plt.xlabel('$t$')
plt.ylabel('$x$')
plt.show()

#plot velocities from both solutions (first column of each solution)
plt.plot(t,XOdeint[:,1])
plt.legend(['Odeint'])
plt.title('Velocities')
plt.xlabel('$t$')
plt.ylabel('$\dot{x}$')
plt.show()

#Save data
np.savetxt('t.txt',t)
np.savetxt('x.txt',XOdeint[:,0])
```

Simulation results