ENG2009 – Modelling of Engineering Systems

Tutorial 3

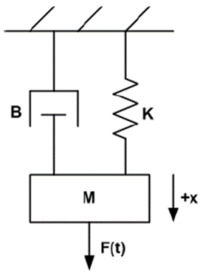
Laplace Transform and Transfer Function

Laplace Transform

Properties of Laplace Transform (2nd order derivatives)

Example: The differential equation of the mass-spring-damper systems is (see Lecture 2):

$$\ddot{x}(t) = \frac{1}{M}(-B\dot{x}(t) - Kx(t) + u(t))$$



Physical Model

Find the Laplace transform of the system, assuming zero initial conditions ($x(0) = 0,\ \dot{x}(0) = 0$)

Solution:

Step 1 and 2: Laplace transform and apply initial condition: $x(0) = 0$ and $\dot{x}(0) = 0$,

Assuming zero initial conditions ($x(0) = 0,\ \dot{x}(0) = 0$):

- Let the applied force as input, i.e. $f(t) = u(t)$, therefore

$$\mathcal{L}(x(t)) = X(s)$$

$$\mathcal{L}(u(t)) = U(s)$$

- Using property $\mathcal{L}\left(\frac{df(t)}{dt}\right) = sF(s) - f(0)$, therefore

$$\mathcal{L}\left(\frac{dx(t)}{dt}\right) = \mathcal{L}(\dot{x}(t)) = sX(s) - x(0) = sX(s)$$

- Using property$\mathcal{L}\left(\frac{d^2f(t)}{dt^2}\right) = s^2F(s) - sf(0) - \dot{f}(0)$, therefore

$$\mathcal{L}\left(\frac{d^2x(t)}{dt^2}\right) = \mathcal{L}(\ddot{x}(t)) = s^2X(s) - sx(0) - \dot{x}(0) = s^2X(s)$$

Step 3: solve for $X(s)$

Therefore

$$\ddot{x}(t) = \frac{1}{M}(-B\dot{x}(t) - Kx(t) + u(t))$$

becomes

$$s^2X(s) = \frac{1}{M}\big(-BsX(s) - kX(s) + U(s)\big)$$

Rearrange to yield

$$X(s)(Ms^2 + Bs + K) = U(s)$$

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Properties of Laplace Transform (Final value theorem)

Example: (Final Value Theorem)

Find the final value of the system corresponding to the following system, where the input $u(t)$ is an unit step i.e. $u(t) = 1$.

$$Y(s) = U(s)\frac{6}{s+2}$$

Hint: use Final Value Theorem

Solution:

System model represented in Laplace as

$$Y(s) = U(s)\frac{6}{s+2}$$

Step 1: Simple check shows that this system is stable as the denominator has a root at left half of s-plane, i.e. $s = -2$. Therefore Final Value Theorem is valid.

Step 2: The input $u(t)$ is an unit step i.e. $u(t) = 1$. From Laplace table

$$\mathcal{L}\big(u(t)\big) = \mathcal{L}(1) = \frac{1}{s}$$

Therefore,

$$Y(s) = \frac{1}{s}\left(\frac{6}{s+2}\right)$$

Step 3: Therefore, the final value of $y(t)$ is:

$$\lim_{t\to\infty}(y(t)) = \lim_{s\to 0}\big(sY(s)\big) = s\frac{1}{s}\left(\frac{6}{s+2}\right)\bigg|_{s=0} = \frac{6}{2} = 3$$

Thus, after the transients have decayed to zero, $y(t)$ will settle to a constant value of 3.

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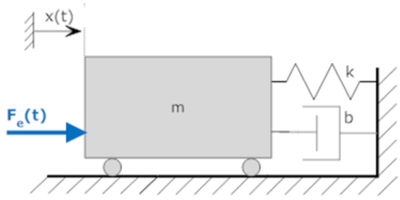
Solving Differential equation

Example: Consider the mechanical systems below, where input=force $f(t) = u(t)$, output=position of mass $x(t)$ and $m = 1,\ b = 2,\ k = 10$:



The differential equation is given by:

$$\ddot{x}(t) + 2\dot{x}(t) + 10x(t) = u(t)$$

Assume that the initial conditions are: $x(0) = 0,\ \dot{x}(0) = 1, u(t) = 0$. *(Note that the input of the system $u(t)$ is zero)*

Find the position $x(t)$ over time t.

Solution:

The differential equation:

$$\ddot{x}(t) + 2\dot{x}(t) + 10x(t) = u(t)$$

Given $x(0) = 0,\ \dot{x}(t) = 1, u(t) = 0$.

Step 1:

- $\mathcal{L}\big(x(t)\big) = X(s)$
- $\mathcal{L}\big(\dot{x}(t)\big) = sX(s) - x(0)$: 1st derivative property
- $\mathcal{L}\big(\ddot{x}(t)\big) = s^2X(s) - sx(0) - \dot{x}(0)$: 2nd derivative property
- $\mathcal{L}\big(u(t)\big) = 0$

Laplace of $\ddot{x}(t) + 2\dot{x}(t) + 10x(t) = 0$:

$$\big(s^2X(s) - sx(0) - \dot{x}(0)\big) + 2\big(sX(s) - x(0)\big) + 10\big(X(s)\big) = 0$$

Step 2: apply initial condition: $x(0) = 0$ and $\dot{x}(0) = 1$,

$$\big(s^2X(s) - sx(0) - \dot{x}(0)\big) + 2\big(sX(s) - x(0)\big) + 10\big(X(s)\big) = 0$$

$$(s^2X(s) - s.0 - 1) + 2(sX(s) - 0) + 10\big(X(s)\big) = 0$$

$$s^2X(s) - 1 + 2sX(s) + 10X(s) = 0$$

Step 3: solve for $X(s)$

$$s^2X(s) - 1 + 2sX(s) + 10X(s) = 0$$

$$s^2X(s) + 2sX(s) + 10X(s) = 1$$

$$X(s)(s^2 + 2s + 10) = 1$$

$$X(s) = \frac{1}{s^2+2s+10}$$

Step 4: get the results from Laplace Transform Table

By completing the square we can rewrite the denominator

$$X(s) = \frac{1}{s^2 + 2s + 10}$$

$$= \frac{1}{(s+1)^2 + 9} = \frac{1}{(s+1)^2 + 3^2}$$

From table of Laplace transform no. (20): $\frac{b}{(s+a)^2+b^2} \overset{\mathcal{L}}{\leftrightarrow} e^{-at}\sin(bt),\ a = 1,\ b = 3$

Therefore

$$\frac{1}{3}\frac{3}{(s+1)^2 + 3^2} \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{3}e^{-1t}\sin(3t)$$

Therefore the solution for the differential equation $\ddot{x}(t) + 2\dot{x}(t) + 10x(t) = u(t)$ is:

$$\boldsymbol{x(t) = \frac{1}{3}e^{-1t}\sin(3t)}$$

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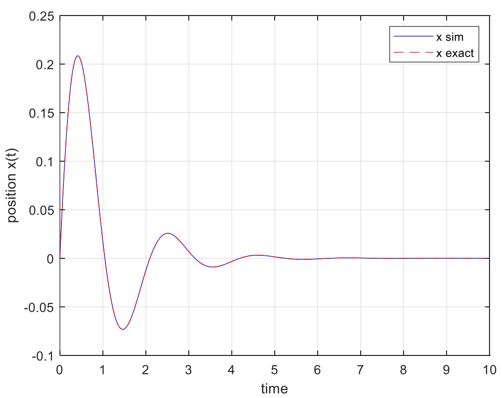
***DIY**: Find the transfer function, is the system is stable? if yes, find final value using FVT and compare the results with the analytical solution x(t) above.*

Example comparison: $x(t)$ odeint (numerical solution) and $x(t)$ from exact solution above.



Exercises:

Q1) Solve the differential equation (using inverse Laplace transform)

$$\ddot{y}(t) + y(t) = 0$$

where $y(0) = 1, \dot{y}(0) = 2$

Q2) Use Laplace transform to solve

$$\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = e^{-3t}$$

where $y(0) = 1$ and $\dot{y}(0) = 1$.

Hint: use *partial fraction*

DIY*: For both systems, find the transfer function. Are the system stable? If yes, find final value using FVT and compare the results with the analytical solution x(t) above.*

Solutions:

Q1) Solution:

Given

$$\ddot{y}(t) + y(t) = 0$$

where $y(0) = 1, \dot{y}(0) = 2$

Step 1:

- $\mathcal{L}\big(y(t)\big) = Y(s)$
- $\mathcal{L}\big(\ddot{y}(t)\big) = s^2Y(s) - sy(0) - \dot{y}(0)$: 2^{nd} derivative property

Laplace of $\ddot{y}(t) + y(t) = 0$:

$$(s^2Y(s) - sy(0) - \dot{y}(0)) + Y(s) = 0$$

Step 2: apply initial condition: $y(0) = 1, \dot{y}(t) = 2,$

$$(s^2Y(s) - sy(0) - \dot{y}(0)) + Y(s) = 0$$

$$(s^2Y(s) - s.1 - 2) + Y(s) = 0$$

$$(s^2Y(s) - s - 2) + Y(s) = 0$$

Step 3: solve for $Y(s)$

$(s^2 + 1)Y(s) = s + 2$

$Y(s) = \frac{s+2}{s^2+1}$

Step 4: get the results from Laplace Transform Table

Rearrange:

$Y(s) = \frac{s+2}{s^2+1}$

$= \frac{s}{s^2+1} + \frac{2}{s^2+1}$

From table of Laplace transform no. (18): $\frac{s}{s^2+1} \overset{\mathcal{L}}{\leftrightarrow} \cos(t)$

From table of Laplace transform no. (17): $\frac{1}{s^2+1} \overset{\mathcal{L}}{\leftrightarrow} \sin(t)$

Therefore

$$\frac{s}{s^2+1} + \frac{2}{s^2+1} \overset{\mathcal{L}}{\leftrightarrow} \cos(t) + 2\sin(t)$$

Therefore the solution for the differential equation $\ddot{y}(t) + y(t) = 0$, where $y(0) = 1, \dot{y}(t) = 2$, is

$$\boldsymbol{y(t) = \cos(t) + 2\sin(t)}$$

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Q2) Solution

Taking Laplace transforms

$$s^2Y(s) - s - 1 + 2(sY(s) - 1) + 5Y(s) = \frac{1}{(s+3)}$$

So $$(s^2+2s+5)Y(s) - s - 3 = \frac{1}{(s+3)}$$

$$\Rightarrow \quad Y(s) = \frac{1}{(s^2+2s+5)(s+3)} + \frac{s+3}{s^2+2s+5}$$

$$\frac{1}{(s^2+2s+5)(s+3)} = \frac{As+B}{s^2+2s+5} + \frac{C}{s+3}$$

$$1 = (As+B)(s+3) + C(s^2+2s+5)$$

at $s=-3$ $\quad 1 = 8C \quad \Rightarrow \quad C = 1/8$

Comparing coeff s^2 $\quad 0 = A + C \quad \Rightarrow \quad A = -1/8$

s^0 $\quad 1 = 3B + 5C \quad \Rightarrow \quad B = 1/8$

$$Y(s) = \frac{1-s}{8\left((s+1)^2+2^2\right)} + \frac{1}{8(s+3)} + \frac{s+3}{(s+1)^2+2^2}$$

$$= \frac{2}{8((s+1)^2+2^2)} - \frac{(s+1)}{8((s+1)^2+2^2)} + \frac{1}{8(s+3)} + \frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}$$

$$= \frac{1}{8}e^{-t}\sin(2t) - \frac{1}{8}e^{-t}\cos(2t) + \frac{1}{8}e^{-3t} + e^{-t}\cos(2t) + e^{-t}\sin(2t)$$

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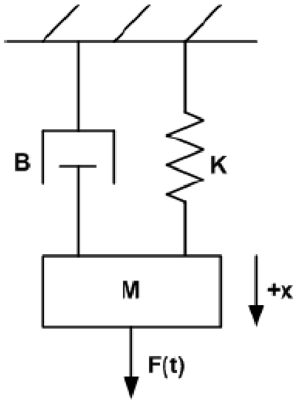
Transfer Function:

Example : Obtain the differential equation of the mass-spring-damper system below (Lecture 2) and evaluate its transfer function. (position $x(t)$ is the output and applied force $u(t)$ is the input):



Physical Model

Solution:

Step 1:

The differential equation of the mass-spring-damper systems in Lecture 2 is (position $x(t)$ is the output and applied force $u(t)$ is the input):

$$\ddot{x}(t) = \frac{1}{M}(-B\dot{x}(t) - Kx(t) + u(t))$$

Step 2:

In Laplace form, assuming *zero initial conditions* ($x(0) = 0,\ \dot{x}(0) = 0$):

$$s^2X(s) = \frac{1}{M}\big(-BsX(s) - kX(s) + U(s)\big)$$

Step 3:

Manipulate to yield the transfer function:

$$\frac{X(s)}{U(s)} = \frac{1}{Ms^2 + Bs + K}$$

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Exercises:

Q3) Find the transfer functions for the following systems. Then analyse if the systems is stable.

a) $\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = 3u(t)$

b) $\dddot{y}(t) - \ddot{y}(t) + 3\dot{y}(t) + 5y(t) = 2\dot{u}(t) + 7u(t)$

Solutions:

Q3 a) Solution:

Step 1:

The differential equation is (where the output is $y(t)$ and input is $u(t)$):

$$\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = 3u(t)$$

Step 2:

In Laplace form, assuming *zero initial conditions* ($y(0) = 0,\ \dot{y}(0) = 0$):

$$s^2Y(s) + 2sY(s) + 5Y(s) = 3U(s)$$

Step 3:

Manipulate to yield the transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{3}{s^2 + 2s + 5}$$

Step 4: Stability analysis

Poles: roots of denominator: $s^2 + 2s + 5 = 0$

Solving quadratic roots yield poles at: $-1 + 2j$ and $-1 - 2j$

Both poles real numbers are negative, therefore system is **stable**.

■

Q3 b) Solution:

Step 1:

The differential equation is (where the output is $y(t)$ and input is $u(t)$):

$$\dddot{y}(t) - \ddot{y}(t) + 3\dot{y}(t) + 5y(t) = 2\dot{u}(t) + 7u(t)$$

Step 2:

In Laplace form, assuming *zero initial conditions* ($y(0) = 0,\ \dot{y}(0) = 0, \ddot{y}(0) = 0$):

$$s^3Y(s) - s^2Y(s) + 3sY(s) + 5Y(s) = 2sU(s) + 7U(s)$$

Step 3:

Manipulate to yield the transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s + 7}{s^3 - s^2 + 3s + 5}$$

Step 4: Stability analysis

Poles: roots of denominator: $s^3 - s^2 + 3s + 5 = 0$

Solving qubic roots yield poles at: $0.8910\ +\ 2.3664j$, $0.8910\ -\ 2.3664j$ and -0.7820.

Two of the poles real numbers are positive, therefore system is **unstable**.

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Table of Laplace Transforms

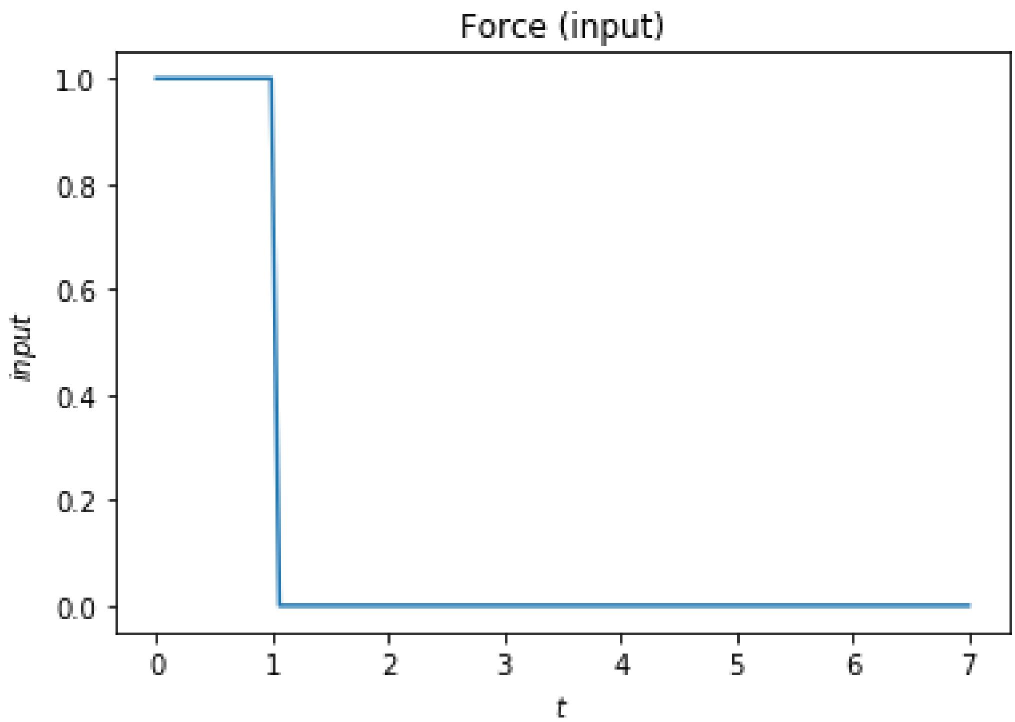
Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$\frac{1}{s}$	$1(t)$
3	$\frac{1}{s^2}$	t
4	$\frac{2!}{s^3}$	t^2
5	$\frac{3!}{s^4}$	t^3
6	$\frac{m!}{s^{m+1}}$	t^m
7	$\frac{1}{(s+a)}$	e^{-at}
8	$\frac{1}{(s+a)^2}$	te^{-at}
9	$\frac{1}{(s+a)^3}$	$\frac{1}{2!}t^2e^{-at}$
10	$\frac{1}{(s+a)^m}$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\frac{a}{s(s+a)}$	$1-e^{-at}$
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at-1+e^{-at})$
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\frac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-at}-ae^{-at}$
17	$\frac{a}{(s^2+a^2)}$	$\sin at$
18	$\frac{s}{(s^2+a^2)}$	$\cos at$
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\frac{a^2+b^2}{s\left[(s+a)^2+b^2\right]}$	$1-e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right)$

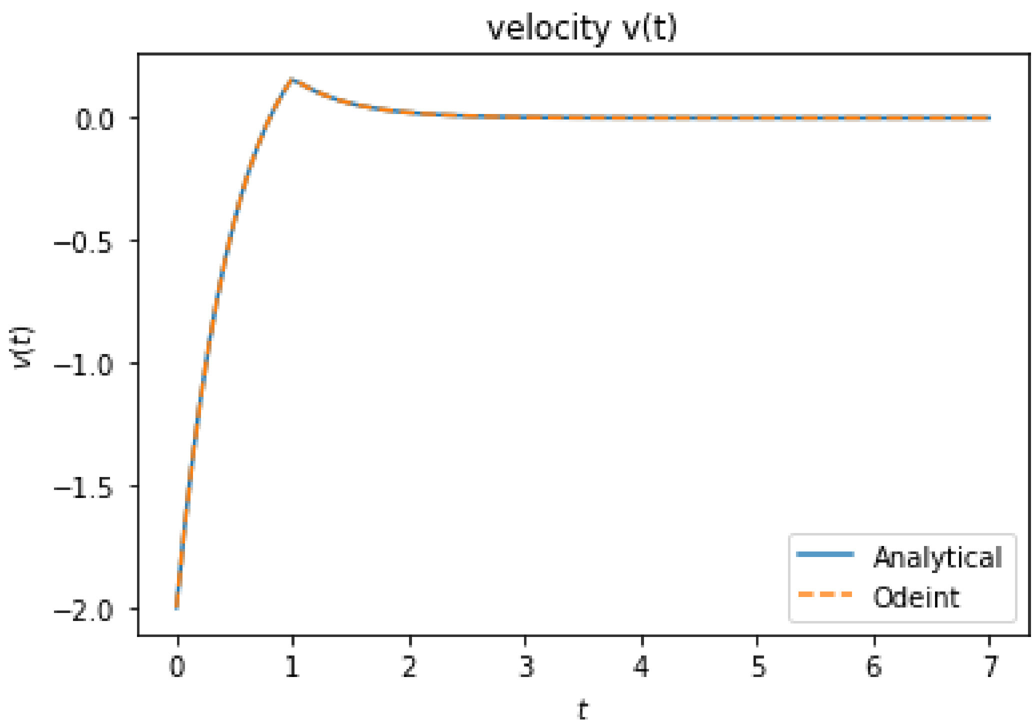
Properties of Laplace Transforms

Number	Laplace Transform	Time Function	Comment
—	$F(s)$	$f(t)$	Transform pair
1	$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Superposition
2	$F(s)e^{-s\lambda}$	$f(t-\lambda)$	Time delay ($\lambda \geq 0$)
3	$\frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right)$	$f(at)$	Time scaling
4	$F(s+a)$	$e^{-at} f(t)$	Shift in frequency
5	$s^m F(s) - s^{m-1} f(0) - s^{m-2} \dot{f}(0) - \cdots - f^{(m-1)}(0)$	$f^{(m)}(t)$	Differentiation
6	$\frac{1}{s} F(s)$	$\int_0^t f(\zeta)\, d\zeta$	Integration
7	$F_1(s) F_2(s)$	$f_1(t) * f_2(t)$	Convolution
8	$\lim_{s\to\infty} sF(s)$	$f(0^+)$	Initial Value Theorem
9	$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$	Final Value Theorem
10	$\frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F_1(\zeta) F_2(s-\zeta) d\zeta$	$f_1(t) f_2(t)$	Time product
11	$\frac{1}{2\pi} \int_{-j\infty}^{+j\infty} Y(-j\omega) U(j\omega)\, d\omega$	$\int_0^\infty y(t) u(t)\, dt$	Parseval's Theorem
12	$-\frac{d}{ds} F(s)$	$t f(t)$	Multiplication by time

Python: Simulation of a mass- damper system (see Lecture 6 example)

```
import math
import numpy as np
from numpy import arange
from scipy import integrate
from matplotlib import pyplot as plt

#mass and damping of the damper
m=1
b=2

#lambda to describe the force as a function of time
#the assigned expression (t<=1) assumes a value of True (1) when t is less than
#1 and False (0) when t is greater than 1
force = lambda t:t<=1

#Function with the rhs of the equation, retuens a list with 2 elements
f = lambda y,t:[y[1],force(t)-2*y[1]]

#Array containing time instances for which the equation is to be solved
ti = np.linspace(0,7,100)

#plot force
plt.plot(ti,force(ti))
plt.title('Force (input)')
plt.xlabel('$t$')
plt.ylabel('$input$')
plt.show()

#solve equation with odeint
vOdeint = integrate.odeint(f,[0,-2],ti)

#expression for the analytical solution
vAnalytical = -2*np.exp(-2*ti)+0.5*(1-np.exp(-2*ti))-(ti>=1)*0.5*(1-np.exp(-2*(ti-1)))

#plot analytical and numerical solution
plt.plot(ti,vAnalytical)
#plt.legend(['Analytical'])
plt.plot(ti,vOdeint[:,1],'--')
plt.legend(['Analytical','Odeint'])
plt.title('velocity v(t)')
plt.xlabel('$t$')
plt.ylabel('$v(t)$')
plt.show()
```


Force (input)
input
t
1.0
0.8
0.6
0.4
0.2
0.0
0 1 2 3 4 5 6 7


velocity v(t)
v(t)
t
0.0
−0.5
−1.0
−1.5
−2.0
0 1 2 3 4 5 6 7
Analytical
Odeint