ENG2009 – Modelling of Engineering Systems

Tutorial 7

ODE – Euler and Runge Kutta 2nd and 4th order

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. (Thermal system)

Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8), \theta(0) = 1200K$$

Find the temperature at $t = 480$ sec using Euler's method.

Assume a step size of $h = 240$ sec

Solution

Step 1:

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

where

$$f(t,\theta) = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

Therefore

$$\begin{aligned}
\theta_{i+1} &= \theta_i + f(t_i,\theta_i)h \\
\theta_1 &= \theta_0 + f(t_0,\theta_0)h \\
&= 1200 + f(0,1200)240 \\
&= 1200 + \left(-2.2067 \times 10^{-12}(1200^4 - 81 \times 10^8)\right)240 \\
&= 1200 + (-4.5579)240 \\
&= 106.09K
\end{aligned}$$

where θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240$$

Therefore

$$\theta(240) \approx \theta_1 = 106.09K$$

Solution

Step 2:

For

$$i = 1, t_1 = 240, \theta_1 = 106.09$$

Therefore

$$\begin{aligned}
\theta_2 &= \theta_1 + f(t_1,\theta_1)h \\
&= 106.09 + f(240,106.09)240 \\
&= 106.09 + \left(-2.2067 \times 10^{-12}(106.09^4 - 81 \times 10^8)\right)240 \\
&= 106.09 + (0.017595)240 \\
&= 110.32K
\end{aligned}$$

where θ_2 is the approximate temperature at

$$t = t_2 = t_1 + h = 240 + 240 = 480$$

Therefore

$$\theta(480) \approx \theta_2 = 110.32K$$

Solution continued…

Note that the exact solution of the ordinary differential equation is given by the solution of a non-linear eq. as

$$0.92593 \ln\frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.00333\theta) = -0.22067 \times 10^{-3}t - 2.9282$$

The solution to this nonlinear equation at $t = 480$ sec is:

$\theta_{exact}(480) = 647.57K$

vs numerical solution:

$\theta_{euler}(480) \approx \theta_2 = 110.32K$

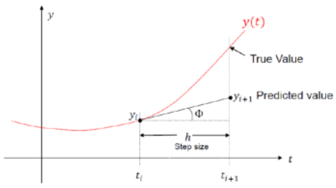

Figure 2. General graphical interpretation of Euler's method

Exact vs Numerical Solutions

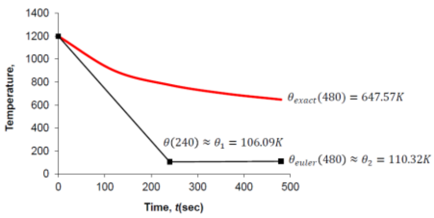

Figure 3. Comparing exact and Euler's method

Effect of step size, h

Table 1. Temperature at 480 seconds as a function of step size, h

Step, h	*No of sim. data*	$\theta(480)$	E_t (error)	$\lvert\epsilon_t\rvert\%$
480	1	−987.81	1635.4	252.54
240	2	110.32	537.26	82.964
120	3	546.77	100.80	15.566
60	4	614.97	32.607	5.0352
30	5	632.77	14.806	2.2864

exact solution: $\theta(480) = 647.57K$

Example calculation:

$$E_t = 647.57 - 110.32 = 537.26$$

$$\begin{aligned}
\lvert\epsilon_t\rvert\% &= \left|\frac{E_t}{exact\ solution} \times 100\right| \\
&= \left|\frac{537.26}{647.57} \times 100\right| = 82.964
\end{aligned}$$

Comparison with exact results

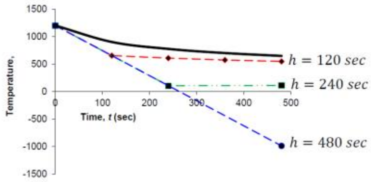

Figure 4. Comparison of Euler's method with exact solution for different step sizes

The smaller the step size h,
the closer to the numerical solution
to the exact solution

Effects of step size h on Euler's Method

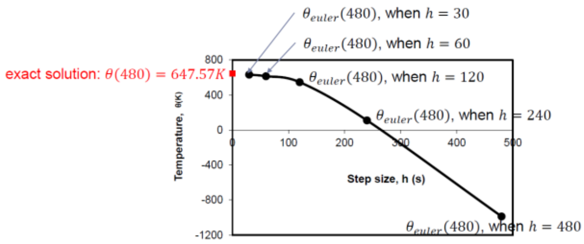

Figure 5. Effect of step size in Euler's method.

Temp θ at 480 sec i.e. $\theta(480)$ for various step size h

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K.

Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8), \theta(0) = 1200K$$

Find the temperature at $t = 480$ sec using Heun's method.

Assume step size of $h = 240$ sec.

Solution

Given

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

therefore

$$f(t, \theta) = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

Heun's method:

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h, y_i + k_1 h)$$

Comparison with exact results

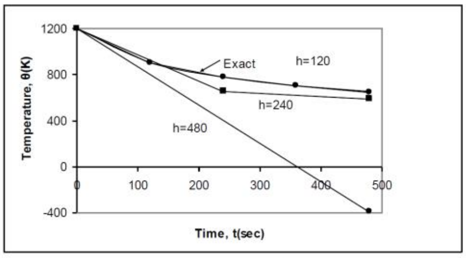

Figure 2. Heun's method results for different step sizes

Euler vs Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\vert \in_t \vert\%$			
	Euler	Heun	Midpoint	Ralston
480	252.54	160.82	86.612	30.544
240	82.964	9.7756	50.851	6.5537
120	15.566	0.58313	6.5823	3.1092
60	5.0352	0.36145	1.1239	0.72299
30	2.2864	0.097625	0.22353	0.15940

$\theta_{exact}(480) = 647.57K$ (exact)

Euler vs Runge-Kutta 2nd Order Methods

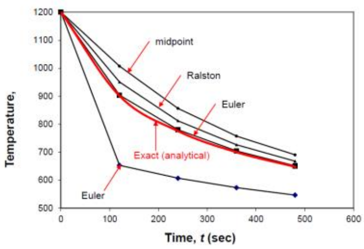

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results. ($h = 120$ sec)

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K.

Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8), \theta(0) = 1200K$$

Find the temperature at $t = 480$ sec using Runge-Kutta 4th order method.

Assume step size of $h = 240$ sec.

Solution

Given

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

therefore

$$f(t,\theta) = -2.2067 \times 10^{-12}(\theta^4 - 81 \times 10^8)$$

Runge-Kutta 4th order method:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

Exact solution

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593 \ln\frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.00333333\theta)$$
$$= -0.22067 \times 10^{-3}t - 2.9282$$

The solution to this nonlinear equation at $t = 480$ sec is

$$\theta_{exact}(480) = 647.57K$$

Compared with Runge-Kutta 4th order method:

$$\theta_{RK4}(480) = 594.91K$$

Comparison with exact results

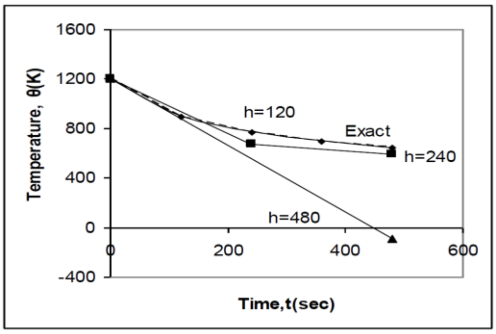

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effects of h on Runge-Kutta 4th Order Method

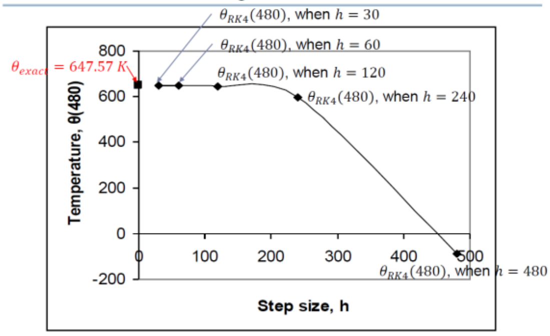

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

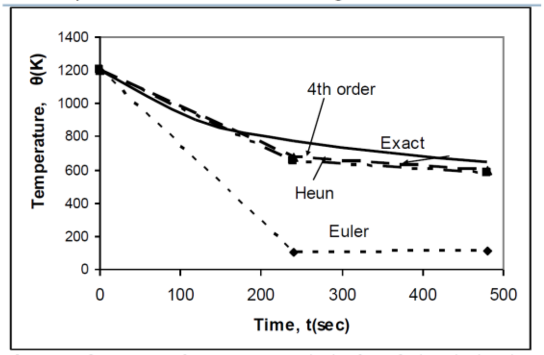

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order ($h = 240$ sec)

Exercises

Q1) Civil engineering example:

A polluted lake has an initial concentration of a bacteria of 10^7 parts/m^3, while the acceptable level is only $5 \times 10^6 \text{parts/m}^3$. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt} + 0.06C = 0,\ C(0) = 10^7$$

Using a step size of 3.5 weeks, find the concentration of the pollutant after 7 weeks. Use the following methods

(a) Euler's methods
(b) Heun's method
(c) Runge-Kutta 4th order methods

Solutions

Q 1) (a) Euler method

$$\frac{dC}{dt} = -0.06C$$
$$f(t, C) = -0.06C$$

The Euler's method reduces to

$$C_{i+1} = C_i + f(t_i, C_i)h$$

For $i = 0$, $t_0 = 0$, $C_0 = 10^7$

$$\begin{aligned}C_1 &= C_0 + f(t_0, C_0)h \\ &= 10^7 + f(0{,}10^7)3.5 \\ &= 10^7 + (-0.06(10^7))3.5 \\ &= 10^7 + (-6 \times 10^5)3.5 \\ &= 7.9 \times 10^6 \text{parts/m}^3\end{aligned}$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5\text{weeks}$$
$$C(3.5) \approx C_1 = 7.9 \times 10^6 \text{ parts/m}^3$$

For $i = 1$, $t_1 = 3.5$, $C_1 = 7.9 \times 10^6$

$$\begin{aligned}C_2 &= C_1 + f(t_1, C_1)h \\ &= 7.9 \times 10^6 + f(3.5, 7.9 \times 10^6)3.5 \\ &= 7.9 \times 10^6 + (-0.06(7.9 \times 10^6))3.5 \\ &= 7.9 \times 10^6 + (-4.74 \times 10^5)3.5 \\ &= 6.241 \times 10^6 \text{ parts/m}^3\end{aligned}$$

C_2 is the approximate concentration of bacteria at

$$t = t_2 = t_1 + h = 3.5 + 3.5 = 7\text{weeks}$$
$$C(7) \approx C_2 = 6.241 \times 10^6 \text{ parts/m}^3$$

The exact solution of the ordinary differential equation is given by

$$C(t) = 1 \times 10^7 e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at $t = 7$weeks is

$$C(7) = 6.5705 \times 10^6 \text{ parts/m}^3$$

Figure 1 compares the exact solution with the numerical solution from Euler's method for the step size of $h = 3.5$.

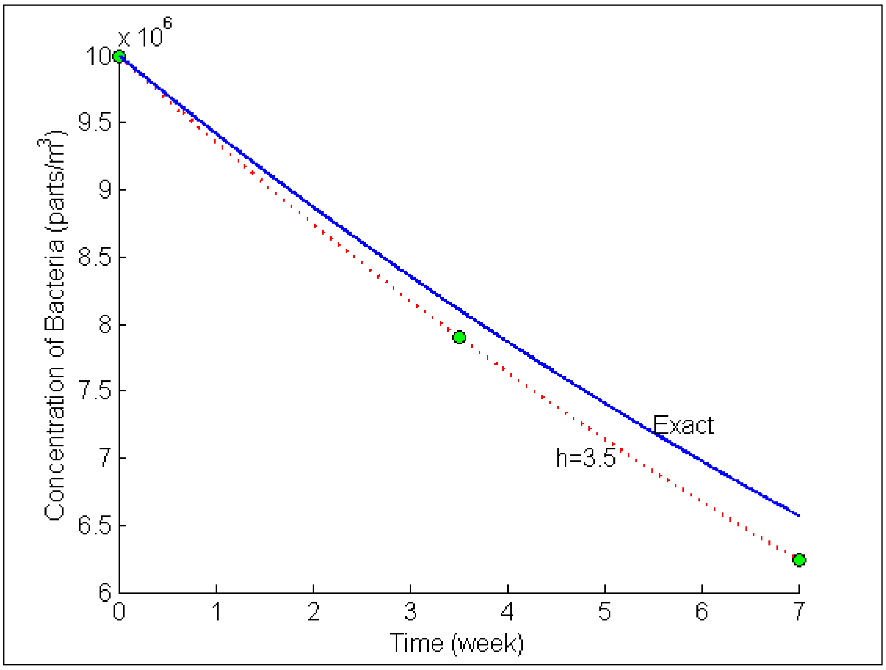

Figure 1 Comparing exact and Euler's method.

The problem was solved again using smaller step sizes. The results are given below in Table 1.

Table 1 Concentration of bacteria after 7 weeks as a function of step size, h.

step size, h	$C(7)$	E_t	$\lvert \in_t \rvert \%$
7	5.8×10^6	770470	11.726
3.5	6.241×10^6	329470	5.0144
1.75	6.4164×10^6	154060	2.3447
0.875	6.4959×10^6	74652	1.1362
0.4375	6.5337×10^6	36763	0.55952

Figure 2 shows how the concentration of bacteria varies as a function of time for different step sizes.

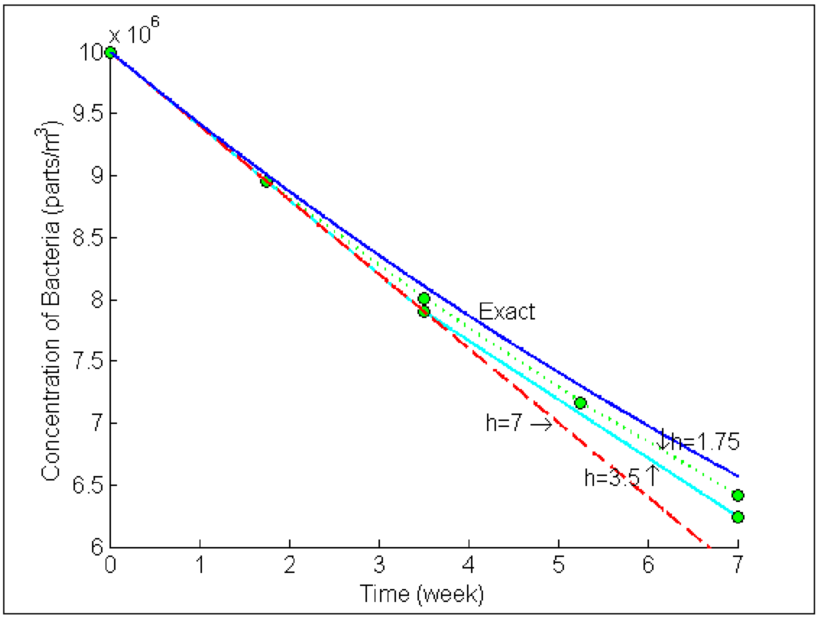

Figure 2 Comparison of Euler's method with exact solution for different step sizes.

While the values of the calculated concentration of bacteria at $t = 7$weeks as a function of step size are plotted in Figure 3.

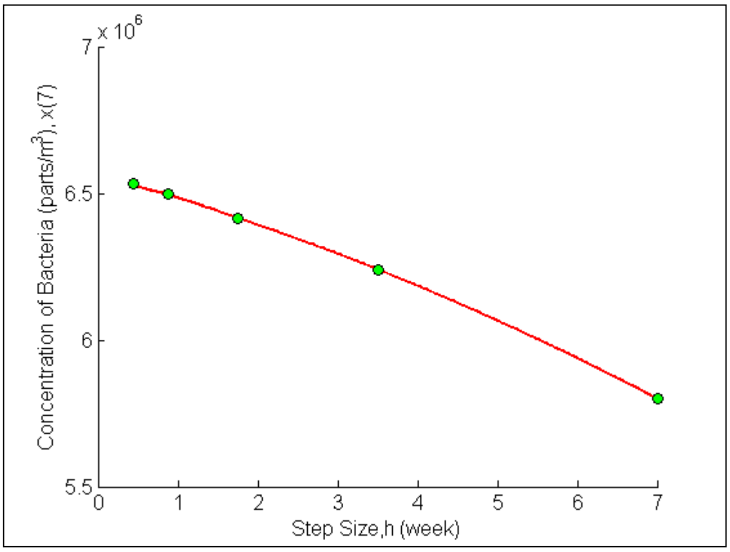

Figure 3 Effect of step size in Euler's method.

Solutions

Q 1) (b) Heun's method

$\frac{dC}{dt} = -0.06C$

$$f(t, C) = -0.06C$$

Per Heun's method

$$C_{i+1} = C_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$k_1 = f(t_i, C_i)$$
$$k_2 = f(t_i + h, C_i + k_1 h)$$

For $i = 0,\ t_0 = 0,\ C_0 = 10^7$

$$\begin{aligned} k_1 &= f(t_0, C_0) \\ &= f(0, 10^7) \\ &= -0.06(10^7) \\ &= -600000 \end{aligned}$$

$$\begin{aligned} k_2 &= f(t_0 + h, C_0 + k_1 h) \\ &= f(0 + 3.5, 10^7 + (-600000)3.5) \\ &= f(3.5, 7.9 \times 10^6) \\ &= -0.06(7.9 \times 10^6) \\ &= -474000 \end{aligned}$$

$$\begin{aligned} C_1 &= C_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 10^7 + \left(\frac{1}{2}(-600000) + \frac{1}{2}(-474000)\right)3.5 \\ &= 10^7 + (-537000)3.5 \\ &= 8.1205 \times 10^6 \text{ parts/m}^3 \end{aligned}$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5\text{weeks}$$
$$C(3.5) \approx C_1 = 8.1205 \times 10^6 \text{ parts/m}^3$$

For $i = 1,\ t_1 = t_0 + h = 0 + 3.5 = 3.5, C_1 = 8.1205 \times 10^6$

$$\begin{aligned} k_1 &= f(t_1, C_1) \\ &= f(3.5, 8.1205 \times 10^6) \\ &= -0.06(8.1205 \times 10^6) \\ &= -487230 \end{aligned}$$

$$\begin{aligned} k_2 &= f(t_1 + h, C_1 + k_1 h) \\ &= f(3.5 + 3.5, 8.1205 \times 10^6 + (-487230)3.5) \\ &= f(7, 6415200) \\ &= -0.06(6415200) \\ &= -384910 \end{aligned}$$

$$\begin{aligned} C_2 &= C_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 8.1205 \times 10^6 + \left(\frac{1}{2}(-487230) + \frac{1}{2}(-384910)\right)3.5 \\ &= 8.1205 \times 10^6 + (-436070)3.5 \\ &= 6.5943 \times 10^6 \text{parts/m}^3 \end{aligned}$$

C_2 is the approximate concentration of bacteria at

$$t = t_2 = t_1 + h = 3.5 + 3.5 = 7\text{weeks}$$
$$C(7) \approx C_2 = 6.5943 \times 10^6 \text{ parts/m}^3$$

The exact solution of the ordinary differential equation is given by

$$C(t) = 1 \times 10^7 e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at $t = 7$weeks is

$$C(7) = 6.5705 \times 10^6 \text{ parts/m}^3$$

The results from Heun's method are compared with exact results in Figure 1.

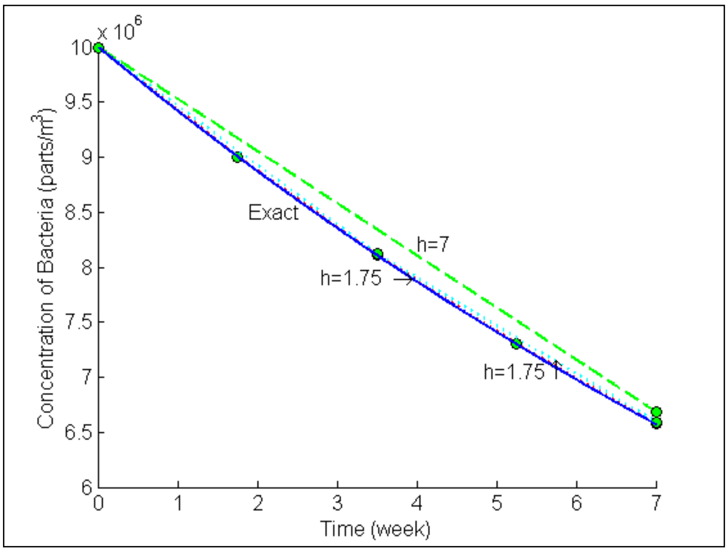

Figure 1 Heun's method results for different step sizes.

Using smaller step size would increases the accuracy of the result as given in Table 1 and Figure 2.

Table 1 Effect of step size for Heun's method.

Step size, h	$C(7)$	E_t	$\lvert\in_t\rvert\%$
7	6.6820×10^6	−111530	1.6975
3.5	6.5943×10^6	−23784	0.36198
1.75	6.5760×10^6	−5489.1	0.083542
0.875	6.5718×10^6	−1318.8	0.020071
0.4375	6.5708×10^6	−323.24	0.0049195

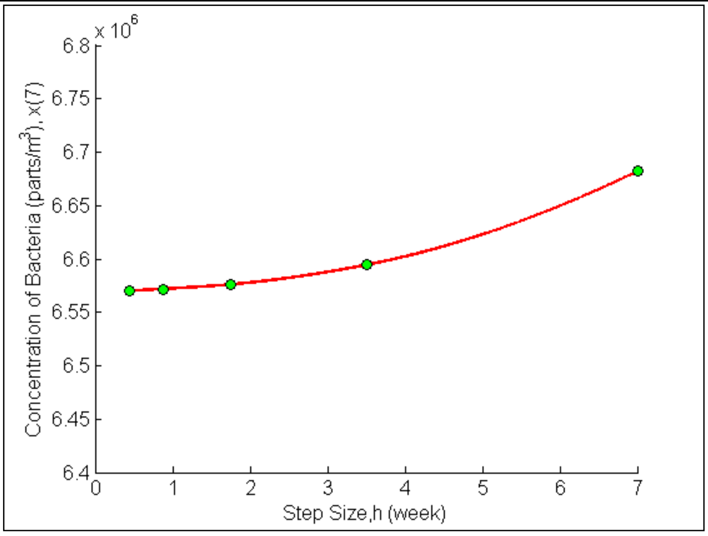

Figure 2 Effect of step size in Heun's method.

In Table 2, the Euler's method and Runge-Kutta 2nd order method results are shown as a function of step size.

Table 2 Comparison of Euler and the Runge-Kutta methods.

Step size, h	$C(7)$			
	Euler	Heun	Midpoint	Ralston
7	5.8000×10^6	6.6820×10^6	6.6820×10^6	6.6820×10^6
3.5	6.2410×10^6	6.5943×10^6	6.5943×10^6	6.5943×10^6
1.75	6.4160×10^6	6.5760×10^6	6.5760×10^6	6.5760×10^6
0.875	6.4960×10^6	6.5718×10^6	6.5718×10^6	6.5718×10^6
0.4375	6.5340×10^6	6.5708×10^6	6.5708×10^6	6.5708×10^6

While in Figure 3, the comparison is shown over the range of time.

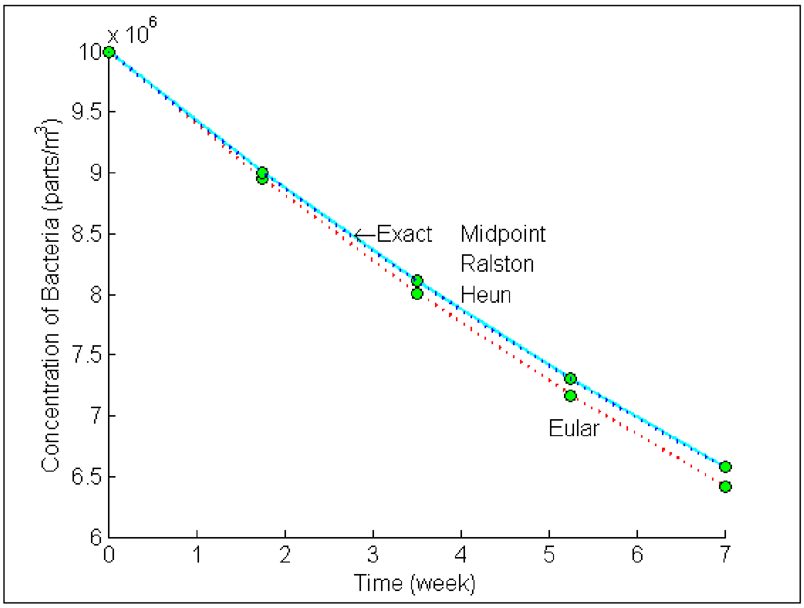

Figure 3 Comparison of Euler and Runge Kutta methods with exact results over time.

Solutions

Q 1) (c) Runge-Kutta 4th order method

$$\frac{dC}{dt} = -0.06C$$
$$f(t, C) = -0.06C$$
$$C_{i+1} = C_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $C_0 = 10^7$

$$\begin{aligned}
k_1 &= f(t_0, C_0) \\
&= f(0, 10^7) \\
&= -0.06(10^7) \\
&= -600000 \\
k_2 &= f\left(t_0 + \tfrac{1}{2}h, C_0 + \tfrac{1}{2}k_1 h\right) \\
&= f\left(0 + \tfrac{1}{2} \times 3.5, 10^7 + \tfrac{1}{2}(-600000)3.5\right) \\
&= f(1.75, 8950000) \\
&= -0.06(8950000) \\
&= -537000 \\
k_3 &= f\left(t_0 + \tfrac{1}{2}h, C_0 + \tfrac{1}{2}k_2 h\right) \\
&= f\left(0 + \tfrac{1}{2}3.5, 10^7 + \tfrac{1}{2}(-537000)3.5\right) \\
&= f(1.75, 9060300) \\
&= -0.06(9060300) \\
&= -543620 \\
k_4 &= f(t_0 + h, C_0 + k_3 h) \\
&= f(0 + 3.5, 10^7 + (-543620)3.5) \\
&= f(3.5, 8097300) \\
&= -0.06(8097300) \\
&= -485840
\end{aligned}$$

$$\begin{aligned}
C_1 &= C_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
&= 10^7 + \frac{1}{6}\big(-600000 + 2(-537000) + 2(-543620) + (-485840)\big)3.5 \\
&= 10^7 + \frac{1}{6}(-3247100)3.5 \\
&= 8.1059 \times 10^6 \text{ parts/m}^3
\end{aligned}$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5 \text{ parts/m}^3$$
$$C(3.5) \approx C_1 = 8.1059 \times 10^6 \text{ parts/m}^3$$

For $i = 1, t_1 = 3.5, C_1 = 8.1059 \times 10^6$

$$\begin{aligned}
k_1 &= f(t_1, C_1) \\
&= f(3.5, 8.1059 \times 10^6) \\
&= -0.06(8.1059 \times 10^6) \\
&= -486350
\end{aligned}$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, C_1 + \frac{1}{2}k_1 h\right)$$
$$= f\left(3.5 + \frac{1}{2} \times 3.5, 8105900 + \frac{1}{2}(-486350)3.5\right)$$
$$= f(5.25, 7254800)$$
$$= -0.06(7254800)$$
$$= -435290$$
$$k_3 = f\left(t_1 + \frac{1}{2}h, C_1 + \frac{1}{2}k_2 h\right)$$
$$= f\left(3.5 + \frac{1}{2} \times 3.5, 8105900 + \frac{1}{2}(-435290)3.5\right)$$
$$= f(5.25, 7344100)$$
$$= -0.06(7344100)$$
$$= -440648$$
$$k_4 = f(t_1 + h, C_1 + k_3 h)$$
$$= f(3.5 + 3.5, 8105900 + (-440648)3.5)$$
$$= f(7, 6563600)$$
$$= -0.06(6563600)$$
$$= -393820$$

$$C_2 = C_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$
$$= 8105900 + \frac{1}{6}\big(-486350 + 2 \times (-435290) + 2 \times (-440648) + (-393820)\big) \times 3.5$$
$$= 8105900 + \frac{1}{6}(-2632000) \times 3.5$$
$$= 6.5705 \times 10^6 \text{ parts/m}^3$$

C_2 is the approximate concentration of bacteria at

$t_2 = t_1 + h$ =3.5 + 3.5 = 7 weeks

$C(7) \approx C_2 = 6.5705 \times 10^6 \text{ parts/m}^3$

The exact solution of the ordinary differential equation is given by

$$C(t) = 1 \times 10^7 e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at $t = 7$weeks is

$$C(7) = 6.5705 \times 10^6 \text{ parts/m}^3$$

Figure 1 compares the exact solution with the numerical solution using the Runge-Kutta 4th order method using different step sizes.

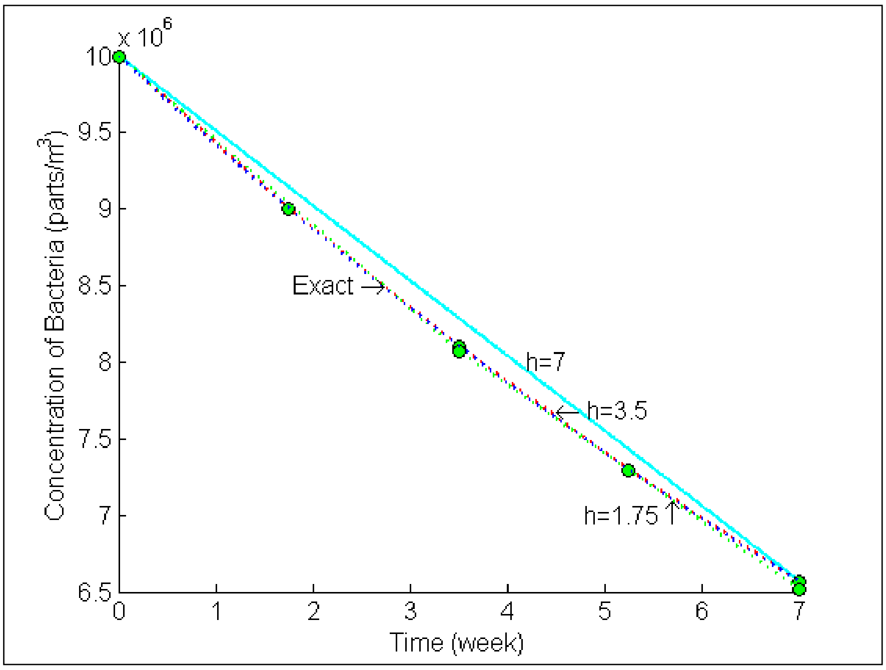

Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Table 1 and Figure 2 shows the effect of step size on the value of the calculated concentration of bacteria at $t = 7$ weeks.

Table 1 Value of concentration of bacteria at 7 weeks for different step sizes.

Step size, h	$C(7)$	E_t	$\vert \in_t \vert\%$
7	6.5715×10^6	-1017.2	0.015481
3.5	6.5705×10^6	-53.301	8.1121×10^{-4}
1.75	6.5705×10^6	-3.0512	4.6438×10^{-5}
0.875	6.5705×10^6	-0.18252	2.7779×10^{-6}
0.4375	6.5705×10^6	-0.011161	1.6986×10^{-7}

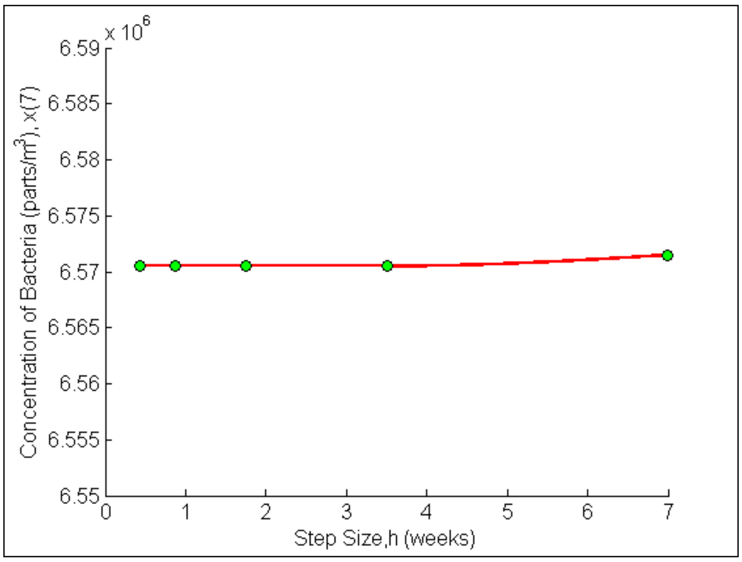

Figure 2 Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method) and the Runge-Kutta 4th order method. ($h = 3.5$)

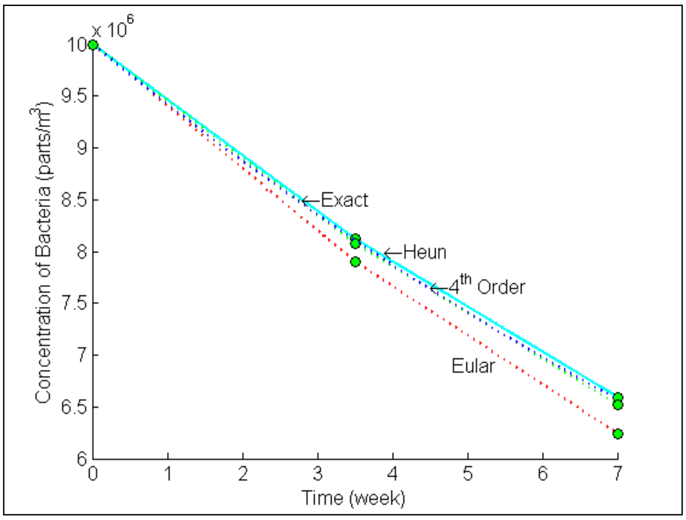

Figure 3 Comparison of Euler, Runge-Kutta methods of 2nd (Heun) and 4th order.