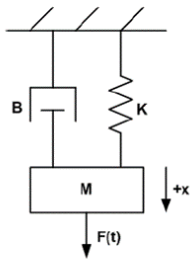
ENG2009 – Modelling of Engineering Systems

Tutorial 4

State space

State Space:

Example 1:
Find the state space for the following mass-spring-damper system:



Physical Model

Solution 1:
Step 1: The differential equation of the mass-spring-damper systems in Lecture 2 is (position $x(t)$ is the output and applied force $u(t)$ is the input):

$$\ddot{x}(t) = \frac{1}{M}(-B\dot{x}(t) - Kx(t) + u(t))$$

Step 2: Identify the states:
Let position as 1st states: $x_1(t) = x(t)$
Let velocity as 2nd states: $x_2(t) = \dot{x}(t)$

Step 3: Derivative of states:
$\dot{x}_1(t) = \dot{x}(t) = x_2(t)$
$\dot{x}_2(t) = \ddot{x}(t) = \frac{1}{M}(-B\dot{x}(t) - Kx(t) + u(t))$

Step 4: Rearrange the derivative of states

$$\begin{aligned} \dot{x}_1(t) &= \quad 0x_1(t) + 1x_2(t) + \ 0\,u(t) \\ \dot{x}_2(t) &= -\frac{K}{M}x_1(t) - \frac{B}{M}x_2(t) + \frac{1}{M}u(t) \end{aligned}$$

Step 5: Rewrite in matrix form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t)$$

where the output equation

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

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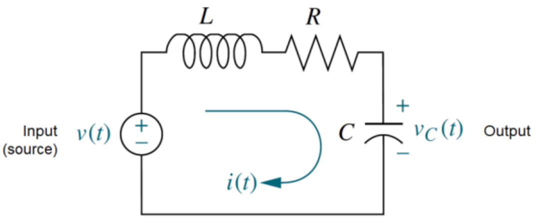
Example 2:
Find the state space equation relating the capacitor voltage $v_c(t)$ (output voltage), to the input voltage, $v(t)$.



Simple RLC circuit

Solution 2:
Step 1: using Kirchoff's voltage law, the differential equation (output: $v_c(t)$, input: $v(t)$):

$$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = v(t) \text{ or}$$

$$\frac{d^2v_c(t)}{dt^2} = \frac{1}{LC}\left(-RC\frac{dv_c(t)}{dt} - v_c(t) + v(t)\right)$$

Step 2: Identify the states and inputs:
Let 1st states: $x_1(t) = v_c(t)$
Let 2nd states: $x_2(t) = \dot{v}_c(t)$
Input: $u(t) = v(t)$

Step 3: Derivative of states:
$\dot{x}_1(t) = \dot{v}_c(t) = x_2(t)$
$\dot{x}_2(t) = \ddot{v}_c(t) = \frac{1}{LC}(-RCx_2(t) - x_1(t) + u(t))$

Step 4: Rearrange the derivative of states

$$\dot{x}_1(t) = \quad 0x_1(t) + 1x_2(t) + \ 0\,u(t)$$

$$\dot{x}_2(t) = -\frac{1}{LC}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{LC}u(t)$$

Step 5: Rewrite in matrix form:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1/LC & -R/L\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1/LC\end{bmatrix}u(t)$$

where the output equation

$$y(t) = [1 \quad 0]\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

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Example 3:
For a DC motor, find the state space equation relating the input voltage $v(t) = e_{in}(t),$ and output shaft speed $\omega(t) = \dot{\theta}(t)$.

Solution 3:
Step 1: Two differential equations (input: $e_{in}(t),$ output: $\dot{\theta}(t)$):

$J\ddot{\theta} + B_r\,\dot{\theta} = \tau_e = K_t i$ (A)
$e_{in} = L\frac{di}{dt} + iR + K_e\dot{\theta}$ (B)

Step 2: Identify the states and input:
Let 1st states: $x_1(t) = \theta(t)$
Let 2nd states: $x_2(t) = \dot{\theta}(t)$
Let 3rd states: $x_3(t) = i(t)$
Input: $u(t) = e_{in}(t)$

Step 3: Derivative of states:
$\dot{x}_1(t) = \dot{\theta}(t) = x_2(t)$
$\dot{x}_2(t) = \ddot{\theta}(t) = \frac{1}{J}(-B_r x_2(t) + K_t x_3(t))$ (from eq (A))
$\dot{x}_3(t) = \frac{di}{dt} = \frac{1}{L}(-Rx_3(t) - K_e x_2(t) + u(t))$ (from eq (B))

Step 4: Rearrange the derivative of states

$$\begin{aligned}\dot{x}_1(t) &= 0x_1(t) + 1x_2(t) + 0x_3(t) + 0\,u(t)\\ \dot{x}_2(t) &= 0x_1(t) - \frac{B_r}{J}x_2(t) + \frac{K_t}{J}x_3(t) + 0\,u(t)\\ \dot{x}_3(t) &= 0x_1(t) - \frac{K_e}{L}x_2(t) - \frac{R}{L}x_3(t) + \frac{1}{L}u(t)\end{aligned}$$

Step 5: Rewrite in matrix form:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & -B_r/J & K_t/J\\ 0 & -K_e/L & -R/L\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1/L\end{bmatrix}u(t)$$

where the output equation

$$y(t) = \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

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Example 4:

Find the eigenvalues for the following system and determine if the system stable?

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Answer:

Eigenvalues λ:

$$\det(A - \lambda I_n) = 0$$

Find λ:

$$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix}\right) = 0$$

$$(-\lambda)(-2-\lambda) - (-1)(1) = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)(\lambda + 1) = 0$$

Therefore eigenvalues (poles) at $\lambda = -1$ and $\lambda = -1$

$\Rightarrow$ Stable system

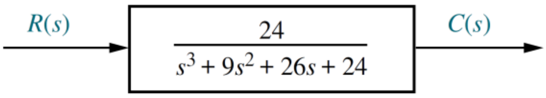
Exercises

Q1) Find the eigenvalues for the following system and determine if the system stable?

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Q2) Find the state-space representation for the transfer function shown below:



Q3) For the translational mechanical system shown below,

(a) Show that the differential equations are given by:

$$M_1 \frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + Kx_1 - Kx_2 = 0$$

$$-Kx_1 + M_2\frac{d^2x_2}{dt^2} + Kx_2 = f(t)$$

(b) Find the state space equations (where the input is $f(t)$ and the output is x_2)