

ENG2009 – Modelling of Engineering Systems

Tutorial 5

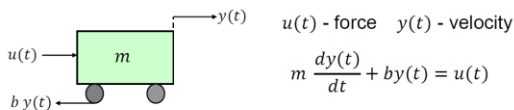
System Responses and Stability

Quick revision: System Response (unit step response) and Stability (Lecture 9-10)

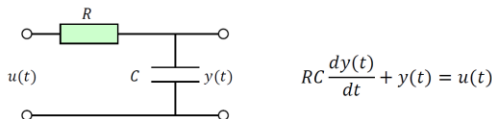
First order system

First-Order Systems

Example:



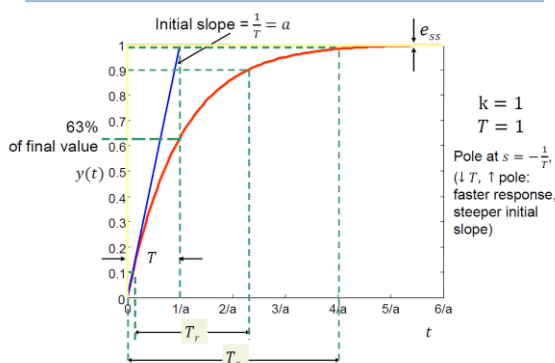
Other 1st order system



First-Order Systems

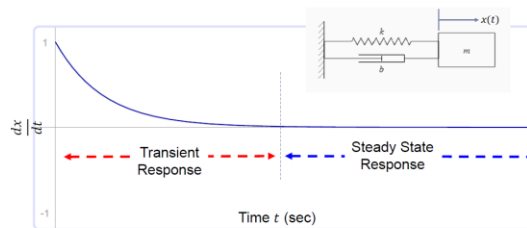
- General form: $T \frac{dy(t)}{dt} + y(t) = ku(t)$
- Transfer function: $U(s) \rightarrow \frac{k}{Ts + 1} \rightarrow Y(s)$
 k – constant (steady-state gain)
 T - time constant
- Pole: $s = -\frac{1}{T}$

Step Response of 1st Order Systems



1st order system: system response

Example: Mass-damper animation



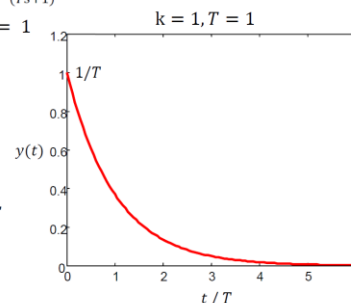
Spring constant $k = 0$
Input: $u(t) = 0$
Output: $y(t)$: velocity
Initial condition: $v(0) = \frac{dx}{dt} = 1$

Step Response of 1st Order Systems

- General output analysis:
- General form: $T \frac{dy(t)}{dt} + y(t) = ku(t) \rightarrow \mathcal{L} \rightarrow \frac{Y(s)}{U(s)} = \frac{k}{(Ts+1)}$
- Pole: $s = -\frac{1}{T}$, ($\downarrow T$, \uparrow pole: faster response)
- Unit-step, $U(s) = \frac{1}{s} \rightarrow Y(s) = \frac{1}{s(Ts+1)}$
- Partial fraction expansion: $Y(s) = \frac{1}{s} - \frac{1}{s+1/T}$
- Solve: Inverse Laplace transform:
 $y(t) = 1 - e^{-(t/T)}$
 $y_{ss}(t)$ $y_t(t)$

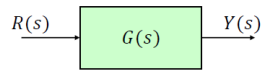
Impulse Response of 1st Order Systems

- General form: $\frac{Y(s)}{U(s)} = \frac{k}{(Ts+1)}$
- Unit-impulse, $R(s) = 1$
 $\rightarrow Y(s) = \frac{1}{Ts+1}$
- Inverse Laplace transform:
 $y(t) = e^{-(t/T)} / T$
- Impulse response:
 $y_{ss}(t) = 0$



Second order system

Second-Order Systems



- Transfer function: $G(s) = \frac{Y(s)}{R(s)} = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
- k = gain of system (sometimes called DC gain)
- ζ = damping ratio - controls the rate of rise or decay of oscillations in the system (unitless)
- ω_n = natural frequency - frequency of oscillation of the system without damping (rad/s)

Second-Order Systems

- Solutions of the characteristic equation (quadratic):

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Roots of characteristic equation (pole of TF)

$$s_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \quad s_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

- We will consider five cases:

- 1) $\zeta = 1$, critically damped (stable system) - last lecture
- 2) $\zeta > 1$, overdamped (stable system) - last lecture
- 3) $0 < \zeta < 1$, underdamped (stable system)
- 4) $\zeta = 0$, undamped (marginally stable system)
- 5) $\zeta < 0$, exponential growth (unstable system)

Step Response of 2nd Order Systems

- Unit-step time response:

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

$y_{ss}(t)$ (steady-state value) $y_r(t)$ (response)

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Stability of LTI systems

An LTI system is said to be **stable** if
all the poles have
negative real parts

(i.e. they are all in the left half of the s-plane).

Note:

poles: the roots of the transfer function denominator polynomial)

Real example: mass-spring-damper system

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = k\omega_n^2 u(t) \quad (\text{general form})$$

$$m \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = u(t)$$

$$\frac{d^2 y(t)}{dt^2} + \left(\frac{b}{m}\right) \frac{dy(t)}{dt} + \left(\frac{k}{m}\right) y(t) = \left(\frac{1}{m}\right) u(t)$$

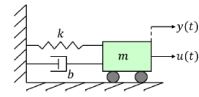
$$\text{Natural frequency, } \omega_n = \sqrt{k/m}$$

$$\text{Damping factor, } \zeta = \frac{b}{2\omega_n m}$$

Therefore

- $\uparrow b \Rightarrow \uparrow \zeta$: (more damper, more damping factor, less oscillatory)
- $\uparrow k \Rightarrow \uparrow \omega_n, \downarrow \zeta$: (more spring, less damping factor, more oscillatory)
- $\uparrow m \Rightarrow \downarrow \omega_n, \downarrow \zeta$: (more weight, less damping factor, more oscillatory)

System output influenced by damping factor ζ ?



Step Response of 2nd Order Systems

Case (3) Underdamped, $0 < \zeta < 1$:

- Two complex poles:

$$s_1 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

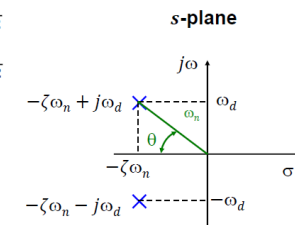
$$s_2 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

Damped natural frequency

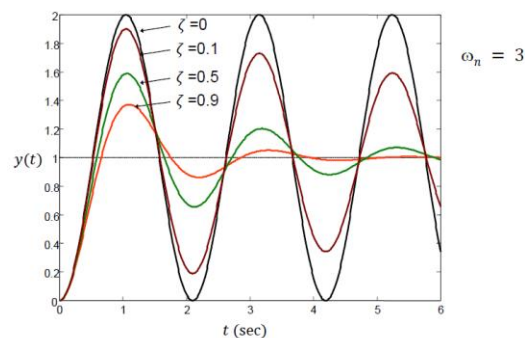
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\cos(\theta) = \zeta$$

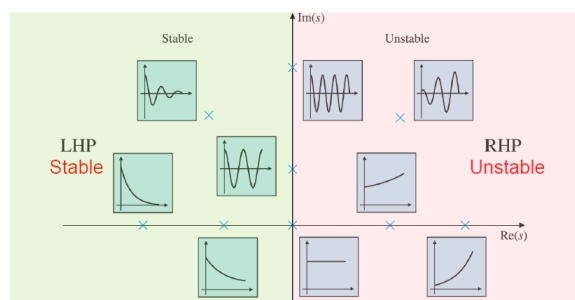


Step Response of 2nd Order Systems



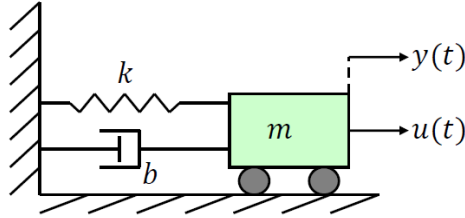
Recall: Stability of LTI systems

Effect of pole location on system response and stability



Example: Mass-spring-damper animation example

Given the differential equation: $m \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = u(t)$ for the mass-spring-damper system below, where $m = 1$, $b = 0.2$, $k = 1$. Assume that the input is a unit impulse $u(t) = \delta$.



Find:

- Transfer function (assume zero initial conditions)
- The order of the system
- Poles and zeros
- Draw the s-plane
- Is the system stable?
- Without solving the differential equation, predict the response of the system.
- Use Python to create a simulation of the system. Assume that the input force $u(t)$ is a unit impulse

Solutions:

(a) Differential eq: $m \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = u(t)$.

Laplace Transform (assuming zero initial conditions):

$$ms^2 Y(s) + bsY(s) + kY(s) = U(s)$$

$$(ms^2 + bs + k)Y(s) = U(s)$$

Transfer function:

$$\frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$

(b) 2nd order system (highest power of denominator)

(c) substituting $m = 1$, $b = 0.2$, $k = 1$ to the transfer function:

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 0.2s + 1}$$

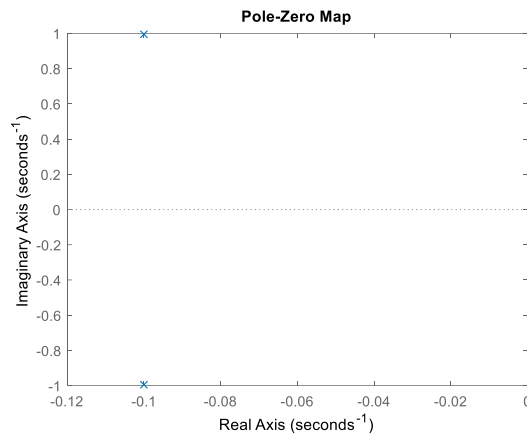
Zeros: roots of numerator: no zero

Poles: roots of denominator

$$s^2 + 0.2s + 1 = 0$$

→ roots: $s = -0.1000 + 0.9950i$, $s = -0.1000 - 0.9950i$ (two complex pair poles)

(d) s-plane



(e) Stable: all poles in LHP

(f) For second order system, analyse damping ratio ζ :

Note that the general transfer function of a second order system is given by

$$G(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s^2 + 0.2s + 1}$$

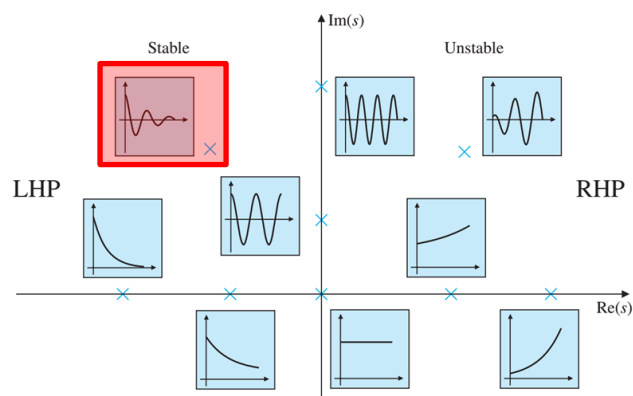
where k is the gain of the system, ζ is the damping ratio and ω is the natural frequency.

Therefore in this case $\omega_n = 1$ and $2\zeta\omega_n = 0.2$

Which yield damping ratio $\zeta = 0.1$ which is underdamped system.

Checking poles: $s = -0.1000 + 0.9950i, s = -0.1000 - 0.9950i$. Two complex pair farly close to the imaginary axis: underdamped system, slightly oscillatory, but converge to zero (due to impulse input).

Example:



(g) Python code example:

```
#import numpy, integrate and pyplot
import numpy as np
from scipy import integrate
from matplotlib import pyplot as plt

#sim data
m=1 #mass of the oscillator
b=0.2 #damper of the oscillator
k=1 #stiffness of the oscillator
omega = (k/m)**0.5 #angular frequency of the oscillator
T=2*np.pi/omega #period of the oscillator

#create 1000 uniformly distributed points in the interval [0,10T]
t = np.linspace(0,10*T,1000)

#lambda returning the force
force = lambda t:t<=0.1

#lambda with the RHS of the equation, it evaluates the force and returns a list
f = lambda X,t: [X[1],1/m*(force(t)-k*X[0]-b*X[1])]

#solve equation using odeint, now the values of X for t=0 are given by a list
XOdeint = integrate.odeint(f,[0,0],t)

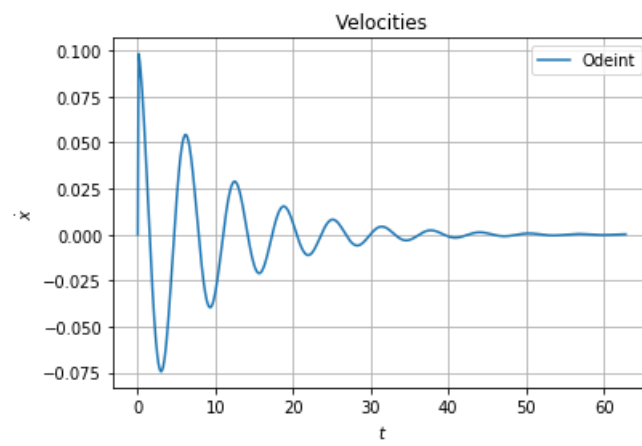
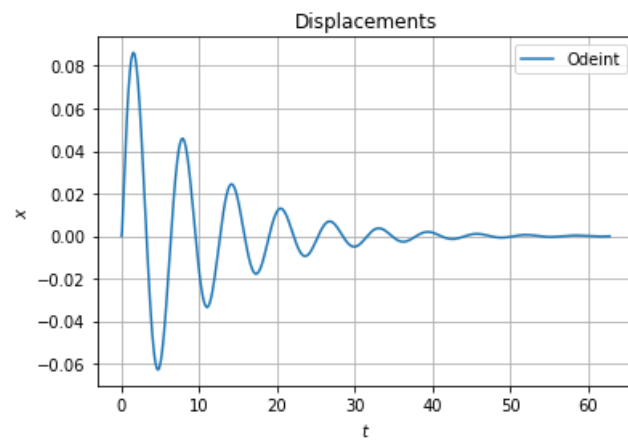
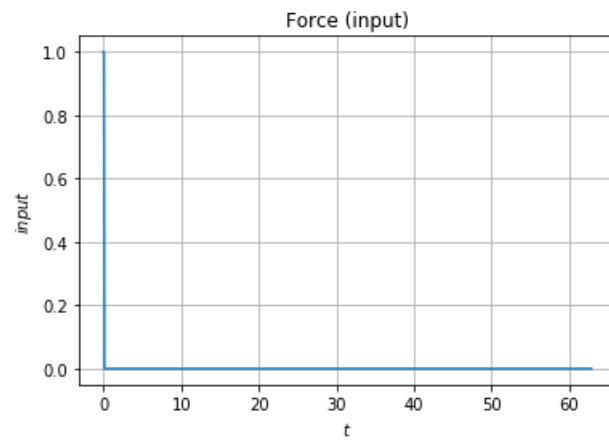
ti = np.linspace(0,10*T,1000)

#plot force
plt.plot(ti,force(ti))
plt.title('Force (input)')
plt.grid()
plt.xlabel('$t$')
plt.ylabel('$input$')
plt.show()

#plot displacements from both solutions (first column of each solution)
plt.plot(t,XOdeint[:,0])
plt.legend(['Odeint'])
plt.title('Displacement')
plt.grid()
plt.xlabel('$t$')
plt.ylabel('$x$')
plt.show()

#plot velocities from both solutions (first column of each solution)
plt.plot(t,XOdeint[:,1])
plt.legend(['Odeint'])
plt.title('Velocitie')
plt.grid()
plt.xlabel('$t$')
plt.ylabel('$\dot{x}$')
plt.show()

#Save data
np.savetxt('t.txt',t)
np.savetxt('x.txt',XOdeint[:,0])
```



Exercises:

For each of the transfer functions shown below, find:

- (a) Order of the system,
- (b) The poles and zeros,
- (c) plot the s-plane,
- (d) state wheather the system stable or unstable,
- (e) without solving for the inverse Laplace transform, state the nature of each response to a step input (state if the system is overdamped, underdamped, and so on).
- (f) DIY: Use Python to create a simulation of the system. Assume that the input force $u(t)$ is a unit impulse

Q1) $G(s) = \frac{2s+1}{s^2+3s+2}$

Q2) $G(s) = \frac{2}{s+2}$

Q3) $G(s) = \frac{10(s+7)}{(s+10)(s+20)}$

Q4) $G(s) = \frac{20}{s^2+6s+144}$

Q5) $G(s) = \frac{s+2}{s^2+9}$