TNRD Smooth Diffusion Equation

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When we modify the original diffusion equation

$$u_t = u_{t-1} - \sum_{i=1}^{N_k} K_t^i \phi_t^i (K_t^i u_{t-1}) + \lambda_t (u_{t-1} - f_n),$$

to the following version

$$u_t = u_{t-1} - \sum_{i=1}^{N_k} \bar{k}_t^i * \phi_t^i (k_t^i * u_{t-1}) + \lambda_t (u_{t-1} - f_n),$$

we find it introduces some imperfections at the image boundary. The basic reason lies in the fact that, in the case of symmetric boundary condition used in our work, $K^T v$ can be interpreted as the convolution with the kernel $\bar{k}_1$ only in the central region of image v. This interpretation does not hold for the image boundary. However, in the diffusion equation (2), the convolution kernel $\bar{k}_i$ is applied to the whole image, thus bringing some artifacts at the boundary. The benefit to use the diffusion equation (2), rather than (1) is that the revised model is more tractable in practice, especially for training, as everything can be done by the convolution operation efficiently.

In order to remove these artifacts, we pad the input image u_{t-1} for stage t, as well as the noisy image f_n, with mirror reflections of itself. This operation is formulated by the sparse "padding" matrix P. After a diffusion step, we only crop the central region of the output image u_t for usage. This operation is formulated by the sparse "cropping" matrix T. When we apply the matrix $PT = P \times T$ to an image u, PTu corresponds to two operations: it first crops the central region of u, then pads it with mirror reflections.

After taking into account the operation of boundary handling, the exact diffusion process is illustrated in Figure 1. There we have $u_t^p = PTu_t$.

In our derivations, we use the symmetric boundary condition for the convolution operation $k * u$ (image $u \in R^{m \times n}, k \in R^{r \times r}$). As we know, it is equivalent to the matrix-vector product formulation Ku, where $K \in R^{N \times N}$ is a highly sparse matrix and u is a column vector $u \in R^N$ with $N = m \times n$. The result $k * u$ can also be interpreted with Uk, where matrix $U \in R^{N \times R}$ is constructed from image u and k is a column vector $k \in R^R$ with $R = r \times r$. This formulation is very helpful for the computation of the gradients of the loss function w.r.t.

the kernel k, as $U^T v$ ($v \in R^N$ is a column vector) can be explicitly interpreted as a convolution operation, which is widely used in classic convolutional neural networks. In the following derivations, we will make use of this equivalence frequently, i.e.,

$$k * u \Leftrightarrow Ku \Leftrightarrow Uk.$$

The derivations also require matrix calculus. We use the denominator layout notation for all the derivations.

Figure 1: Proposed nonlinear diffusion process with careful boundary handling operation. Note that $u_t^p = PTu_t$.

In this paper, we assume Ω to be an open, connected, bounded subset of R^2 with Lipschitz boundary $\partial\Omega$, and set $\Omega_T = \Omega \times (0,T)$ for some fixed time $T > 0$. We also denote by $\sup_\Omega u$ (resp., $\inf_\Omega u$) the $esssup_\Omega u$ (resp., $essinf_\Omega u$), if a function u belongs to $L^p(\Omega)$ for some $1 \leq p \leq \infty$.

The RPM model proposed by Catté et al. [?] is given as follows:

$$\frac{\partial u}{\partial t} = div\left(\frac{\nabla u}{1 + |\nabla u_\sigma|^2}\right) in \Omega_T,$$

$$\frac{\partial u}{\partial n} = 0 on \partial\Omega \times (0,T),$$

$$u = f on \Omega \times \{t = 0\},$$

where $u_\sigma = G_\sigma * u$, $G_\sigma(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{|x|^2}{2\sigma^2}}$ and f is the noisy image. The Gaussian convolution in the diffusion coefficient increases the robustness of the denoising model by making the edge detector ignore small details caused by additive noise. When $|\nabla u_\sigma|$ is large, the diffusion will be slow and edges will be kept. When $|\nabla u_\sigma|$ is small, the diffusion will be faster to effectively remove the noise in flat areas. Despite its advantages in removing additive Gaussian noise [?], the RPM model cannot handle the multiplicative noise case, as illustrated in a numerical example in Sect. 2.3.

Generally, most mathematical methods for image denoising deal with the gray-scale image. Dong et al. [?] proposed a diffusion coefficient function $c(u)$, called gray level indicator, in a convex adaptive total variation model to remove multiplicative noise. It takes into account the gray level of noisy image and controls the speed of diffusion at different regions. As discussed in [?], the gray scale of image is useful information to steer the diffusion progress, especially for the multiplicative noise removal problem. Inspired by the gray level indicator, Zhou et al. [?] developed a doubly degenerated (DD) diffusion model to address the multiplicative noise removal problem:

$$\frac{\partial u}{\partial t} = div(g(u, |\nabla u|)\nabla u) in \Omega_T,$$

$$\frac{\partial u}{\partial n} = 0 on \partial\Omega \times (0, T),$$
$$u = f on \Omega \times \{t = 0\}.$$

The diffusion coefficient is given as follows:

$$g(u, |\nabla u|) = \frac{2|u|^\alpha}{M^\alpha + |u|^\alpha} \cdot \frac{1}{(1 + |\nabla u|^2)^{(1-\beta)/2}},$$

where $0 < \alpha, 0 < \beta < 1$ and $M = \sup_{x\in\Omega} u(x,t)$. $a(u) := \frac{2|u|^\alpha}{M^\alpha+|u|^\alpha}$ is a gray level indicator, which satisfies $0 \leq a(u) \leq 1$. $b(|\nabla u|) := \frac{1}{(1+|\nabla u|^2)^{(1-\beta)/2}}$ is devoted to an edge detection function. However, the authors did not perform a thorough theoretical analysis because of the degeneration in the edge detection function, i.e., $b(|\nabla u|) \to 0$ when $|\nabla u| \to \infty$, which brings difficulty in studying the theoretical properties of the denoising model.

0.1 The Proposed Model

In this paper, we propose a model that incorporates the strengths of the RPM and DD models for noise removal. The diffusion depends on the local information of the image, not only the gradient but also the gray value information. Moreover, the choice of the diffusion coefficient yields a model which we can show is theoretically sound. To this end, the proposed model is given as follows:

$$\frac{\partial u}{\partial t} = div(g(u_\sigma, |\nabla u_\sigma|)\nabla u) in \Omega_T,$$

$$\frac{\partial u}{\partial n} = 0 on \partial\Omega \times (0, T),$$
$$u = f on \Omega \times \{t = 0\},$$

where

$$g(u_\sigma, |\nabla u_\sigma|) = \left(\frac{u_\sigma}{M}\right)^\alpha \frac{1}{1 + |\nabla u_\sigma|^\beta},$$

where $M = \max_{x\in\Omega} |u_\sigma(x,t)|$ and f is the initial image, and α, β, and σ are positive constants. Next, the motivation of our diffusion model is summarized.

As pointed out in Zhou et al. [**?**], the mathematical model of multiplicative noise, $f = u\eta$, indicates that the noise level is related to the original image. The higher the gray level of u is, the stronger the noise will be. Thus, we should consider the information of gray level and introduce $a(u_\sigma) := \left(\frac{u_\sigma}{M}\right)^\alpha$ as a gray level indicator to deal with noise in varying gray level regions. $a(u_\sigma)$ satisfies $0 \leq a(u) \leq 1$. On the other hand, the Gaussian convolution used in the RPM model brings a lot of benefits, not only robustness in the denoising aspect, but also well-posedness in the theoretical aspect. Therefore, we also introduce the Gaussian convolution in the following edge detector $b(|\nabla u_\sigma|) := \frac{1}{1+|\nabla u_\sigma|^\beta}$ with a varying parameter β. $b(|\nabla u_\sigma|)$ indicates the diffusion speed is related to the gradient of the image. The larger the value is, the lower the diffusion speed will be, resulting in better edge protection.

The introduced Gaussian kernel function in our diffusion coefficient brings a lot of advantages. It makes the diffusion model powerful in edge detection and avoids the degeneration arising in the DD model. In addition, it also leads to a smooth solution of our nonlinear diffusion model, which may not be expected in image denoising. We will show that our nonlinear smooth diffusion model is efficient in removing high-level multiplicative Gamma noise and also protects the main edges of the image.

Original TNRD

$$u_t = u_{t-1} - \sum_{i=1}^{N_k} (k_i^T \phi_i (k_i u_{t-1}) + \lambda(u_{t-1} - f_n)) \tag{1}$$

additive noise

The Proposed Model

$$u_t = u_{t-1} - \sum_{i=1}^{N_k} (\bar{k}_i^T \phi_i (\bar{k}_i u_{t-1}) + \lambda(u_{t-1} - f_n)) \tag{2}$$

$$v_t = u_{t-1} - \sum_{i=1}^{N_k} (k_i^T \phi_i (k_i v_{t-1}) + \lambda(v_{t-1} - f_n)) \tag{3}$$

Fractional Perona-Malik Equation

$$u_t^\alpha = div(C(\nabla u)\nabla u) \tag{4}$$

$$\sigma_\alpha(u_t) = \sum_{i=1}^{k-1} (\omega_i^\alpha * u_i) + \sum_{i=1}^{k-1} (\omega_i^\alpha * u_i) \tag{5}$$

$$= div(C(\nabla u)\nabla u) \tag{6}$$

$$k_i \nabla u_{t-1} = \sum_{i=1}^{k} (\omega_i^\alpha * u_i) + \sum_{i=1}^{k-1} (\omega_i^\alpha * u_i) + 2\frac{\partial}{\partial x} div(C(\nabla u)\nabla u) \tag{7}$$

$$u_\phi = l^{1-\alpha} \left((k-1)^{1-\alpha}\right) TNRDwith(U/M)$$

with additive noise (u - f)
PSNR = 24.64
with multiplicative noise (U / M) (u - f / ($c^2 + 1$))
PSNR = 19.01, = 2 18.82, = 1

(8)