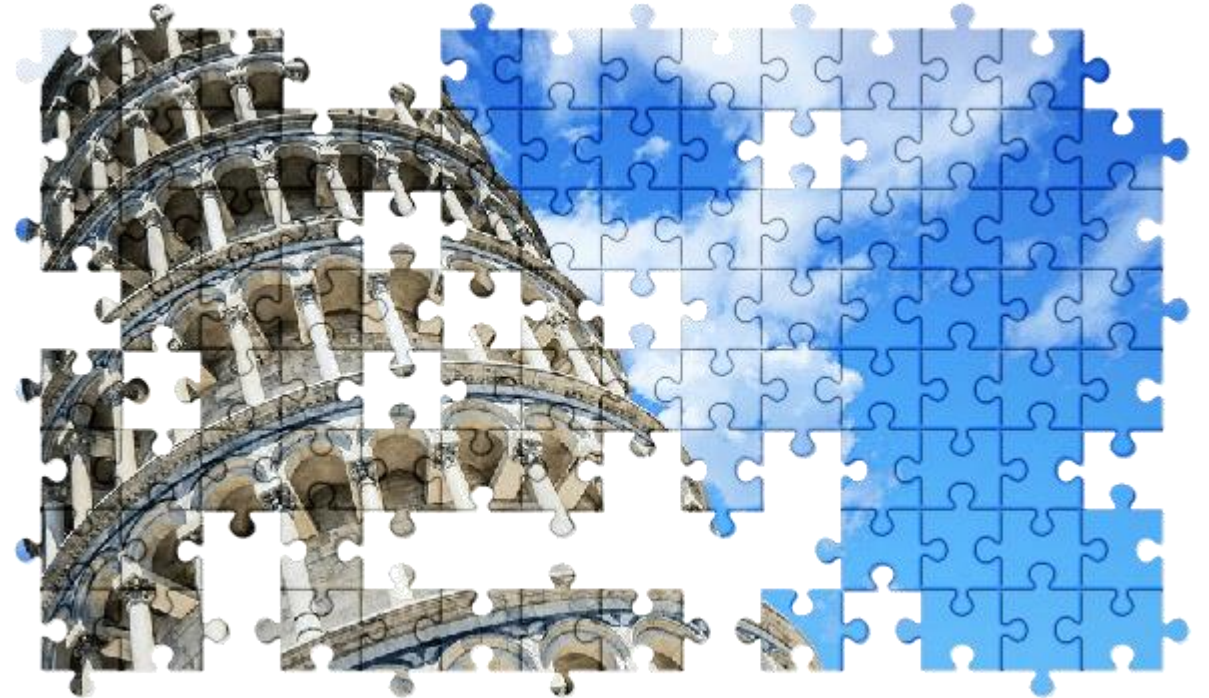


An Invitation to 3D Vision: **Solving Problems**

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Getting Started with a Simple Problem

- Jane bought 3 apples and 4 oranges and paid 10,000 KRW.
- Daniel bought 5 apples and 2 orange and paid 12,000 KRW.

- **Question) How much is an apple and an orange, respectively?**

- **Answer) Calculating the price of apple and orange**

- Unknown: the price apple x and the price of orange y
- Constraints: $3x + 4y = 10000$ and $5x + 2y = 12000$
- Solution using an inverse matrix

$$\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10000 \\ 12000 \end{bmatrix} \rightarrow A\mathbf{x} = \mathbf{b}$$

$$\text{Solve } A\mathbf{x} = \mathbf{b} \rightarrow \mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2000 \\ 1000 \end{bmatrix}$$

Therefore, an apple is 2,000 KRW, and an orange is 1,000 KRW.

Getting Started with Line Fitting

Q) How about finding a line from three points?

(1, 4)



(4, 2)



(7, 1)



- Line representation: $y = ax + b$ ($y = -\frac{2}{3}x + \frac{14}{3}$)
- Slope $a = \frac{2-4}{4-1} = -\frac{2}{3}$
- Y intercept $b = 4 - a \cdot 1 = \frac{14}{3}$

Solving Linear Equations using Linear Algebra

- **Example) Line fitting from two points, (1, 4) and (4, 2)** Q) How about finding a line from three points?

- Unknown: Line parameters a and b (line representation: $y = ax + b$)
- Constraints: $a \cdot 1 + b = 4$ and $a \cdot 4 + b = 2$
- Solution

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \rightarrow \mathbf{Ax} = \mathbf{b}$$
$$\therefore \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{3} \begin{bmatrix} -2 \\ 14 \end{bmatrix}$$

- **Example) Line fitting from two points, (1, 4) and (4, 2), using [NumPy](#)**

```
import numpy as np
```

```
A = np.array([[1., 1.], [4., 1.]])
```

```
b = np.array([[4.], [2.]])
```

```
A_inv = np.linalg.inv(A)
```

```
print(A_inv * b) # [[-1.33333333  1.33333333]
                  # [ 2.66666667 -0.66666667]] Note) broadcast
```

```
print(A_inv @ b) # [[-0.66666667]
                  # [ 4.66666667]] Note) A_inv.matmul(b)
```

Solving Linear Equations using Linear Algebra

- **Example) Line fitting from more than two points such as (1, 4), (4, 2), and (7, 1)**

- Unknown: Line parameters a and b (line representation: $y = ax + b$)
- Constraints: $a \cdot 1 + b = 4$, $a \cdot 4 + b = 2$, and $a \cdot 7 + b = 1$
- Solution

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \rightarrow A\mathbf{x} = \mathbf{b}$$

$\therefore \mathbf{x} = A^\dagger \mathbf{b}$ where A^\dagger is a pseudo-inverse (Note: $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{Ax} - \mathbf{b}\|_2^2$)

- **Note) Pseudo-inverse** (~ a generalized matrix inverse)

- **Not necessarily square** (A : $m \times n$ matrix, A^\dagger : $n \times m$ matrix)
- Left inverse ($A^\dagger A = I_n$): $A^\dagger = (A^\top A)^{-1} A^\top$
 - If A has linearly independent columns ($\operatorname{rank}(A) = n$)
- Right inverse ($AA^\dagger = I_m$): $A^\dagger = A^\top (AA^\top)^{-1}$
 - If A has linearly independent rows ($\operatorname{rank}(A) = m$)

Solving Linear Equations using Linear Algebra

- **Example) Line fitting from more than two points such as (1, 4), (4, 2), and (7, 1)**

- Unknown: Line parameters a and b (line representation: $y = ax + b$)
- Constraints: $a \cdot 1 + b = 4$, $a \cdot 4 + b = 2$, and $a \cdot 7 + b = 1$
- Solution

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \rightarrow A\mathbf{x} = \mathbf{b}$$

$\therefore \mathbf{x} = A^\dagger \mathbf{b}$ where A^\dagger is a pseudo-inverse (Note: $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|A\mathbf{x} - \mathbf{b}\|_2^2$)

- **Example) Line fitting from more than two points such as (1, 4), (4, 2), and (7, 1), using [NumPy](#)**

```
import numpy as np
```

```
A = np.array([[1., 1.], [4., 1.], [7., 1.]])
```

```
b = np.array([[4.], [2.], [1.]])
```

```
A_inv = np.linalg.pinv(A)
```

```
print(A_inv @ b) # [[-0.5], [ 4.33333333]]
```

Getting Started with Line Fitting

(1, 4)

(4, 2)

- Line representation: $y = ax + b$ ($y = -\frac{2}{3}x + \frac{14}{3}$)
- Slope $a = \frac{2-4}{4-1} = -\frac{2}{3}$
- Y intercept $b = 4 - a \cdot 1 = \frac{14}{3}$

Q) Can it represent a vertical line such as $x = 1$?

- Line representation: $ax + by + c = 0$
($2x + 3y - 14 = 0$; $4x + 6y - 28 = 0$)
→ additional constraint $a^2 + b^2 = 1$
- Its shorter form: $\mathbf{n}^T \mathbf{x} + c = 0$
($\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$)

Normal vector

Solving Linear Equations using Linear Algebra

- **Example) Line fitting from two points, (1, 4) and (4, 2)**

- Unknown: Line parameters a , b , and c (line representation: $ax + by + c = 0$)
- Constraints: $a \cdot 1 + b \cdot 4 + c = 0$ and $a \cdot 4 + b \cdot 2 + c = 0$
- Solution

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow A\mathbf{x} = \mathbf{0} \text{ (homogeneous equation) } \quad \cancel{\mathbf{x} = A^{-1}\mathbf{b}}$$

$$\therefore \mathbf{x} = \text{null}(A) = [2, 3, -14]^T \rightarrow 2x + 3y - 14 = 0$$

- **Note) Null space** (a.k.a. [kernel](#))

- A set of vectors which map A (m -by- n matrix) to the zero vector

$$\text{null}(A) = \{ \mathbf{v} \in K^n \mid A\mathbf{v} = \mathbf{0} \}$$

- [Rank-nullity theorem](#): $\text{rank}(A) + \text{nullity}(A) = n$

Solving Linear Equations using Linear Algebra

- **Example) Line fitting from two points, (1, 4) and (4, 2)**

- Unknown: Line parameters a , b , and c (line representation: $ax + by + c = 0$)
- Constraints: $a \cdot 1 + b \cdot 4 + c = 0$ and $a \cdot 4 + b \cdot 2 + c = 0$
- Solution

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow A\mathbf{x} = \mathbf{0} \text{ (homogeneous equation) } \mathbf{x} = A^\perp \mathbf{b}$$

$$\therefore \mathbf{x} = \text{null}(A) = [2, 3, -14]^T \rightarrow 2x + 3y - 14 = 0$$

- **Example) Line fitting from two points, (1, 4) and (4, 2), using [NumPy](#)**

```
import numpy as np
from scipy import linalg
```

```
A = np.array([[1., 4., 1.], [4., 2., 1.]])
x = linalg.null_space(A)
print(x / x[0]) # [[1.], [1.5], [-7.]]
```

Q) How about finding a line from three points?

A) No null space for $A = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 2 & 1 \\ 7 & 1 & 1 \end{bmatrix}$ (\because full rank)

Solving Linear Equations using Linear Algebra

- **Example) Line fitting from more than two points such as (1, 4), (4, 2), and (7, 1)**

- Unknown: Line parameters a , b , and c (line representation: $ax + by + c = 0$)
- Constraints: $a \cdot 1 + b \cdot 4 + c = 0$, $a \cdot 4 + b \cdot 2 + c = 0$, and $a \cdot 7 + b \cdot 1 + c = 0$
- Solution

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 2 & 1 \\ 7 & 1 & 1 \end{bmatrix} = USV^T \text{ using } \text{singular value decomposition (SVD)}$$

$\therefore \mathbf{x}$ is the last row of V^T . (Note: $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{Ax}\|^2$ with $\|\mathbf{x}\|^2 = 1$ condition)

- **Example) Line fitting from more than two points such as (1, 4), (4, 2), and (7, 1), using [NumPy](#)**

```
import numpy as np
```

```
A = np.array([[1., 4., 1.], [4., 2., 1.], [7., 1., 1.]])
```

```
U, S, Vt = np.linalg.svd(A, full_matrices=True)
```

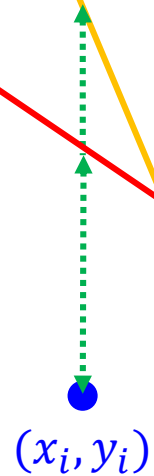
```
x = Vt[-1].T
```

```
print(x / -x[1]) # [[-0.50824973], [-1.], [4.37835787]]
```

```
# Note) [[a], [b]] = [[-0.5], [ 4.33333333]]
```

Getting Started with Line Fitting

- Line representation: $y = ax + b$
- Algebraic distance $d_a = (ax_i + b) - y_i$
(signed distance)
- Line fitting using $\hat{\mathbf{x}} = \mathbf{A}^\dagger \mathbf{b} \rightarrow \hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$
 $\hat{a}, \hat{b} = \underset{a,b}{\operatorname{argmin}} \sum_i (ax_i + b - y_i)^2$



Q) Which line is more closer to the point?

Getting Started with Line Fitting

- Line representation: $ax + by + c = 0$
- Geometric distance $d_g = \frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}} = \frac{\mathbf{n}^T \mathbf{x}_i + c}{\|\mathbf{n}\|}$
(signed distance)
- Line fitting using $\hat{a}, \hat{b}, \hat{c} = \operatorname{argmin}_{a,b,c} \sum_i \left(\frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}} \right)^2$

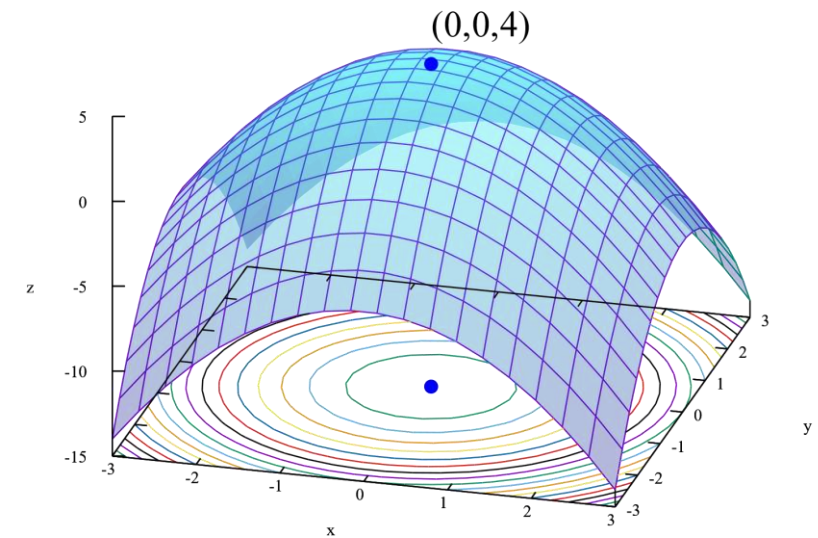


(x_i, y_i)

Q) Which line is more closer to the point?

Solving Nonlinear Equation using Nonlinear Optimization

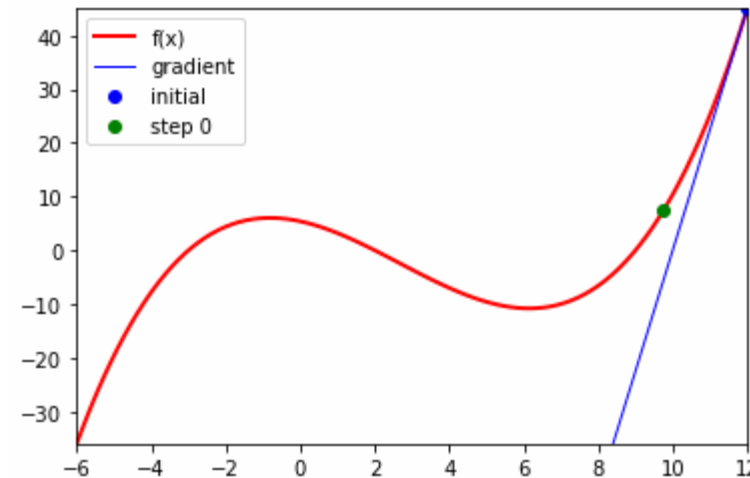
- Nonlinear optimization is the process of solving an optimization problem where some of the constraints or the objective function are **nonlinear**.
 - Alias: Nonlinear programming (NLP)
 - Mathematically, $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} f(\mathbf{x})$ subject to $g_i(\mathbf{x}) \leq 0$ for each $i \in \{1, \dots, m\}$
 $h_j(\mathbf{x}) = 0$ for each $j \in \{1, \dots, p\}$
 $\mathbf{x} \in X$ (X is a subset of \mathbb{R}^n)
 - $f(\mathbf{x})$: The real-valued objective function
 - $g_i(\mathbf{x})$: The i -th real-valued inequality constraint function
 - $h_j(\mathbf{x})$: The j -th real-valued equality constraint function
 - Example) The objective function $f(x, y) = 4 - (x^2 + y^2)$ is nonlinear.



Nonlinear Optimization

- Gradient descent

- A **first-order iterative algorithm** for finding a local minimum of a differentiable function by pursuing to the opposite direction of the gradient of the function at the current point
- Mathematically, $x_{t+1} = x_t - \gamma f'(x_t)$
 - γ : The step size (a.k.a. learning rate)



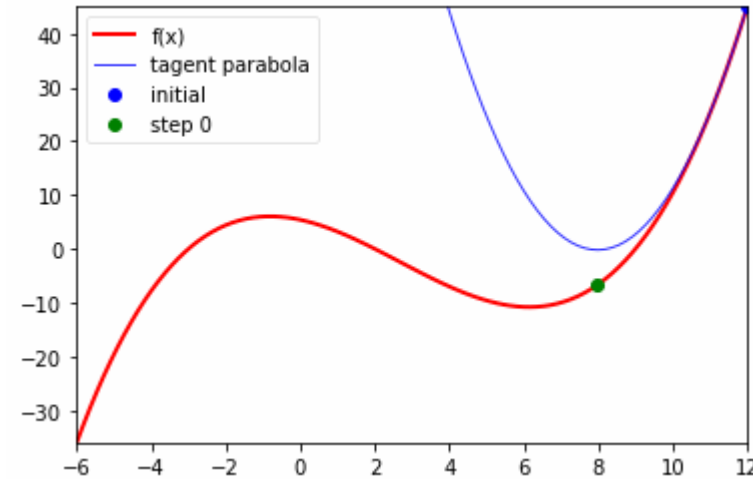
- Note) Stochastic gradient descent (SGD)

- SGD uses an approximated gradient (calculated from a randomly selected subset of the given data) instead of the actual gradient (calculated from the entire data).
- SGD variants: AdaGrad, RMSProp, Adam, ...

Nonlinear Optimization

▪ Newton's method

- A **second-order iterative algorithm** for finding a local minimum of a differentiable function by pursuing the minima of the locally approximated parabola of the function at the current point
- Mathematically, $x_{t+1} = x_t - \frac{f'(x_t)}{f''(x_t)}$
 - The step size is **not** required.



▪ Note) Gauss-Newton method

- A special case for non-linear least squares problems
 - When the function has a form of $f(x) = r^2(x)$,
 - Newton's method becomes $x_{t+1} = x_t - \frac{r(x_t)}{r'(x_t)}$ (without the 2nd-order derivative)

Solving Nonlinear Equation using Nonlinear Optimization

- **Example) Line fitting from more than two points such as (1, 4), (4, 2), and (7, 1), using [SciPy](#)**
 - Unknown: Line parameters a , b , and c (line representation: $ax + by + c = 0$)
 - Cost function: $f(a, b, c) = \sum_i \left(\frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}} \right)^2$
 - Optimizer: [Gauss-Newton method](#) (least squares)

```
import numpy as np
from scipy.optimize import least_squares

def geometric_error(line, pts):
    a, b, c = line
    err = [(a*x + b*y + c) / np.sqrt(a*a + b*b) for (x, y) in pts]
    return err

pts = [(1, 4), (4, 2), (7, 1)]
line_init = [1, 1, 0]
result = least_squares(geometric_error, line_init, args=(pts,))
line = result['x'] / -result['x'][1] # [-0.50372575, -1., 4.34823633]
```

Summary (Line Fitting Problems)

▪ Linear equations

- Inhomogeneous equations $A\mathbf{x} = \mathbf{b} \rightarrow \mathbf{x} = A^\dagger \mathbf{b}$ where A^\dagger is a pseudo-inverse
 - Formulation) $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|A\mathbf{x} - \mathbf{b}\|_2^2$
- Homogeneous equation $A\mathbf{x} = \mathbf{0} \rightarrow \mathbf{x}$ is the last row of V^\top where $A = USV^\top$ (from singular value decomposition)
 - Formulation) $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|A\mathbf{x}\|^2$ with $\|\mathbf{x}\|^2 = 1$ condition

▪ Nonlinear equations

- Nonlinear optimization: ~~Gradient descent, Newton's method, and Gauss-Newton method (for $f(x) = r^2(x)$)~~
 - Formulation) $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} f(\mathbf{x})$
 - Note) Let's use magic tools such as [scipy.optimize](#) and [Ceres Solver](#) for better performance.

Solving Linear Equations in 3D Vision

▪ Affine transformation estimation

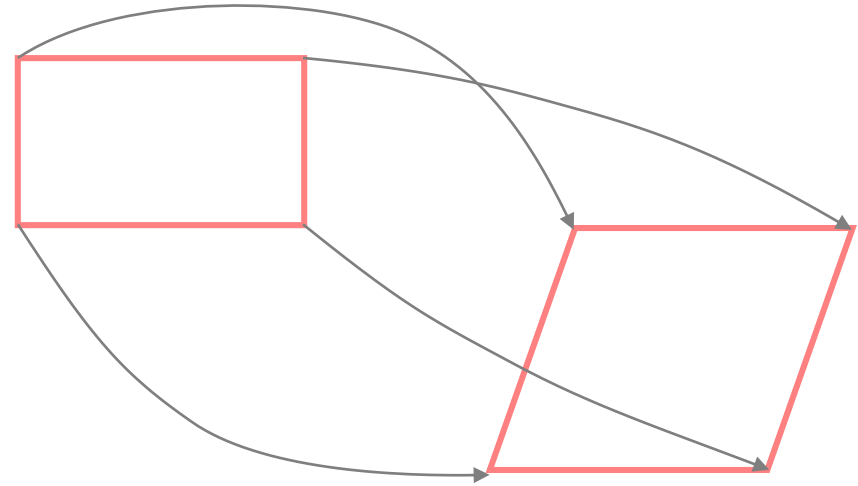
- Unknown: Affine transformation H (6 DOF)
- Given: Point correspondence $(\mathbf{x}_1, \mathbf{x}'_1), \dots, (\mathbf{x}_n, \mathbf{x}'_n)$
- Constraints: n x affine transformation $\mathbf{x}'_i = H\mathbf{x}_i$
- Solutions ($n \geq 3$)
 - OpenCV: `cv.getAffineTransform()`

• Affine transformation estimation

– Affine transformation:
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} x'_i \\ y'_i \\ 1 \end{bmatrix}$$

– For n pairs of points,
$$\begin{bmatrix} x_1 & y_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_1 & y_1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n & y_n & 0 & 0 & 1 & 0 \\ 0 & 0 & x_n & y_n & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x'_1 \\ y'_1 \\ \vdots \\ x'_n \\ y'_n \end{bmatrix} \rightarrow \mathbf{Ax} = \mathbf{b}$$

- Solve $\mathbf{Ax} = \mathbf{b}$ and organize H from \mathbf{x}



Solving Linear Equations in 3D Vision

- Affine transformation estimation [affine_estimation_implement.py]

```
import numpy as np
import cv2 as cv
```

```
def getAffineTransform(src, dst):
    if len(src) == len(dst):
        # Solve 'Ax = b'
        A, b = [], []
        for p, q in zip(src, dst):
            A.append([p[0], p[1], 0, 0, 1, 0])
            A.append([0, 0, p[0], p[1], 0, 1])
            b.append(q[0])
            b.append(q[1])
        x = np.linalg.pinv(A) @ b

        # Reorganize `x` as a matrix
        H = np.array([[x[0], x[1], x[4]], [x[2], x[3], x[5]]])
        return H
```

```
if __name__ == '__main__':
    src = np.array([[115, 401], [776, 180], [330, 793]], dtype=np.float32)
    dst = np.array([[0, 0], [900, 0], [0, 500]], dtype=np.float32)

    my_H = getAffineTransform(src, dst)
    cv_H = cv.getAffineTransform(src, dst) # Note) It accepts only 3 pairs of points.
    ...
```

$$\begin{bmatrix} x_1 & y_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_1 & y_1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n & y_n & 0 & 0 & 1 & 0 \\ 0 & 0 & x_n & y_n & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x'_1 \\ y'_1 \\ \vdots \\ x'_n \\ y'_n \end{bmatrix}$$

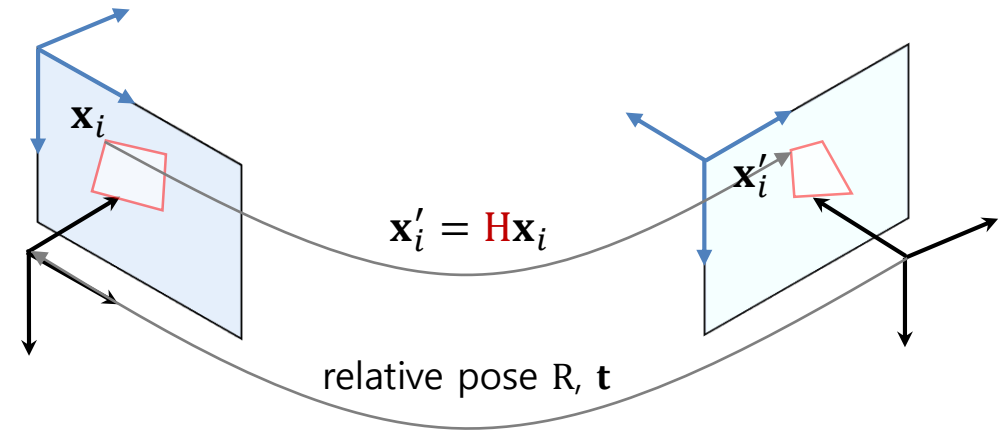
Solving Linear Equations in 3D Vision

Planar homography estimation

- Unknown: **Planar homography** H (8 DOF \leftarrow up to scale)
- Given: Point correspondence $(\mathbf{x}_1, \mathbf{x}'_1), \dots, (\mathbf{x}_n, \mathbf{x}'_n)$
- ~~Constraints: $n \times$ projective transformation $\mathbf{x}'_i = H\mathbf{x}_i$~~

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \rightarrow \mathbf{x}'_i = \begin{bmatrix} x'_i \\ y'_i \\ 1 \end{bmatrix} = \frac{1}{w_i} \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}$$

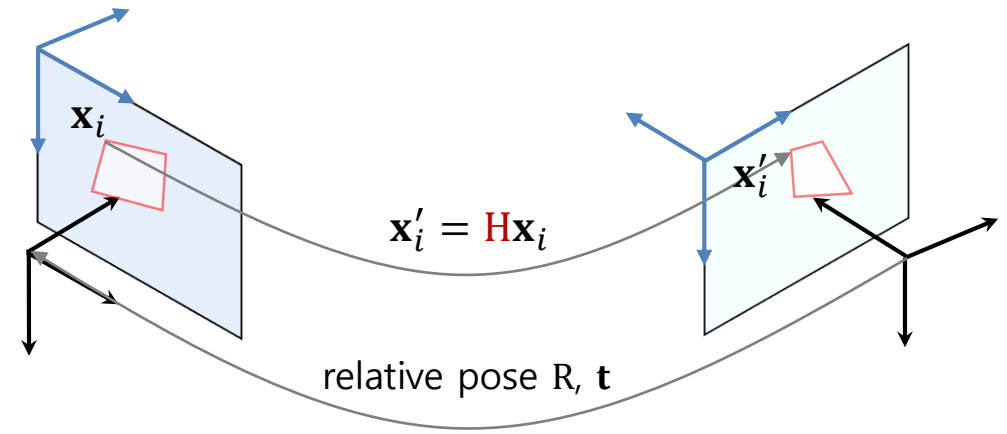
- Modified constraints: $n \times$ transformation $\mathbf{x}'_i \sim H\mathbf{x}_i$ (similarity) or $\mathbf{x}'_i = \lambda_i H\mathbf{x}_i$ or $\mathbf{x}'_i \times (H\mathbf{x}_i) = \mathbf{0}$
 - Note) The last element of \mathbf{x}_i and \mathbf{x}'_i in homogeneous coordinates are not necessary to be 1.
- Solutions ($n \geq 4$) \rightarrow 4-point algorithm
 - OpenCV: `cv.getPerspectiveTransform()` and `cv.findHomography()`



Solving Linear Equations in 3D Vision

Planar homography estimation

- Unknown: **Planar homography** H (8 DOF \leftarrow up to scale)
- Given: Point correspondence $(\mathbf{x}_1, \mathbf{x}'_1), \dots, (\mathbf{x}_n, \mathbf{x}'_n)$
- Constraints: n x transformation $\mathbf{x}'_i \times (H\mathbf{x}_i) = \mathbf{0}$
- Solutions ($n \geq 4$) \rightarrow 4-point algorithm
 - OpenCV: `cv.getPerspectiveTransform()` and `cv.findHomography()`
 - 4-point algorithm**



$$[x'_i \quad y'_i \quad w'_i] \times \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{aligned} w'_i(x_i h_{21} + y_i h_{22} + w_i h_{23}) - y'_i(x_i h_{31} + y_i h_{32} + w_i h_{33}) &= 0 \\ w'_i(x_i h_{11} + y_i h_{12} + w_i h_{13}) - x'_i(x_i h_{31} + y_i h_{32} + w_i h_{33}) &= 0 \\ y'_i(x_i h_{11} + y_i h_{12} + w_i h_{13}) - x'_i(x_i h_{21} + y_i h_{22} + w_i h_{23}) &= 0 \end{aligned}$$

$$\text{For } n \text{ pairs, } \begin{bmatrix} 0 & 0 & 0 & w'_1 x_1 & w'_1 y_1 & w'_1 w_1 & -y'_1 x_1 & -y'_1 y_1 & -y'_1 w_1 \\ w'_1 x_1 & w'_1 y_1 & w'_1 w_1 & 0 & 0 & 0 & -x'_1 x_1 & -x'_1 y_1 & -x'_1 w_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & w'_n x_n & w'_n y_n & w'_n w_n & -y'_n x_n & -y'_n y_n & -y'_n w_n \\ w'_n x_n & w'_n y_n & w'_n w_n & 0 & 0 & 0 & -x'_n x_n & -x'_n y_n & -x'_n w_n \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = 0 \rightarrow A\mathbf{x} = 0$$

- Solve $A\mathbf{x} = 0$ and reorganize H (3x3 matrix)

Solving Linear Equations in 3D Vision

- Planar homography estimation [homography_estimation_implement.py]

```
import numpy as np
import cv2 as cv

def getPerspectiveTransform(src, dst):
    if len(src) == len(dst):
        # Make homogeneous coordinates if necessary
        if src.shape[1] == 2:
            src = np.hstack((src, np.ones((len(src), 1), dtype=src.dtype)))
        if dst.shape[1] == 2:
            dst = np.hstack((dst, np.ones((len(dst), 1), dtype=dst.dtype)))

        # Solve 'Ax = 0'
        A = []
        for p, q in zip(src, dst):
            A.append([0, 0, 0, q[2]*p[0], q[2]*p[1], q[2]*p[2], -q[1]*p[0], -q[1]*p[1], -q[1]*p[2]])
            A.append([q[2]*p[0], q[2]*p[1], q[2]*p[2], 0, 0, 0, -q[0]*p[0], -q[0]*p[1], -q[0]*p[2]])
        _, _, Vt = np.linalg.svd(A, full_matrices=True)
        x = Vt[-1]

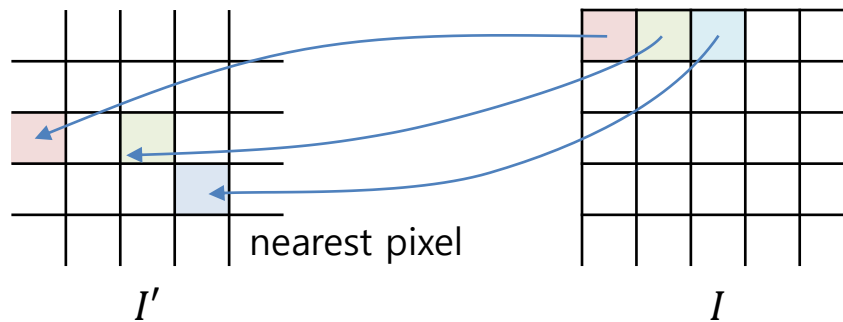
        # Reorganize `x` as a matrix
        H = x.reshape(3, -1) / x[-1] # Normalize the last element as 1
        return H

if __name__ == '__main__':
    ...
```

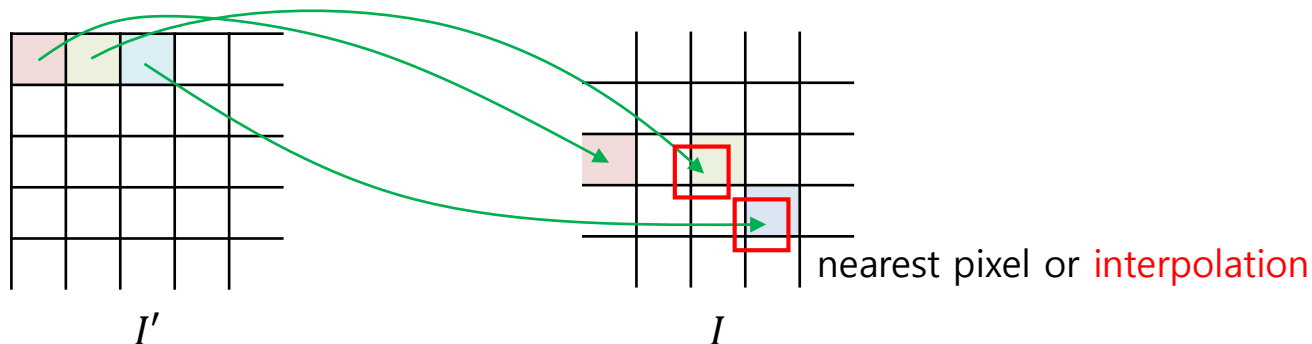
Appendix) Image Warping

Example) Image warping using homography

- Target: Transformed image I'
- Source: Original image I
- Relationship: $I'(\mathbf{x}') = I(\mathbf{x})$ where $\mathbf{x}' = H\mathbf{x}$
- Method #1) Select a pair of points, $\mathbf{x} \in \{(0,0), (0,1), \dots, (1,0), (1,1), \dots\}$ and $\mathbf{x}' = H\mathbf{x}$



- Method #2) Select a pair of points, $\mathbf{x}' \in \{(0,0), (0,1), \dots, (1,0), (1,1), \dots\}$ and $\mathbf{x} = H^{-1}\mathbf{x}'$



▪ Example) Image warping using homography [image_warping_implement.py]

```
import cv2 as cv
```

```
...
```

```
def warpPerspective1(src, H, dst_size):
```

```
    # Generate an empty image
```

```
    width, height = dst_size
```

```
    channel = src.shape[2] if src.ndim > 2 else 1
```

```
    dst = np.zeros((height, width, channel), dtype=src.dtype)
```

```
    # Copy a pixel from `src` to `dst`
```

```
    for py in range(img.shape[0]):
```

```
        for px in range(img.shape[1]):
```

```
            q = H @ [px, py, 1]
```

```
            qx, qy = int(q[0]/q[-1] + 0.5), int(q[1]/q[-1] + 0.5)
```

```
            if qx >= 0 and qy >= 0 and qx < width and qy < height:
```

```
                dst[qy, qx] = src[py, px]
```

```
    return dst
```

```
def warpPerspective2(src, H, dst_size):
```

```
    # Generate an empty image
```

```
    width, height = dst_size
```

```
    channel = src.shape[2] if src.ndim > 2 else 1
```

```
    dst = np.zeros((height, width, channel), dtype=src.dtype)
```

```
    # Copy a pixel from `src` to `dst`
```

```
    H_inv = np.linalg.inv(H)
```

```
    for qy in range(height):
```

```
        for qx in range(width):
```

```
            p = H_inv @ [qx, qy, 1]
```

```
            px, py = int(p[0]/p[-1] + 0.5), int(p[1]/p[-1] + 0.5)
```

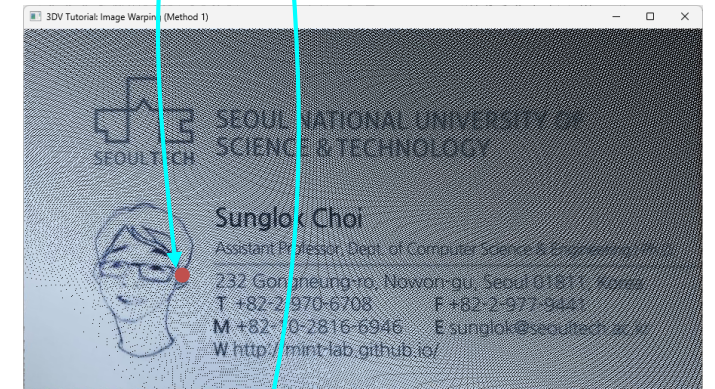
```
            if px >= 0 and py >= 0 and px < img.shape[1] and py < img.shape[0]:
```

```
                dst[qy, qx] = src[py, px]
```

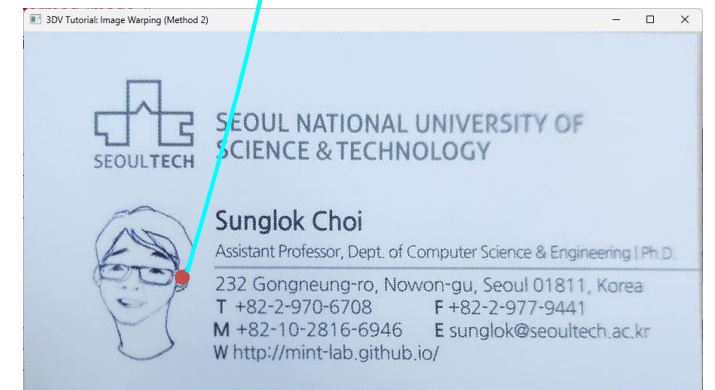
```
    return dst
```



$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$



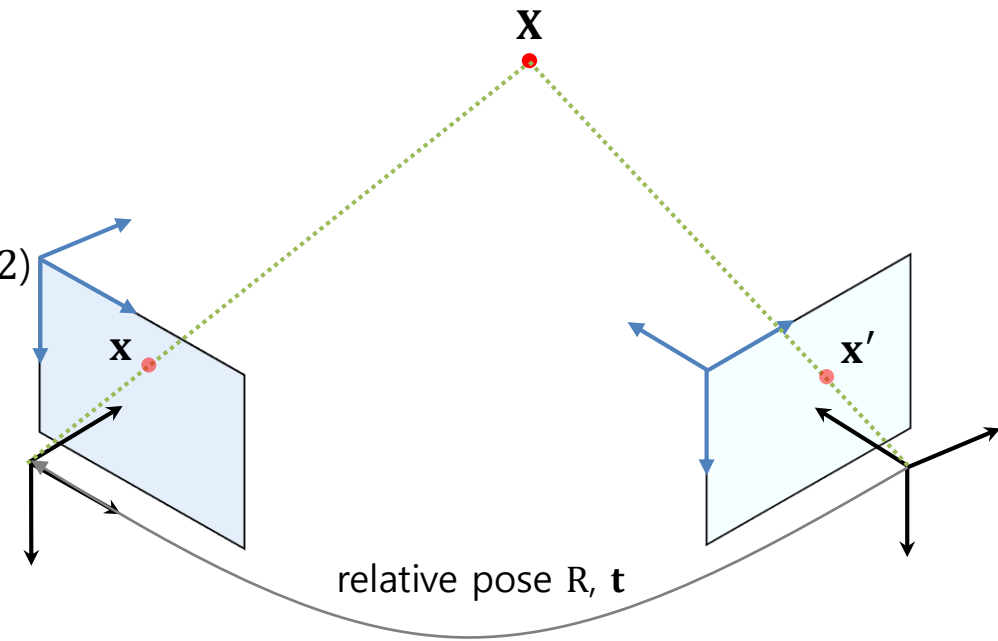
$$\mathbf{x} = \mathbf{H}^{-1}\mathbf{x}'$$



Solving Linear Equations in 3D Vision

▪ Fundamental matrix estimation

- Unknown: **Fundamental matrix** F (7 DOF \leftarrow up to scale, $\text{rank}(F) = 2$)
- Given: Point correspondence $(\mathbf{x}_1, \mathbf{x}'_1), \dots, (\mathbf{x}_n, \mathbf{x}'_n)$
- Constraints: n x epipolar constraint $\mathbf{x}'^T F \mathbf{x} = 0$
- Solutions ($n \geq 7$) \rightarrow 7-point and 8-point algorithms
 - OpenCV: `cv.findFundamentalMat()`
 - **8-point algorithm**



- Epipolar constraint: $[x'_i \ y'_i \ w'_i] \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ w_i \end{bmatrix} = 0$
- For n pairs of points, $\begin{bmatrix} x'_1 x_1 & x'_1 y_1 & x'_1 w_1 & y'_1 x_1 & y'_1 y_1 & y'_1 w_1 & w'_1 x_1 & w'_1 y_1 & w'_1 w_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_n x_n & x'_n y_n & x'_n w_n & y'_n x_n & y'_n y_n & y'_n w_n & w'_n x_n & w'_n y_n & w'_n w_n \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0 \rightarrow A\mathbf{x} = 0$
- Solve $A\mathbf{x} = 0$ and reorganize F (3x3 matrix)
- Enforce $\text{rank}(F) = 2$ using [singular value decomposition \(SVD\)](#)

- Fundamental matrix estimation [fundamental_mat_estimation_implement.py]

```
import numpy as np
import cv2 as cv

def findFundamentalMat(pts1, pts2):
    if len(pts1) == len(pts2):
        # Make homogeneous coordinates if necessary
        if pts1.shape[1] == 2:
            pts1 = np.hstack((pts1, np.ones((len(pts1), 1), dtype=pts1.dtype)))
        if pts2.shape[1] == 2:
            pts2 = np.hstack((pts2, np.ones((len(pts2), 1), dtype=pts2.dtype)))

        # Solve 'Ax = 0'
        A = []
        for p, q in zip(pts1, pts2):
            A.append([q[0]*p[0], q[0]*p[1], q[0]*p[2], q[1]*p[0], q[1]*p[1], q[1]*p[2], q[2]*p[0], q[2]*p[1], q[2]*p[2]])
        _, _, Vt = np.linalg.svd(A, full_matrices=True)
        x = Vt[-1]

        # Reorganize `x` as `F` and enforce 'rank(F) = 2'
        F = x.reshape(3, -1)
        U, S, Vt = np.linalg.svd(F)
        S[-1] = 0
        F = U @ np.diag(S) @ Vt
        return F / F[-1, -1] # Normalize the last element as 1

if __name__ == '__main__':
    ...
```

Solving Linear Equations in 3D Vision

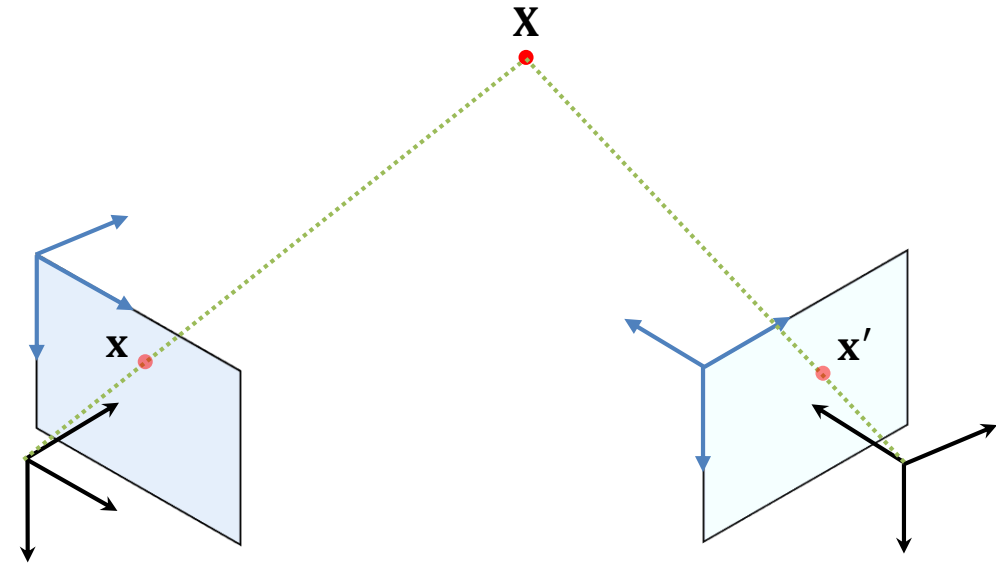
▪ Triangulation (point localization)

- Unknown: **Position of a 3D point \mathbf{X}** (3 DoF)
- Given: Point correspondence $(\mathbf{x}, \mathbf{x}')$ and projection matrices (P, P')
- Constraints: $\mathbf{x} = P\mathbf{X}$, $\mathbf{x}' = P'\mathbf{X}$
- Solutions
 - OpenCV: `cv.triangulatePoints()`

• Linear triangulation

- Projection: $\mathbf{x} = P\mathbf{X} \rightarrow \mathbf{x} \times (P\mathbf{X}) = \mathbf{0} \rightarrow$
$$\begin{aligned} x(\mathbf{p}_3^\top \mathbf{X}) - w(\mathbf{p}_1^\top \mathbf{X}) &= 0 \\ y(\mathbf{p}_3^\top \mathbf{X}) - w(\mathbf{p}_2^\top \mathbf{X}) &= 0 \\ x(\mathbf{p}_2^\top \mathbf{X}) - y(\mathbf{p}_1^\top \mathbf{X}) &= 0 \end{aligned}$$
 where $P = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top \end{bmatrix}$

- Solve $A\mathbf{X} = \mathbf{0}$ where $A = \begin{bmatrix} x\mathbf{p}_3^\top - w\mathbf{p}_1^\top \\ y\mathbf{p}_3^\top - w\mathbf{p}_2^\top \\ x'\mathbf{p}_3'^\top - w'\mathbf{p}_1'^\top \\ y'\mathbf{p}_3'^\top - w'\mathbf{p}_2'^\top \end{bmatrix}$



- **Triangulation** (point localization) [triangulation_implement.py]

```
import numpy as np
import cv2 as cv
```

```
def triangulatePoints(P0, P1, pts0, pts1):
    Xs = []
    for (p, q) in zip(pts0.T, pts1.T):
        # Solve 'AX = 0'
        A = np.vstack((p[0] * P0[2] - P0[0],
                        p[1] * P0[2] - P0[1],
                        q[0] * P1[2] - P1[0],
                        q[1] * P1[2] - P1[1]))
        _, _, Vt = np.linalg.svd(A, full_matrices=True)
        Xs.append(Vt[-1])
    return np.vstack(Xs).T
```

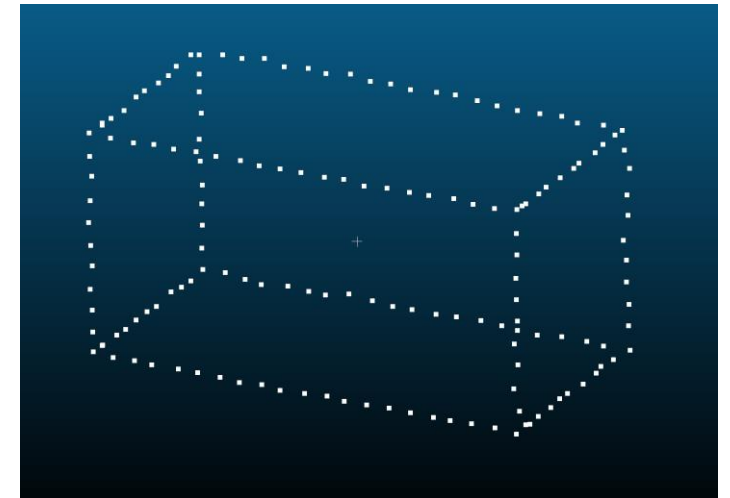
```
if __name__ == '__main__':
    f, cx, cy = 1000., 320., 240.
    pts0 = np.loadtxt('../data/image_formation0.xyz')[:, :2]
    pts1 = np.loadtxt('../data/image_formation1.xyz')[:, :2]
    output_file = 'triangulation_implement.xyz'

    # Estimate relative pose of two view
    F, _ = cv.findFundamentalMat(pts0, pts1, cv.FM_8POINT)
    K = np.array([[f, 0, cx], [0, f, cy], [0, 0, 1]])
    E = K.T @ F @ K
    _, R, t, _ = cv.recoverPose(E, pts0, pts1)
```

$$A = \begin{bmatrix} x\mathbf{p}_3^\top - w\mathbf{p}_1^\top \\ y\mathbf{p}_3^\top - w\mathbf{p}_2^\top \\ x'\mathbf{p}'_3^\top - w'\mathbf{p}'_1^\top \\ y'\mathbf{p}'_3^\top - w'\mathbf{p}'_2^\top \end{bmatrix}$$

- **Triangulation** (point localization) [triangulation_implement.py]

```
if __name__ == '__main__':  
    f, cx, cy = 1000., 320., 240.  
    pts0 = np.loadtxt('../data/image_formation0.xyz')[:, :2]  
    pts1 = np.loadtxt('../data/image_formation1.xyz')[:, :2]  
    output_file = 'triangulation_implement.xyz'  
  
    # Estimate relative pose of two view  
    F, _ = cv.findFundamentalMat(pts0, pts1, cv.FM_8POINT)  
    K = np.array([[f, 0, cx], [0, f, cy], [0, 0, 1]])  
    E = K.T @ F @ K  
    _, R, t, _ = cv.recoverPose(E, pts0, pts1)  
  
    # Reconstruct 3D points (triangulation)  
    P0 = K @ np.eye(3, 4, dtype=np.float32)  
    Rt = np.hstack((R, t))  
    P1 = K @ Rt  
    X = triangulatePoints(P0, P1, pts0.T, pts1.T)  
    X /= X[3]  
    X = X.T  
  
    # Write the reconstructed 3D points  
    np.savetxt(output_file, X)
```

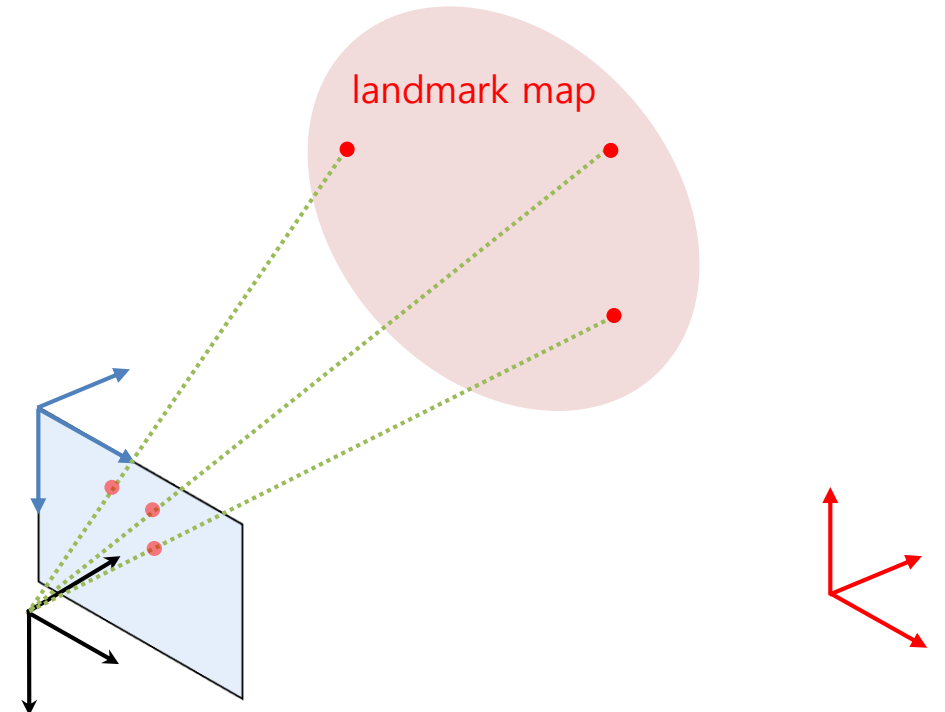


triangulation_implement.xyz

Solving Nonlinear Equations in 3D Vision

- **Absolute camera pose estimation** ([perspective-n-point](#); PnP)

- Unknown: **Camera pose R and t** (6 DOF)
- Given: 3D points $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$, their projected points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, and camera matrix K
- ~~Constraints: $n \times$ projection $\mathbf{x}_i = K[R|t]\mathbf{X}_i$~~
- More general constraints: $\mathbf{x}_i = \text{proj}(\mathbf{X}_i; R, t)$ where $\mathbf{x}_i = [x_i, y_i]^T$
 - Note) The projection proj generates 2D points on the image plane (not in homogeneous coordinates) considering nonlinear lens distortion.
- Solutions ($n \geq 3$) \rightarrow 3-point algorithm
 - OpenCV: `cv.solvePnP()` and `cv.solvePnPRansac()`
 - **Iterative 3-point algorithm** using local optimization
 - Cost function
$$\hat{R}, \hat{t} = \underset{R, t}{\operatorname{argmin}} \sum_{i=1}^n \|\mathbf{x}_i - \text{proj}(\mathbf{X}_i; R, t)\|_2^2$$
 - Optimizer: Gauss-Newton method



Solving Nonlinear Equations in 3D Vision

- **Absolute camera pose estimation** ([perspective-n-point](#); PnP) [pose_estimation_implement.py]

```
import numpy as np
from scipy.optimize import least_squares
from scipy.spatial.transform import Rotation
import cv2 as cv
```

```
def project_no_distort(X, rvec, t, K):
    R = Rotation.from_rotvec(rvec.flatten()).as_matrix()
    XT = X @ R.T + t # Transpose of 'X = R @ X + t'
    xT = XT @ K.T # Transpose of 'x = KX'
    xT = xT / xT[:, -1].reshape((-1, 1)) # Normalize
    return xT[:, 0:2]
```

```
def reproject_error_pnp(unknown, X, x, K):
    rvec, tvec = unknown[:3], unknown[3:]
    xp = project_no_distort(X, rvec, tvec, K)
    err = x - xp
    return err.ravel()
```

```
def solvePnP(obj_pts, img_pts, K):
    unknown_init = np.array([0, 0, 0, 0, 0, 1.]) # Sequence: rvec(3), tvec(3)
    result = least_squares(reproject_error_pnp, unknown_init, args=(obj_pts, img_pts, K))
    return result['success'], result['x'][:3], result['x'][3:]
```


Solving Nonlinear Equations in 3D Vision

▪ Camera calibration

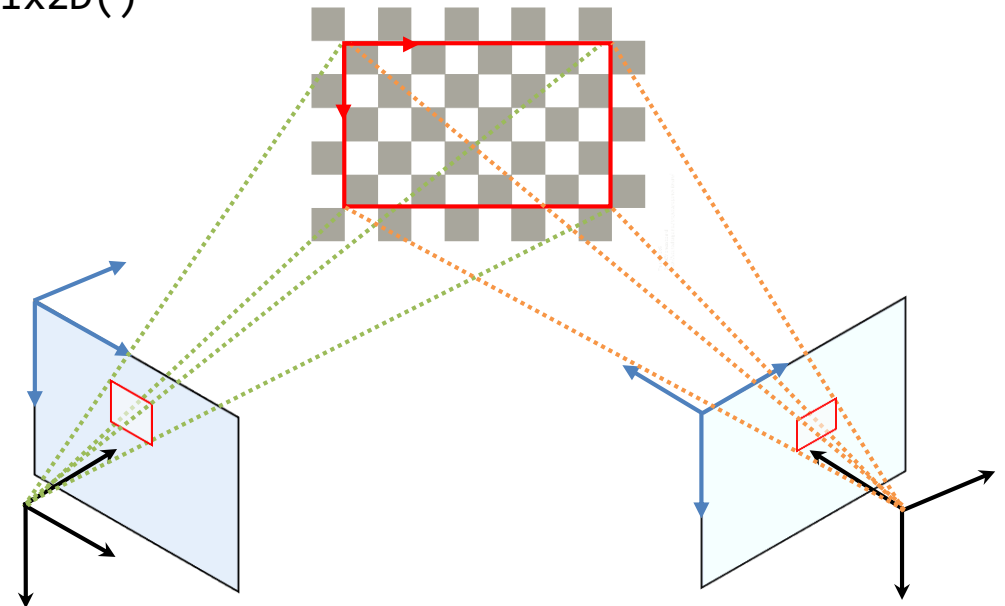
- Unknown: **Intrinsic + m x extrinsic parameters** ($3^* + m \times 6$ DOF)
- Given: 3D points $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ and their projected points, \mathbf{x}_i^j , on the j -th image
 - Note) m : the number of images, n : the number of 3D points
- ~~Constraints: $m \times n \times$ projection $\mathbf{x}_i^j = \mathbf{K} [\mathbf{R}_j | \mathbf{t}_j] \mathbf{X}_i$~~
- More general constraints: $\mathbf{x}_i^j = \text{proj}(\mathbf{X}_i; \mathbf{K}_j, \mathbf{R}_j, \mathbf{t}_j)$ where $\mathbf{x}_i = [x_i, y_i]^\top$
- Solutions [\[Tools\]](#)

- OpenCV: `cv.calibrateCamera()` and `cv.initCameraMatrix2D()`
- **Camera calibration** using local optimization

- Cost function

$$\hat{\mathbf{K}}, \hat{\mathbf{R}}_{\dots}, \hat{\mathbf{t}}_{\dots} = \underset{\mathbf{K}, \mathbf{R}_{\dots}, \mathbf{t}_{\dots}}{\operatorname{argmin}} \sum_{j=1}^m \sum_{i=1}^n \left\| \mathbf{x}_i^j - \text{proj}(\mathbf{X}_i; \mathbf{K}, \mathbf{R}_j, \mathbf{t}_j) \right\|_2^2$$

- Optimizer: Gauss-Newton method



- **Camera calibration** [camera_calibration_implement.py]

```
import numpy as np
```

```
...
```

```
def fxcycy_to_K(f, cx, cy):  
    return np.array([[f, 0, cx], [0, f, cy], [0, 0, 1]])
```

```
def reproject_error_calib(unknown, Xs, xs):  
    K = fxcycy_to_K(*unknown[0:3])  
    err = []  
    for j in range(len(xs)):  
        offset = 3 + 6 * j  
        rvec, tvec = unknown[offset:offset+3], unknown[offset+3:offset+6]  
        xp = project_no_distort(Xs[j], rvec, tvec, K)  
        err.append(xs[j] - xp)  
    return np.vstack(err).ravel()
```

```
def calibrateCamera(obj_pts, img_pts, img_size):  
    img_n = len(img_pts)  
    unknown_init = np.array([img_size[0], img_size[0]/2, img_size[1]/2] \  
                             + img_n * [0, 0, 0, 0, 0, 1.]) # Sequence: f, cx, cy, img_n * (rvec, tvec)  
    result = least_squares(reproject_error_calib, unknown_init, args=(obj_pts, img_pts))  
    K = fxcycy_to_K(*result['x'][0:3])  
    rvecs = [result['x'][(6*i+3):(6*i+6)] for i in range(img_n)]  
    tvecs = [result['x'][(6*i+6):(6*i+9)] for i in range(img_n)]  
    return result['cost'], K, np.zeros(5), rvecs, tvecs
```

Summary (3D Vision Problems)

▪ Linear equations

- Inhomogeneous equations $A\mathbf{x} = \mathbf{b} \rightarrow \mathbf{x} = A^\dagger \mathbf{b}$ where A^\dagger is a pseudo-inverse
 - Formulation) $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{Ax} - \mathbf{b}\|_2^2$
 - Example) Affine transformation estimation
- Homogeneous equation $A\mathbf{x} = \mathbf{0} \rightarrow \mathbf{x}$ is the last row of V^T where $A = USV^T$ (from singular value decomposition)
 - Formulation) $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{Ax}\|^2$ with $\|\mathbf{x}\|^2 = 1$ condition
 - Example) Planar homography estimation
 - Example) Fundamental matrix estimation
 - Example) Triangulation

▪ Nonlinear equations

- Nonlinear optimization: Magic tools such as [scipy.optimize](#) and [Ceres Solver](#)
 - Formulation) $\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} f(\mathbf{x})$
 - Example) Absolute camera pose estimation (PnP)
 - Example) Camera calibration