Note) Slides and example codes are available: https://github.com/mint-lab/3dv_tutorial

An Invitation to 3D Vision: Finding Correspondence

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Table of Contents: Finding Correspondence

- **Feature Points**
  - Q) How can we detect salient points (or parts)?
- **Feature Descriptors**
  - Q) How can we distinguish the points each other?
- **Feature Matching**
  - Q) How can we associate the points across different images?
- **Feature Tracking**
  - Q) How can we associate the points across their next image?
- **Outlier Rejection**
  - Q) How can we select correctly matched points?

Image: pixabay

Getting Started with a Quiz

- **Q) What is it?**

600 pixels

Getting Started with a Quiz

- **Q) What is it?**

600 pixels

Getting Started with a Quiz

- **Q) What is it? “Duck”**

600 pixels

1095 pixels

600 pixels

- **Q) Why corners (junction) instead of edges or blobs?**
  - Human visual systems understand objects better from corners than edges.
  - a.k.a. feature points, keypoints, salient points, and interest points

Getting Started with a Quiz

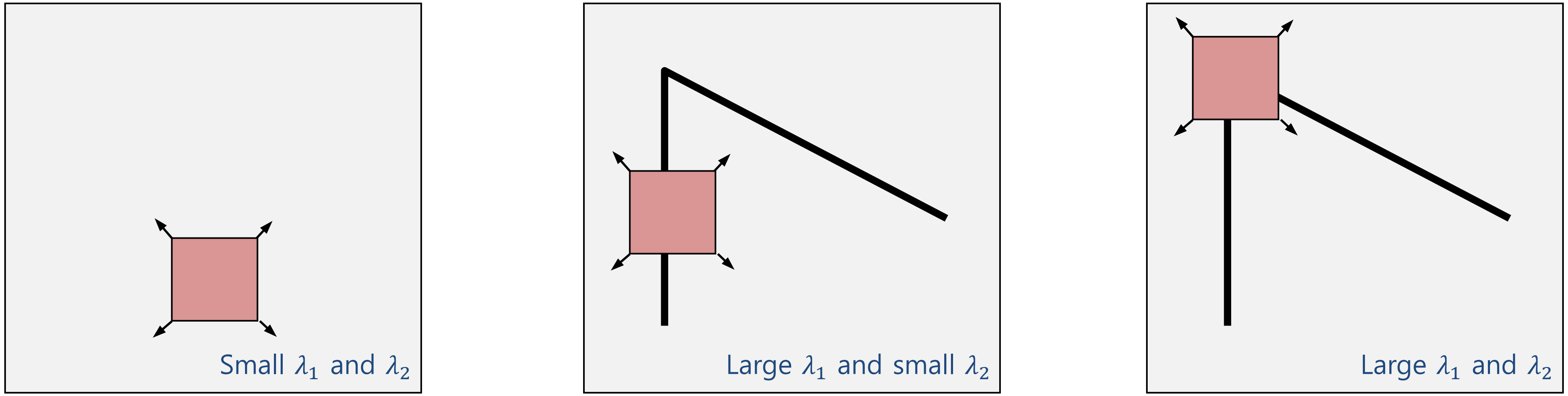
- **Q) Where is it?**

- **Q) What is a good feature? (requirements of visual features**
  - **Repeatability** (invariance/robustness to transformation and noise)
  - **Distinctiveness** (easy to distinguish or match)
  - **Locality** (due to occlusion)
  - Quantity, accuracy (localization), efficiency (computing time), …

Image: [OpenCV Tutorials]

Feature Point) **Harris Corner (1988)**

- Key idea: **Sliding window**



"flat" region:
no change
in all directions

"edge":
no change
along the edge direction

"corner":
significant change
in all directions

– Formulation

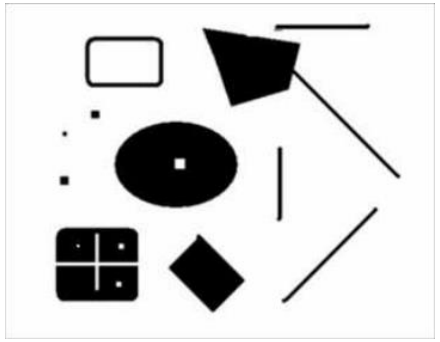
- $I(x+\Delta_x, y+\Delta_y) \approx I(x,y) + [I_x(x,y) \quad I_y(x,y)]\begin{bmatrix}\Delta_x \\ \Delta_y\end{bmatrix}$ where $I_x = \frac{\partial I}{\partial x}$ and $I_y = \frac{\partial I}{\partial y}$
- $D(\Delta_x, \Delta_y) = \sum_{(x,y)\in W}\left(I(x+\Delta_x, y+\Delta_y) - I(x,y)\right)^2 \approx [\Delta_x \quad \Delta_y]\underbrace{\begin{bmatrix}\sum_W I_x^2 & \sum_W I_x I_y \\ \sum_W I_x I_y & \sum_W I_y^2\end{bmatrix}}_{\mathrm{M}}\begin{bmatrix}\Delta_x \\ \Delta_y\end{bmatrix}$

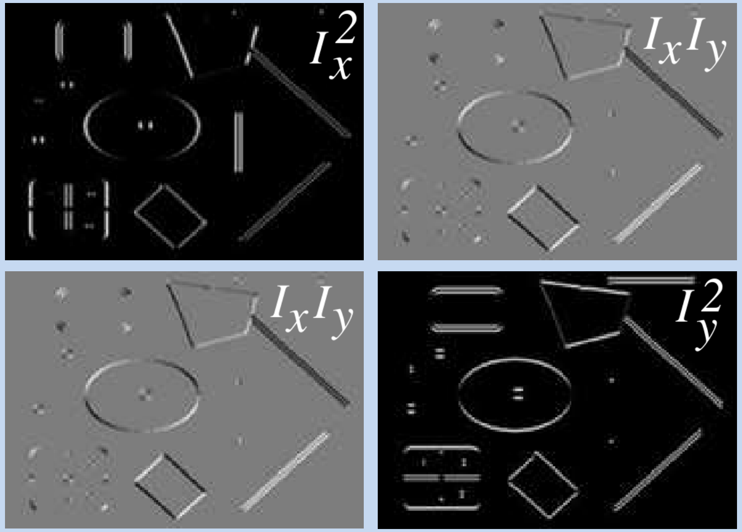
Image: Frolova and Simakov, **Matching with Invariant Features** (Lecture Slides), 2004

Feature Point) Harris Corner (1988)

- Key idea: **Sliding window**
  - Formulation
    - $I(x+\Delta_x, y+\Delta_y) \approx I(x,y) + [I_x(x,y) \quad I_y(x,y)] \begin{bmatrix} \Delta_x \\ \Delta_y \end{bmatrix}$ where $I_x = \frac{\partial I}{\partial x}$ and $I_y = \frac{\partial I}{\partial y}$
    - $D(\Delta_x, \Delta_y) = \sum_{(x,y)\in W} \left(I(x+\Delta_x, y+\Delta_y) - I(x,y)\right)^2 \approx [\Delta_x \quad \Delta_y] \underbrace{\begin{bmatrix} \sum_W I_x^2 & \sum_W I_x I_y \\ \sum_W I_x I_y & \sum_W I_y^2 \end{bmatrix}}_{\mathrm{M}} \begin{bmatrix} \Delta_x \\ \Delta_y \end{bmatrix}$
  - **Harris corner response**
    - $\text{cornerness} = \det(\mathrm{M}) - k\ \mathrm{trace}(\mathrm{M})^2$
    - Note) $\det(\mathrm{M}) = \lambda_1 \lambda_2$, $\mathrm{trace}(\mathrm{M}) = \lambda_1 + \lambda_2$, and $k \in [0.04, 0.06]$
  - Note) **Good-Feature-to-Track** (a.k.a. GFTT or Shi-Tomasi corner; 1994)
    - $\text{cornerness} = \min(\lambda_1, \lambda_2)$

$I(x,y)$





M

$\det(\mathrm{M}) - k\ \mathrm{trace}(\mathrm{M})^2$

Image: Tuytelaars et al., **Local Invariant Feature Detectors: A Survey**, Foundations and Trends in Computer Graphics and Vision, 2008

Feature Point) Harris Corner (1988)

- Properties
  - Invariant to translation, rotation, and intensity shift ($I \rightarrow I + b$) ~~intensity scaling ($I \rightarrow aI$)~~
  - Variant to **image scaling**

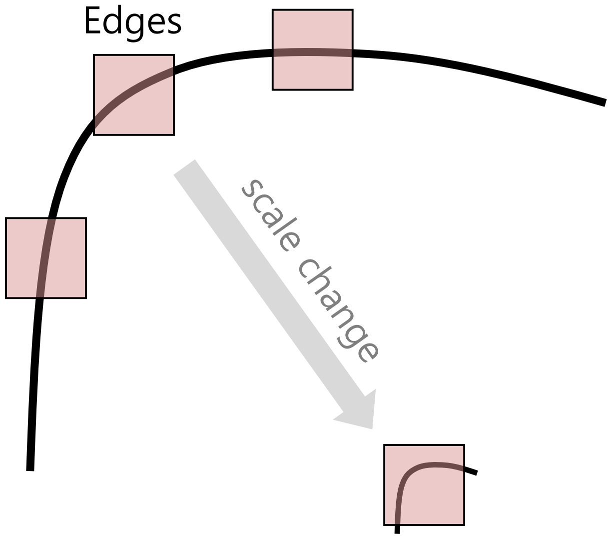
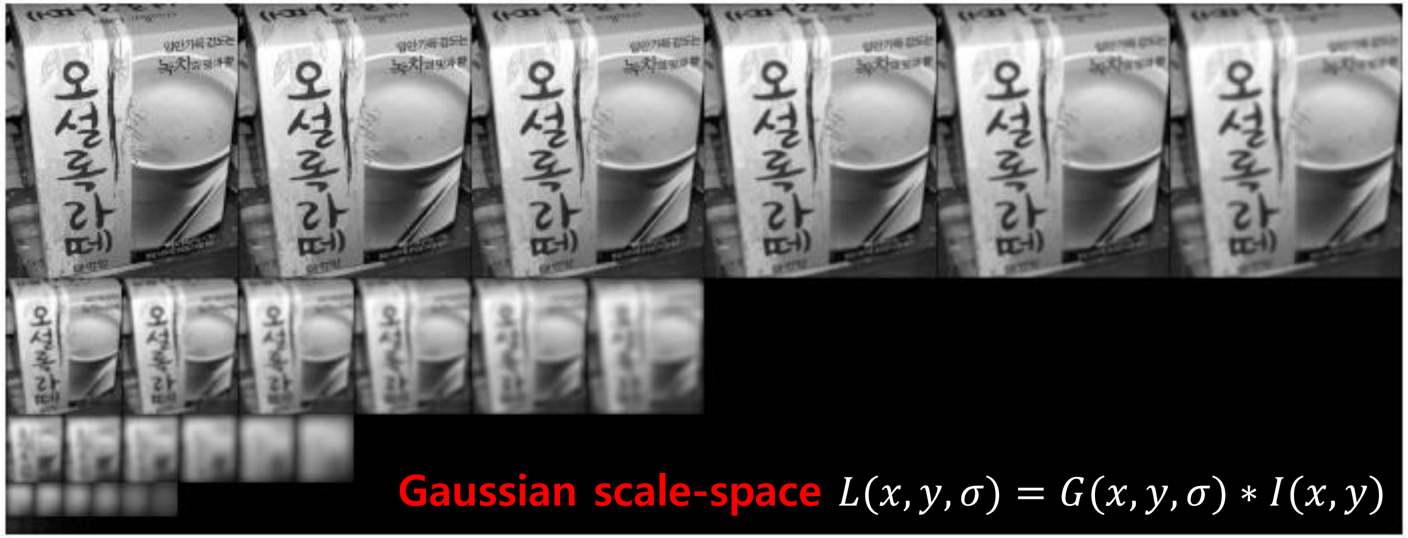
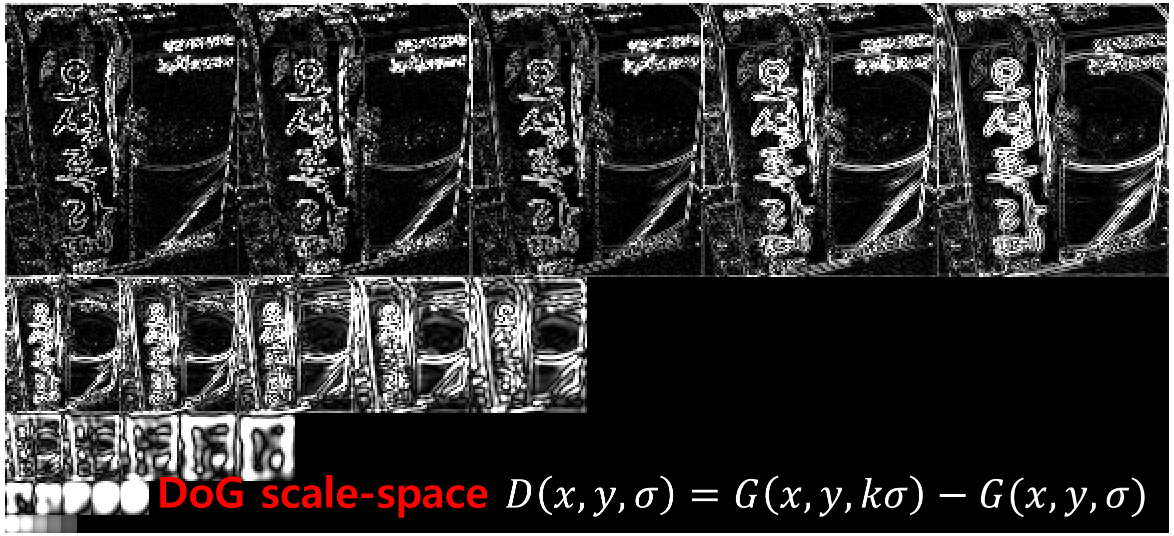


Image: Frolova and Simakov, **Matching with Invariant Features** (Lecture Slides), 2004

Feature Point) SIFT (Scale-Invariant Feature Transform; 1999)

- Key idea: **Scale-space** (~ image pyramid) **and DoG** (difference of Gaussian)



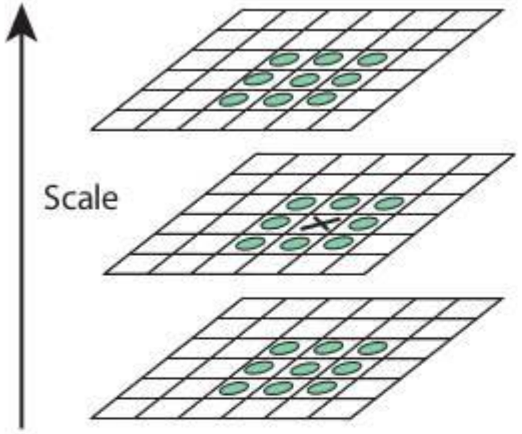


Note) The patent for SIFT feature and descriptors was expired at March 6, 2020.

Feature Point) SIFT (Scale-Invariant Feature Transform; 1999)

- Key idea: **Scale-space** (~ image pyramid) **and DoG** (difference of Gaussian)
- Part #1) **Feature point detection**
  1. Find **local extrema** (minima and maxima) in DoG scale-space
  2. Localize their position accurately (sub-pixel level) using 3D quadratic function
  3. Eliminate **low contrast candidates**, $|D(\mathbf{x})| < \tau$
  4. Eliminate **candidates on edges**,

$$\frac{\mathrm{trace}(H)^2}{\det(H)} < \frac{(r+1)^2}{r} \quad \text{where} \quad H = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{xy} & D_{yy} \end{bmatrix}$$



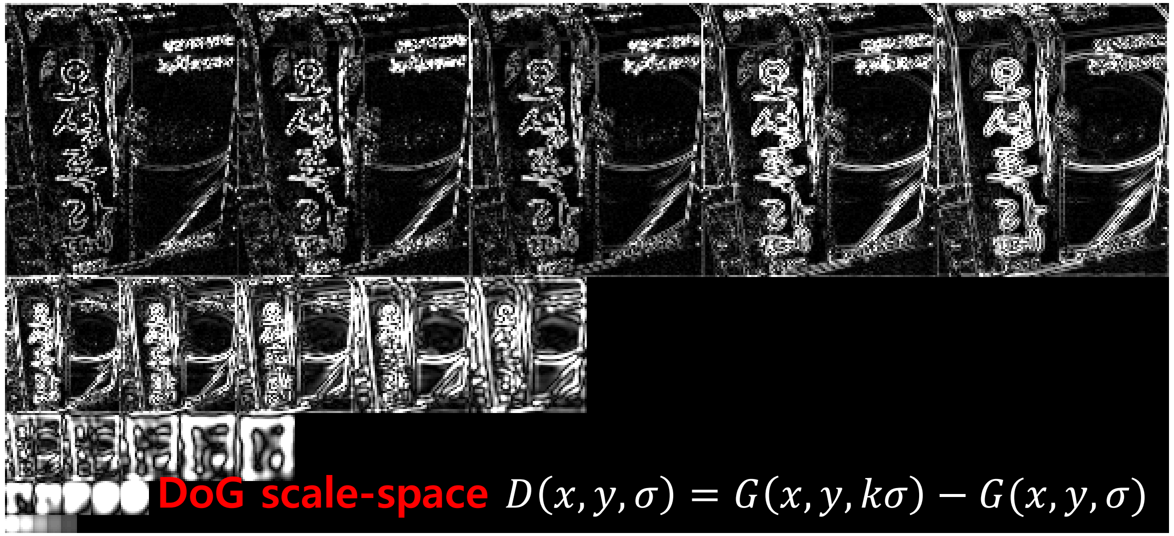


Image: David Lowe, **Distinctive Image Features from Scale-invariant Keypoints**, IJCV, 2004

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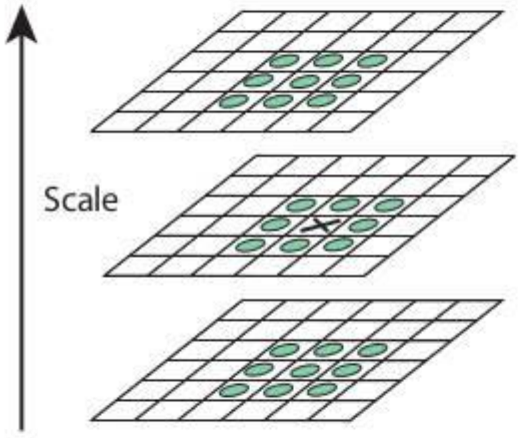
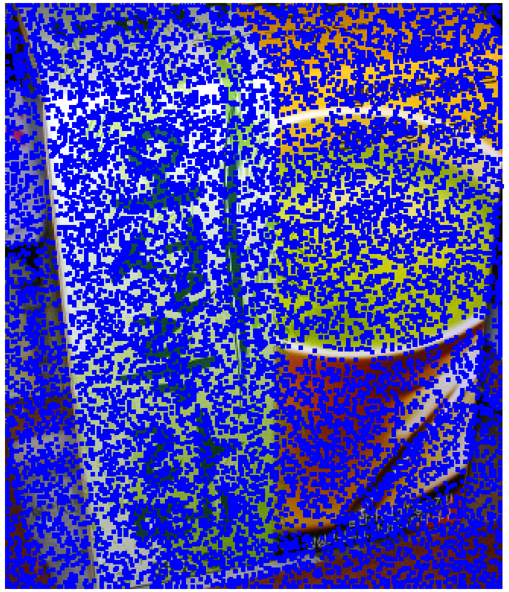


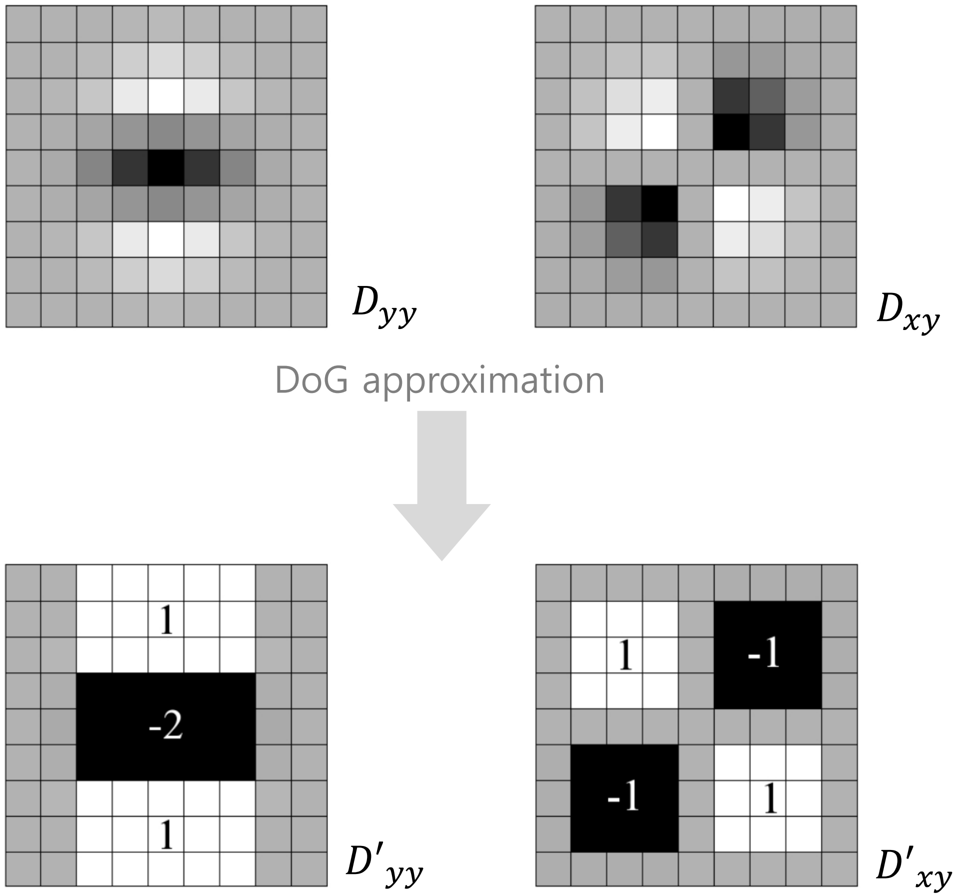


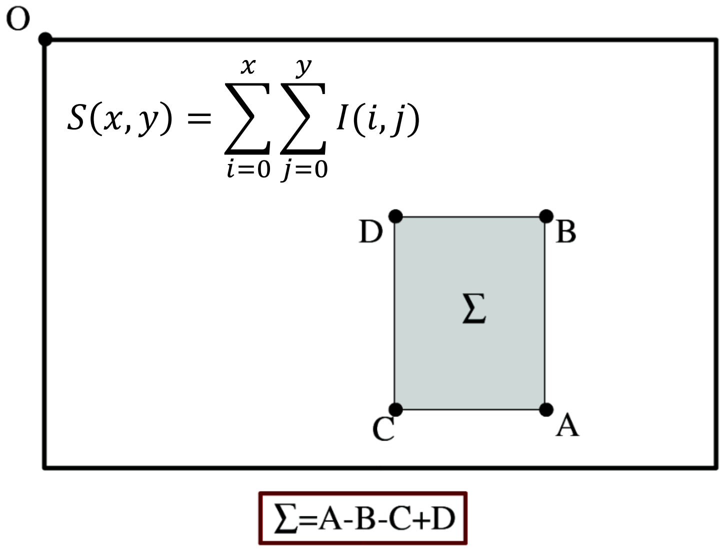
Local extrema (N: 11479)

Feature points (N: 971)

Feature Point) SURF (Speeded Up Robust Features; 2006)

- Key idea: **Approximation of SIFT**
  - e.g. DoG approximation Haar-like features and **integral image**





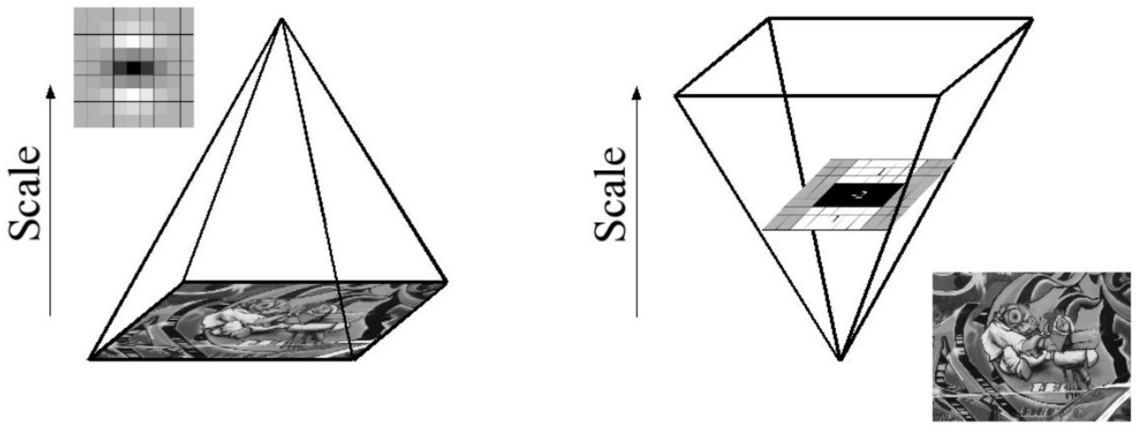


Image: Bay et. al., **SURF: Speeded Up Robust Features**, CVIU, 2008

Feature Descriptor) **SIFT (Scale-Invariant Feature Transform; 1999)**

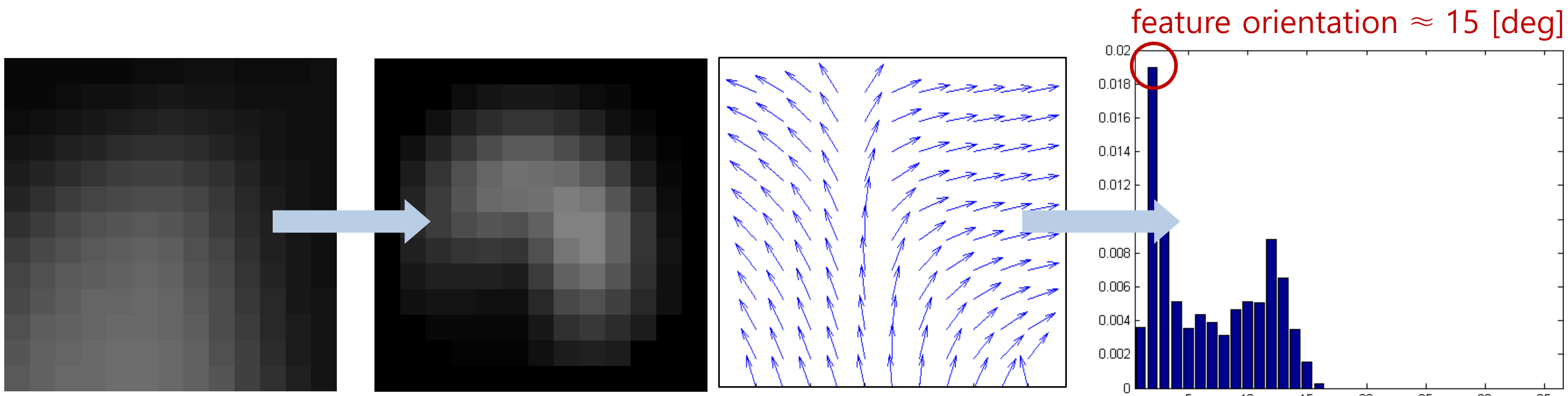
Feature scales and orientations

- Part #2) **Orientation assignment**
  1. Derive magnitude and orientation of gradient of each patch

$$m(x,y) = \sqrt{\left(L(x+1,y) - L(x-1,y)\right)^2 + \left(L(x,y+1) - L(x,y-1)\right)^2}$$

$$\theta(x,y) = \tan^{-1}\frac{L(x,y+1) - L(x,y-1)}{L(x+1,y) - L(x-1,y)}$$

  2. Find the strongest orientation
     - Histogram voting (36 bins) with Gaussian-weighted magnitude



feature patch | gradient magnitude $m(x,y)$ | gradient orientation $\theta(x,y)$ | histogram voting (36 bins)

Feature Descriptor) SIFT (Scale-Invariant Feature Transform; 1999)

- Part #3) **Feature descriptor extraction**
  1. Build a 4x4 gradient histogram (8 bins) from each patch (16x16 pixels)
     - Use Gaussian-weighted magnitude again
     - Use relative angles w.r.t. the assigned feature orientation
  2. Encode the histogram into a 128-dimensional vector
     - Apply normalization to be an unit vector


Feature scales and orientations

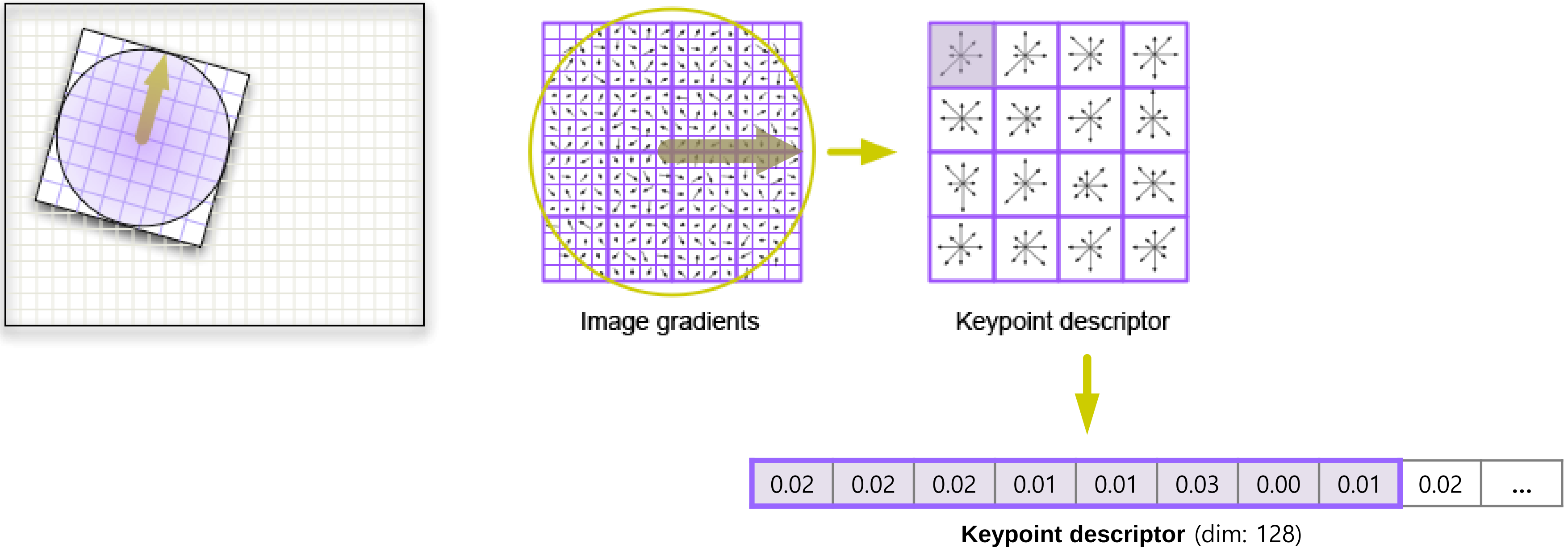
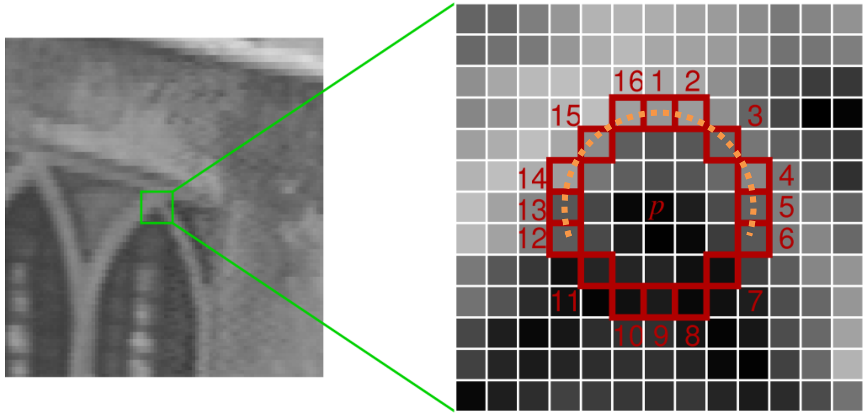


Image: Jonas Hurrelmann, ~~Lecture Note~~ (not available now), 2006

Feature Point) FAST (Features from Accelerated Segment Test; 2006)

- Key idea: **Intensity check of continuous arc of n pixels**
  - Is this point p a corner? (I_p: intensity at p, t: the intensity threshold)
    - Is a segment of n continuous pixels brighter than $I_p + t$? (OR) Is the segment darker than $I_p - t$?
    - Note) High-speed non-corner rejection: Checking intensity values at 1, 9, 5, and 13 (n: 12)



  - Too many corners! → Non-maximum suppression
- Versions
  - FAST-9 (n: 9; `cv.FastFeatureDetector_TYPE_9_16`), FAST-12 (n: 12), …
  - FAST-**ER**: Training a decision tree to <u>e</u>nhance <u>r</u>epeatability with more pixels

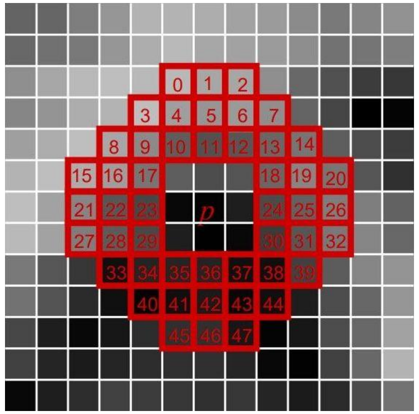
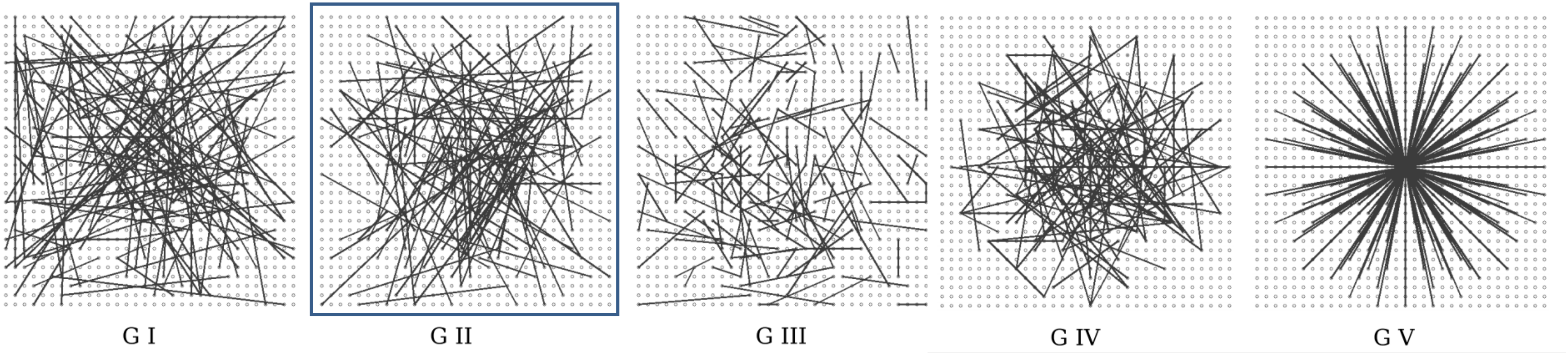


Image: Rosten et al., **FASTER and Better: A Machine Learning Approach to Corner Detection**, T-PAMI, 2010

Feature Descriptor) BRIEF (Binary Robust Independent Elementary Features; 2010)

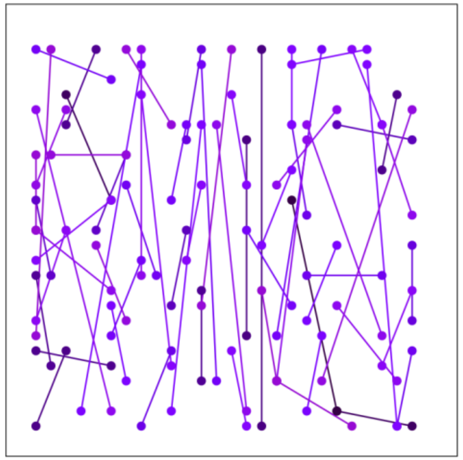
- Key idea: **Intensity comparison of a sequence of random pairs (binary test)**
  - Path size: 31 x 31 pixels (Note: Applying smoothing for stability and repeatability)



  - Descriptor size: 128 tests (128 bits) → 16 bytes
    - Note) SIFT: 128-dimensional vector → 512 bytes
- Versions: The number of tests
  - BRIEF-32, BRIEF-64, BRIEF-128 (16 bytes), BRIEF-256 (32 bytes), BRIEF-512 (64 bytes), …
- Combination examples
  - SIFT feature points + BRIEF descriptors
  - FAST feature points + BRIEF descriptors

Image: Calonder et. al., **BRIEF: Computing a Local Binary Descriptor Very Fast**, T-PAMI, 2012

Feature Point and Descriptor) **ORB (Oriented FAST and rotated BRIEF, 2011)**



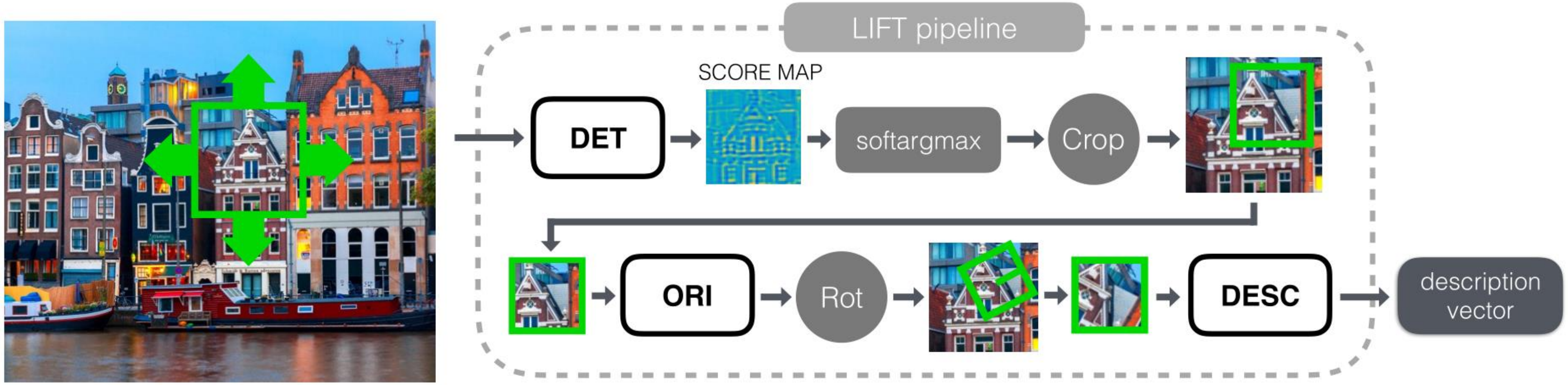
- Key idea: **Adding rotation invariance to BRIEF**
  - **Oriented FAST**
    - Generate scale pyramid for scale invariance
    - Detect *FAST-9* points (filtering with Harris corner response)
    - Calculate feature orientation by *intensity centroid* $C = (\frac{m_{10}}{m_{00}}, \frac{m_{01}}{m_{00}})$

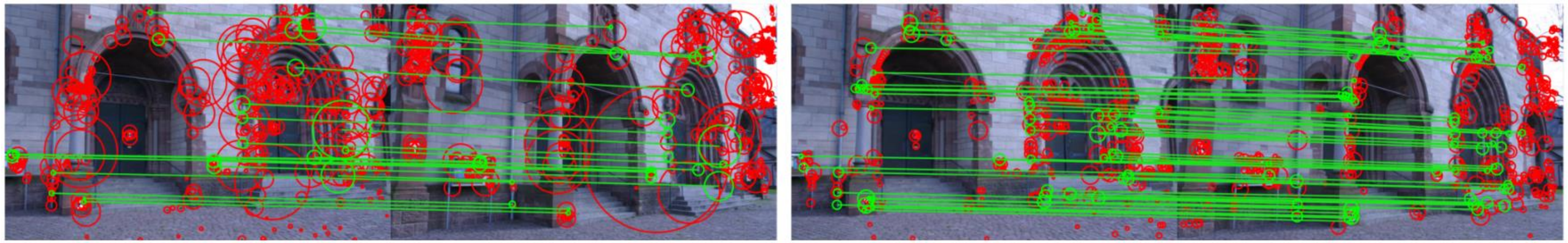
      $\theta = \tan^{-1}\frac{m_{01}}{m_{10}}$ where $m_{pq} = \sum_{x,y} x^p y^q I(x,y)$
  - **Rotation-aware BRIEF**
    - Extract BRIEF descriptors w.r.t. the known orientation
    - Use better comparison pairs trained by greedy search
- Combination (default): **ORB**
  - FAST-9 detector (with orientation) + BRIEF-256 descriptor (with trained pairs)
- Computing time
  - ORB: **15.3 [msec]** / SURF: 217.3 [msec] / SIFT: 5228.7 [msec] @ 24 images (640x480) in Pascal dataset

Image: Rublee et. al., **ORB: An Efficient Alternative to SIFT or SURF**, ICCV, 2011

Feature Point and Descriptor) LIFT (Learned Invariant Feature Transform; 2016)

- Key idea: **Deep neural network**
  - CNN network: **DET** (feature detector) + **ORI** (orientation estimator) + **DESC** (feature descriptor)
  - Training data: Photo Tourism dataset with VisualSFM (SIFT)





SIFT LIFT

Image: Yi et. al., **LIFT: Learned Invariant Feature Transform**, ECCV, 2016 arXiv Github

Feature Point and Descriptor) SuperPoint (2020)

- Key idea: **Self-supervised training with homography transformation**
  - CNN network: Encoder (~ VGG) + Decoders (for point and descriptor)

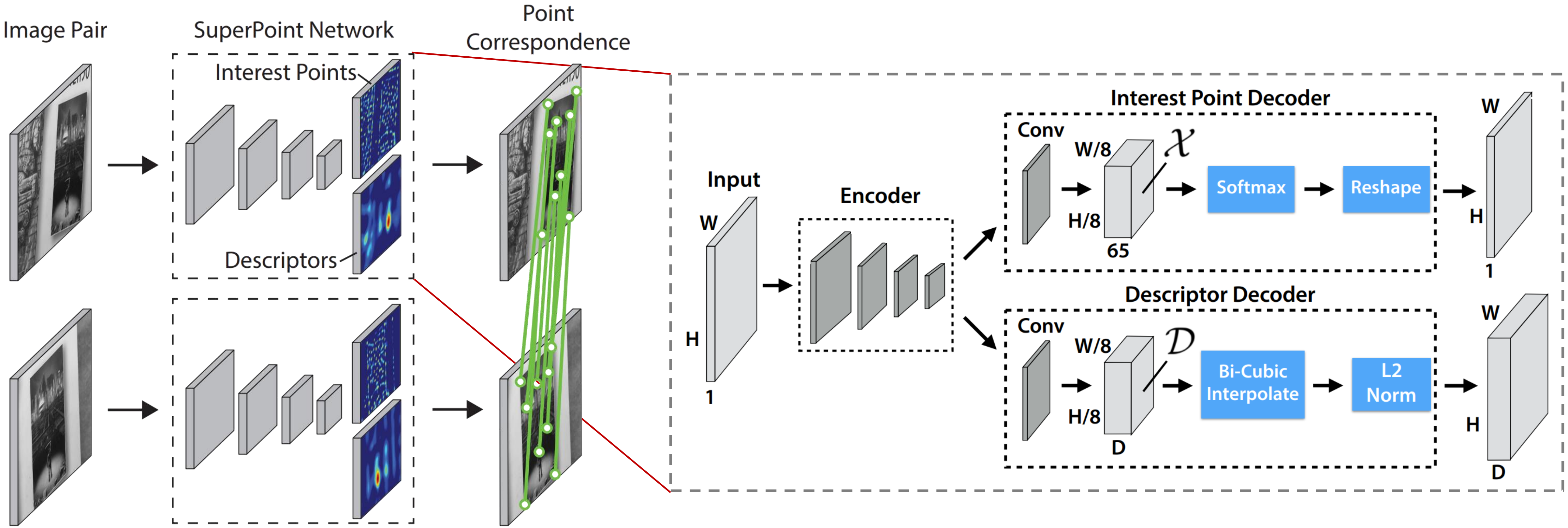
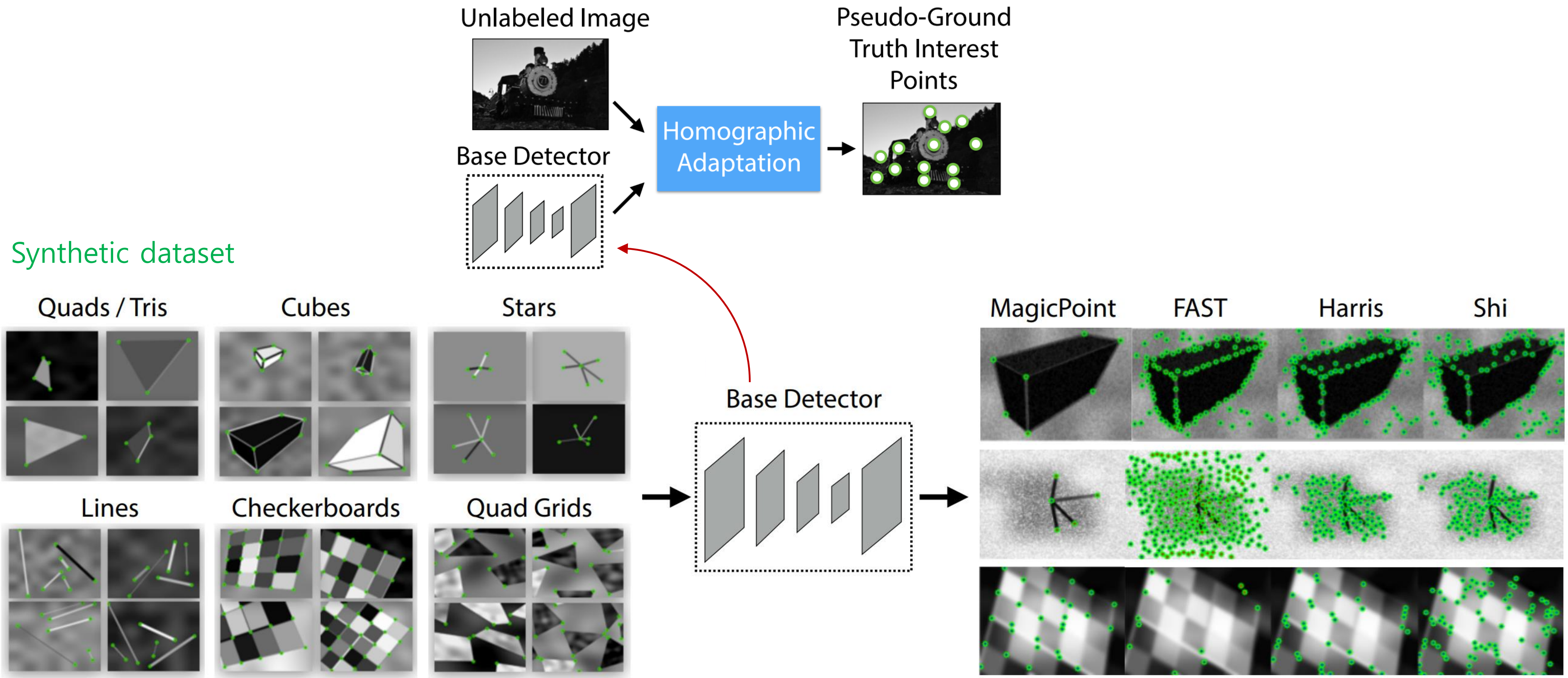


Image: DeTone et. al., **SuperPoint: Self-Supervised Interest Point Detection and Description**, CVPR, 2020 arXiv Github

Feature Point and Descriptor) SuperPoint (2020)

- Key idea: **Self-supervised training with homography transformation**
  - Training data generation: The base detector, *MagicPoint* → Ground truth interest points



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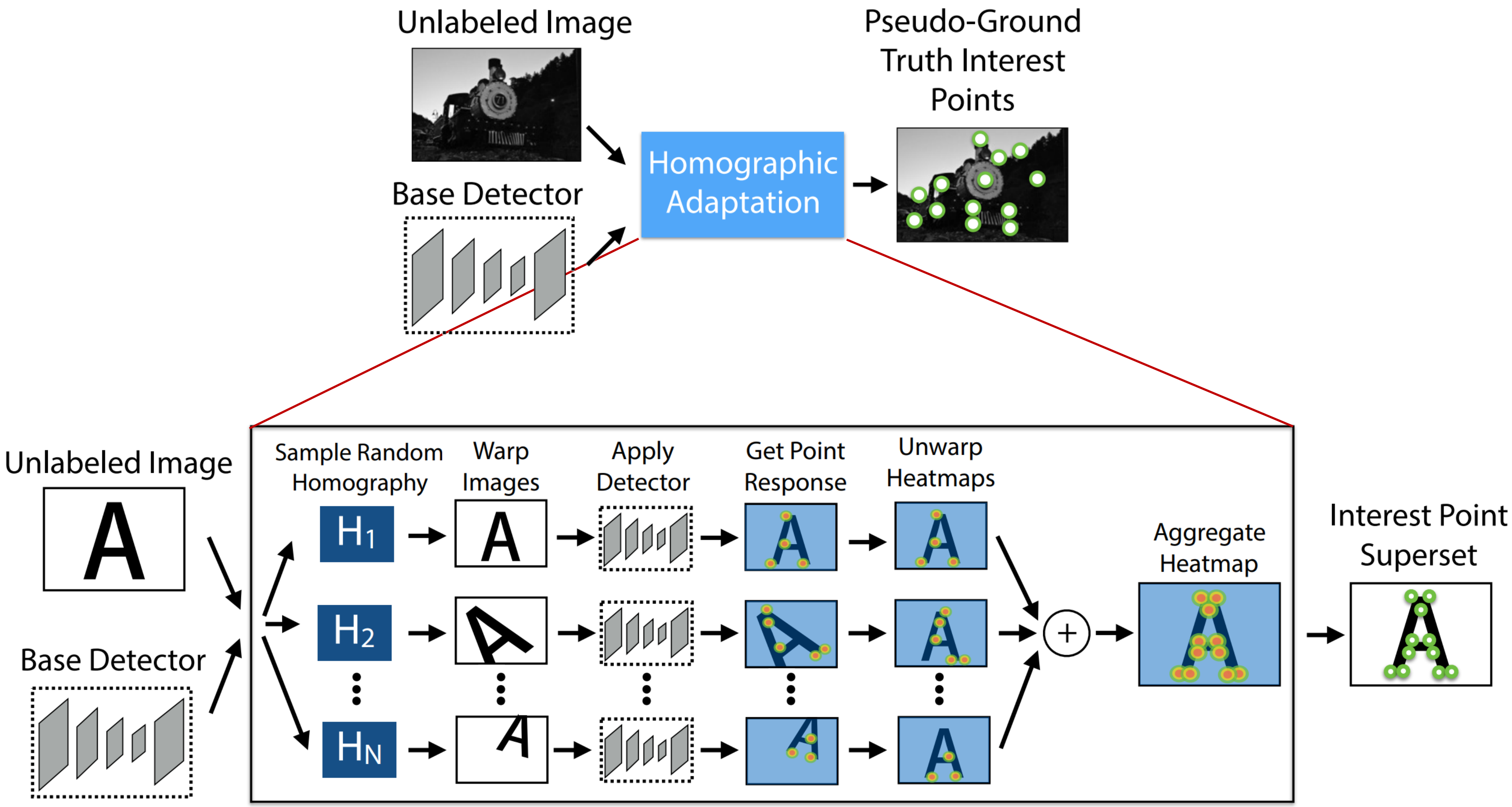


Image: DeTone et. al., **SuperPoint: Self-Supervised Interest Point Detection and Description**, CVPR, 2020 arXiv Github

Feature Point and Descriptor) SuperPoint (2020)

- Key idea: **Self-supervised training with homography transformation**
  - Training data augmentation: Random homography transformation
  - Loss functions: Interest Point Loss (X, Y) + Interest Point Loss (X’, Y’) + Descriptor Loss (Y, Y’)

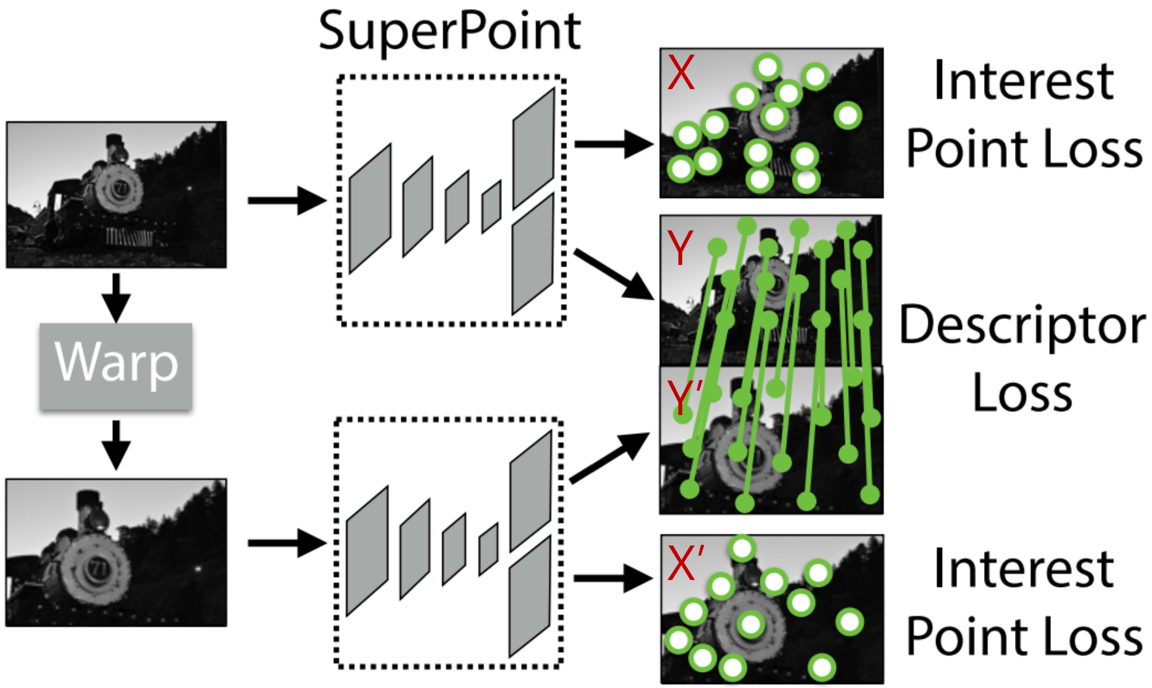


Image: DeTone et. al., **SuperPoint: Self-Supervised Interest Point Detection and Description**, CVPR, 2020 arXiv Github

Feature Point and Descriptor) SuperPoint (2020)

- Key idea: **Self-supervised training with homography transformation**
  - Real-time performance: **70 FPS** (13 msec) on 480 x 640 images with NVIDIA Titan X GPU

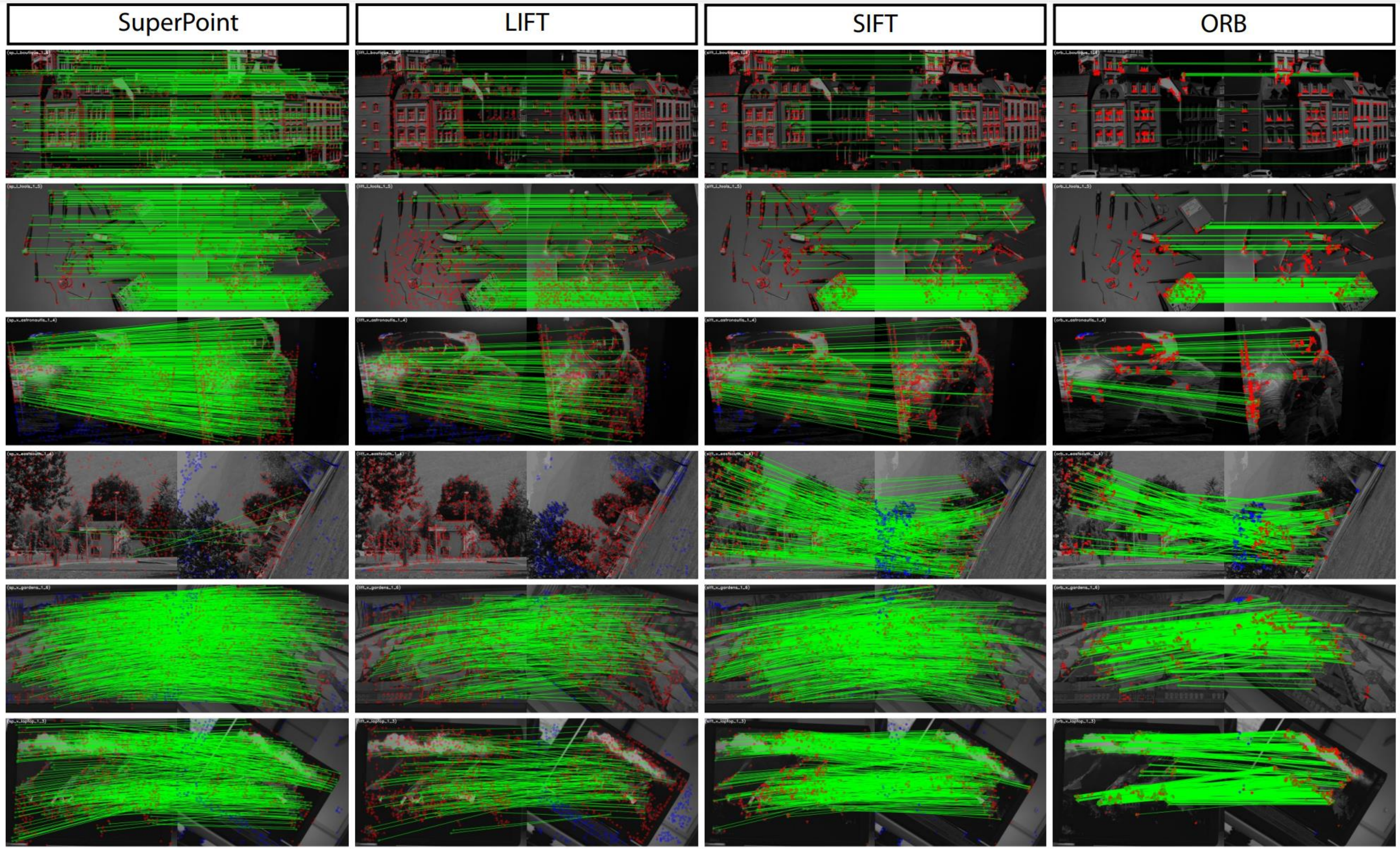


Image: DeTo ... hub

Summary) Feature Points and Descriptors

<table>
<tr><td>Feature Points</td><td>Gradient-based
▪ Harris
▪ GFTT (a.k.a. Shi-Tomasi)
▪ SIFT
▪ SURF</td><td>Intensity-based
▪ FAST</td><td>DL-based
▪ LIFT
▪ SuperPoint</td></tr>
<tr><td>Feature Descriptor</td><td>Real-valued
▪ SIFT
▪ SURF</td><td>Binary-valued
▪ BRIEF
▪ ORB (FAST+BRIEF)</td><td>DL-based (Real-valued)
▪ LIFT
▪ SuperPoint</td></tr>
<tr><td>Advantages Disadvantages</td><td>(+) Accurate
(–) Slow</td><td>(+) Fast
(–) Inaccurate
(+) Less storage</td><td>(+) Accurate
(+) Fast
(–) GPU requirement</td></tr>
</table>

Feature Matching) For Real-valued Descriptors

- **Real-valued descriptors**
  - Distance measures
    - Euclidean distance: $l_2(\mathbf{d}, \mathbf{d}') = \|\mathbf{d} - \mathbf{d}'\|_2$
    - Cosine similarity: $s_c(\mathbf{d}, \mathbf{d}') = \frac{\mathbf{d} \cdot \mathbf{d}'}{\|\mathbf{d}\| \|\mathbf{d}'\|}$
    - Note) Matching measures can be combined or advanced.
      - e.g. The ratio of the best and second best similarity > threshold
        - It may select more distinguishable feature matching.
  - Matching algorithms
    - Brute-force search
      - Time complexity: $O(N)$ for N descriptors
    - Approximated nearest neighborhood search (ANN search)
      - Time complexity: $O(\log N)$ or less for N descriptors
      - Note) Big-ANN Competition (recent: NeurIPS 2023)

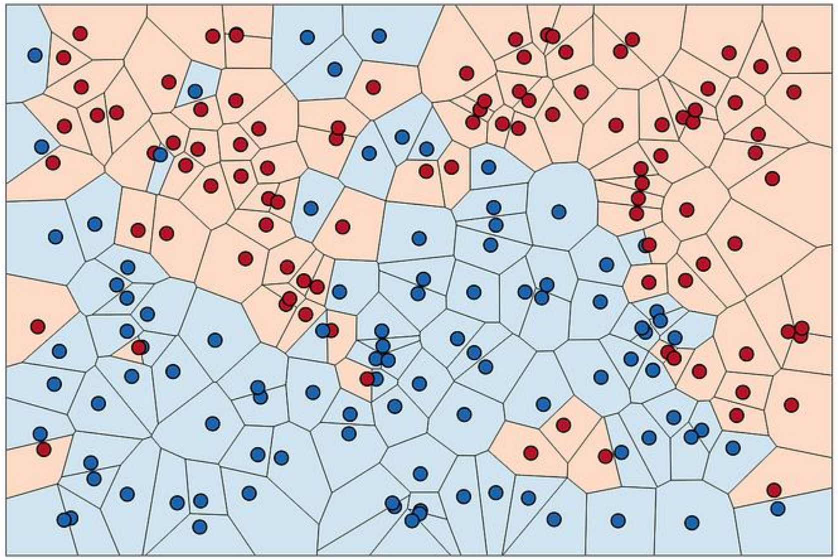


Image: George Williams, Medium

Feature Matching) For Binary-valued Descriptors and Image Patches

- **Binary-valued descriptors**
  - Distance measures
    - Hamming distance: $l_h(\mathbf{d}, \mathbf{d}') = \sum_i (d_i \neq d_i')$
      - e.g. (0110, 1110) = 1 vs. (6, 14)
      - e.g. (0110, 0111) = 1 vs. (6, 7)
      - e.g. (0110, 0101) = 2 vs. (6, 5)
      - Note) Hamming distance is the L_1-norm with binary-valued descriptors.
  - Matching algorithms
    - Brute-force search

- Note) **Raw image patches** can be used as descriptors.
  - Distance measure
    - SAD (sum of absolute difference): $l_1(\mathbf{d}, \mathbf{d}') = \|\mathbf{d} - \mathbf{d}'\|_1$
    - SSD (sum of squared difference): $l_2(\mathbf{d}, \mathbf{d}') = \|\mathbf{d} - \mathbf{d}'\|_2$
    - ZNCC (zero-mean normalized cross-correlation)

Feature Tracking) Lukas-Kanade Optical Flow (1981)

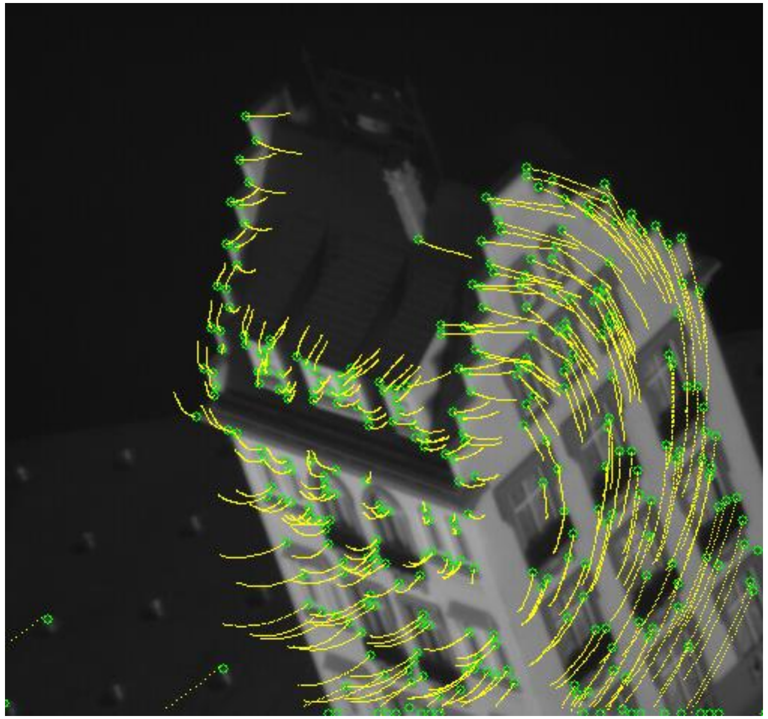
- Key idea: **Finding movement of a patch whose pixel values are same**
  - Brightness constancy constraint: $I(x, y, t) = I(x + \Delta_x, y + \Delta_y, t + \Delta_t)$

$$I_x \frac{\Delta_x}{\Delta_t} + I_y \frac{\Delta_y}{\Delta_t} + I_t = 0 \quad \leftarrow \quad I(x + \Delta_x, y + \Delta_y, t + \Delta_t) \approx I(x, y, t) + I_x \Delta_x + I_y \Delta_y + I_t \Delta_t$$

$\mathrm{A}\mathbf{x} = \mathbf{b}$ where $\mathrm{A} = [I_x \quad I_y]$, $\mathbf{x} = [\Delta_x, \ \Delta_y]^\intercal$, and $\mathbf{b} = [-I_t]$ $(\Delta_t = 1)$

$$\therefore \mathbf{x} = \mathrm{A}^\dagger \, \mathbf{b}$$

- Combination: **KLT tracker**
  - GFTT detector (a.k.a. Shi-**T**omasi) + **L**ukas-**K**anade optical flow

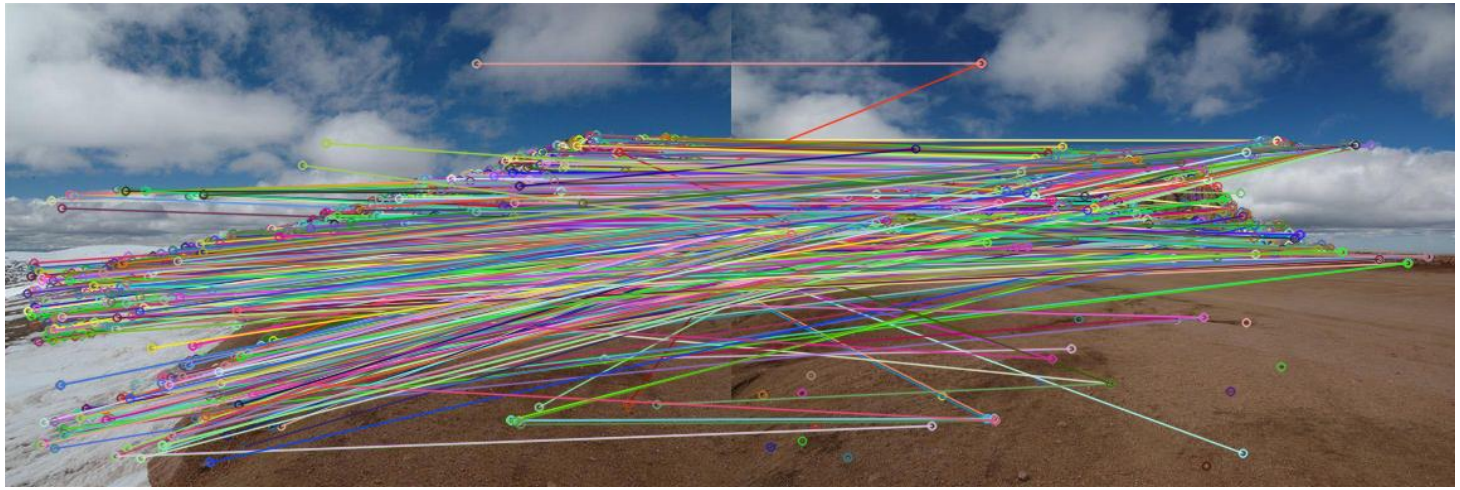


- Advantages and disadvantages (feature tracking vs. matching)
  - (+) No descriptor required (→ fast and compact)
  - (–) Continuous feature tracking causes drift errors.
  - (–) Not working in wide-baseline cases
  - (+) Able to control matching range

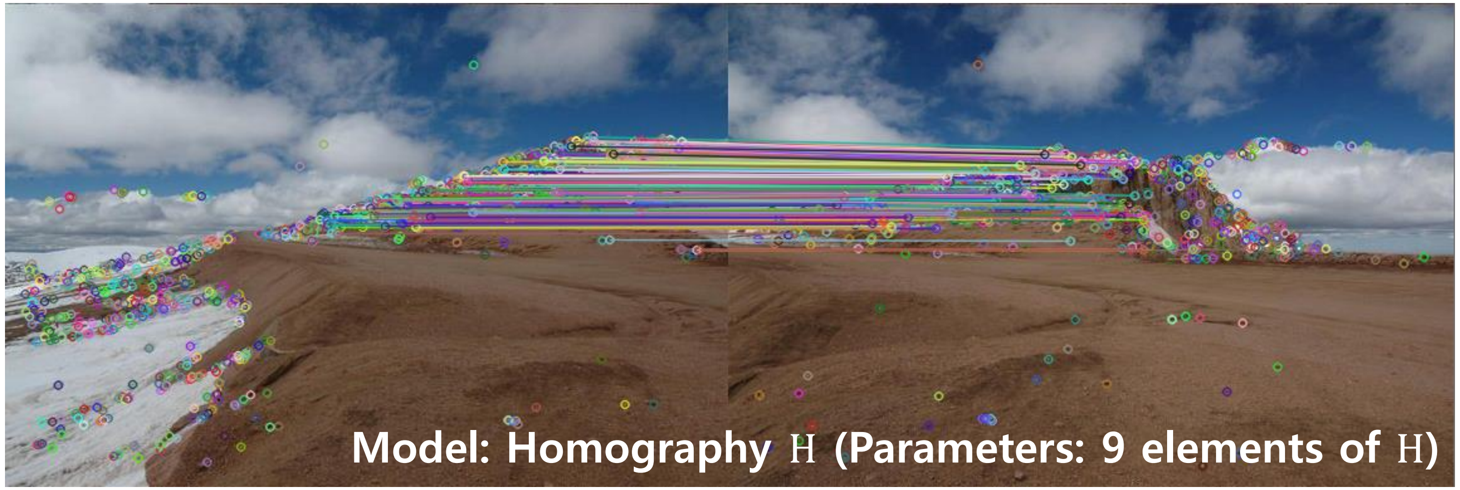
Image: Nirmal Amirthalingam, Github

Why Outliers?

Putative feature matches (inliers + outliers)

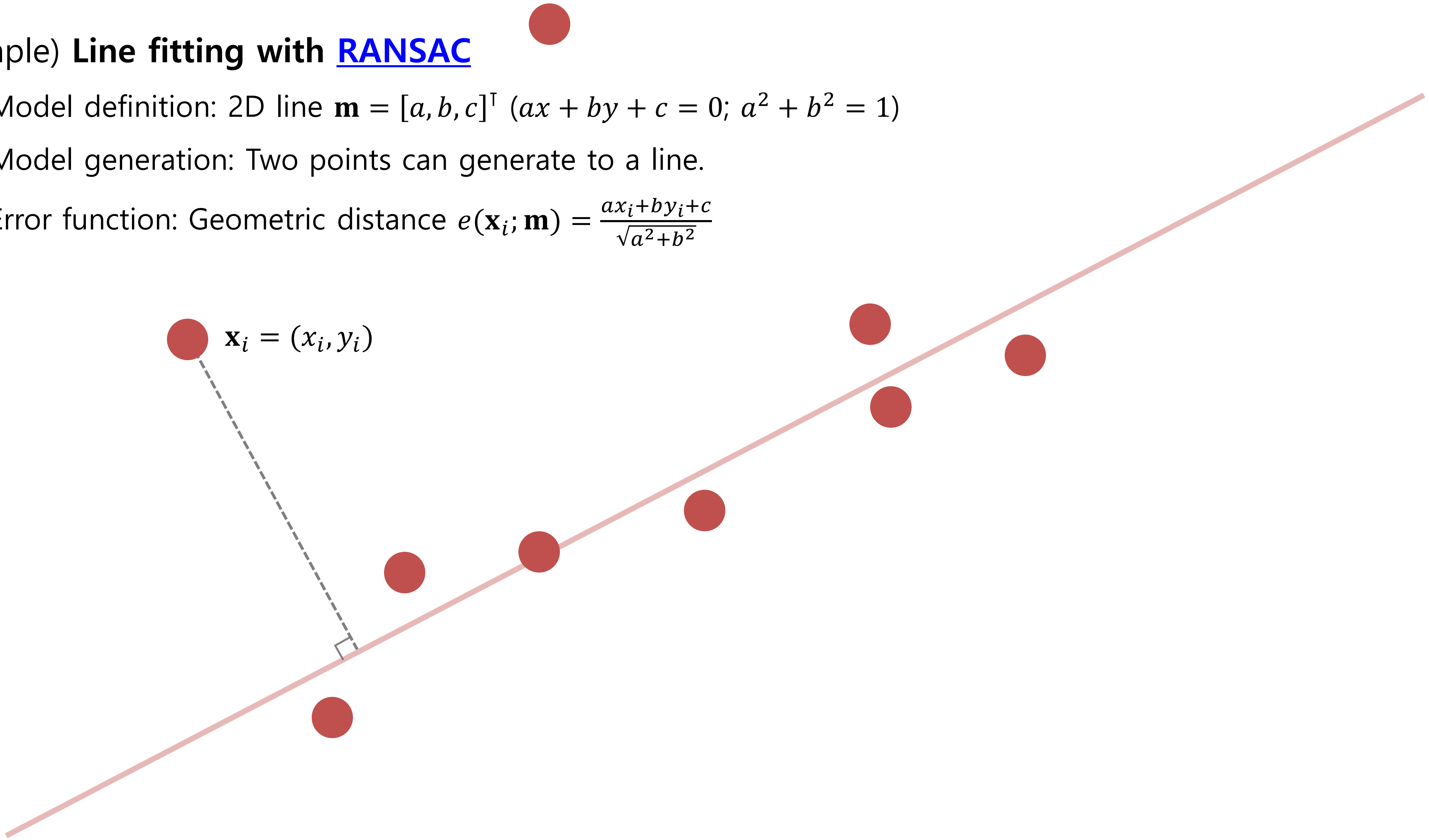


After applying RANSAC (inliers)



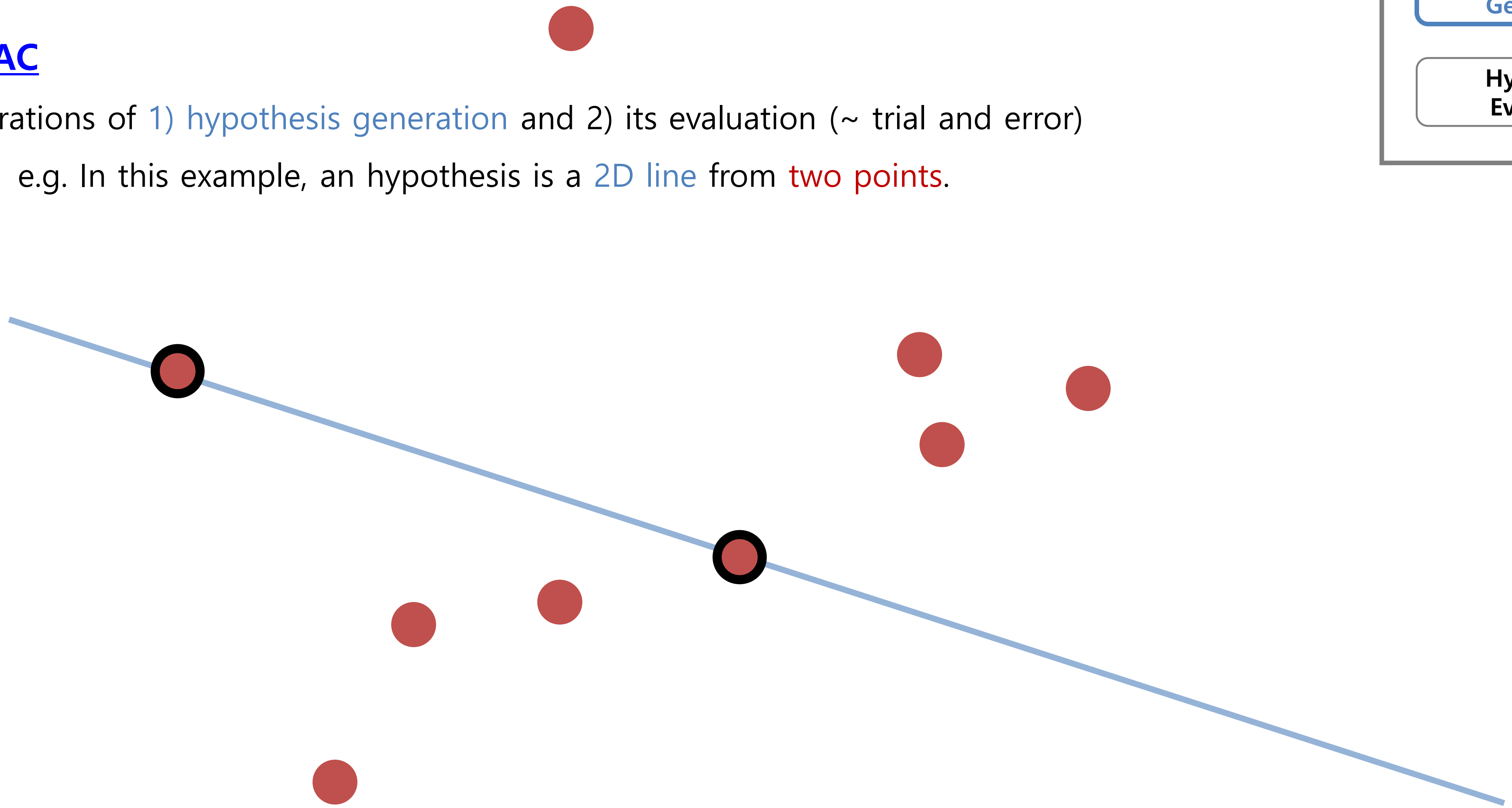
Outlier Rejection) RANSAC (Random Sample Consensus; 1981)

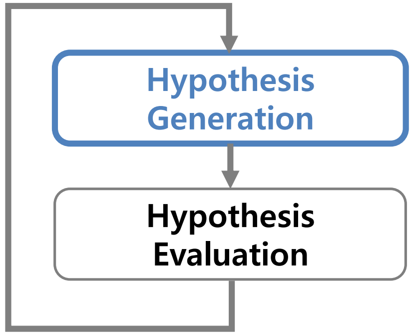
- Example) **Line fitting with** [RANSAC]
  - Model definition: 2D line $\mathbf{m} = [a, b, c]^\intercal$ $(ax + by + c = 0;\ a^2 + b^2 = 1)$
  - Model generation: Two points can generate to a line.
  - Error function: Geometric distance $e(\mathbf{x}_i; \mathbf{m}) = \frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}}$



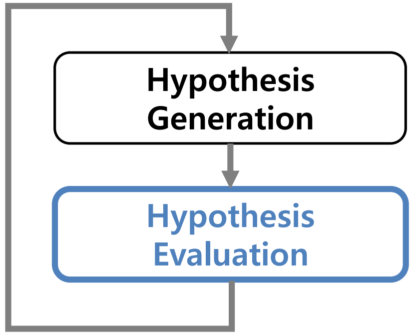
Outlier Rejection) RANSAC (Random Sample Consensus; 1981)

- RANSAC
  - Iterations of 1) hypothesis generation and 2) its evaluation (~ trial and error)
    - e.g. In this example, an hypothesis is a 2D line from two points.

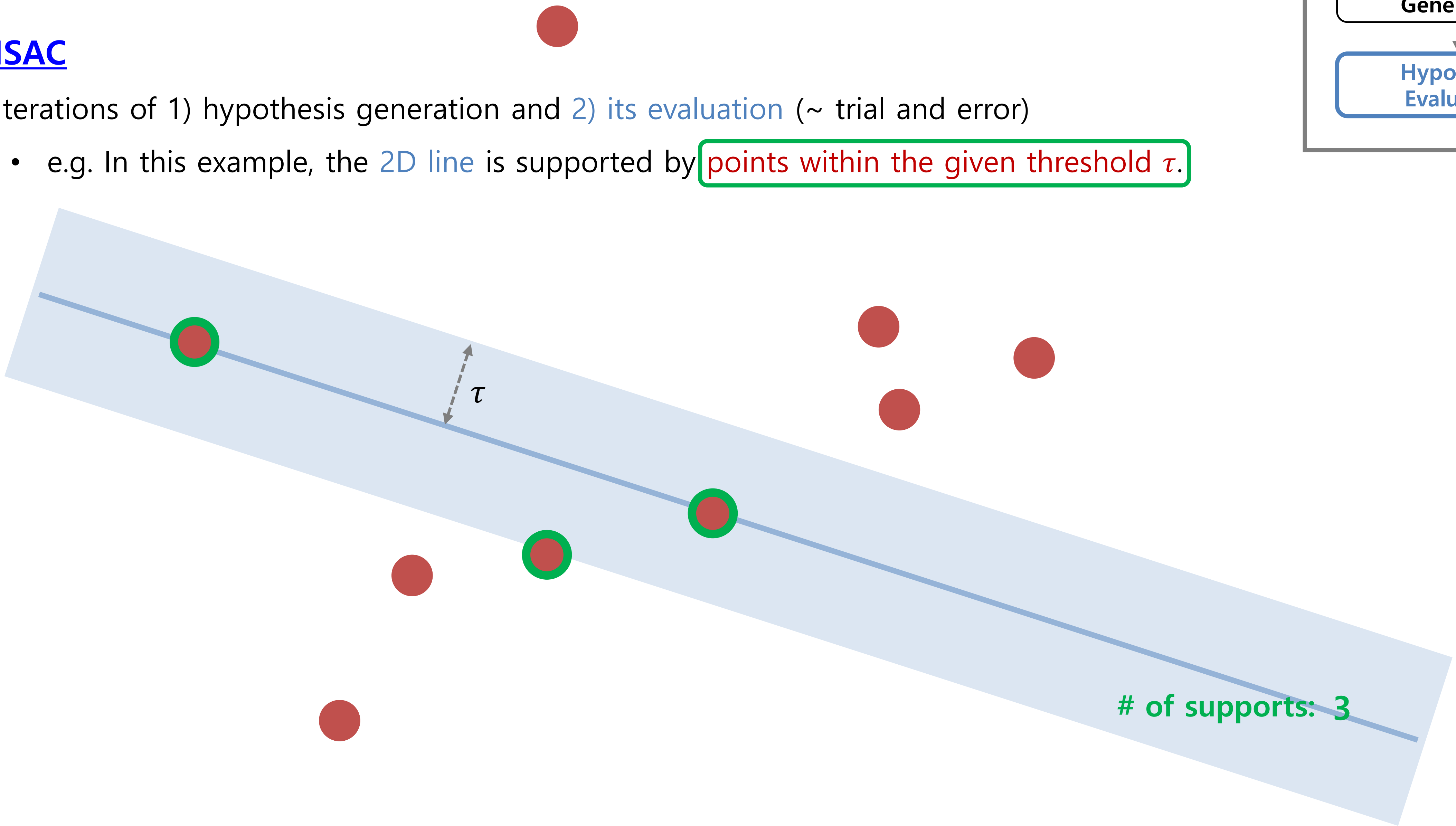




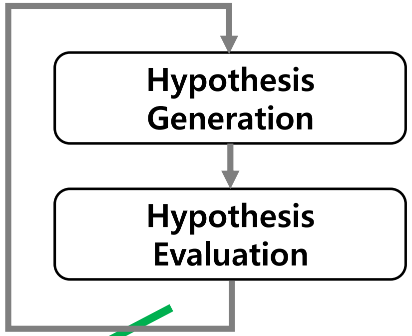
Outlier Rejection) RANSAC (Random Sample Consensus; 1981)



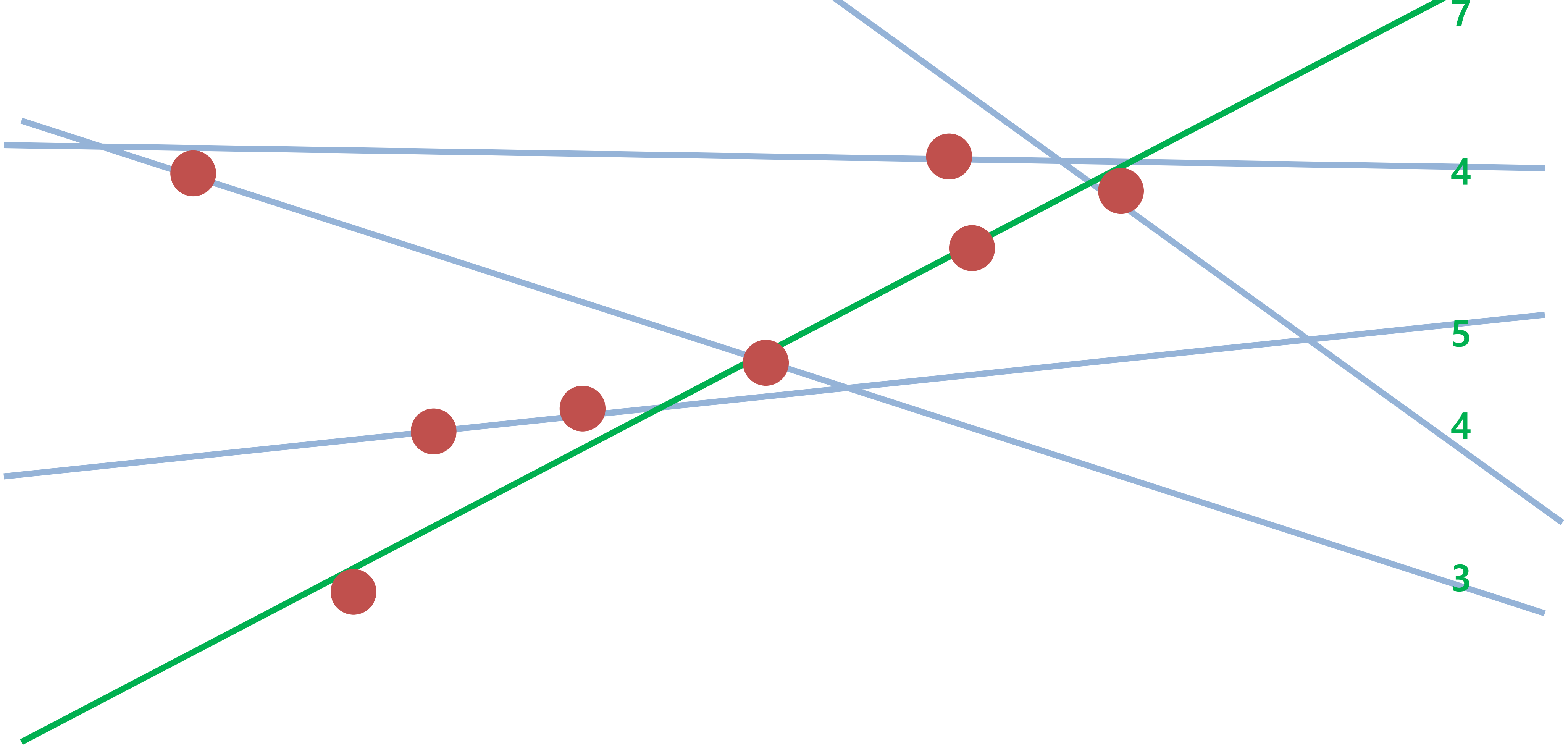
- **RANSAC**
  - Iterations of 1) hypothesis generation and 2) its evaluation (~ trial and error)
    - e.g. In this example, the 2D line is supported by points within the given threshold τ.



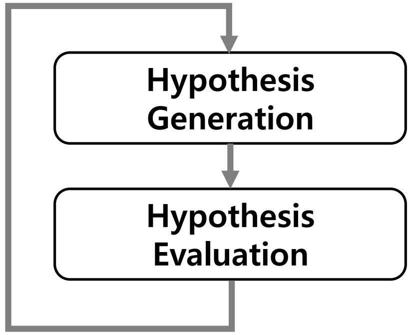
Outlier Rejection) RANSAC (Random Sample Consensus; 1981)



- RANSAC
  - Iterations of 1) hypothesis generation and 2) its evaluation (~ trial and error)
  - The best hypothesis is the final model and its supports are regarded as inliers.



Outlier Rejection) RANSAC (Random Sample Consensus; 1981)



- **RANSAC**
  - Iterations of 1) hypothesis generation and 2) its evaluation (~ trial and error)
  - How many iterations t are required?
    - Parameters and assumptions
      - s: Success probability (confidence level)
      - d: The number of samples for model generation
      - γ: Inlier ratio
    - Success criteria: **Selecting d samples from inliers within t iterations**
      - Failure sampling: $1 - \gamma^d$
      - Failure probability: $1 - s = \left(1 - \gamma^d\right)^t$
    - **The minimum number of iteration for success**: $t = \frac{\log(1-s)}{\log(1-\gamma^d)}$

- Example) **Line fitting with RANSAC** [line_fitting_ransac.cpp]

```cpp
#include "opencv2/opencv.hpp"

// Convert a line format, [n_x, n_y, x_0, y_0] to [a, b, c]
// Note) A line model in OpenCV: n_x * (x - x_0) = n_y * (y - y_0)
#define CONVERT_LINE(line) (cv::Vec3d(line[0], -line[1], -line[0] * line[2] + line[1] * line[3]))

int main()
{
    cv::Vec3d truth(1.0 / sqrt(2.0), 1.0 / sqrt(2.0), -240.0); // The line model: a*x + b*y + c = 0 (a^2 + b^2 = 1)
    int ransac_trial = 50, ransac_n_sample = 2;
    double ransac_thresh = 3.0; // 3 x 'data_inlier_noise'
    int data_num = 1000;
    double data_inlier_ratio = 0.5, data_inlier_noise = 1.0;

    // Generate data
    std::vector<cv::Point2d> data;
    cv::RNG rng;
    for (int i = 0; i < data_num; i++)
    {
        if (rng.uniform(0.0, 1.0) < data_inlier_ratio)
        {
            double x = rng.uniform(0.0, 480.0);
            double y = (truth(0) * x + truth(2)) / -truth(1);
            x += rng.gaussian(data_inlier_noise);
            y += rng.gaussian(data_inlier_noise);
            data.push_back(cv::Point2d(x, y)); // Inlier
        }
        else data.push_back(cv::Point2d(rng.uniform(0.0, 640.0), rng.uniform(0.0, 480.0))); // Outlier
    }

    // Estimate a line using RANSAC ...
    // Estimate a line using least-squares method (for reference) ...

    // Display estimates
    printf("* The Truth: %.3f, %.3f, %.3f\n", truth[0], truth[1], truth[2]);
    printf("* Estimate (RANSAC): %.3f, %.3f, %.3f (Score: %d)\n", best_line[0], best_line[1], ..., best_score);
    printf("* Estimate (LSM): %.3f, %.3f, %.3f\n", lsm_line[0], lsm_line[1], lsm_line[2]);
    return 0;
}
```

$$t > \frac{\log(1-s)}{\log(1-\gamma^d)} = \frac{\log(1-0.999)}{\log(1-0.5^2)} = 24$$

```cpp
// Estimate a line using RANSAC
int best_score = -1;
cv::Vec3d best_line;
for (int i = 0; i < ransac_trial; i++)
{
    // Step 1: Hypothesis generation
    std::vector<cv::Point2d> sample;
    for (int j = 1; j < ransac_n_sample; j++)
    {
        int index = rng.uniform(0, int(data.size()));
        sample.push_back(data[index]);
    }
    cv::Vec4d nnxy;
    cv::fitLine(sample, nnxy, CV_DIST_L2, 0, 0.01, 0.01);
    cv::Vec3d line = CONVERT_LINE(nnxy);

    // Step 2: Hypothesis evaluation
    int score = 0;
    for (size_t j = 0; j < data.size(); j++)
    {
        double error = fabs(line(0) * data[j].x + line(1) * data[j].y + line(2));
        if (error < ransac_thresh) score++;
    }

    if (score > best_score)
    {
        best_score = score;
        best_line = line;
    }
}

// Estimate a line using least squares method (for reference)
cv::Vec4d nnxy;
cv::fitLine(data, nnxy, CV_DIST_L2, 0, 0.01, 0.01);
cv::Vec3d lsm_line = CONVERT_LINE(nnxy);
```

Line Fitting Result

```
* The Truth: 0.707, 0.707, -240.000
* Estimate (RANSAC): 0.712, 0.702, -242.170 (Score: 434)
* Estimate (LSM): 0.748, 0.664, -314.997
```

Outlier Rejection) Least Squares Method, RANSAC, and M-estimator

- **Least Squares Method**
  - Find a model while minimizing sum of squared errors, $\underset{\mathbf{m}}{\operatorname{argmin}} \sum_i e(\mathbf{x}_i; \mathbf{m})^2$
- **RANSAC**
  - Find a model while maximizing the number of supports (~ inlier candidates)
    ~ minimizing the number of outlier candidates
- **Q) Why RANSAC was robust to outliers?**
  - Problem formulation with a **loss function** ρ: $\underset{\mathbf{m}}{\operatorname{argmin}} \sum_i \rho\big(e(\mathbf{x}_i; \mathbf{m})\big)$
  - Loss function of least squares method: $\rho(x) = x^2$
  - Loss function of RANSAC: $\rho(x) = \begin{cases} 0 & \text{if } |x| < \tau \\ 1 & \text{otherwise} \end{cases}$
- **M-estimator** (~ weighted least squares) **| MSAC** (in OpenCV)
  - Find a model while minimizing sum of (squared) errors **with a truncated loss function**
  - Loss function of M-estimator and MSAC: $\rho(x) = \begin{cases} x^2 & \text{if } |x| < \tau \\ \tau^2 & \text{otherwise} \end{cases}$

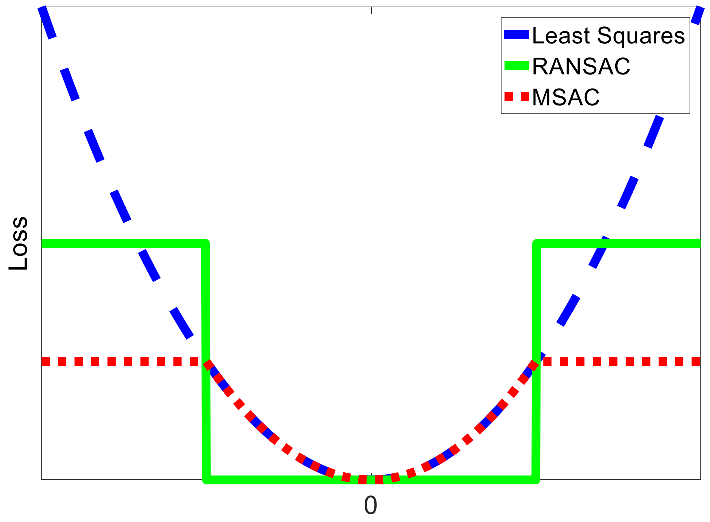
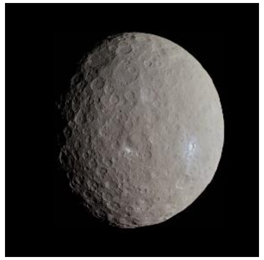


Image: Choi et al., **Performance Evaluation of RANSAC Family**, BMVC, 2009 [PDF]

One-page Tutorial for Ceres Solver

- **What is Ceres Solver?** [Homepage]
  - An **open source C++** library for modelling and solving large and complicated **optimization** problems.
  - Problem types: 1) **Non-linear least squares** (with bounds), 2) General unconstrained minimization

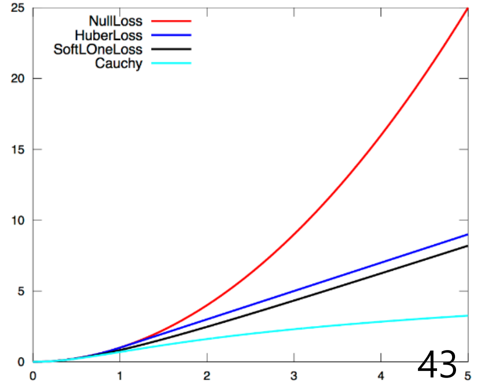


Ceres (an asteroid)

- **Solving Non-linear Least Squares**: $\hat{\mathrm{x}} = \underset{\mathrm{x}}{\operatorname{argmin}} \sum_i \rho_i(\|r_i(\mathrm{x})\|^2)$
  1. Define residual functions (or cost function or error function)
  2. Instantiate `ceres::Problem` and add residuals using its member function, `AddResidualBlock()`
     - Instantiate each residual r_i in the form of `ceres::CostFunction` and add it
       - Select how to calculate its derivative (Jacobian)

         (`ceres::AutoDiffCostFunction` or `ceres::NumericDiffCostFunction` or `ceres::SizedCostFunction`)
     - Instantiate its `ceres::LossFunction` ρ_i and add it (if the problem needs robustness against outliers)
  3. Instantiate `ceres::Solver::Options` (and also `ceres::Solver::Summary`) and configure the option
  4. Run `ceres::Solve()`

loss functions



- **Solving General Minimization**: $\hat{\mathrm{x}} = \underset{\mathrm{x}}{\operatorname{argmin}} f(\mathrm{x})$
  - `ceres::CostFunction` → `ceres::FirstOrderFunction`, `ceres::GradientFunction`
  - `ceres::Problem` → `ceres::GradientProblem`
  - `ceres::Solver` → `ceres::GradientProblemSolver`

▪ Example: **Line fitting with M-estimator** [line_fitting_m_estimator.cpp]

```cpp
#include "opencv2/opencv.hpp"
#include "ceres/ceres.h"
...
struct GeometricError
{
    GeometricError(const cv::Point2d& pt) : datum(pt) { }
    template<typename T>
    bool operator()(const T* const line, T* residual) const
    {
        residual[0] = (line[0] * T(datum.x) + line[1] * T(datum.y) + line[2]) / sqrt(line[0] * line[0] + line[1] * line[1]);
        return true;
    }
private:
    const cv::Point2d datum;
};

int main()
{
    ...
    // Estimate a line using M-estimator
    cv::Vec3d opt_line(1, 0, 0);
    ceres::Problem problem;
    for (size_t i = 0; i < data.size(); i++)
    {
        ceres::CostFunction* cost_func = new ceres::AutoDiffCostFunction<GeometricError, 1, 3>(new GeometricError(data[i]));
        ceres::LossFunction* loss_func = NULL;
        if (loss_width > 0) loss_func = new ceres::CauchyLoss(loss_width);
        problem.AddResidualBlock(cost_func, loss_func, opt_line.val);
    }
    ceres::Solver::Options options;
    options.linear_solver_type = ceres::ITERATIVE_SCHUR;
    options.num_threads = 8;
    options.minimizer_progress_to_stdout = true;
    ceres::Solver::Summary summary;
    ceres::Solve(options, &problem, &summary);
    std::cout << summary.FullReport() << std::endl;
    opt_line /= sqrt(opt_line[0] * opt_line[0] + opt_line[1] * opt_line[1]); // Normalize
    ...
    return 0;
}
```

1) Define a residual as C++ generic function (T ~ double)
   Note) The generic is necessary for automatic differentiation.

c.f. $e(x, y; a, b, c) = \dfrac{ax + by + c}{\sqrt{a^2 + b^2}}$

2) Instantiate a problem and add a residual for each datum

The dimension of a residual

The dimension of the first model parameter

3) Instantiate options and configure it

4) Solve the minimization problem

Summary) Outlier Rejection (Robust Parameter Estimation)

- **Bottom-up Approaches (~ Voting)**
  - ~~**Hough transform**~~
    - A single datum votes multiple model parameter candidates.
      - Note) The parameter space is maintained as a multi-dimensional histogram (discretization).
    - Score: The number of hits by data
  - **RANSAC family**
    - A sample of data votes a single model parameter candidate.
    - Score: The number of supports (whose error is within threshold)
  - Note) Application examples: Line fitting, homography estimation, relative pose estimation, PnP
- **Top-down Approaches**
  - **Optimization with truncated loss functions (e.g. M-estimator)**
    - The inital model parameter moves along with the gradient of the given cost function with whole data.
    - Score: A cost function
      - Note) The cost function includes a truncated loss function.
  - Note) Application examples: Camera calibration, visual SLAM, SfM

Summary

- **Feature Points**
  - Gradient-based: Harris corner, GFTT corner, SIFT, SURF
  - Intensity-based: FAST
  - DL-based: LIFT, SuperPoint
- **Feature Descriptors**
  - Real-valued: SIFT, SURF (DL-based: LIFT, SuperPoint)
  - Binary-valued: BRIEF, ORB
- **Feature Matching**
  - Distance measures
  - Matching methods
- **Feature Tracking**
  - LK optical flow → KLT tracker
- **Outlier Rejection**
  - RANSAC → MSAC
  - Least squares method → M-estimation

Image: pixabay