Note) Slides and example codes are available:
https://github.com/mint-lab/3dv_tutorial

An Invitation to 3D Vision:
Single-View Geometry

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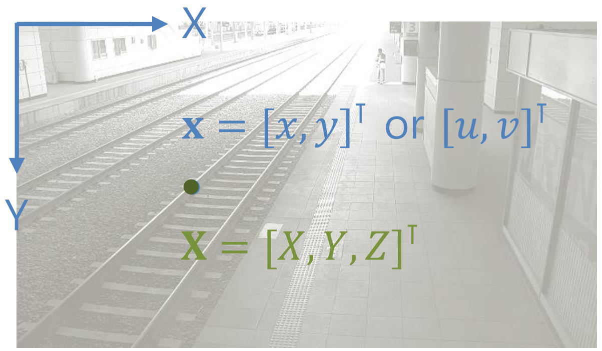
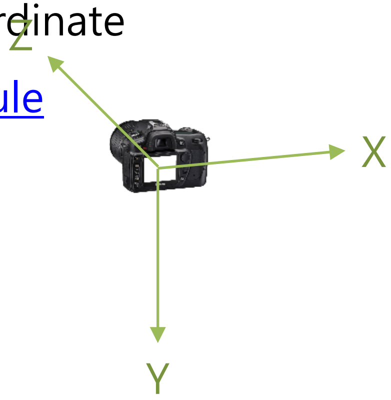
Getting Started with 2D

- **Image coordinate** (unit: [pixel])



- **Camera coordinate** (unit: [meter])
  - Position: Focal point
  - X/Y direction: Image coordinate
  - Z direction: Right-hand rule



Getting Started with 2D

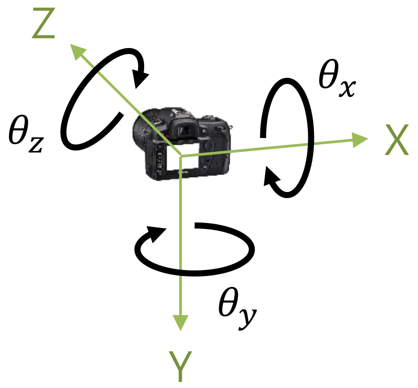
- **2D rotation matrix**
  - Rotational direction: Right-hand rule

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Properties of a rotation matrix

- $R^{-1} = R^{\mathsf{T}}$ (orthogonal matrix)
- $\det(R) = 1$

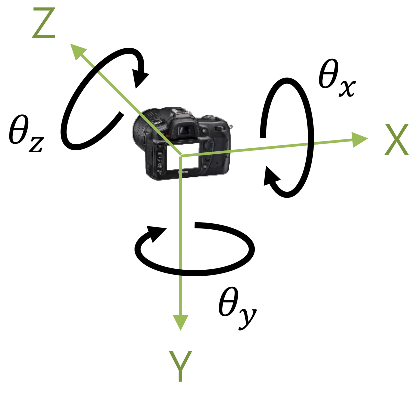
- **3D rotation matrix**
  - Rotational direction: Right-hand rule

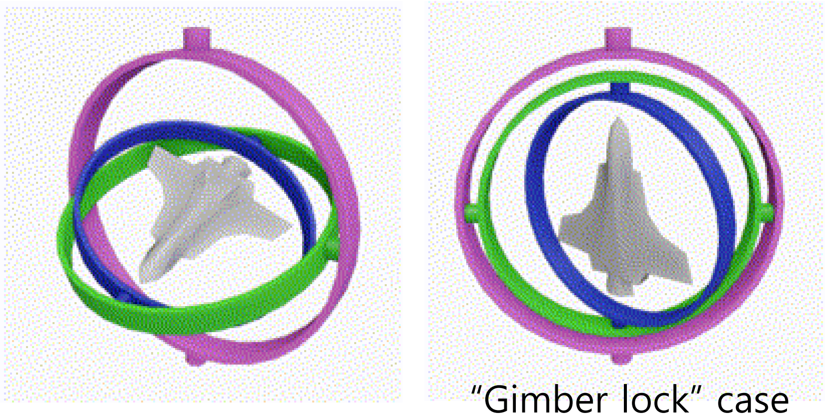


	Cameras	Vehicles	Airplanes	Telescopes
θ_x	Tilt	Pitch	Attitude	Elevation
θ_y	Pan	Yaw	Heading	Azimuth
θ_z	Roll	Roll	Bank	Horizon

Getting Started with 2D

- **3D rotation representation** (3 DOF)
  - **3D rotation matrix** (9 parameters)
    - Notation: 3x3 matrix
      - e.g. $\mathrm{R} = \mathrm{R}_z(\theta_z)\,\mathrm{R}_y(\theta_y)\,\mathrm{R}_x(\theta_x) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}$
    - Properties: $\mathrm{R}^{-1} = \mathrm{R}^\intercal$ (orthogonal matrix), $\det(\mathrm{R}) = 1$
  - **Euler angle** (3 parameters)
    - Notation: $[\theta_x, \theta_y, \theta_z]$
    - Issues: Not unique, not continuous, Gimbar lock (loss of DOF)

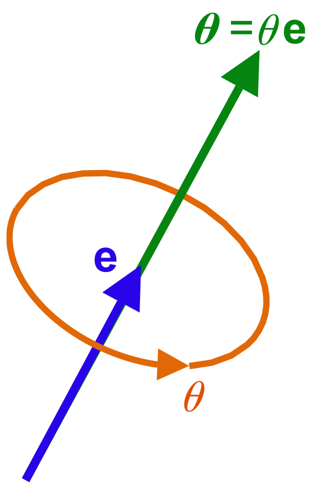




"Gimber lock" case

Image: Wikipedia

Getting Started with 2D



- **3D rotation representation** (3 DOF)
  - **Axis-angle representation** (3 parameters; a.k.a. *rotation vector*, *Rodrigues notation*)
    - Notation: $\boldsymbol{\theta} = \theta\mathbf{e}$
      - e.g. Axis (unit vector): $\mathbf{e} = [0, 0, 1]$, angle: $\theta = {}^{\pi}/_{2} \rightarrow \boldsymbol{\theta} = [0, 0, {}^{\pi}/_{2}]$
    - Properties: Log map of SO(3), dual ($-\mathbf{e}$ with $-\theta \rightarrow \boldsymbol{\theta}$), reverse angle ($-\boldsymbol{\theta}$)
    - Note) The standard notation in OpenCV with `cv.Rodrigues()` (`R` ↔ `rvec`)
  - (Unit) **Quaternion** (4 parameters)
    - Notation: $\mathbf{q} = [q_w, q_x, q_y, q_z]$ or $[q_x, q_y, q_z, q_w]$
      - Meaning: $\mathbf{q} = \cos\frac{\theta}{2} + (e_x\mathbf{i} + e_y\mathbf{j} + e_z\mathbf{k})\sin\frac{\theta}{2}$
    - Property: ${q_x}^2 + {q_y}^2 + {q_z}^2 + {q_w}^2 = 1$, dual ($-\mathbf{q}$), reverse angle ($\bar{\mathbf{q}}$; conjugate)

- Note) 3D rotation conversion
  - Python: `scipy.spatial.transform.Rotation`
  - Web apps: NinjaCalc, Glowbuzzer, Andre Gaschler

Image: Wikipedia

Getting Started with 2D

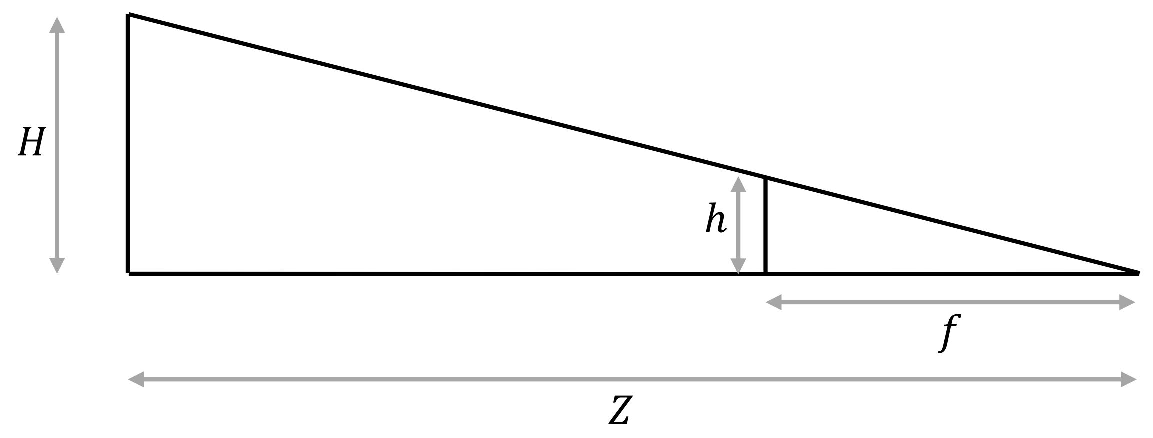
- Example) **3D rotation conversion** [3d_rotation_conversion.py]

```
import numpy as np
from scipy.spatial.transform import Rotation

# The given 3D rotation
euler = (45, 30, 60) # Unit: [deg] in the XYZ-order

# Generate 3D rotation object
robj = Rotation.from_euler('zyx', euler[::-1], degrees=True)

# Print other representations
print('\n## Euler Angle (ZYX)')
print(np.rad2deg (robj.as_euler('zyx'))) # [60, 30, 45] [deg] in the ZYX-order
print('\n## Rotation Matrix')
print(robj.as_matrix())
print('\n## Rotation Vector')
print(robj.as_rotvec())                   # [0.97, 0.05, 1.17]
print('\n## Quaternion (XYZW)')
print(robj.as_quat())                     # [0.44, 0.02, 0.53, 0.72]
```

Getting Started with 2D

- **Similarity**

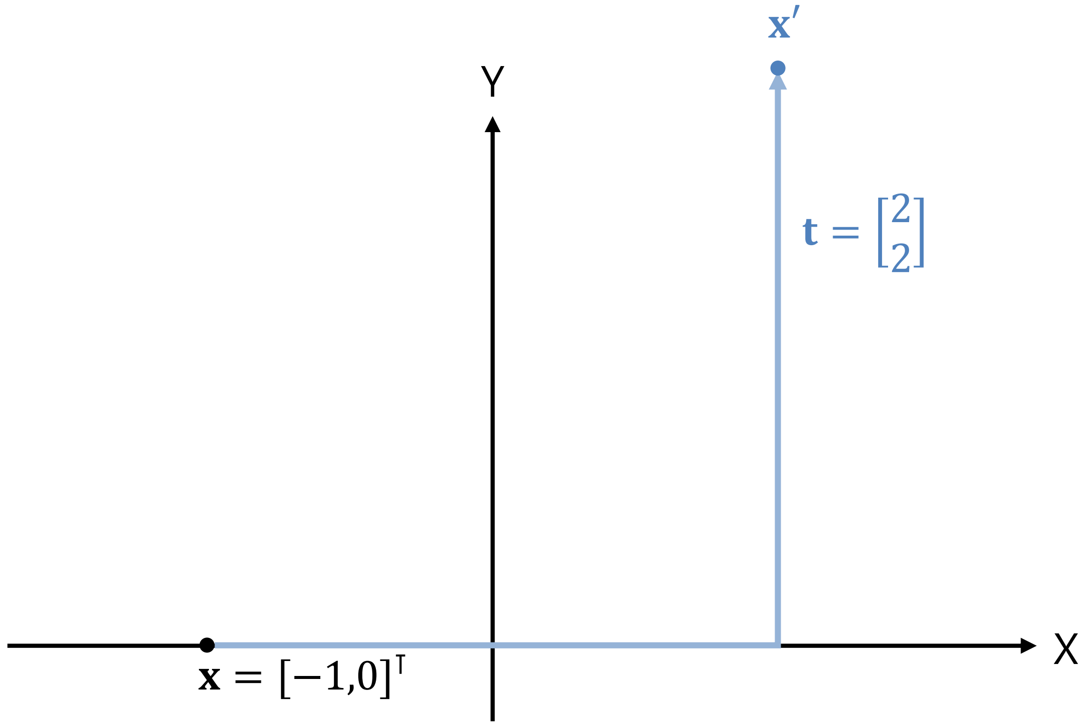
$$\frac{h}{H} = \frac{f}{Z} \quad \text{or} \quad \frac{h}{f} = \frac{H}{Z} \quad \rightarrow \quad h = f\,\frac{H}{Z}$$



Getting Started with 2D

- **Point translation**

  $\mathbf{x}' = \mathbf{x} + \mathbf{t}$



Getting Started with 2D

- **Coordinate translation**

  Valid for the following point transformation?

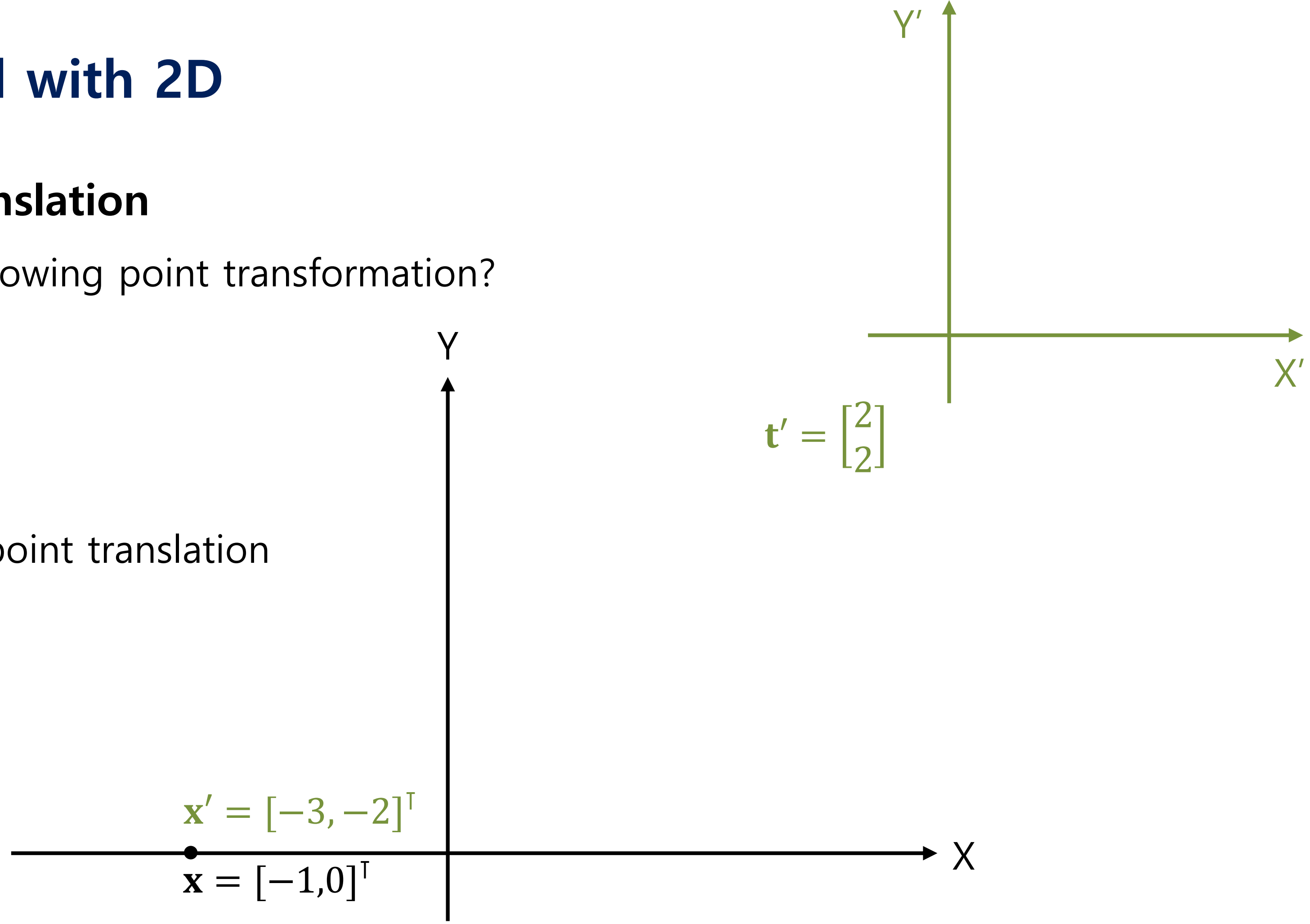
  $\mathbf{x}' = \mathbf{x} + \mathbf{t}'$?

  No!

  The **inverse** of point translation

  $\mathbf{x} = \mathbf{x}' + \mathbf{t}'$

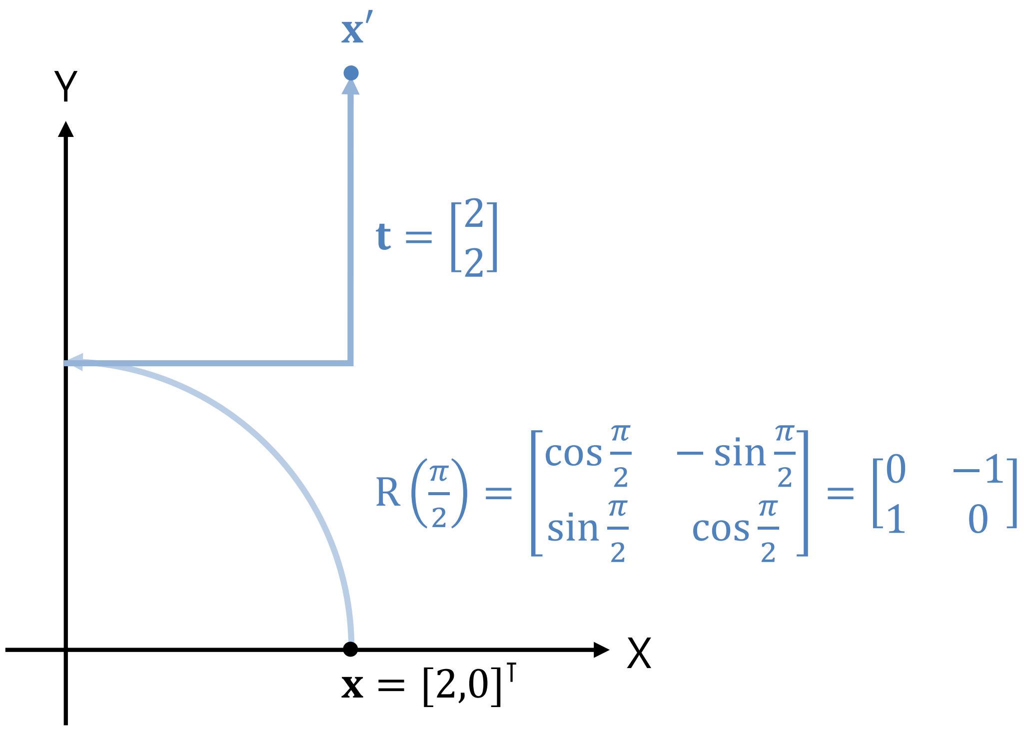
  $\mathbf{x}' = \mathbf{x} - \mathbf{t}'$



Getting Started with 2D

- **Point transformation**

$$\mathbf{x}' = \mathrm{R}\mathbf{x} + \mathbf{t} = [\mathrm{R} \mid \mathbf{t}] \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$$

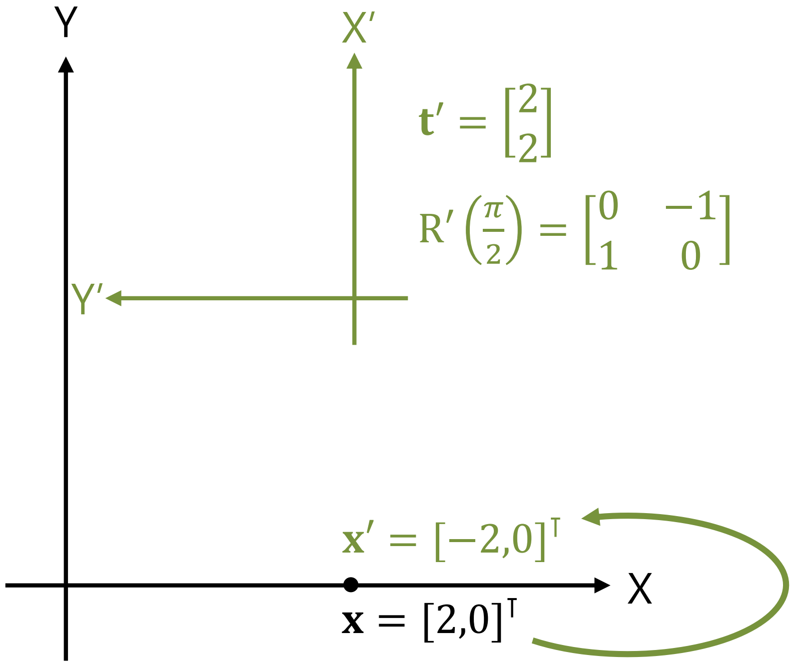


Getting Started with 2D

- **Coordinate transformation**

  Valid for the following point transformation?

  $\mathbf{x}' = \mathrm{R}'\mathbf{x} + \mathbf{t}'$?



Getting Started with 2D

- **Coordinate transformation**

Valid for the following point transformation?

$\mathbf{x}' = \mathrm{R}'\mathbf{x} + \mathbf{t}'$?

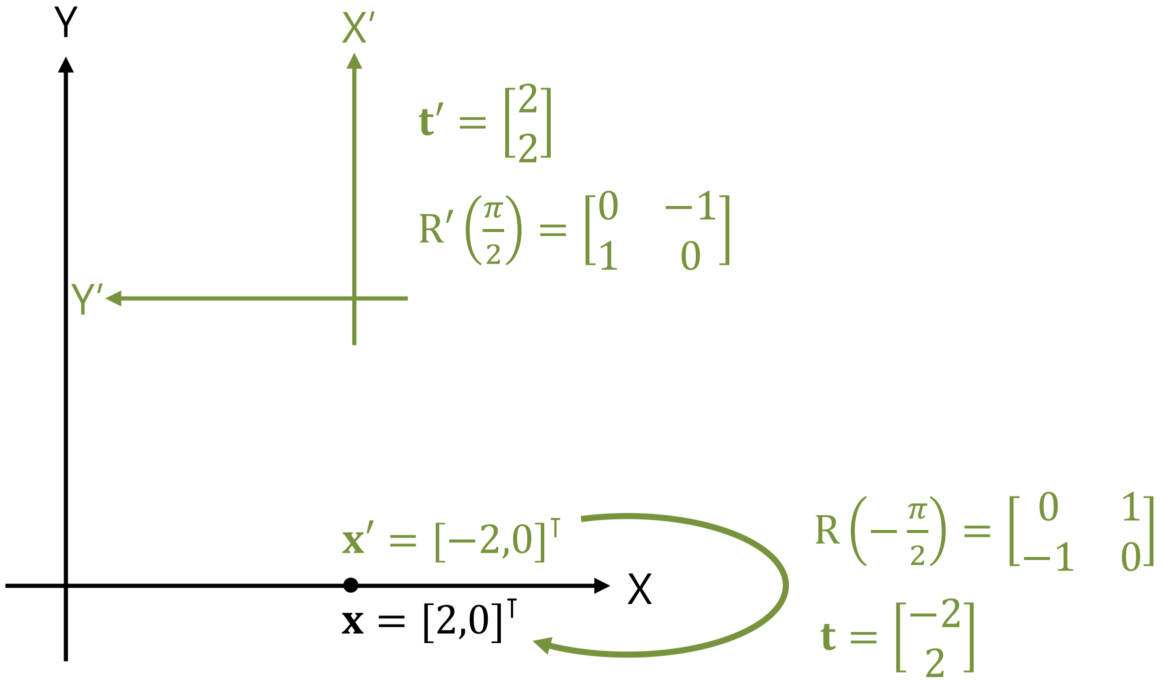
No!

The **inverse** of point transformation

$\mathbf{x} = \mathrm{R}'\mathbf{x}' + \mathbf{t}'$

$\downarrow \mathrm{R}'^{\top}(\mathbf{x} - \mathbf{t}') = \mathbf{x}'$

$\mathbf{x}' = \mathrm{R}\mathbf{x} + \mathbf{t}$ ($\mathrm{R} = \mathrm{R}'^{\top}$ and $\mathbf{t} = -\mathrm{R}'^{\top}\mathbf{t}'$)



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  - Similarity
  - Point transformation, coordinate transformation: inverse relationship
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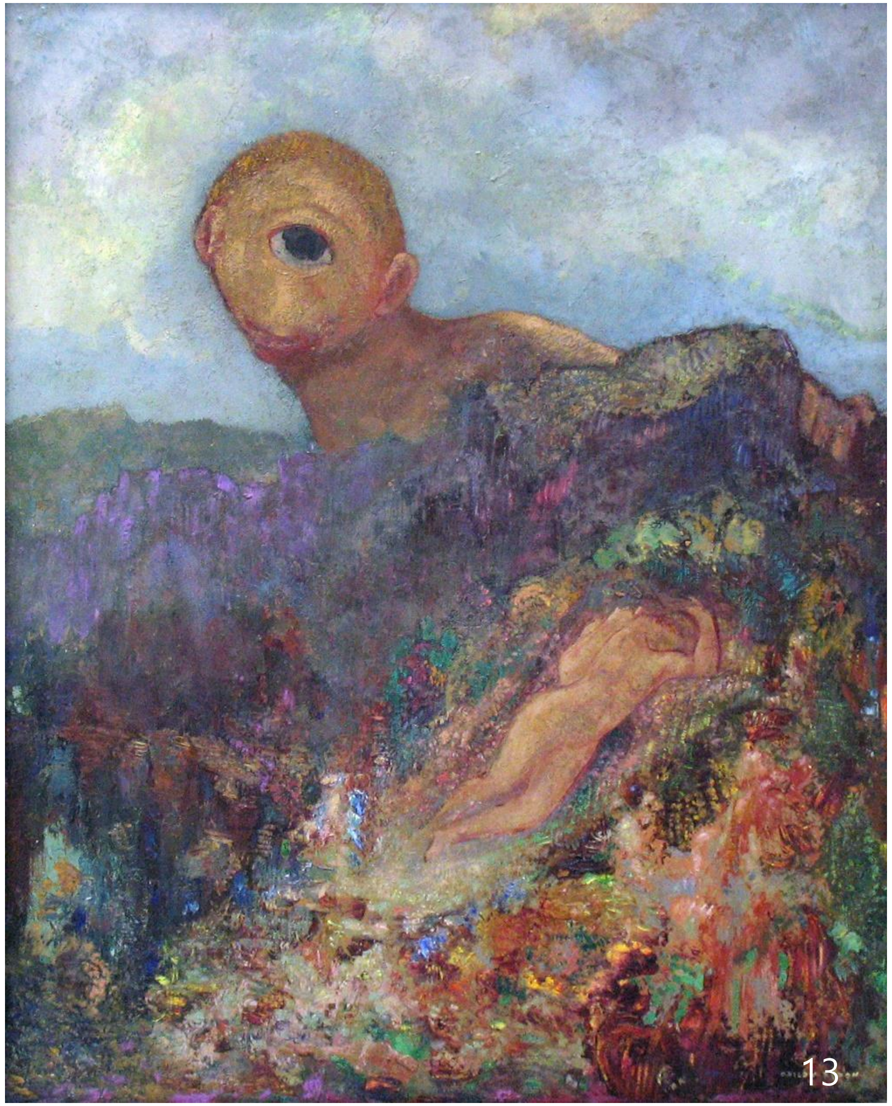
  - Pinhole camera model
  - Geometric distortion models
- **Camera Calibration**
- **Absolute Camera Pose Estimation**

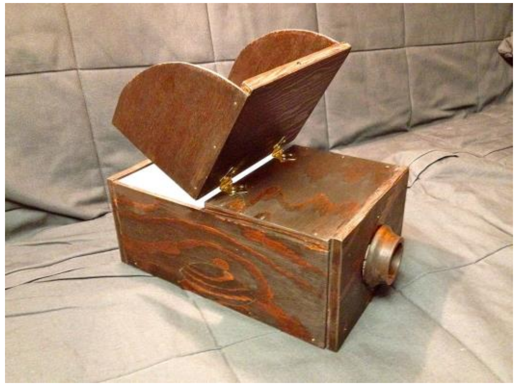


The Cyclops, gouache and oil by Odilon Redon

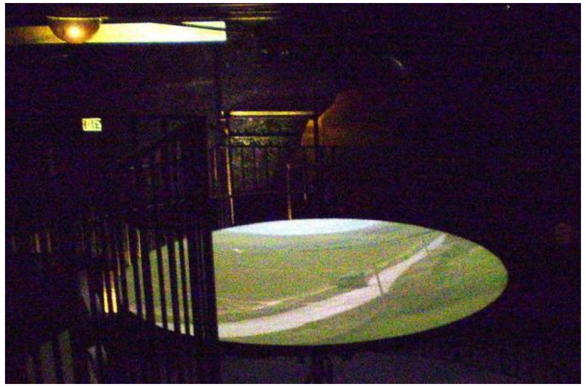
Pinhole Camera Model

- **Pinhole camera model**

A large-scale camera obscura at San Francisco, California



A modern-day camera obscura



An Image in camera obscura at Portslade, England

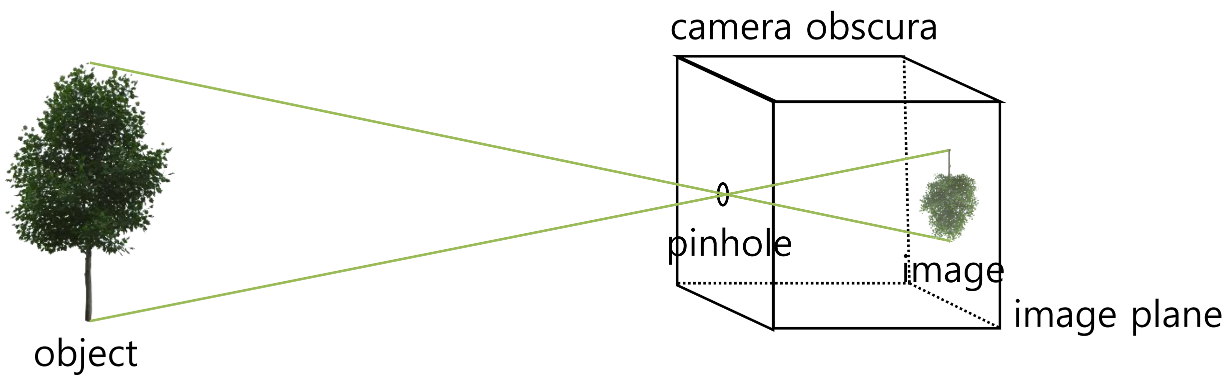
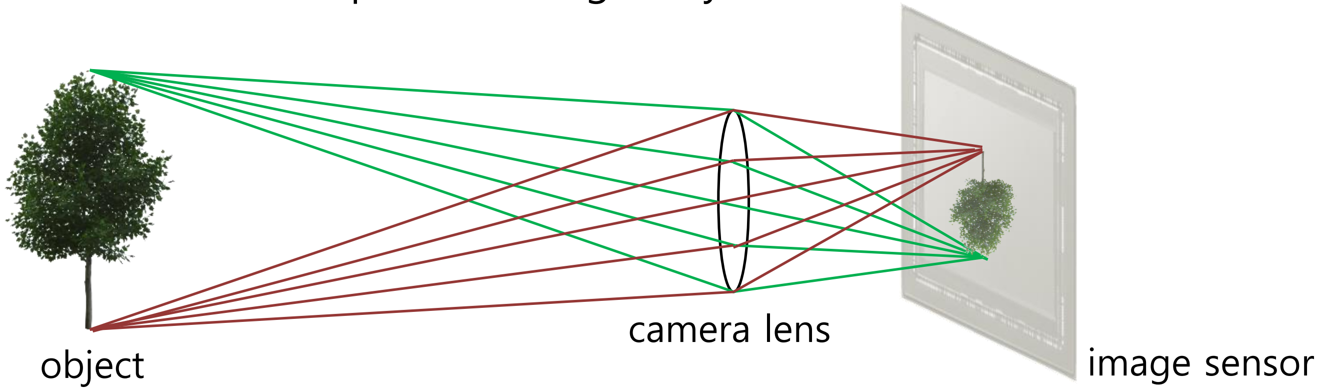


Image: Wikipedia, PNG EGG

Pinhole Camera Model

- **Real camera with a lens**
  - Q) Why does a camera use a lens? To acquire more light rays



- **[Pinhole camera model]**

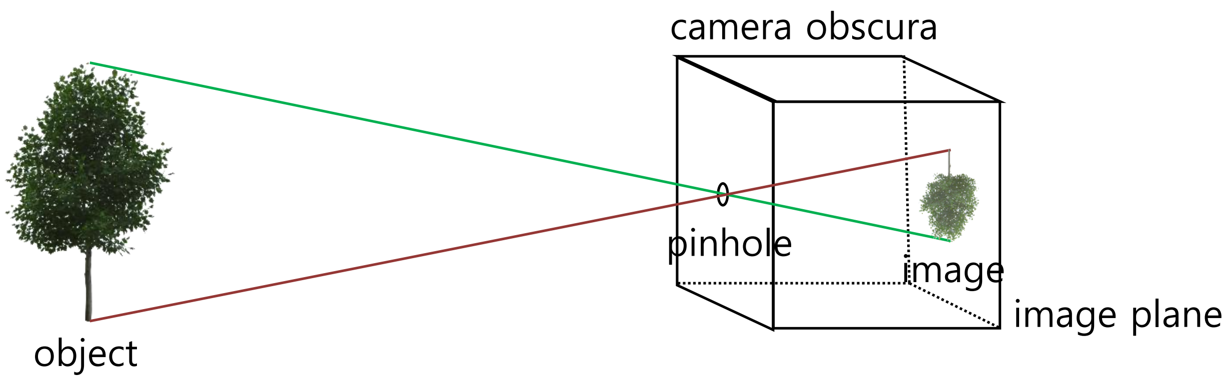
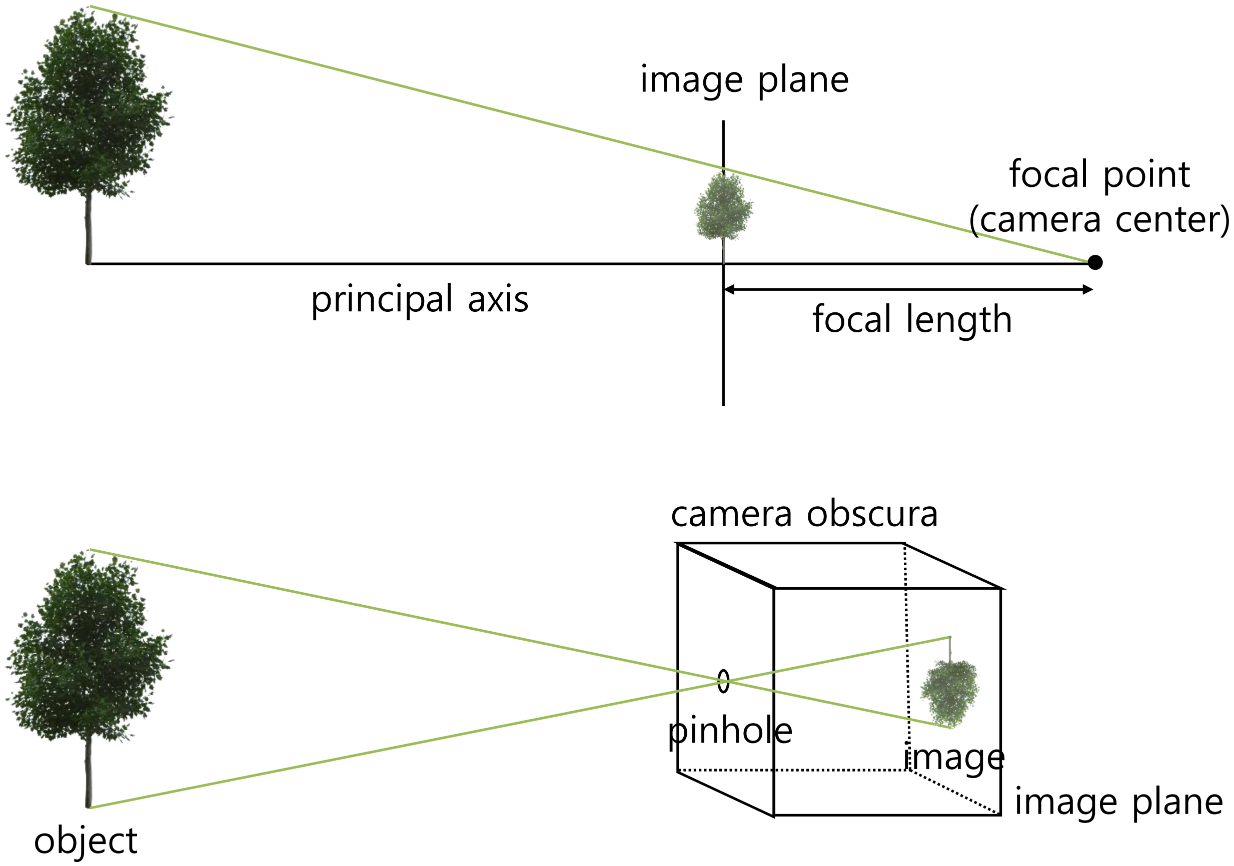


Image: PNG EGG

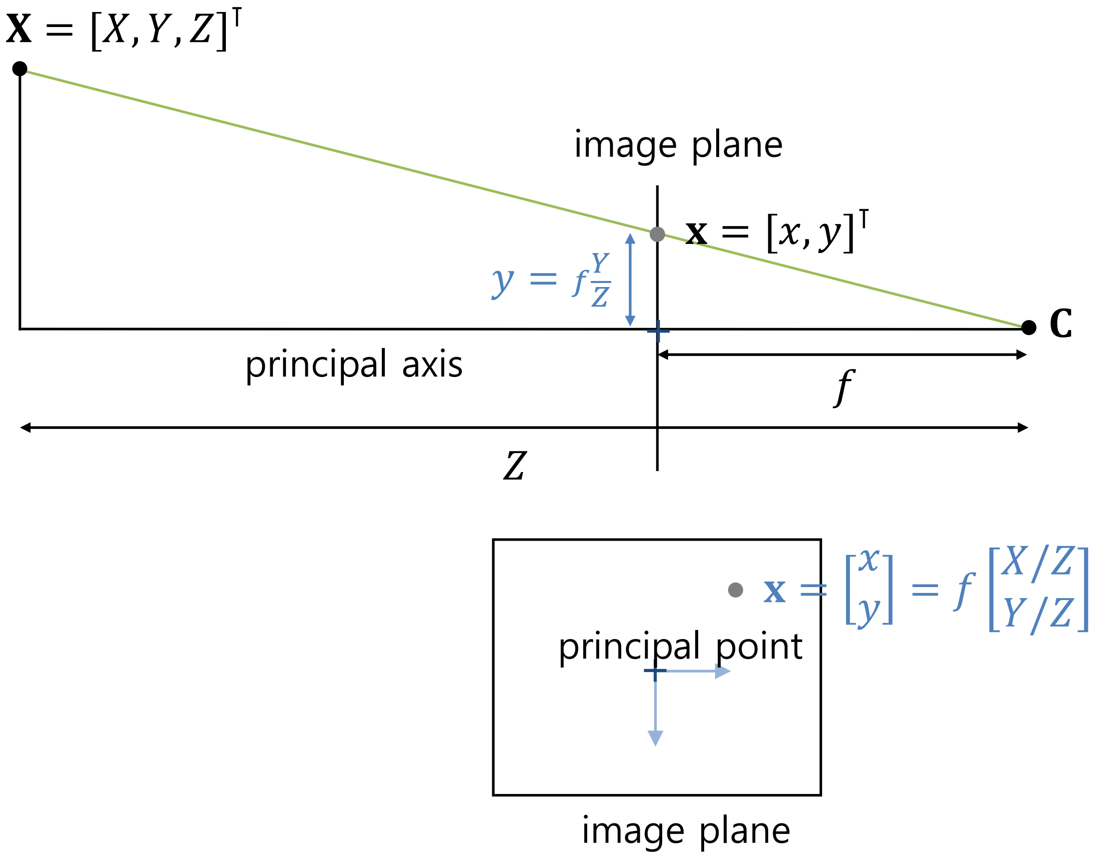
Pinhole Camera Model

- **Pinhole camera model**
  - In conclusion (without lens distortion), $\mathbf{x} = \mathrm{P}\mathbf{X}$ $(\mathrm{P} = \mathrm{K}[\mathrm{R} \mid \mathbf{t}])$



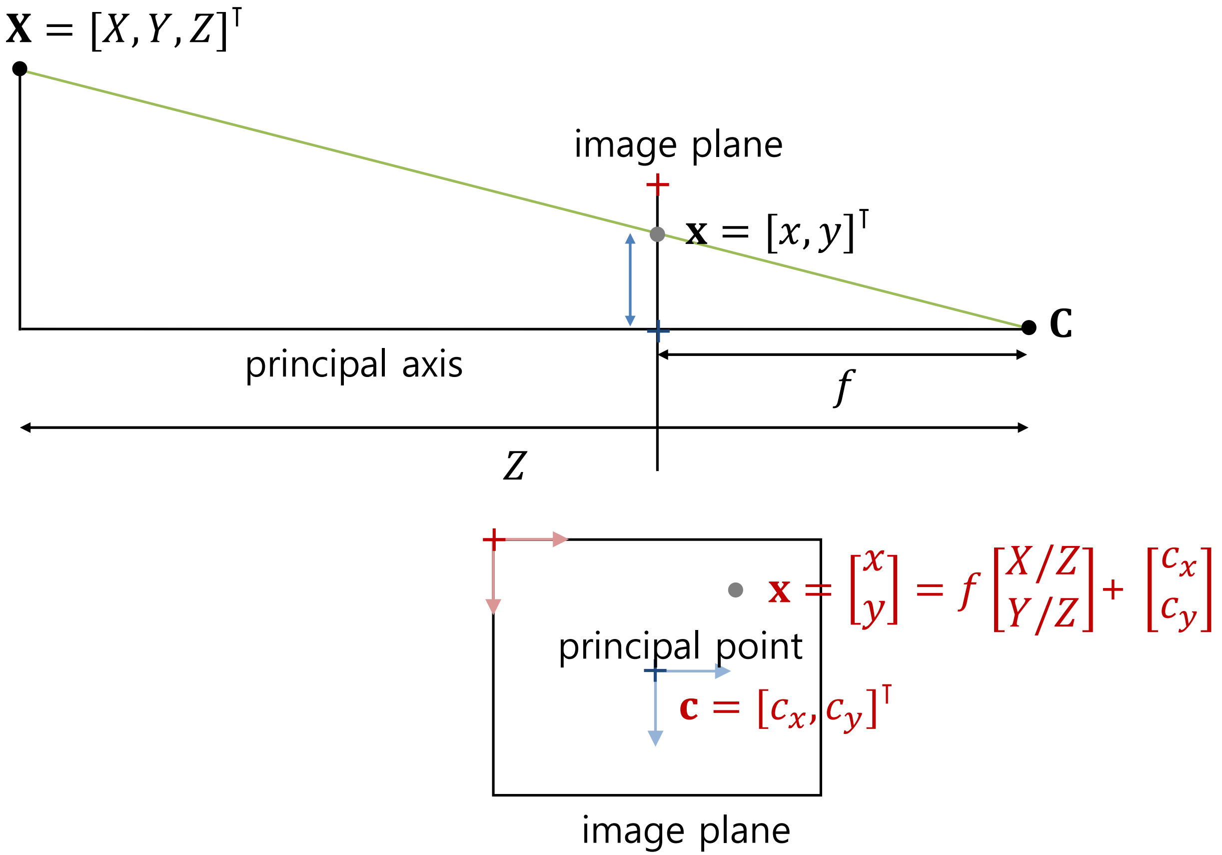
Pinhole Camera Model

- Pinhole camera model



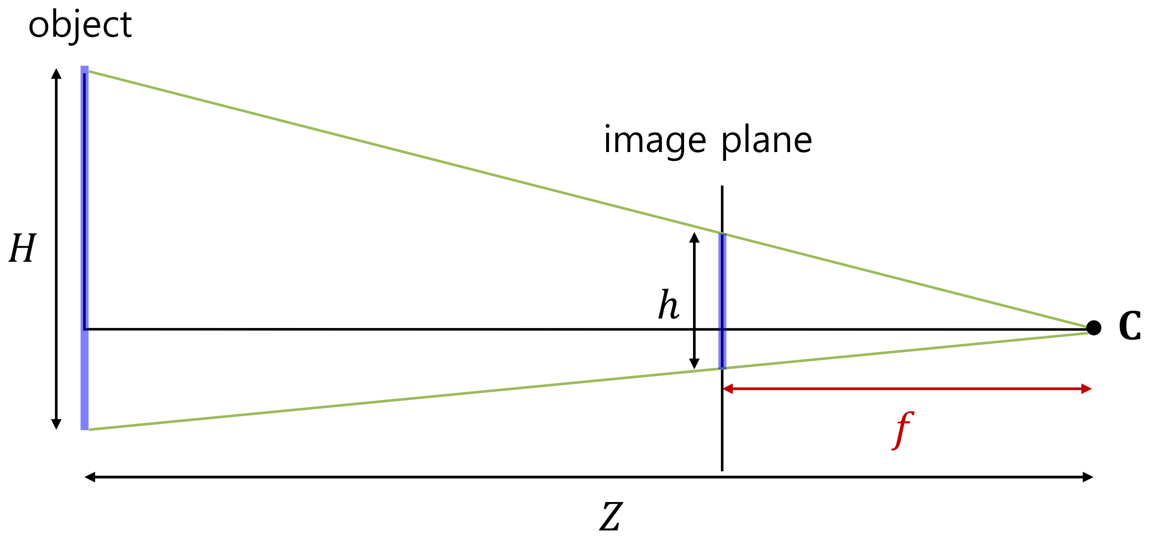
Pinhole Camera Model

- Pinhole camera model



Pinhole Camera Model

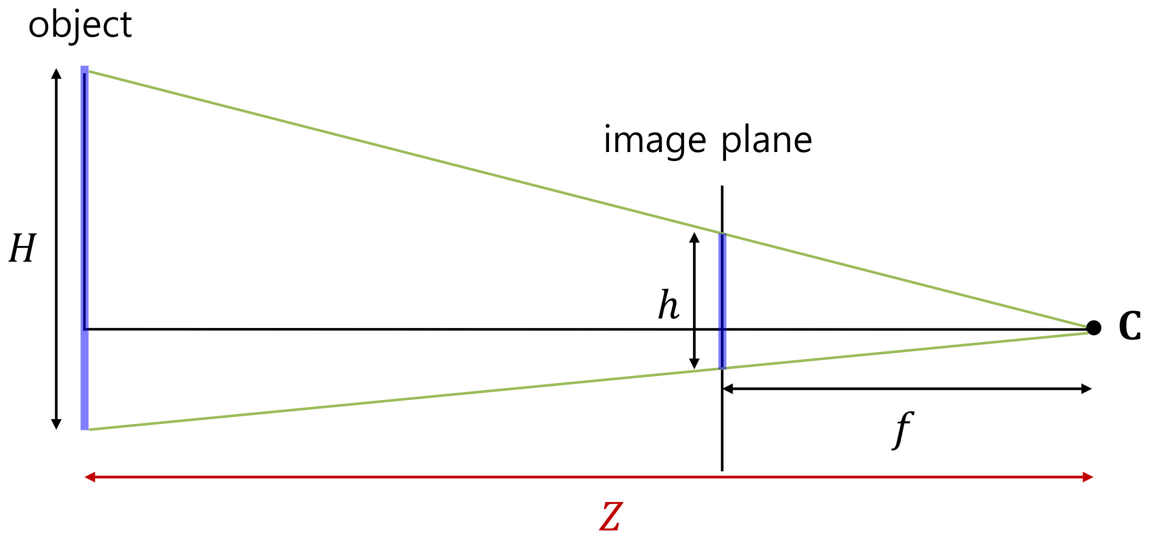
- Example) **Simple camera calibration**
  - Unknown: Focal length (f) of the camera (unit: [pixel])
  - Given: The observed object height (h) on the image plane (unit: [pixel])
  - Assumptions
    - The object height (H) and distance (Z) from the camera are known.
    - The object is aligned with the image plane.



$$\therefore f = \frac{Z}{H} h$$

Pinhole Camera Model

- Example) **Simple depth estimation** (object localization)
  - Unknown: Object distance (Z) from the camera (unit: [m])
  - Given: The observed object height (h) on the image plane (unit: [pixel])
  - Assumptions
    - The object height (H) and focal length (f) are known.
    - The object is aligned with the image plane.

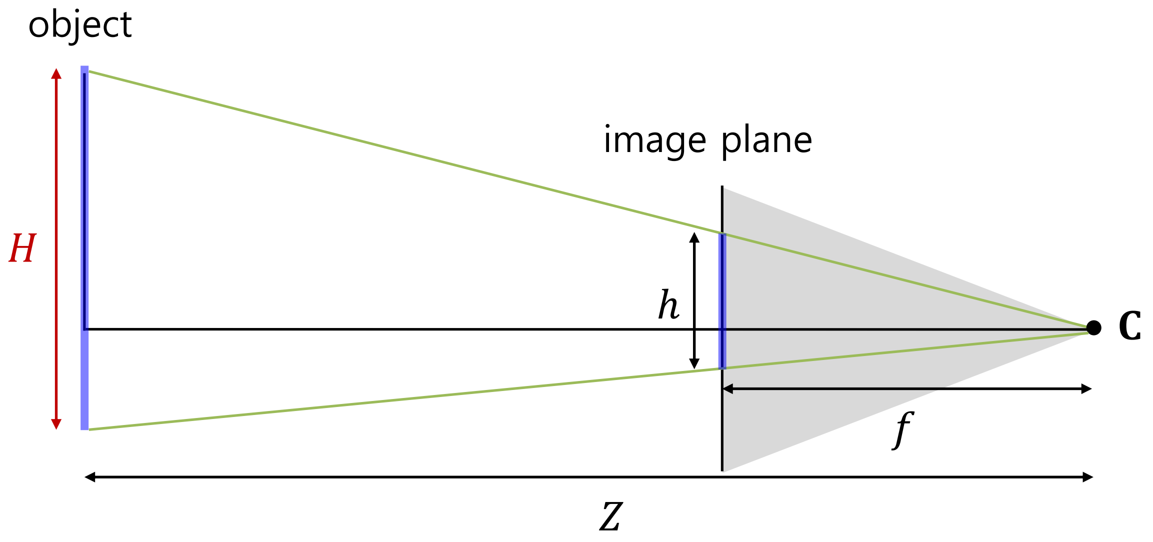


$$\therefore Z = \frac{f}{h} H$$

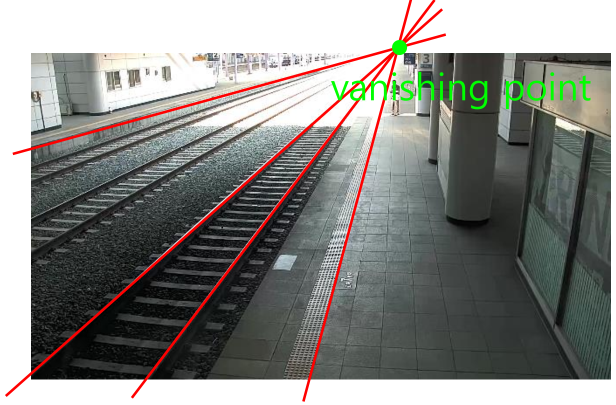
Pinhole Camera Model

- Example) **Simple object measurement**
  - Unknown: Object height (H) (unit: [m])
  - Given: The observed object height (h) on the image plane (unit: [pixel])
  - Assumptions
    - The object distance (Z) from the camera and focal length (f) are known.
    - The object is aligned with the image plane.

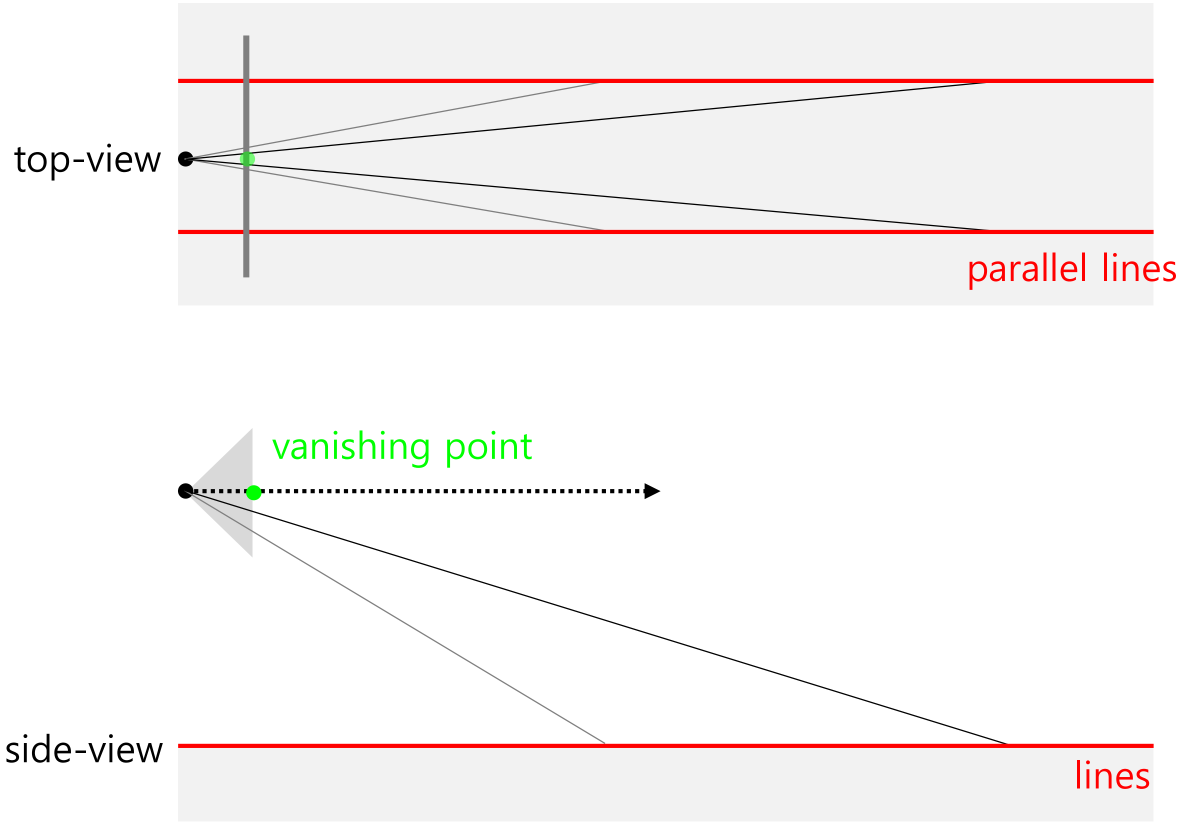
$$\therefore H = \frac{h}{f} Z$$



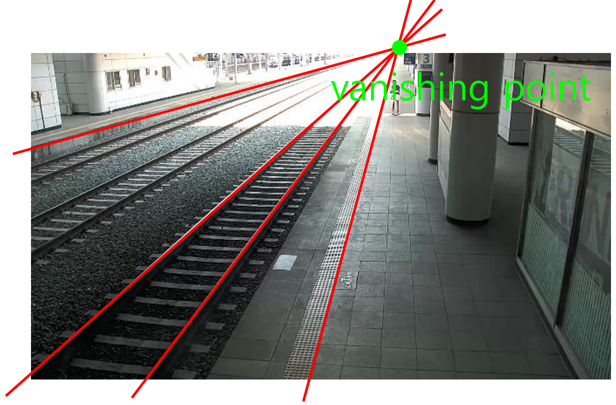
Pinhole Camera Model



- **Vanishing points**
  - A point on the image plane where mutually parallel lines in 3D spac
    - A vector to the vanishing point is parallel to the lines.
    - A vector to the vanishing point is parallel to the reference plane made by the lines.



Pinhole Camera Model



- [Vanishing points]
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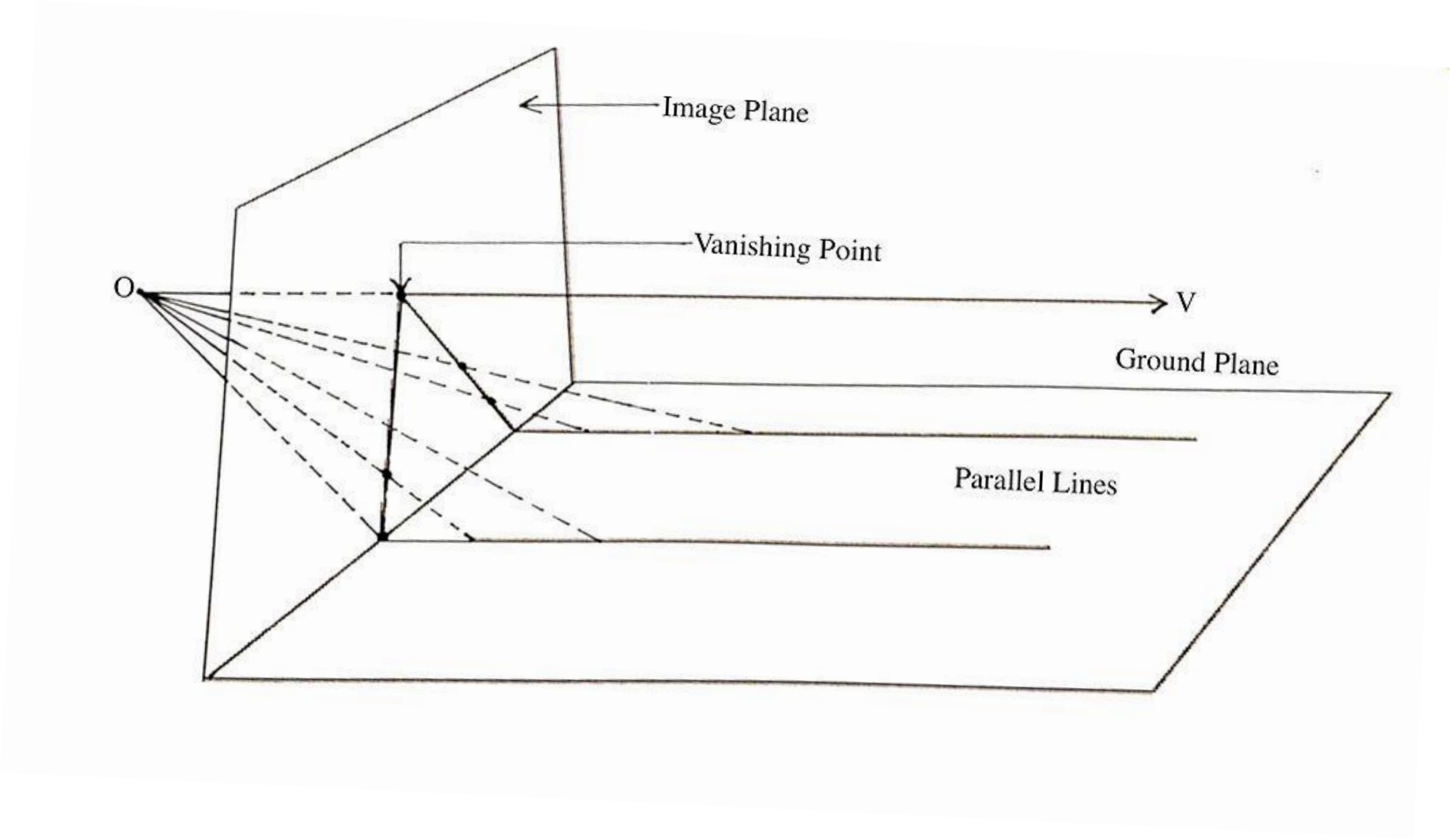
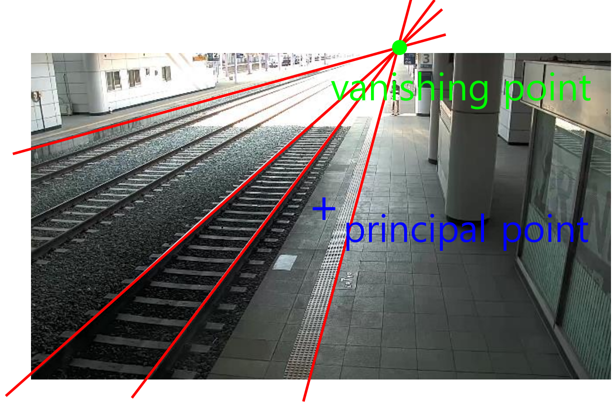


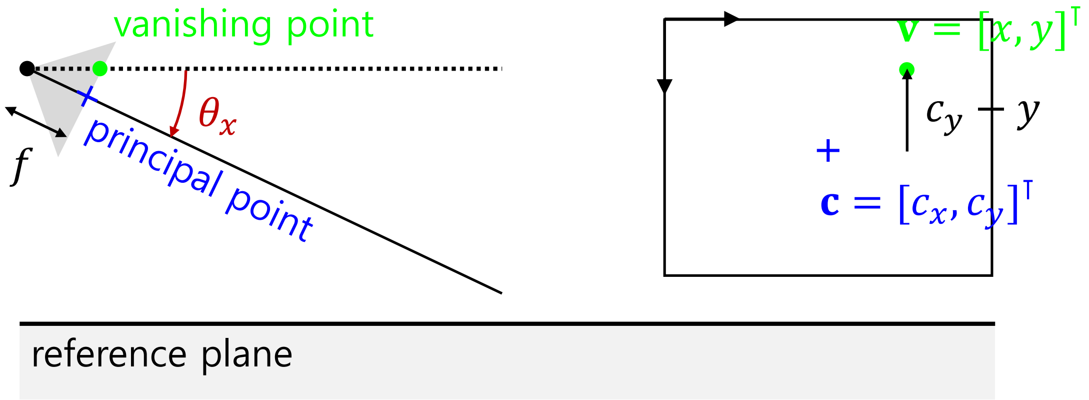
Image: Wikipedia

Pinhole Camera Model

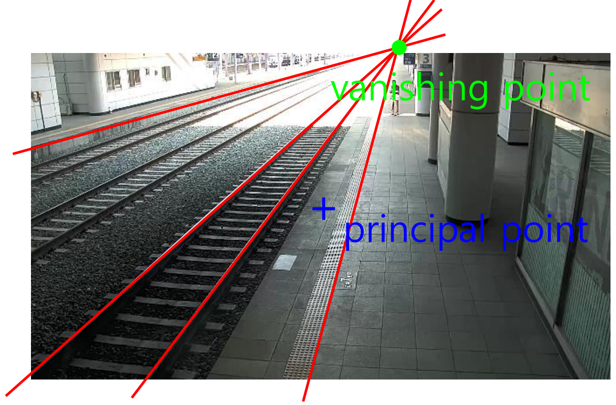


- Example) **Simple camera pose estimation**
  - Unknown: Tilt angle (θ_x) of the camera w.r.t. the reference plane (unit: [rad])
  - Given: A vanishing point (x, y) from the reference plane
  - Assumptions
    - The focal length (f) is known.
    - The principal point (c_x, c_y) is known or selected as the center of images.
    - The camera has no roll, $\theta_z = 0$.
  - Note) The tilt angle in this page is defined as the opposite direction of the common notation.

$$\therefore \theta_x = \tan^{-1} \frac{c_y - y}{f}$$

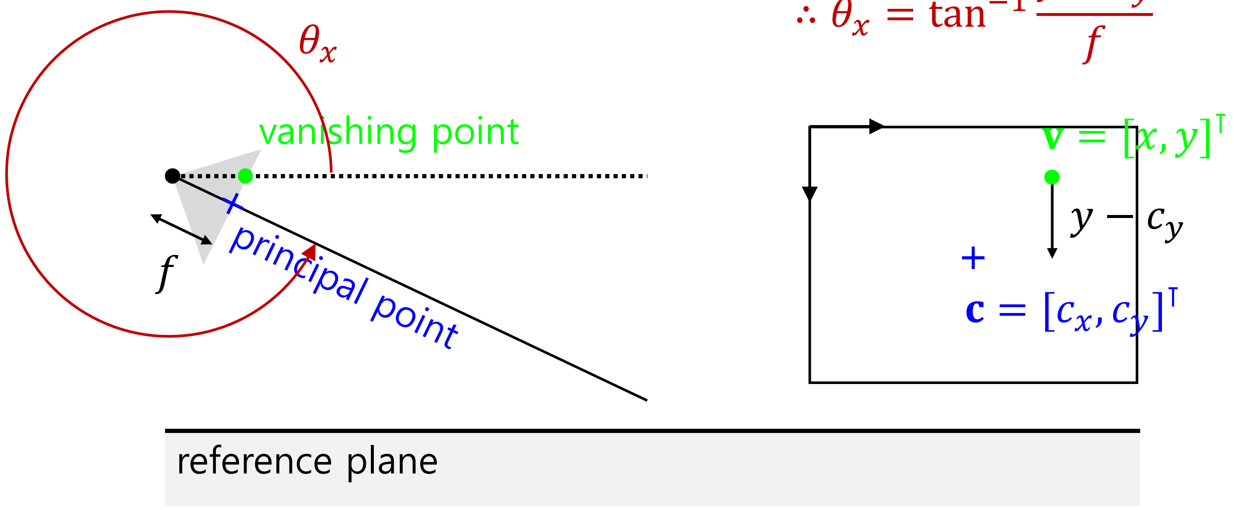


Pinhole Camera Model



- Example) **Simple camera pose estimation**
  - Unknown: Tilt angle (θ_x) of the camera w.r.t. the reference plane (unit: [rad])
  - Given: A vanishing point (x, y) from the reference plane
  - Assumptions
    - The focal length (f) is known.
    - The principal point (c_x, c_y) is known or selected as the center of images.
    - The camera has no roll, $\theta_z = 0$.
  - Note) The pan angle (θ_y) w.r.t. rails can be calculated similarly using x instead of y.

$$\therefore \theta_x = \tan^{-1} \frac{y - c_y}{f}$$

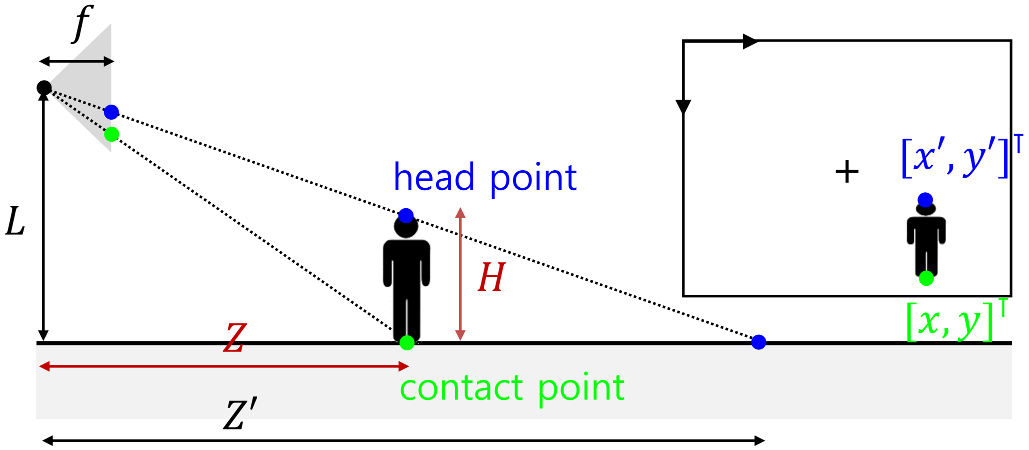


Pinhole Camera Model

- Example) **Object localization #1**
  - Unknown: Object position and height (unit: [m])
  - Given: The object's contact and head points on the image (unit: [pixel])
  - Assumptions
    - The focal length, principal points, and camera height, are known.
    - The camera is aligned to the reference plane.
    - The object is on the reference plane.

$$\therefore Z = \frac{\bar{z}}{\bar{y}} L \qquad X = \frac{\bar{x}}{\bar{y}} L \qquad H = \left(\frac{\bar{y}}{\bar{z}} - \frac{\bar{y}'}{\bar{z}'} \right) Z$$

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} x - c_x \\ y - c_y \\ f \end{bmatrix}$$



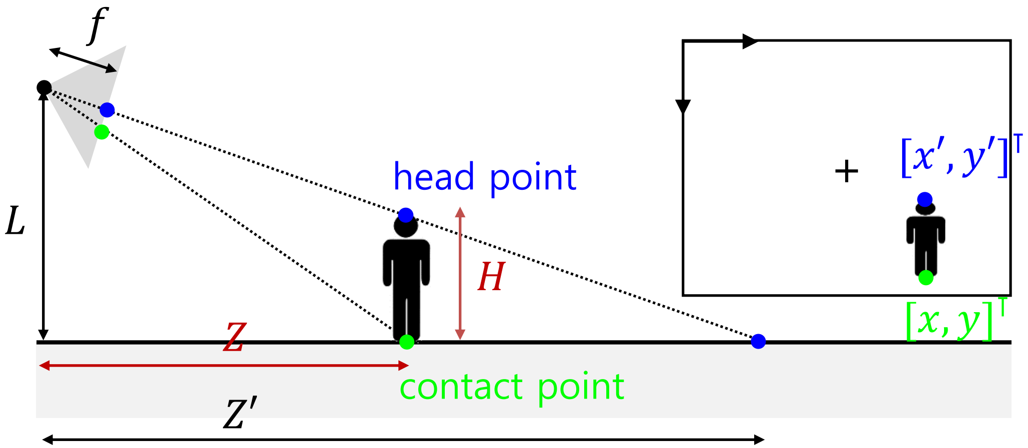
Pinhole Camera Model

- Example) **Object localization #2**
  - Unknown: Object position and height (unit: [m])
  - Given: The object's contact and head points on the image (unit: [pixel])
  - Assumptions
    - The focal length, principal points, and camera height are known.
    - ~~The camera is aligned to the reference plane.~~ The camera orientation (R) is known.
    - The object is on the reference plane.

$$\therefore Z = \frac{\bar{z}}{\bar{y}} L \qquad X = \frac{\bar{x}}{\bar{y}} L \qquad H = \left(\frac{\bar{y}}{\bar{z}} - \frac{\bar{y}'}{\bar{z}'} \right) Z$$

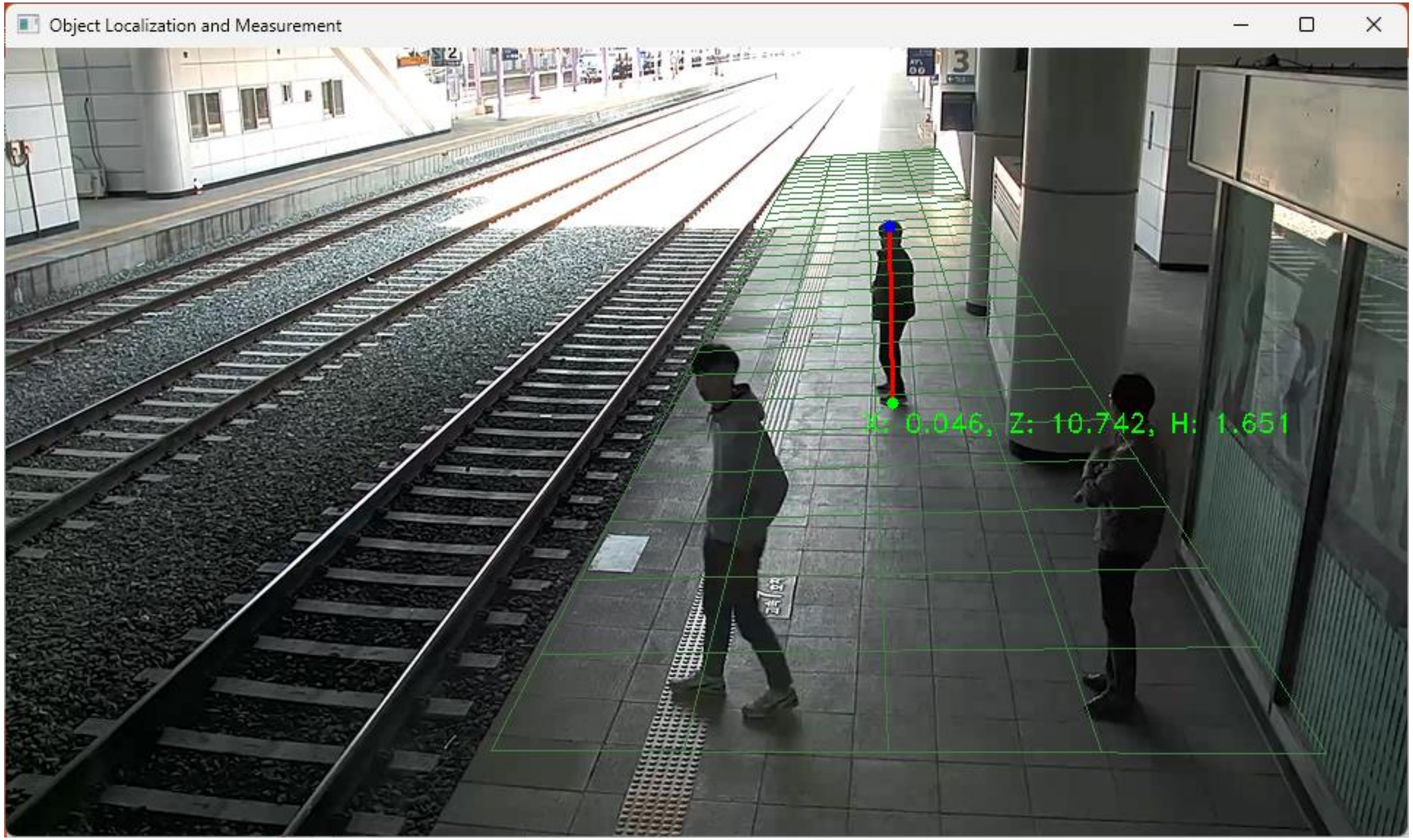
$\because$ same camera center

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \mathrm{R}^{\intercal} \begin{bmatrix} x - c_x \\ y - c_y \\ f \end{bmatrix}$$



Pinhole Camera Model

- Example) **Object localization #2** `[object_localization.py]`



Pinhole Camera Model

- Example) **Object localization #2** [object_localization.py]

```
if __name__ == '__main__':
    ...
    while True:
        img_copy = img.copy()
        if mouse_state['xy_e'][0] > 0 and mouse_state['xy_e'][1] > 0:
            # Calculate object location and height
            c = R.T @ [mouse_state['xy_s'][0] - cx, mouse_state['xy_s'][1] - cy, f]
            h = R.T @ [mouse_state['xy_e'][0] - cx, mouse_state['xy_e'][1] - cy, f]
            if c[1] < 1e-6:
                continue
            X = c[0] / c[1] * L                    # Object location X [m]
            Z = c[2] / c[1] * L                    # Object location Y [m]
            H = (c[1] / c[2] - h[1] / h[2]) * Z  # Object height [m]

            # Draw the head/contact points and location/height
            cv.line(img_copy, mouse_state['xy_s'], mouse_state['xy_e'], (0, 0, 255), 2)
            cv.circle(img_copy, mouse_state['xy_e'], 4, (255, 0, 0), -1) # Head point
            cv.circle(img_copy, mouse_state['xy_s'], 4, (0, 255, 0), -1) # Contact point
            info = f'X: {X:.3f}, Z: {Z:.3f}, H: {H:.3f}'
            cv.putText(img_copy, info, np.array(mouse_state['xy_s']) + (-20, 20), cv.FONT_HERSHEY_DUPLEX, 0.6, (0, 2

        cv.imshow('Object Localization and Measurement', img_copy)
        key = cv.waitKey(10)
        if key == 27: # ESC
            break

    cv.destroyAllWindows()
```

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \mathrm{R}^{\intercal} \begin{bmatrix} x - c_x \\ y - c_y \\ f \end{bmatrix}$$

$$X = \frac{\bar{x}}{\bar{y}} L \quad Z = \frac{\bar{z}}{\bar{y}} L \quad H = \left(\frac{\bar{y}}{\bar{z}} - \frac{\bar{y}'}{\bar{z}'} \right) Z$$

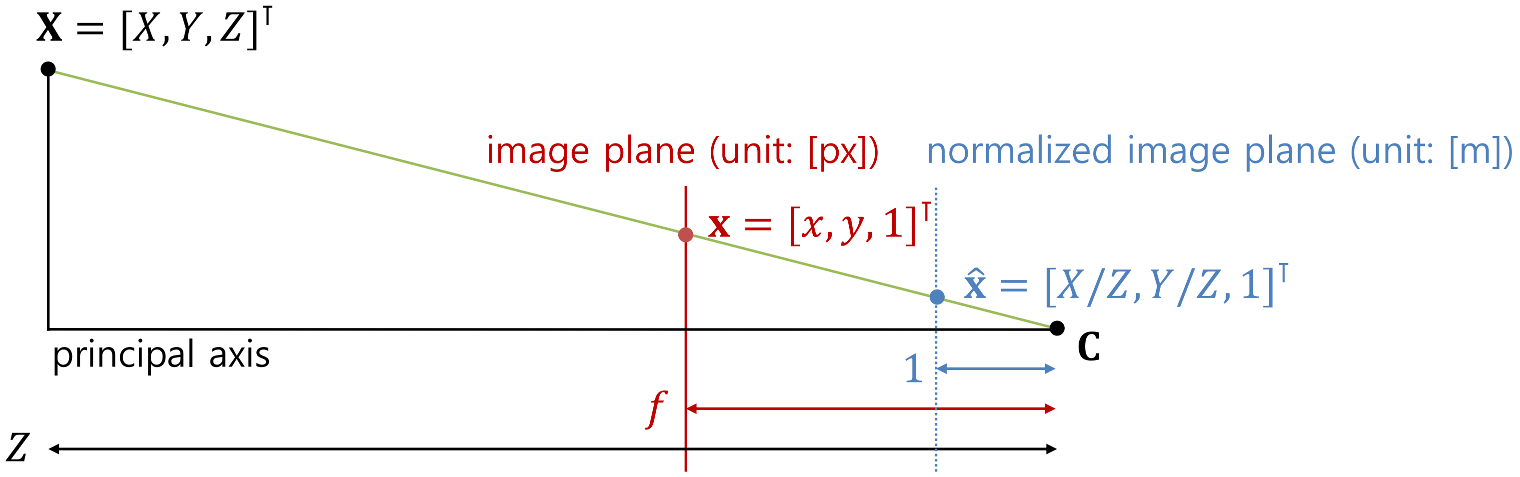
Pinhole Camera Model

- **Camera matrix** K

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = f \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix} \rightarrow \mathbf{x} = \mathrm{K}\hat{\mathbf{x}} \quad \text{where} \quad \mathrm{K} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}, \; \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \text{ and } \hat{\mathbf{x}} = \begin{bmatrix} X/Z \\ Y/Z \\ 1 \end{bmatrix}$$

Simplified as $\mathrm{K} = \begin{bmatrix} f & 0 & w/2 \\ 0 & f & h/2 \\ 0 & 0 & 1 \end{bmatrix}$ (w: image width, h: image height)

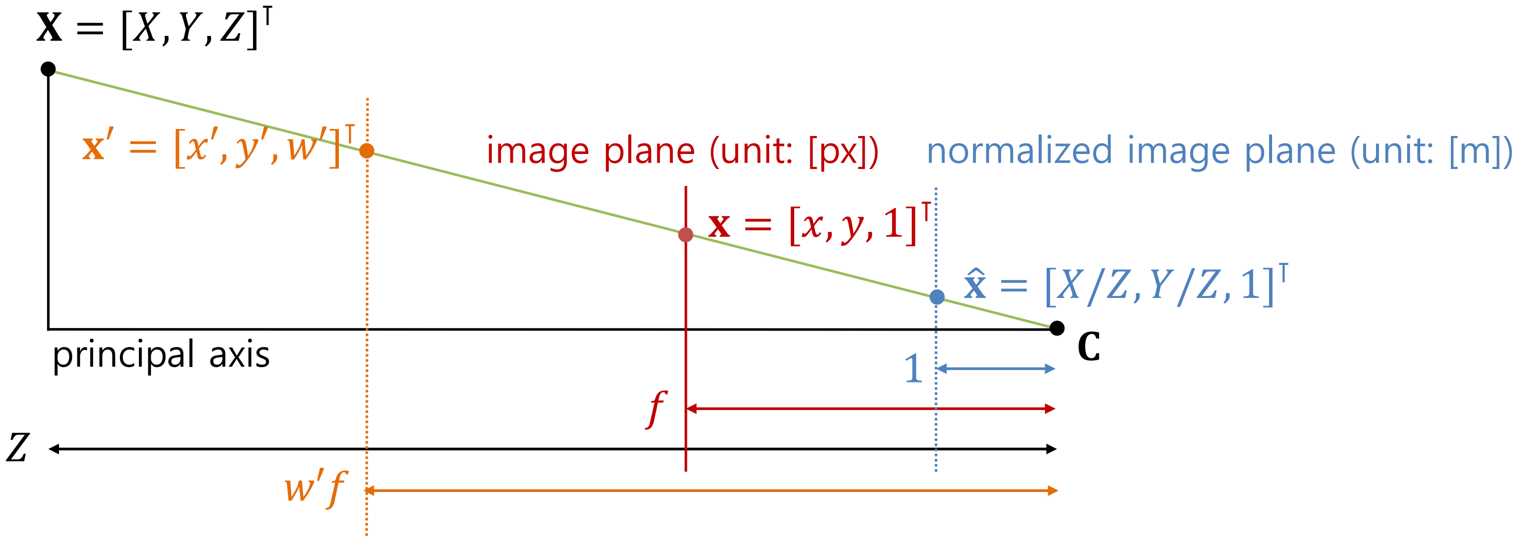
Generalized as $\mathrm{K} = \begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$ (s: skew parameter)



Pinhole Camera Model

- Homogeneous coordinates (a.k.a. projective coordinates)
  - It describes n-dimensional project space as $n+1$-dimensional coordinate system.
  - It hold non-conventional equivalence relationship: $(x_1, x_2, \ldots, x_{n+1}) \sim (\lambda x_1, \lambda x_2, \ldots, \lambda x_{n+1})$ such that $(0 \neq \lambda \in \mathbb{R})$.
    - e.g. (5, 12) is written as (5, 12, 1) which is also equal to (10, 24, 2) or (15, 36, 3) or …
  - On the previous slide, $\mathbf{x} = \mathrm{K}\hat{\mathbf{x}}$ where $\mathrm{K} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, and $\hat{\mathbf{x}} = \begin{bmatrix} X/Z \\ Y/Z \\ 1 \end{bmatrix}$

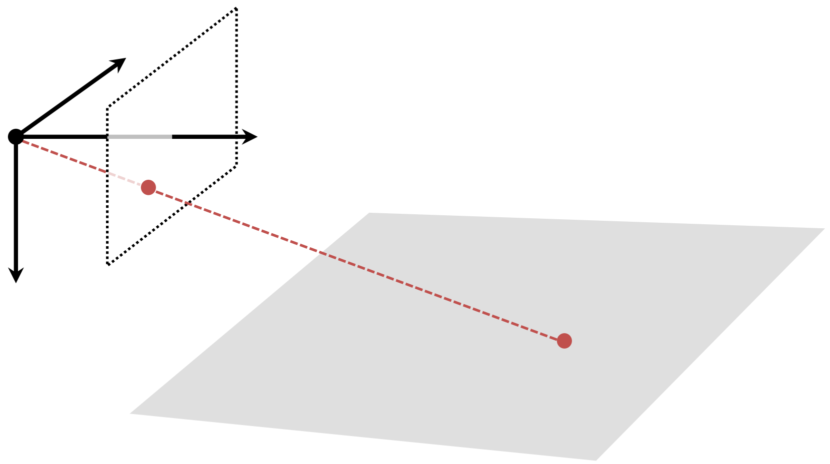
    $\mathbf{x}' = \mathrm{K}\mathbf{X}$ where $\mathrm{K} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x}' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$, and $\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ (Note: $\mathbf{x} = \frac{1}{w'}\mathbf{x}'$)

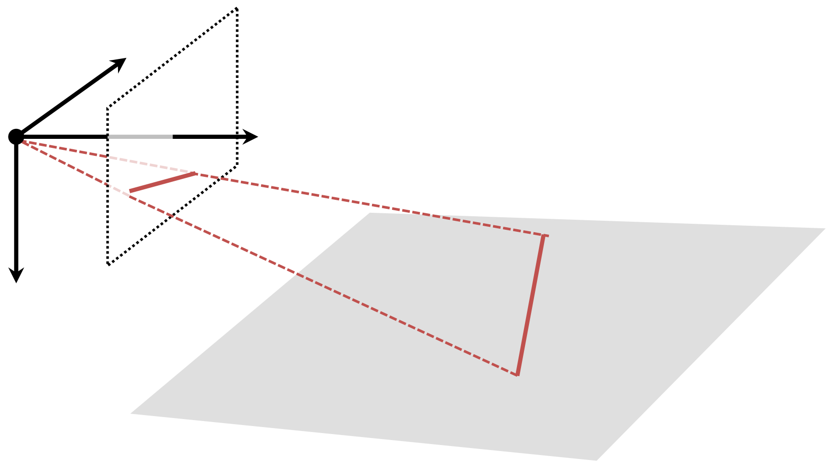


Pinhole Camera Model

- **Why homogeneous coordinates?**
  - An affine transformation ($\mathbf{y} = \mathrm{A}\mathbf{x} + \mathbf{b}$) is formulated by a single matrix multiplication.
  - A point at infinity (a.k.a. ideal point) is numerically represented by $w = 0$.
  - A point and line ($ax + by + c = 0$) are described beautifully as like $\mathbf{l}^\intercal \mathbf{x} = 0$ or $\mathbf{x}^\intercal \mathbf{l} = 0$ ($\mathbf{l} = [a, b, c]^\intercal$).
    - Intersection of two lines: $\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$
    - A line by two points: $\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$

    } *Duality* of a point and line
  - A light ray (line at the camera center) is observed as a point on the image plane.
    - A plane at the camera center is observed as a line on the image plane.
    - A conic whose peak is at the camera center is observed as a conic section on the image plane.

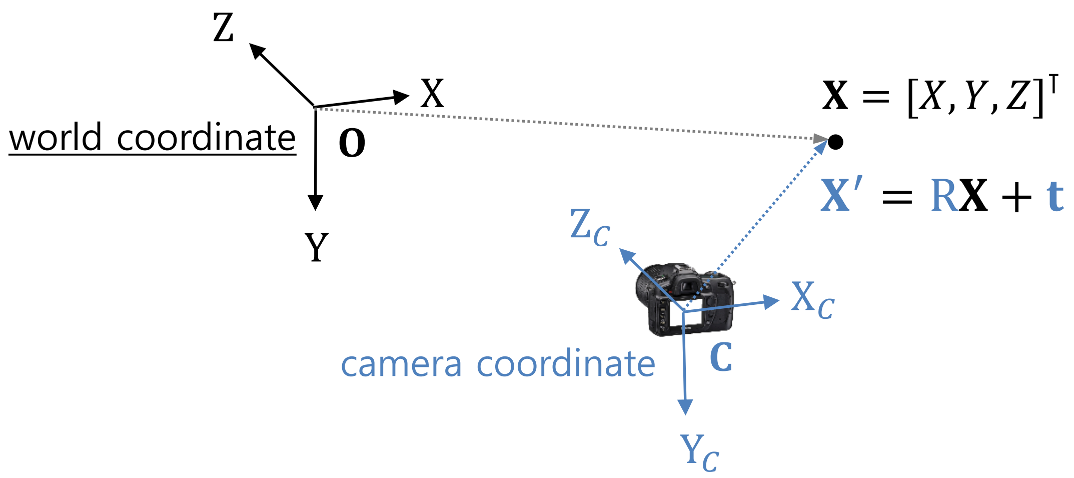




Pinhole Camera Model

- **Projection matrix** P
  - Generally, a point $\mathbf{X}$ is not based on the camera coordinate so that it need be transformed to the camera coordinate.

$$\mathbf{X}' = \mathrm{R}\mathbf{X} + \mathbf{t} \quad \rightarrow \quad \mathbf{X}' = [\mathrm{R} \mid \mathbf{t}] \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$



  - The whole camera projection from the world coordinate to the image coordinate:

$$\mathbf{x} = \mathrm{K}\mathbf{X}' = \mathrm{K}(\mathrm{R}\mathbf{X} + \mathbf{t}) = \mathrm{K}\,[\mathrm{R} \mid \mathbf{t}] \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

$\rightarrow$ $\mathbf{x} = \mathrm{P}\mathbf{X}$ where $\mathrm{P} = \mathrm{K}\,[\mathrm{R} \mid \mathbf{t}]$ (3x4 matrix), $\mathbf{x}$ and $\mathbf{X}$ in homogenous coordinates

  - Note) The camera pose ($\mathrm{R}^\top$ and $-\mathrm{R}^\top\mathbf{t}$) can be derived from the inverse of point transformation (R and $\mathbf{t}$).

$$\mathrm{T}^{-1} = \begin{bmatrix} \mathrm{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathrm{R}^\top & -\mathrm{R}^\top\mathbf{t} \\ 0 & 1 \end{bmatrix}$$

Pinhole Camera Model

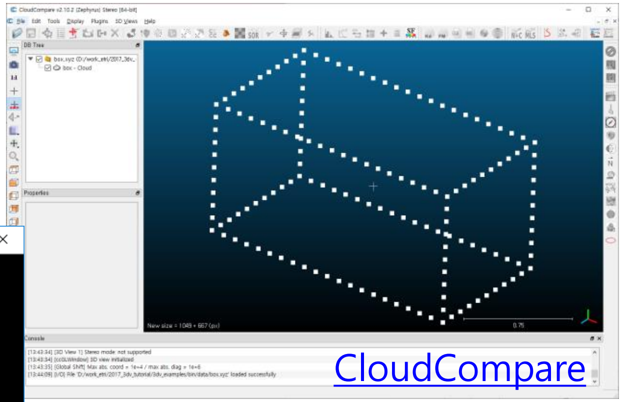
- **Camera parameters** ~ projection matrix P

$$\mathrm{P} = \mathrm{K}\,[\mathrm{R} \mid \mathbf{t}]$$

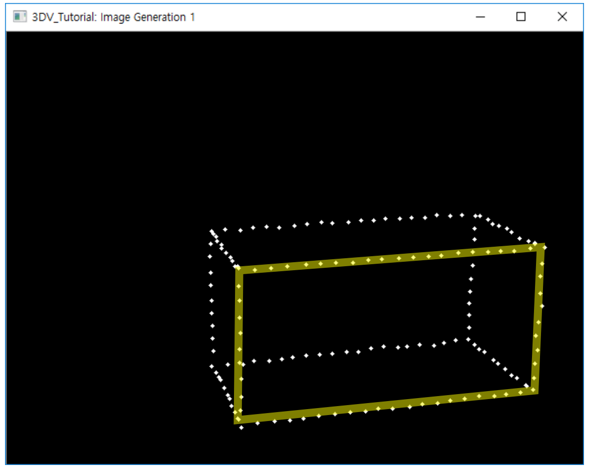
  - **Intrinsic parameters** ~ camera matrix K
    - e.g. Focal length, principle point, skew, distortion coefficient, …
  - **Extrinsic parameters** ~ point transformation R and $\mathbf{t}$
    - e.g. Rotation and translation

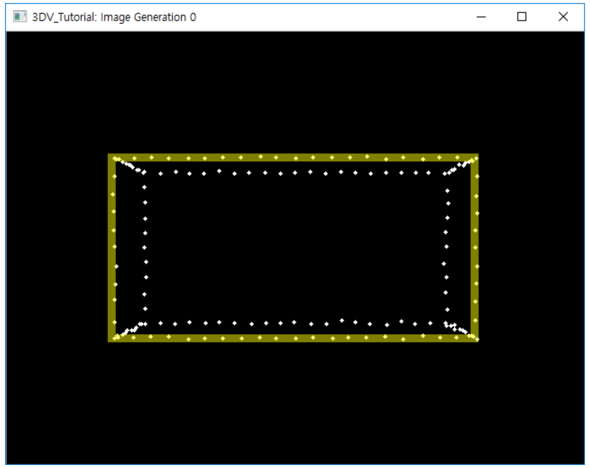
Pinhole Camera Model

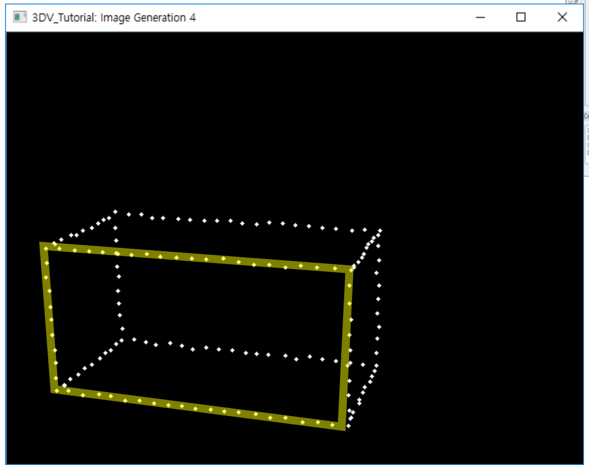
A point cloud: `data/box.xyz`

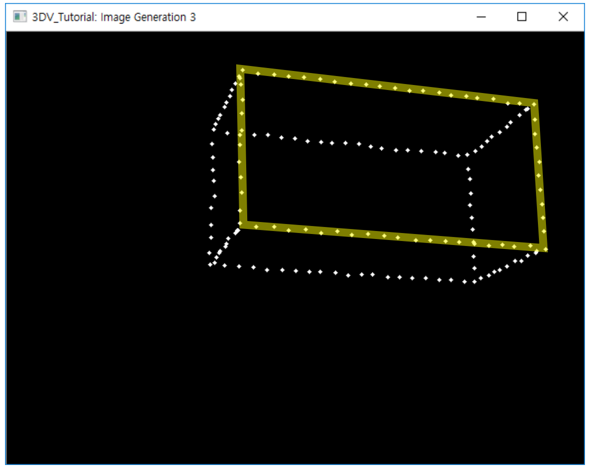


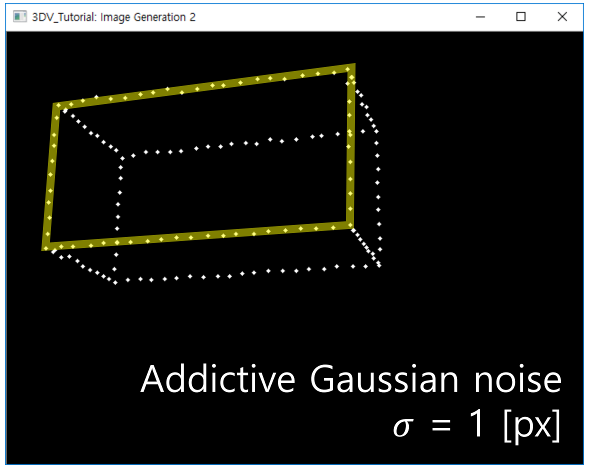
- Example) **Image formation** `[image_formation.py]`











Pinhole Camera Model

- Example) **Image formation** [image_formation.py]

```python
from scipy.spatial.transform import Rotation

# The given camera configuration: Focal length, principal point, image resolution, position, and orientation
f, cx, cy, noise_std = 1000, 320, 240, 1
img_res = (640, 480)
cam_pos = [[0, 0, 0], [-2, -2, 0], [2, 2, 0], [-2, 2, 0], [2, -2, 0]]          # Unit: [m]
cam_ori = [[0, 0, 0], [-15 , 15, 0], [15, -15, 0], [15, 15, 0], [-15, -15, 0]] # Unit: [deg]

# Load a point cloud in the homogeneous coordinate
X = np.loadtxt('../data/box.xyz') # Size: N x 3

# Generate images for each camera pose
K = np.array([[f, 0, cx], [0, f, cy], [0, 0, 1]])
for i, (pos, ori) in enumerate(zip(cam_pos, cam_ori)):
    # Derive 'R' and 't'
    Rc = Rotation.from_euler('zyx', ori[::-1], degrees=True).as_matrix()
    R = Rc.T
    t = -Rc.T @ pos

    # Project the points (Alternative: `cv.projectPoints()`)
    x = K @ (R @ X.T + t.reshape(-1, 1)) # Size: 3 x N
    x /= x[-1]

    # Add Gaussian noise
    noise = np.random.normal(scale=noise_std, size=(2, len(X)))
    x[0:2,:] += noise
```

$$\mathbf{K} = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{R} = \mathrm{R}_c^\top$ and $\mathbf{t} = -\mathrm{R}_c^\top \mathbf{t}_c$

$$\mathbf{x} = \mathrm{K}(\mathrm{R}\mathbf{X} + \mathbf{t})$$

$$\mathbf{x} = \begin{bmatrix} x/w \\ y/w \\ w/w \end{bmatrix}$$

Geometric Distortion Models

Q) How to represent such geometric distortion?

- **Geometric distortion models**
  - A camera lens generates geometric distortion, which can be approximated (modeled) as a nonlinear function f_d.
  - Geometric distortion models f_d are mostly defined on the normalized image plane.
  - Camera projection with geometric distortion: $\mathbf{x} = \mathrm{proj}(\mathbf{X}; \mathrm{K}, \mathrm{R}, \mathbf{t}, d)$ where d is a set of distortion coefficients.

Note) $\mathbf{x} = \mathrm{K}(\mathrm{R}\mathbf{X} + \mathbf{t})$ without distortion and normalization

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \longrightarrow \mathbf{X}' = \mathrm{R}\mathbf{X} + \mathbf{t} \longrightarrow \hat{\mathbf{x}} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} X'/Z' \\ Y'/Z' \end{bmatrix} \longrightarrow \hat{\mathbf{x}}_d = f_d(\hat{\mathbf{x}}) \longrightarrow \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_x \hat{x}_d + c_x \\ f_y \hat{y}_d + c_y \end{bmatrix}$$

3D point (the world coordinate) → 3D point (the camera coordinate) → 2D point (the normalized image plane) → **geometric distortion** → pinhole camera projection

Image: Shawn Becker

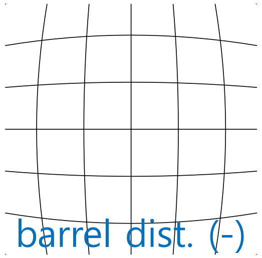
Image: [Camera Calibration Toolbox for MATLAB](), [Wikipedia](), [MathWorks]()

Geometric Distortion Models

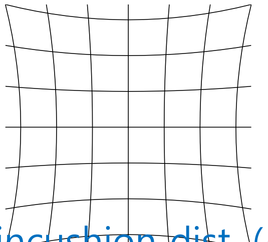
- **Geometric distortion models**
  - **Polynomial distortion model** (a.k.a. Brown-Conrady model; 1919)

$$\begin{bmatrix} \hat{x}_d \\ \hat{y}_d \end{bmatrix} = (1 + k_1 r^2 + k_2 r^4 + \cdots) \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} + (1 + p_3 r^2 + p_4 r^4 + \cdots) \begin{bmatrix} 2 p_1 \hat{x}\hat{y} + p_2 (r^2 + 2\hat{x}^2) \\ 2 p_2 \hat{x}\hat{y} + p_1 (r^2 + 2y^2) \end{bmatrix} \quad \text{where} \quad r^2 = \hat{x}^2 + \hat{y}^2$$

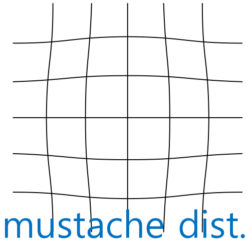
    - OpenCV (default): `cv.projectPoints()` ↔ `cv.undistortPoints()`



barrel dist. (-)

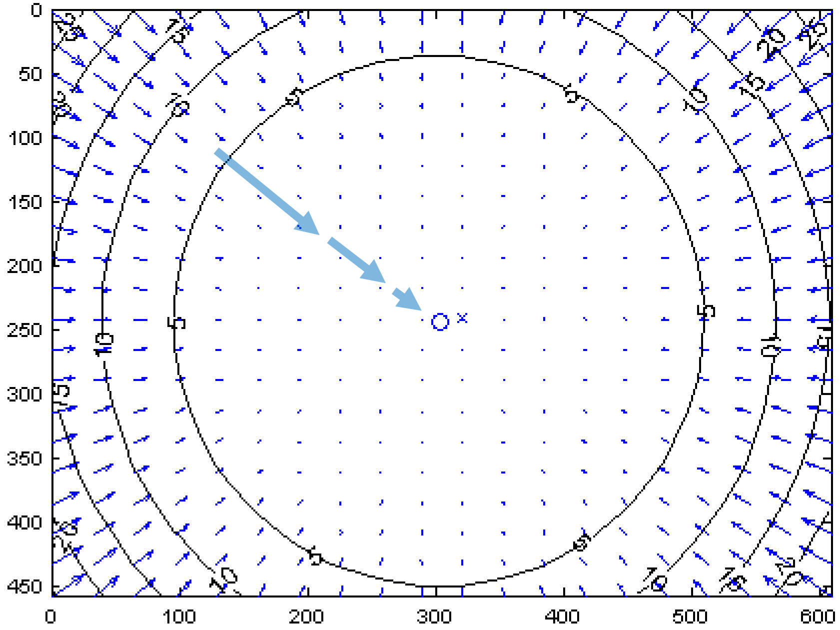


pincushion dist. (+)



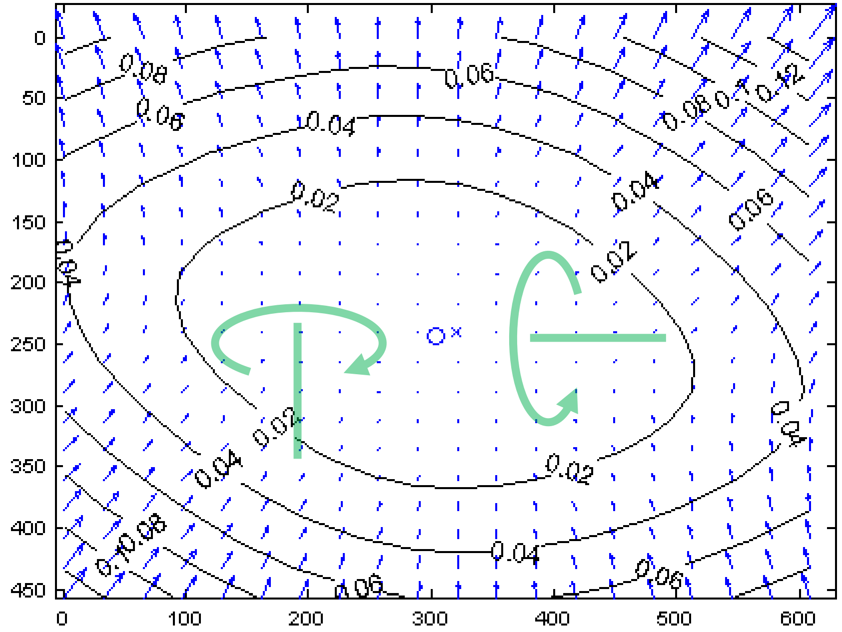
mustache dist.

Radial distortion

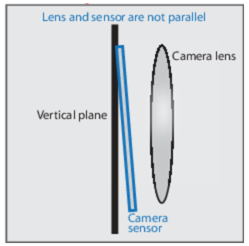


```
Pixel error            = [0.1174, 0.1159]
Focal Length           = (657.303, 657.744)    +/- [0.2849, 0.2894]
Principal Point        = (302.717, 242.334)    +/- [0.5912, 0.5571]
Skew                   = 0.0004198             +/- 0.0001905
Radial coefficients    = (-0.2535, 0.1187, 0)  +/- [0.00231, 0.009418, 0]
Tangential coefficients = (-0.0002789, 5.174e-005)  +/- [0.0001217, 0.0001208]
```

Tangential distortion



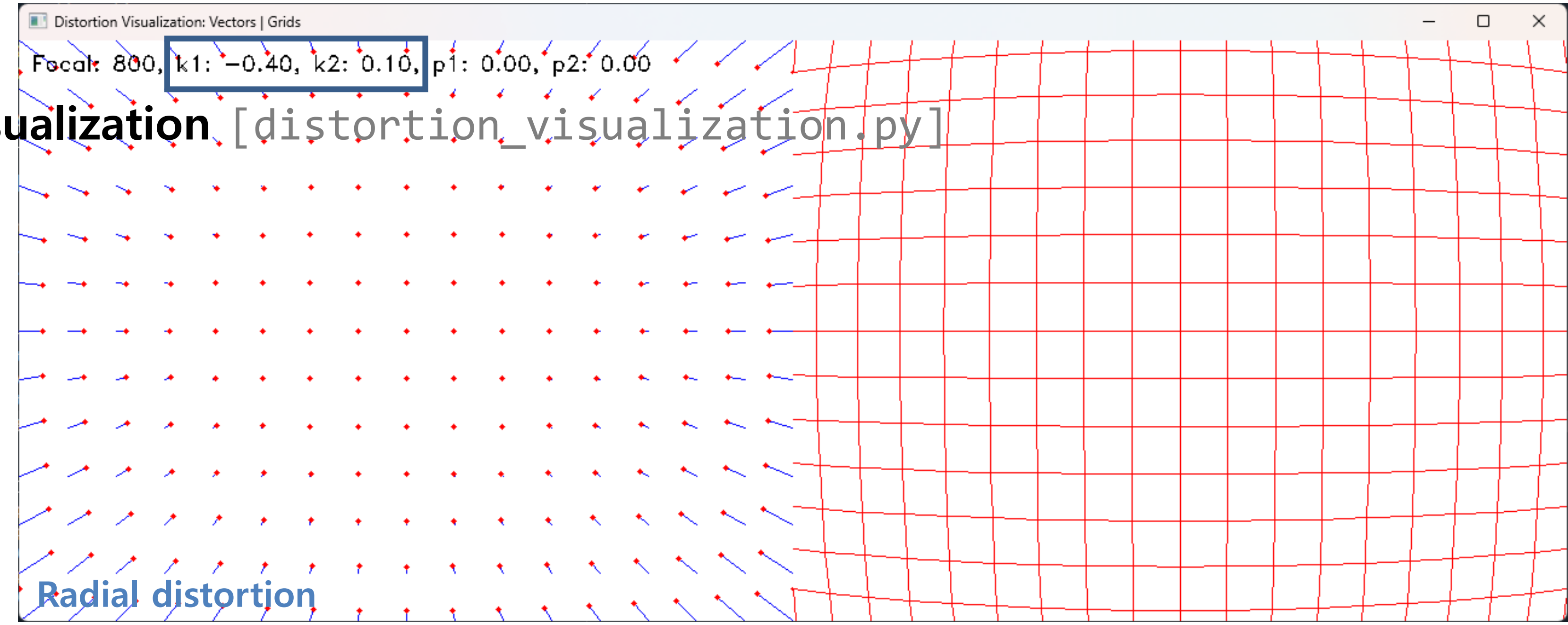
```
Pixel error            = [0.1174, 0.1159]
Focal Length           = (657.303, 657.744)    +/- [0.2849, 0.2894]
Principal Point        = (302.717, 242.334)    +/- [0.5912, 0.5571]
Skew                   = 0.0004198             +/- 0.0001905
Radial coefficients    = (-0.2535, 0.1187, 0)  +/- [0.00231, 0.009418, 0]
Tangential coefficients = (-0.0002789, 5.174e-005)  +/- [0.0001217, 0.0001208]
```

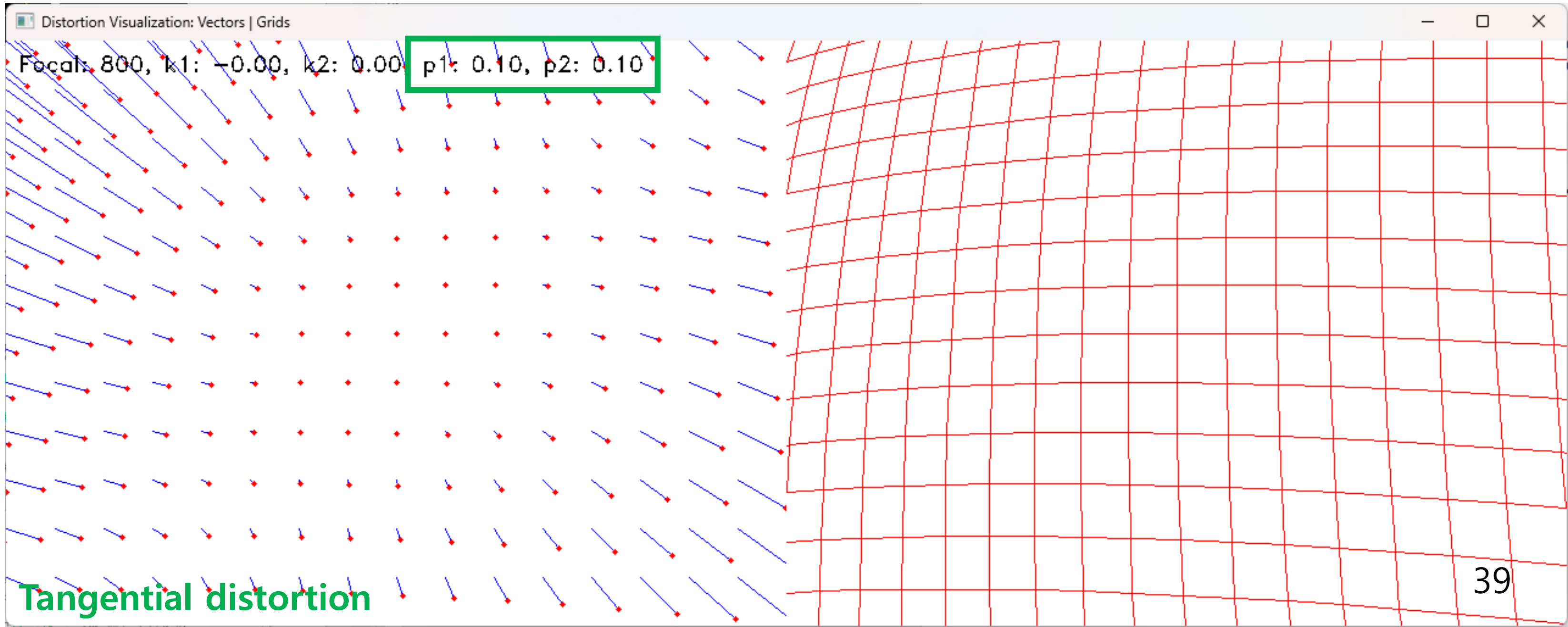


Why?
Lens and sensor are not parallel.
(usually negligible)

Geometric Distortion Models

- Example) **Geometric distortion visualization** [distortion_visualization.py]





Geometric Distortion Models

- Example) **Geometric distortion visualization** [distortion_visualization.py]

```python
# The initial camera configuration
img_w, img_h = (640, 480)
K = np.array([[800, 0, 320],
              [0, 800, 240],
              [0, 0, 1.]])
dist_coeff = np.array([-0.2, 0.1, 0, 0])
grid_x, grid_y, grid_z = (-18, 19), (-15, 16), 20

obj_pts = np.array([[x, y, grid_z] for y in range(*grid_y) for x in range(*grid_x)], dtype=np.float32)
while True:
    # Project 3D points with/without distortion
    dist_pts, _ = cv.projectPoints(obj_pts, np.zeros(3), np.zeros(3), K, dist_coeff)
    zero_pts, _ = cv.projectPoints(obj_pts, np.zeros(3), np.zeros(3), K, np.zeros(4))

    # Draw vectors
    img_vector = np.full((img_h, img_w, 3), 255, dtype=np.uint8)
    for zero_pt, dist_pt in zip(zero_pts, dist_pts):
        cv.line(img_vector, np.int32(zero_pt.flatten()), np.int32(dist_pt.flatten()), (255, 0, 0))
    for pt in dist_pts:
        cv.circle(img_vector, np.int32(pt.flatten()), 1, (0, 0, 255), -1)

    # Draw grids
    img_grid = np.full((img_h, img_w, 3), 255, dtype=np.uint8)
    dist_pts = dist_pts.reshape(len(range(*grid_y)), -1, 2)
    for pts in dist_pts:
        cv.polylines(img_grid, [np.int32(pts)], False, (0, 0, 255))
```

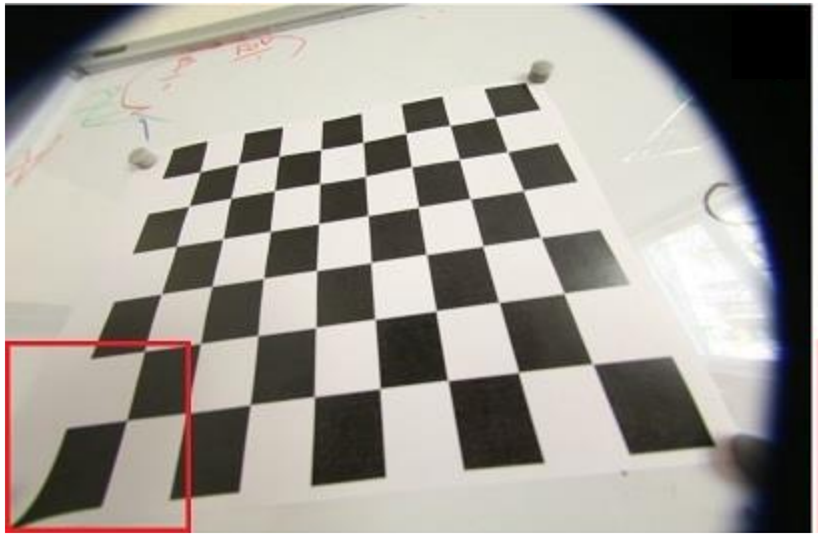
Camera Projection Model

- **Geometric distortion models**
  - **Fisheye lens model** (a.k.a. Kannala-Brandt model; T-PAMI 2006)

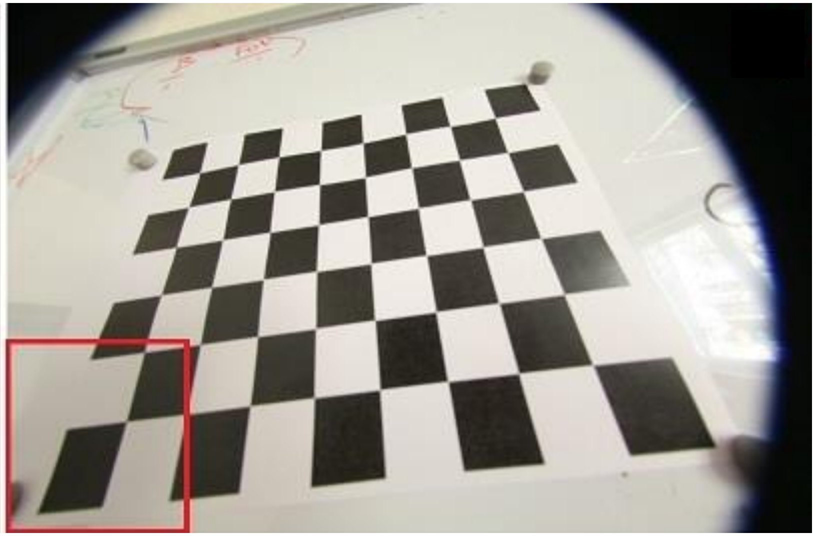
$$\begin{bmatrix} \hat{x}_d \\ \hat{y}_d \end{bmatrix} = (1 + k_1\theta^2 + k_2\theta^4 + \cdots)\frac{\theta}{r}\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \quad \text{where} \quad r^2 = \hat{x}^2 + \hat{y}^2 \quad \text{and} \quad \theta = \tan^{-1} r$$

    - The fisheye lens model can describe strong barrel distortion especially around image boundary.
    - OpenCV: `cv.fisheye.projectPoints()` ↔ `cv.fisheye.undistortPoints()`

Original fisheye image



Polynomial distortion model



Fisheye lens model

Image: OpenCV

Geometric Distortion Models

- **Geometric distortion correction**
  - Input: The original image
  - Output: Its rectified image (without geometric distortion)
  - Given: Its camera matrix and distortion coefficient
  - Solutions for the polynomial distortion model
    - OpenCV `cv.undistort()` and `cv.undistortPoints()` (Note: included in `imgproc` module)
      ↔ `cv.projectPoints()` (Note: included in `calib3d` module)

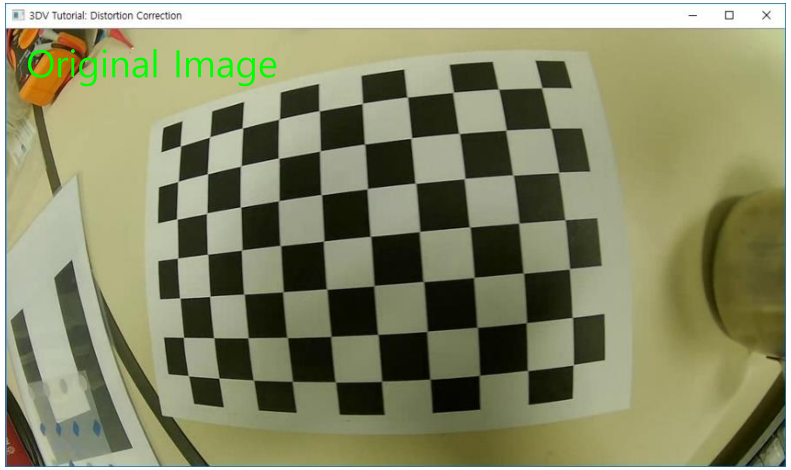
distortion correction →

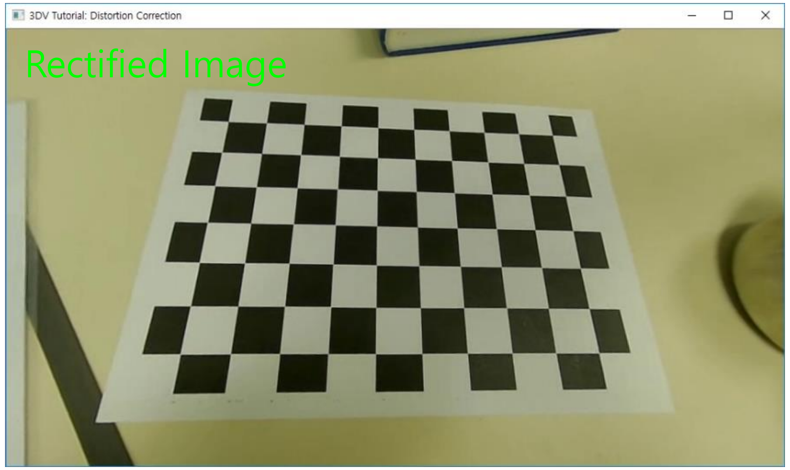
```
K1: 1.105763E-01
K2: 1.886214E-02
K3: 1.473832E-02
P1:-8.448460E-03
P2:-7.356744E-03
```

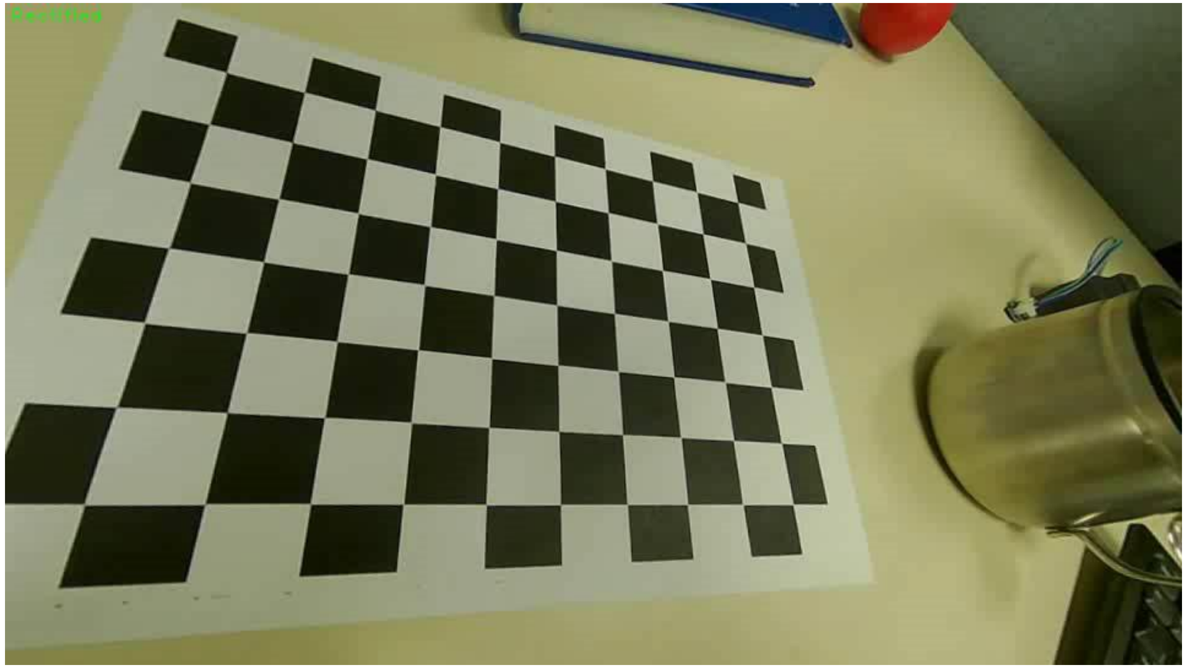
Image: [Shawn Becker]

Geometric Distortion Models

- Example) **Geometric distortion correction** `[distortion_correction.py]`







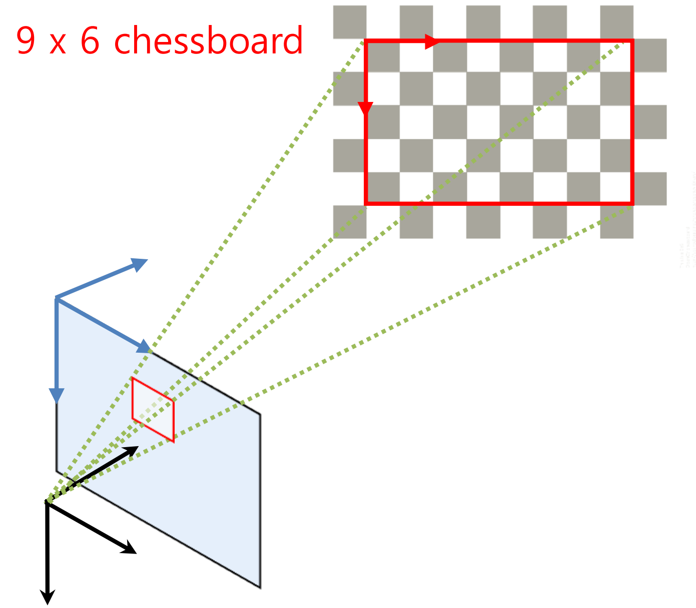
Geometric Distortion Models

- Example) **Geometric distortion correction** [distortion_correction.py]

```python
# The given video and calibration data
video_file = '../data/chessboard.avi'
K = np.array([[432.7390364738057, 0, 476.0614994349778],
              [0, 431.2395555913084, 288.7602152621297],
              [0, 0, 1]]) # Derived from `calibrate_camera.py`
dist_coeff = np.array([-0.2852754904152874, 0.1016466459919075, -0.0004420196146339175, ...])

# Open a video
video = cv.VideoCapture(video_file)

# Run distortion correction
show_rectify = True
map1, map2 = None, None
while True:
    # Read an image from the video
    valid, img = video.read()
    ...

    # Rectify geometric distortion (Alternative: `cv.undistort()`)
    info = "Original"
    if show_rectify:
        if map1 is None or map2 is None:
            map1, map2 = cv.initUndistortRectifyMap(K, dist_coeff, None, None, (img.shape[1], img.shape[0]), cv.CV_3
        img = cv.remap(img, map1, map2, interpolation=cv.INTER_LINEAR)
        info = "Rectified"
    cv.putText(img, info, (10, 25), cv.FONT_HERSHEY_DUPLEX, 0.6, (0, 255, 0))
```

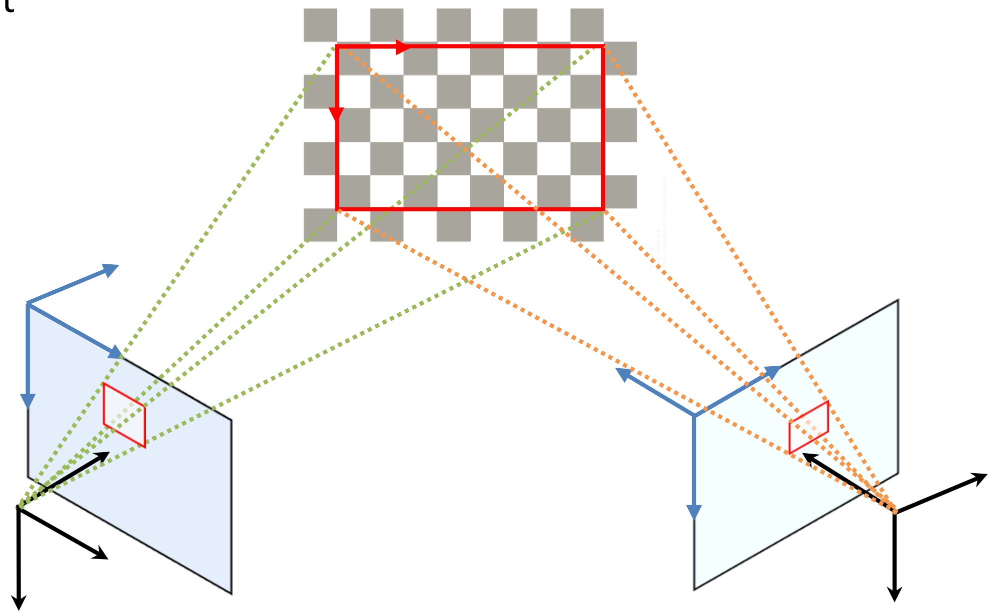
Camera Calibration

- **Camera calibration**
  - Unknown: Intrinsic + extrinsic parameters (5* + 6 DOF)
    - Note) The number of intrinsic parameters* can be varied according to user configuration.
  - Given: 3D points $\mathbf{X}_1$, $\mathbf{X}_2$, …, $\mathbf{X}_n$ and their projected points $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$
  - Constraints: n x projection $\mathbf{x}_i = \mathrm{K}\,[\,\mathrm{R}\,|\,\mathbf{t}\,]\,\mathbf{X}_i$



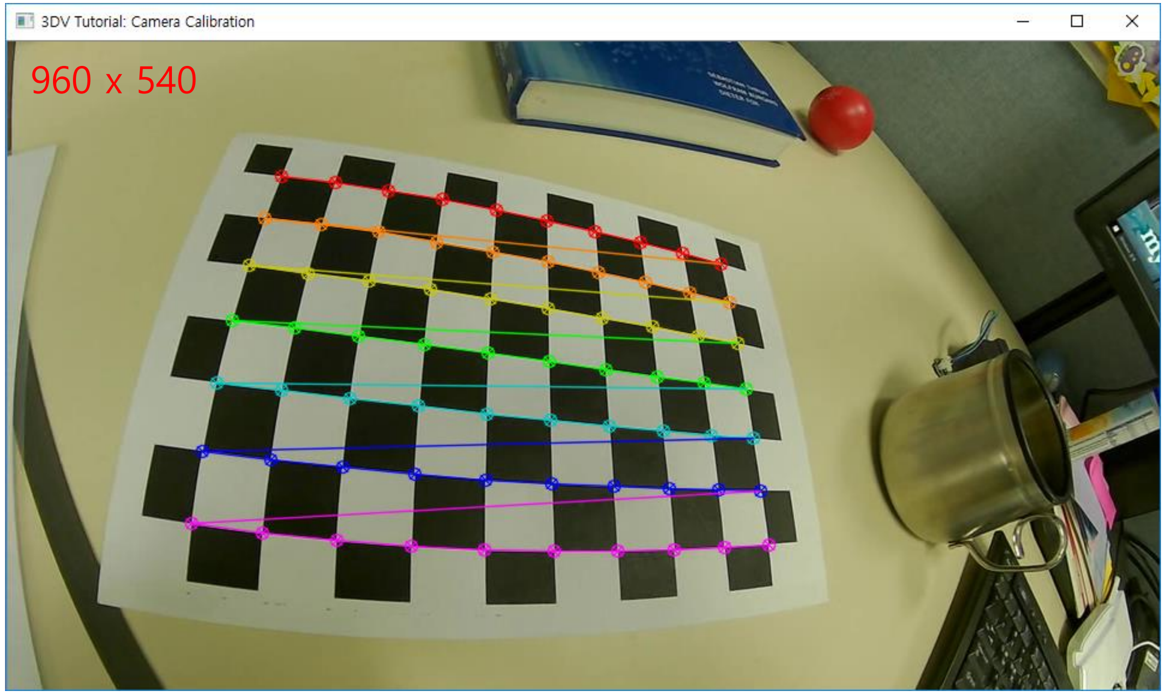
Camera Calibration

- **Camera calibration**
  - Unknown: Intrinsic + m x extrinsic parameters (5* + m x 6 DOF)
  - Given: 3D points $\mathbf{X}_1$, $\mathbf{X}_2$, …, $\mathbf{X}_n$ and their projected points, $\mathbf{x}_i^j$, on the jth image
    - Note) m: the number of images, n: the number of 3D points
  - Constraints: m x n x projection $\mathbf{x}_i^j = \mathrm{K}\left[\,\mathrm{R}_j \,|\, \mathbf{t}_j\,\right]\mathbf{X}_i$
  - Solutions [Tools]
    - OpenCV: `cv.calibrateCamera()` and `cv.initCameraMatrix2D()`
    - Camera Calibration Toolbox for MATLAB, Jean-Yves Bouguet
    - DarkCamCalibrator, 다크 프로그래머
  - Note) How to get calibration boards
    - Print out the pattern and stick it on a hard board
    - Calibration Checkerboard Collection, Mark Jones
    - Pattern Generator, calib.io



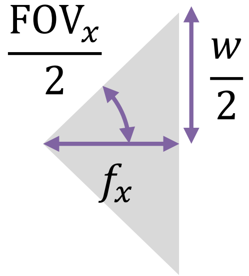
Camera Calibration

- Example) **Camera calibration** [camera_calibration.py]



Note) **Field-of-view (FOV)** = focal length (w/o distortion)

- Horizontal: $2 \times \tan^{-1} \frac{w/2}{f_x} = 96°$
- Vertical: $2 \times \tan^{-1} \frac{h/2}{f_y} = 64°$




```
## Camera Calibration Results
* The number of applied images = 22
* RMS error = 0.473353
* Camera matrix (K) =
  [432.7390364738057, 0, 476.0614994349778]
  [0, 431.2395555913084, 288.7602152621297]
  [0, 0, 1]
* Distortion coefficient (k1, k2, p1, p2, k3, ...) =
  [-0.2852754904152874, 0.1016466459919075, -0.0004420196146339175, 0.0001149909868437517, -0.01803978785585194]
```

Note) Close to the center of the image, (480, 270)

Note) Close to zero (usually negligible)

Camera Calibration

- Example) **Camera calibration** [camera_calibration.py]

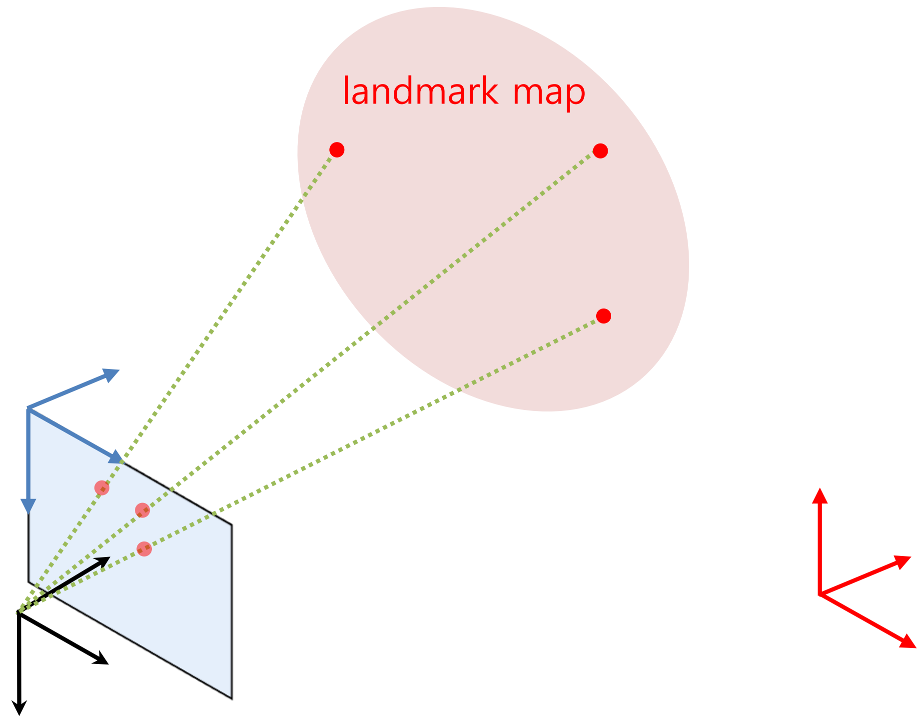
```
def select_img_from_video(video_file, board_pattern, select_all=False, wait_msec=10):
    # Open a video
    video = cv.VideoCapture(video_file)

    # Select images
    img_select = []
    ...
    return img_select

def calib_camera_from_chessboard(images, board_pattern, board_cellsize, K=None, dist_coeff=None, calib_flags=None):
    # Find 2D corner points from given images
    img_points = []
    for img in images:
        gray = cv.cvtColor(img, cv.COLOR_BGR2GRAY)
        complete, pts = cv.findChessboardCorners(gray, board_pattern)
        if complete:
            img_points.append(pts)
    assert len(img_points) > 0, 'There is no set of complete chessboard points!'

    # Prepare 3D points of the chess board
    obj_pts = [[c, r, 0] for r in range(board_pattern[1]) for c in range(board_pattern[0])]
    obj_points = [np.array(obj_pts, dtype=np.float32) * board_cellsize] * len(img_points) # Must be `np.float32`

    # Calibrate the camera
    return cv.calibrateCamera(obj_points, img_points, gray.shape[::-1], K, dist_coeff, flags=calib_flags)
```

$\mathbf{x}_i^j$

$\mathbf{X}_i$

Absolute Camera Pose Estimation

- **Absolute camera pose estimation** (perspective-n-point; PnP)
  - Unknown: Camera pose R and $\mathbf{t}$ (6 DOF)
  - Given: 3D points $\mathbf{X}_1$, $\mathbf{X}_2$, …, $\mathbf{X}_n$, their projected points $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$, and camera matrix K
  - Constraints: n x projection $\mathbf{x}_i = \mathrm{K}\,[\,\mathrm{R}\,|\,\mathbf{t}\,]\,\mathbf{X}_i$
  - Solutions ($n \geq 3$) → 3-point algorithm
    - OpenCV: `cv.solvePnP()` and `cv.solvePnPRansac()`



Absolute Camera Pose Estimation

- Example) **Pose estimation (chessboard)** `[pose_estimation_chessboard.py]`

- Prepare a 3D box (AR model)
- Prepare 3D points (chessboard)

↓

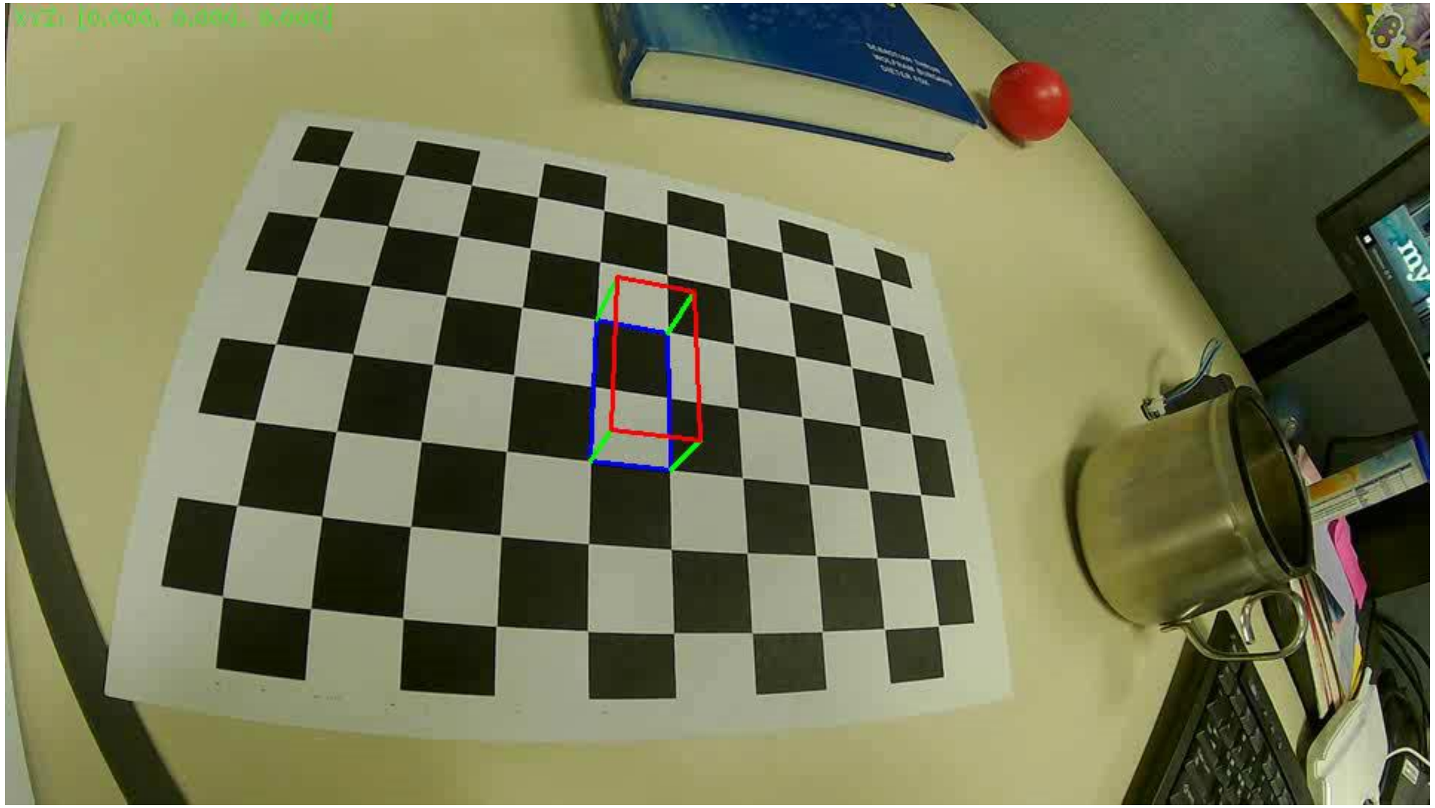
- Grab an image
- Find a chessboard on the image

↓

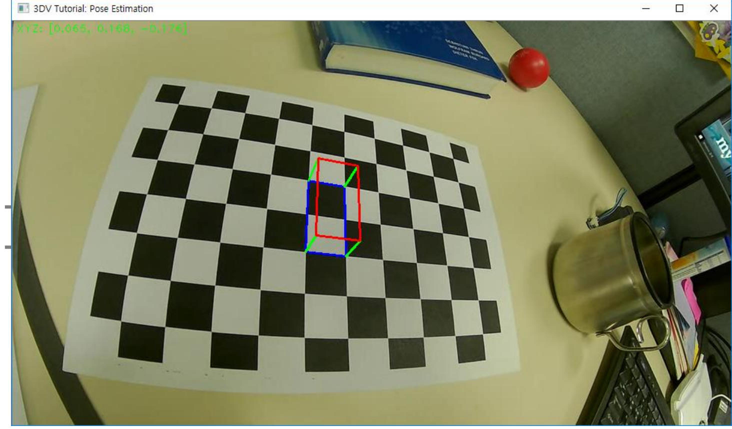
[Camera pose estimation]
- Estimate camera pose using `cv.solvePnP()`

↓

[AR rendering]
- Draw the 3D box on the image using `cv.projectPoints()`



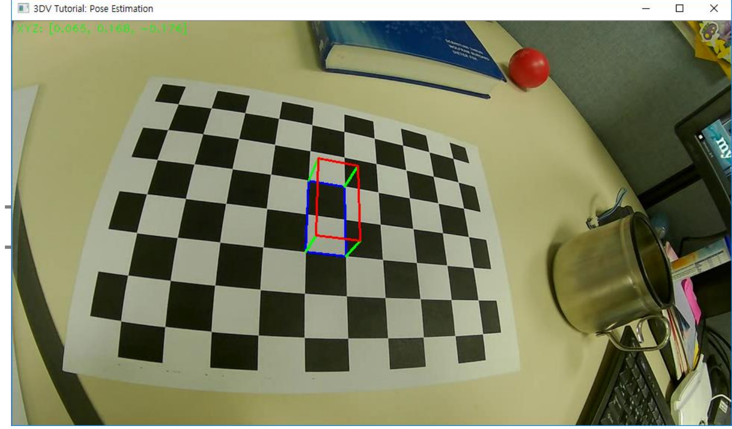
Absolute Camera Pose Estimation



- Example) **Pose estimation (chessboard)** [pose_estimation_chessboard.py]

```
# Open a video
video = cv.VideoCapture(video_file)

# Prepare a 3D box for simple AR
box_lower = board_cellsize * np.array([[4, 2,  0], [5, 2,  0], [5, 4,  0], [4, 4,  0]])
box_upper = board_cellsize * np.array([[4, 2, -1], [5, 2, -1], [5, 4, -1], [4, 4, -1]])

# Prepare 3D points on a chessboard
obj_points = board_cellsize * np.array([[c, r, 0] for r in range(board_pattern[1]) for c in range(board_pattern[0])]

# Run pose estimation
while True:
    # Read an image from the video
    valid, img = video.read()
    if not valid:
        break

    # Estimate the camera pose
    complete, img_points = cv.findChessboardCorners(img, board_pattern, board_criteria)
    if complete:
        ret, rvec, tvec = cv.solvePnP(obj_points, img_points, K, dist_coeff)

        # Draw the box on the image
        line_lower, _ = cv.projectPoints(box_lower, rvec, tvec, K, dist_coeff)
        line_upper, _ = cv.projectPoints(box_upper, rvec, tvec, K, dist_coeff)
        cv.polylines(img, [np.int32(line_lower)], True, (255, 0, 0), 2)
```

Absolute Camera Pose Estimation



- Example) **Pose estimation (chessboard)** [pose_estimation_chessboard.py]

```python
# Open a video
# Prepare a 3D box for simple AR
# Prepare 3D points on a chessboard

# Run pose estimation
while True:
    # Read an image from the video

    # Estimate the camera pose
    complete, img_points = cv.findChessboardCorners(img, board_pattern, board_criteria)
    if complete:
        ret, rvec, tvec = cv.solvePnP(obj_points, img_points, K, dist_coeff)

        # Draw the box on the image
        line_lower, _ = cv.projectPoints(box_lower, rvec, tvec, K, dist_coeff)
        line_upper, _ = cv.projectPoints(box_upper, rvec, tvec, K, dist_coeff)
        cv.polylines(img, [np.int32(line_lower)], True, (255, 0, 0), 2)
        cv.polylines(img, [np.int32(line_upper)], True, (0, 0, 255), 2)
        for b, t in zip(line_lower, line_upper):
            cv.line(img, np.int32(b.flatten()), np.int32(t.flatten()), (0, 255, 0), 2)

        # Print the camera position
        R, _ = cv.Rodrigues(rvec) # Alternative) `scipy.spatial.transform.Rotation`
        p = (-R.T @ tvec).flatten()
        info = f'XYZ: [{p[0]:.3f} {p[1]:.3f} {p[2]:.3f}]'
        cv.putText(img, info, (10, 25), cv.FONT_HERSHEY_DUPLEX, 0.6, (0, 255, 0))
```

Absolute Camera Pose Estimation

- Example) **Pose estimation (book)** `[pose_estimation_book1.py]`

- Load the cover image
- Extract ORB features
- Build **3D object points** $\mathbf{X}_1, \mathbf{X}_2, \ldots$

- Grab an input image
- Extract **ORB features** $\mathbf{x}_1, \mathbf{x}_2, \ldots$
- Match them to the object features

Pass tentative matches

[Outlier Rejection]
- Remove wrong matches using `cv::solvePnPRansac()`

Pass correct matches

[Camera Pose Estimation]
- Estimate camera pose using `cv::solvePnP()`

[Camera Pose Update]
- Intrinsic: K (given and fixed)
- Extrinsic: R, $\mathbf{t}$

Draw the box using `cv::projectPoints()`

(due to unknown and changing K; autofocus)

Absolute Camera Pose Estimation

- Example) **Pose estimation (book) + camera calibration** [pose_estimation_book2.py]

- Load the cover image
- Extract ORB features
- Build **3D object points** $\mathbf{X}_1, \mathbf{X}_2, \ldots$

- Grab an input image
- Extract **ORB features** $\mathbf{x}_1, \mathbf{x}_2, \ldots$
- Match them to the object features

Pass tentative matches

[Outlier Rejection]
- Remove wrong matches using `cv::solvePnPRansac()`

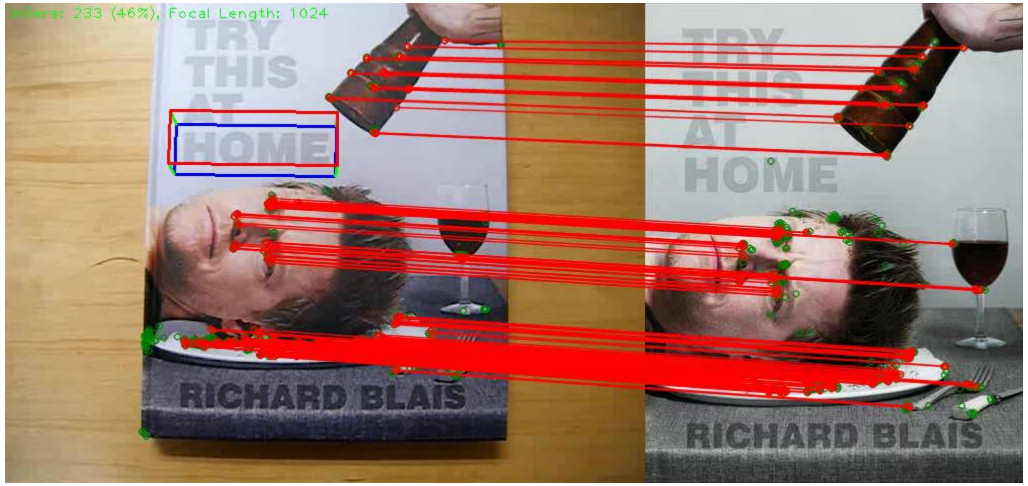
Pass correct matches

[Camera Pose Estimation]
- Estimate camera parameters using `cv::calibrateCamera()`

[Camera Pose Update]
- Intrinsic: K (initially given)
- Extrinsic: R, $\mathbf{t}$

Draw the box using `cv::projectPoints()`

(due to wrong initial K)



Absolute Camera Pose Estimation

- Example) **Pose estimation (book) + camera calibration – initially given** K `[pose_estimation_book3.py]`

(due to wrong initial K)

- Load the cover image
- Extract ORB features
- Build **3D object points** $\mathbf{X}_1, \mathbf{X}_2, \ldots$

- Grab an input image
- Extract **ORB features** $\mathbf{x}_1, \mathbf{x}_2, \ldots$
- Match them to the object features

Pass tentative matches

[Outlier Rejection]
- Remove wrong matches using `cv::findHomography()`

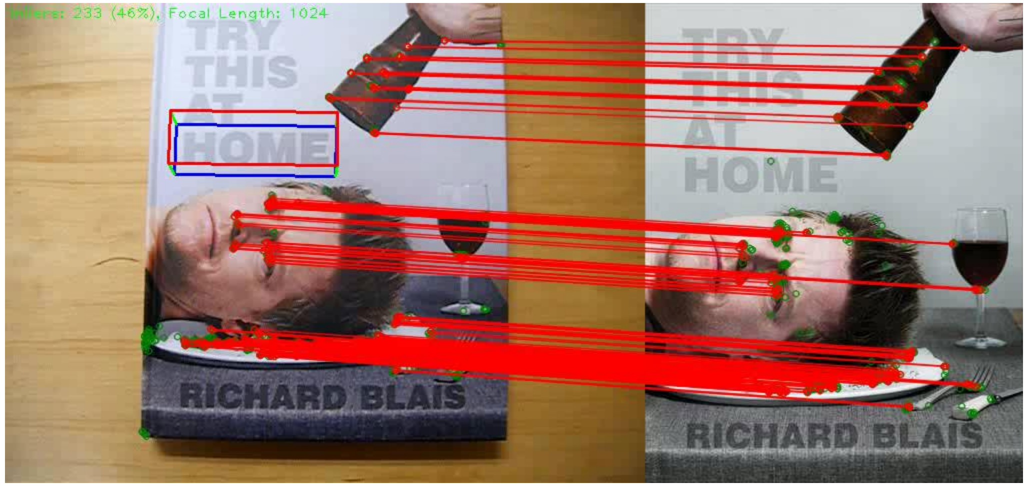
(free from K)

Pass correct matches

[Camera Pose Estimation]
- Estimate camera parameters using `cv::calibrateCamera()`

[Camera Pose Update]
- Intrinsic: K (initially known)
- Extrinsic: R, $\mathbf{t}$

Draw the box using `cv::projectPoints()`



Summary) Single-view Geometry

- **Camera Projection Models:** $\mathbf{x} = \mathrm{proj}(\mathbf{X}; \mathrm{K}, \mathrm{R}, \mathbf{t}, d)$
  - **Pinhole camera model**: $\mathbf{x} = \mathrm{K}(\mathrm{R}\mathbf{X} + \mathbf{t})$
    - Note) Homogeneous coordinate
    - Example) Object localization / image formation
  - **Geometric distortion models**: $\hat{\mathbf{x}}_d = f_d(\hat{\mathbf{x}})$ on the normalized image plane ($\hat{\mathbf{x}}$; $\hat{z} = 1$)
    - e.g. Polynomial distortion model: Radial distortion and tangential distortion
    - Example) Distortion visualization / distortion correction
- **Camera Calibration**
  - Problem) Finding camera intrinsic parameters (K, distortion coefficients) and extrinsic parameters (R and $\mathbf{t}$)
  - Example) Camera calibration
- **Absolute Camera Pose Estimation (PnP)**
  - Problem) Finding camera extrinsic parameters (R and $\mathbf{t}$) → camera pose ($\mathrm{R}^\intercal$ and $-\mathrm{R}^\intercal\mathbf{t}$)
  - Example) Pose estimation (chessboard)
  - Example) Pose estimation (book) as three versions
    - Q) What is RANSAC? What is homography?