Note) Slides and example codes are available:
https://github.com/mint-lab/3dv_tutorial

An Invitation to 3D Vision: Two-View Geometry

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Review) Absolute Camera Pose Estimation

- Example) **Pose estimation (book) + camera calibration – initially given** K `[pose_estimation_book3.py]`

(due to wrong initial K)

- Load the cover image
- Extract ORB features
- Build **3D object points** $\mathbf{X}_1, \mathbf{X}_2, \ldots$

Pass tentative matches

Pass correct matches

- Grab an input image
- Extract **ORB features** $\mathbf{x}_1, \mathbf{x}_2, \ldots$
- Match them to the object features

[Outlier Rejection]
- Remove wrong matches using `cv::findHomography()`

(free from K)

[Camera Pose Estimation]
- Estimate camera parameters using `cv::calibrateCamera()`

[Camera Pose Update]
- Intrinsic: K (initially known)
- Extrinsic: R, $\mathbf{t}$

Draw the box using `cv::projectPoints()`

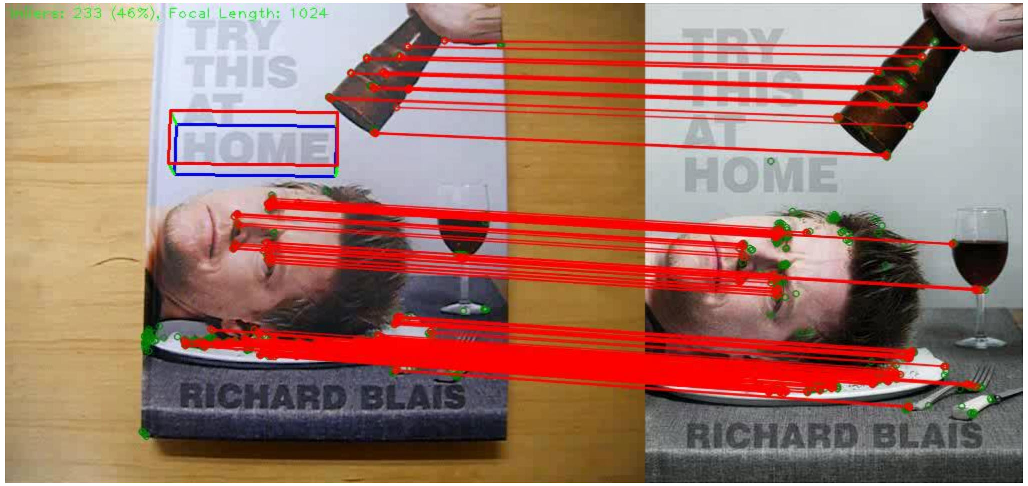
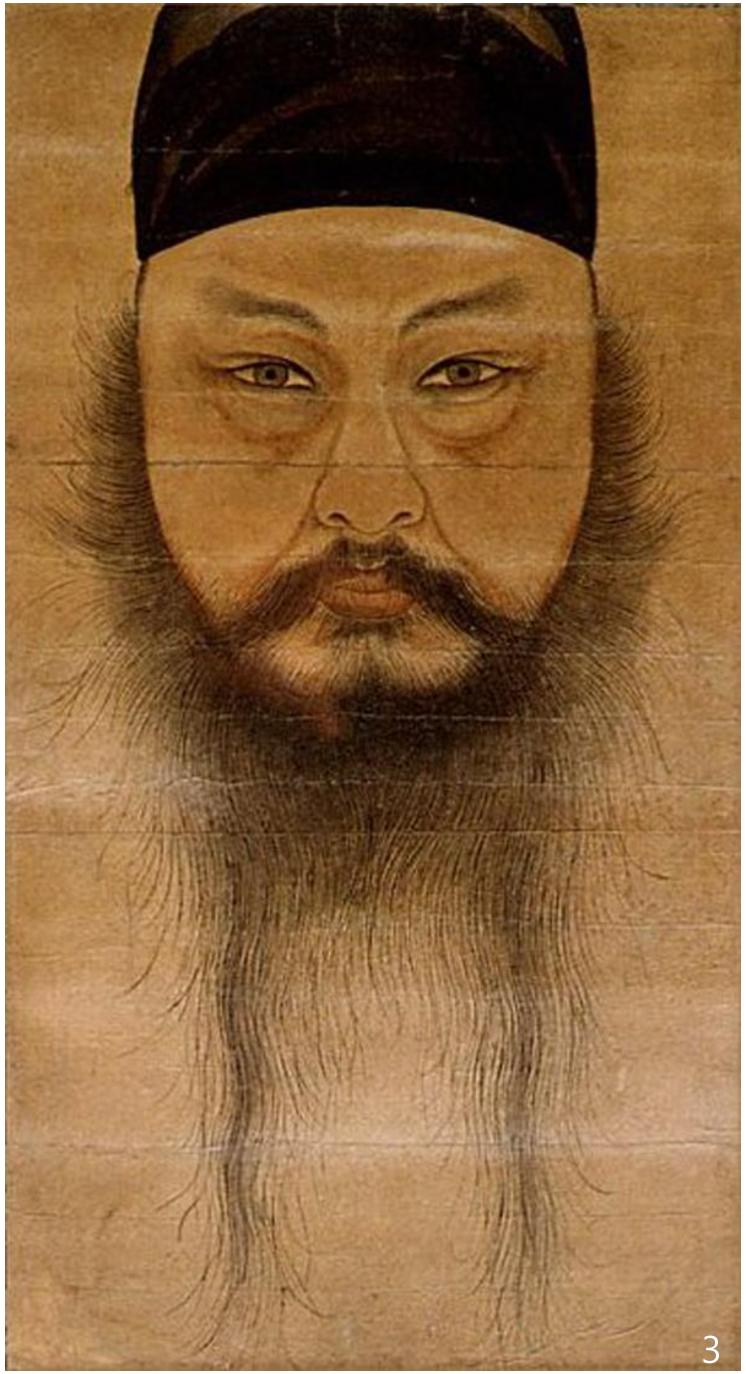


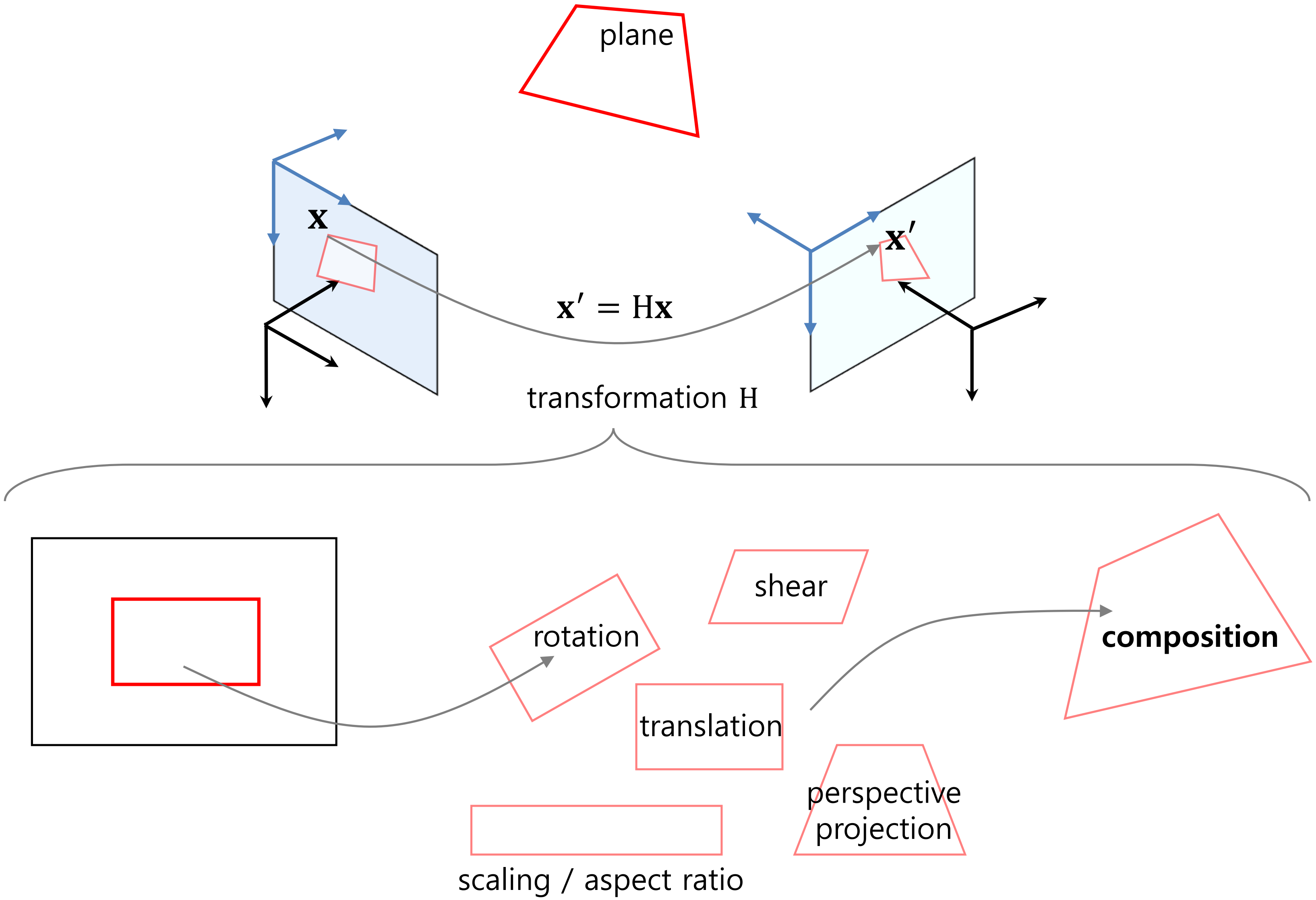
Table of Contents: Two-view Geometry

- **Planar Homography**
- **Epipolar Geometry**
  - Epipolar constraint
  - Fundamental and essential matrix
- **Relative Camera Pose Estimation**
- **Triangulation**

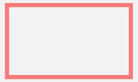
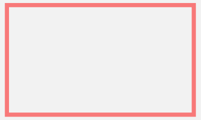


윤두서(1668-1715) 자화상, 국보 제240호
Korean National Treasure No. 240

Planar Homography



Planar Homography

	**Euclidean Transformation** (a.k.a. Rigid Transform) 	**Similarity Transformation** 	**Affine Transformation** 	**Projective Transformation** (a.k.a. Planar Homography) 
Matrix Forms H	$\begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix}$
DOF	3	4	6	8
Transformations - rotation - translation - scaling - aspect ratio - shear - perspective projection	 O O X X X X	 O O O X X X	 O O O O O X	 O O O O O O
Invariants - length - angle - ratio of lengths - parallelism - incidence - cross ratio	 O O O O O O	 X O O O O O	 X X X O O O	 X X X X O O

OpenCV APIs			`cv.getAffineTransform()`
			`cv.estimateRigidTransform()`
			-
			`cv.warpAffine()`
				`cv.getPerspectiveTransform()`
				-
				`cv.findHomography()`
				`cv.warpPerspective()`

Note) Similarly **3D transformations** (3D-3D geometry) are represented as **4x4 matrices**.

Note) Affine Transformation

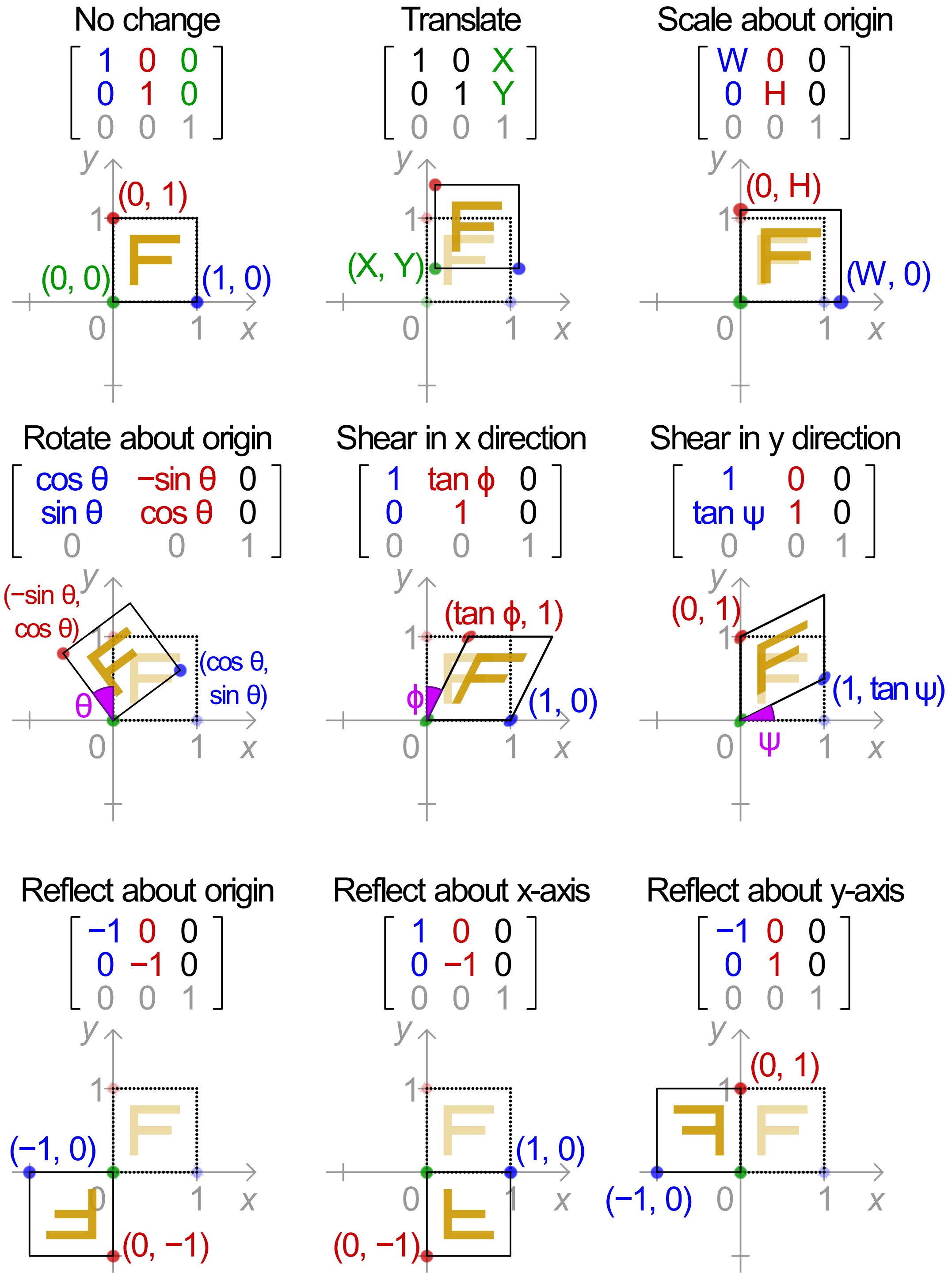
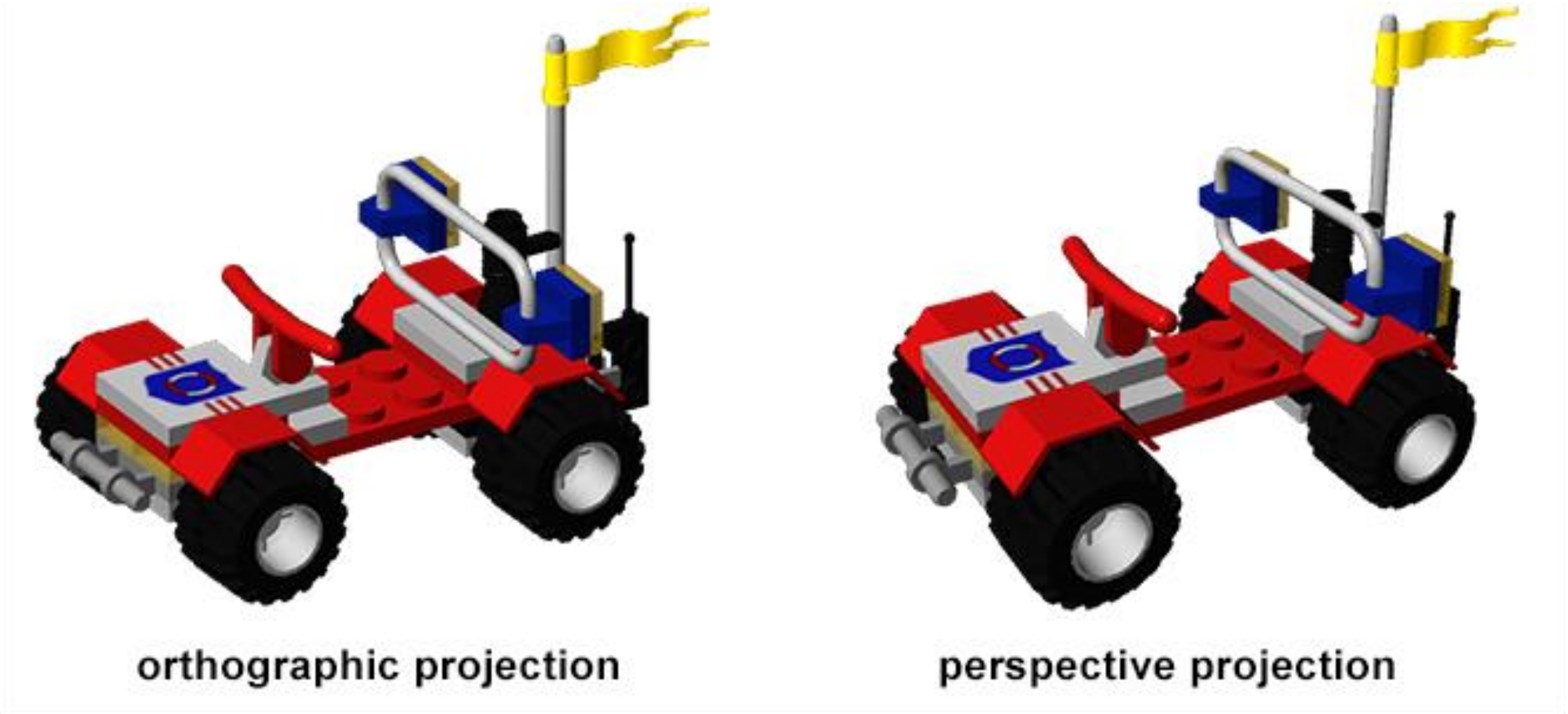
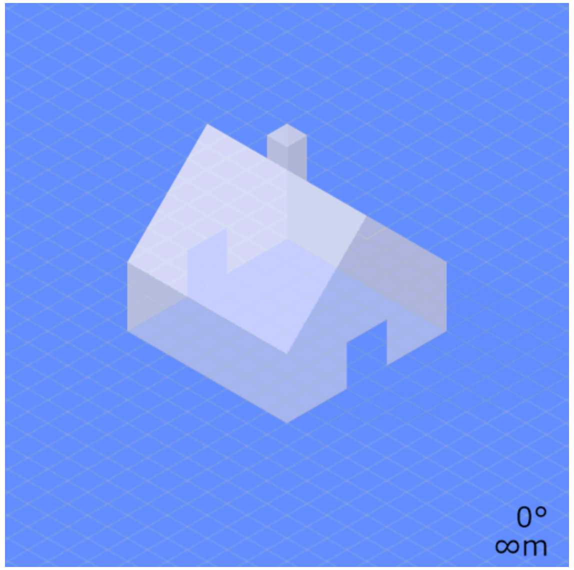


Image: [Wikipedia]()

Note) Affine Transformation



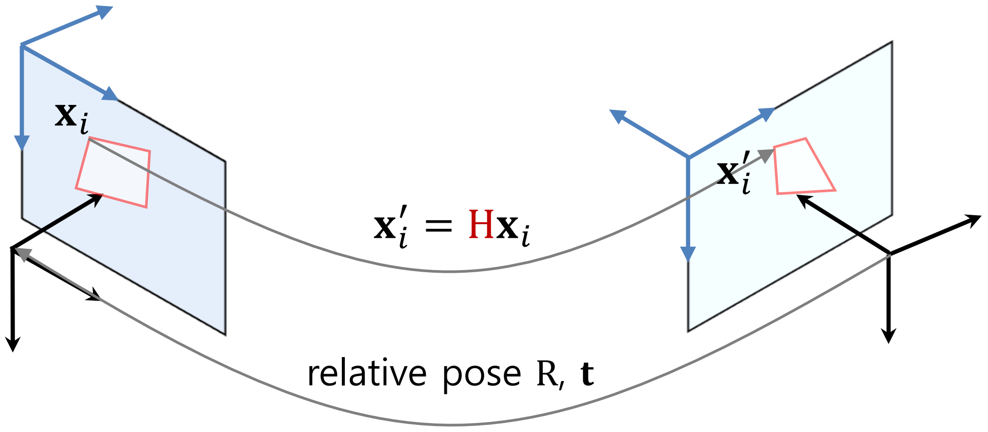


- **Q) What is difference between two images?**
  - Parallel projection vs. Perspective projection
    - Affine camera (a.k.a. weak perspective camera; $f = \infty$)
    - Less natural but (sometimes) useful in technical visualization

Image: Scatchpixel, Wikipedia

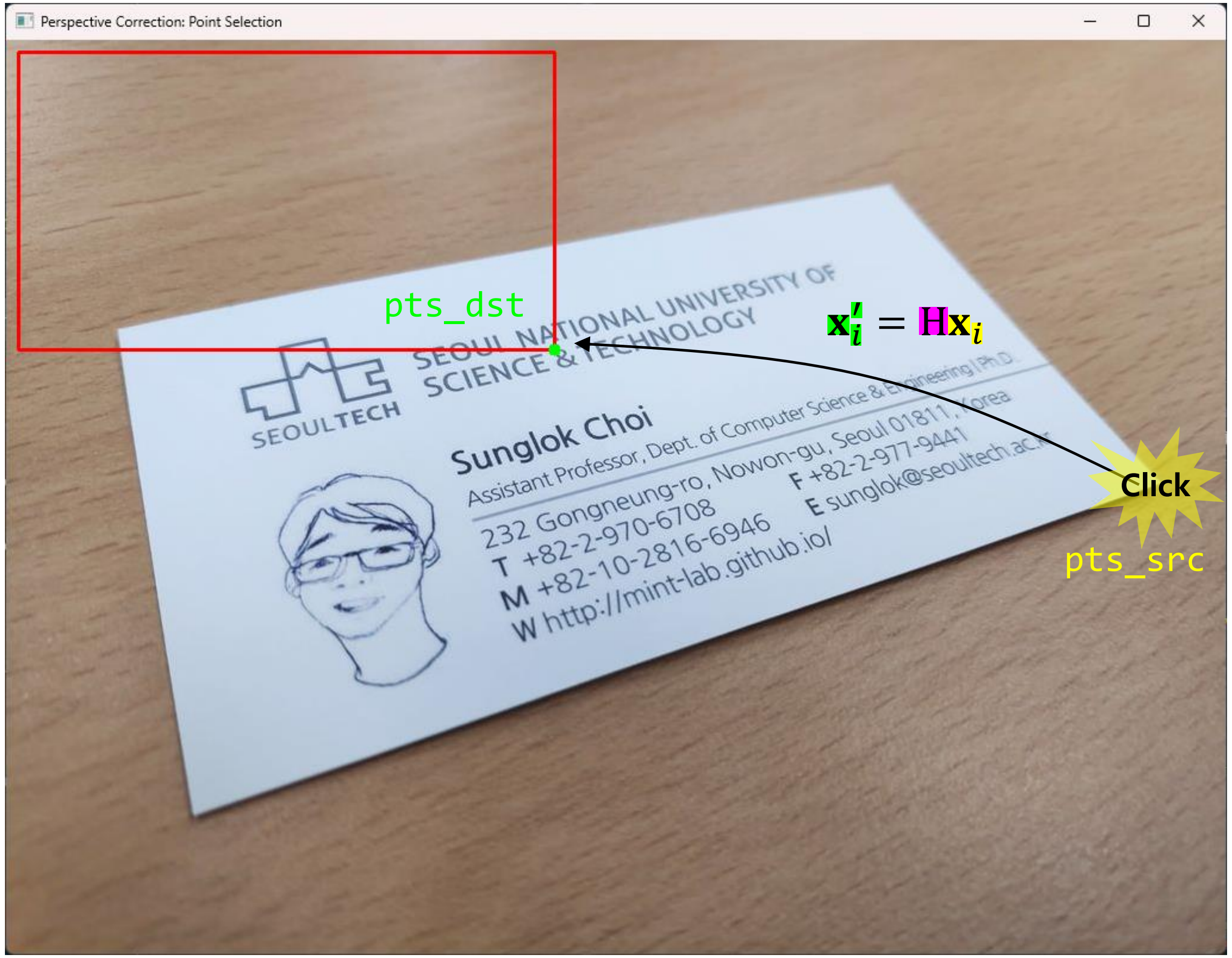
Planar Homography

- **Planar homography estimation**
  - Unknown: Planar homography H (8 DOF)
  - Given: Point correspondence $(\mathbf{x}_1, \mathbf{x}'_1)$, …, $(\mathbf{x}_n, \mathbf{x}'_n)$
  - Constraints: n x projective transformation $\mathbf{x}'_i = \mathrm{H}\mathbf{x}_i$
  - Solutions ($n \geq 4$) → 4-point algorithm
    - OpenCV: `cv.getPerspectiveTransform()` and `cv.findHomography()`
    - Note) More simplified transformations need less number of minimal correspondence.
      - Affine ($n \geq 3$), similarity ($n \geq 2$), Euclidean ($n \geq 2$)
  - Note) Planar homography can be decomposed as relative camera pose.
    - OpenCV: `cv.decomposeHomographyMat()`
    - The decomposition needs to know camera matrices.



Planar Homography

- Example) **Perspective distortion correction** `[perspective_correction.py]`



Planar Homography

- Example) **Perspective distortion correction** [perspective_correction.py]

```python
def mouse_event_handler(event, x, y, flags, param):
    if event == cv.EVENT_LBUTTONDOWN:
        param.append((x, y))

if __name__ == '__main__':
    img_file = '../data/sunglok_card.jpg'
    card_size = (450, 250)
    offset = 10

    # Prepare the rectified points
    pts_dst = np.array([[0, 0], [card_size[0], 0], [0, card_size[1]], [card_size[0], card_size[1]]])

    # Load an image
    img = cv.imread(img_file)

    # Get the matched points from mouse clicks
    pts_src = []
    wnd_name = 'Perspective Correction: Point Selection'
    cv.namedWindow(wnd_name)
    cv.setMouseCallback(wnd_name, mouse_event_handler, pts_src)
    while len(pts_src) < 4:
        img_display = img.copy()
        cv.rectangle(img_display, (offset, offset), (offset + card_size[0], offset + card_size[1]), (0, 0, 255), 2)
        idx = min(len(pts_src), len(pts_dst))
        cv.circle(img_display, offset + pts_dst[idx], 5, (0, 255, 0), -1)
        cv.imshow(wnd_name, img_display)
```

Planar Homography

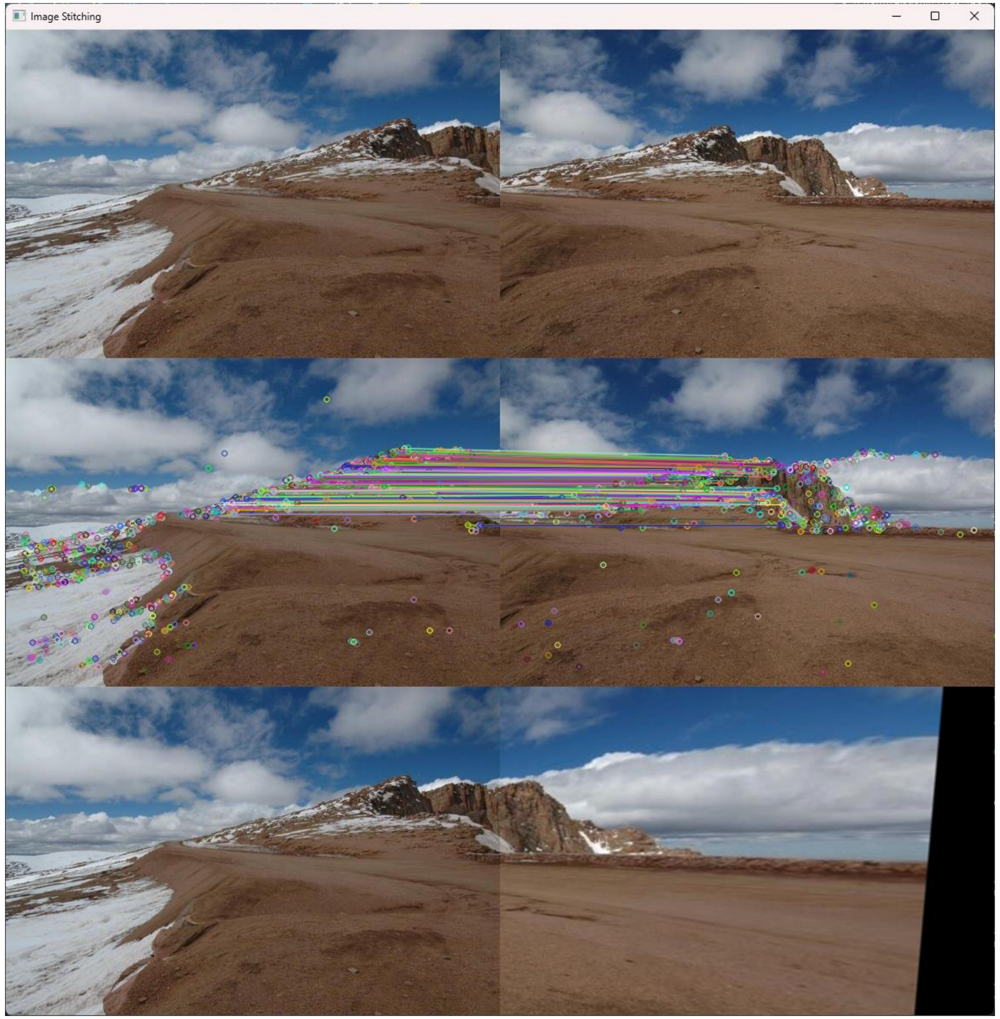
- Example) **Perspective distortion correction** [perspective_correction.py]

```
if __name__ == '__main__':
    img_file = '../data/sunglok_card.jpg'
    card_size = (450, 250)
    offset = 10

    # Prepare the rectified points
    pts_dst = np.array([[0, 0], [card_size[0], 0], [0, card_size[1]], [card_size[0], card_size[1]]])

    # Load an image
    img = cv.imread(img_file)

    # Get the matched points from mouse clicks
    pts_src = []
    ...

    if len(pts_src) == 4:
        # Calculate planar homography and rectify perspective distortion
        H, _ = cv.findHomography(np.array(pts_src), pts_dst)
        img_rectify = cv.warpPerspective(img, H, card_size)

        # Show the rectified image
        cv.imshow('Perspective Correction: Rectified Image', img_rectify)
        cv.waitKey(0)

    cv.destroyAllWindows()
```

Planar Homography

- Example) **Planar image stitching** [image_stitching.py]

```python
# Load two images
img1 = cv.imread('../data/hill01.jpg')
img2 = cv.imread('../data/hill02.jpg')

# Retrieve matching points
brisk = cv.BRISK_create()
keypoints1, descriptors1 = brisk.detectAndCompute(img1, None)
keypoints2, descriptors2 = brisk.detectAndCompute(img2, None)

fmatcher = cv.DescriptorMatcher_create('BruteForce-Hamming')
match = fmatcher.match(descriptors1, descriptors2)

# Calculate planar homography and merge them
pts1, pts2 = [], []
for i in range(len(match)):
    pts1.append(keypoints1[match[i].queryIdx].pt)
    pts2.append(keypoints2[match[i].trainIdx].pt)
pts1 = np.array(pts1, dtype=np.float32)
pts2 = np.array(pts2, dtype=np.float32)

H, inlier_mask = cv.findHomography(pts2, pts1, cv.RANSAC)
img_merged = cv.warpPerspective(img2, H, (img1.shape[1]*2, img1.shape[0]))
img_merged[:,:img1.shape[1]] = img1 # Copy

# Show the merged image
img_matched = cv.drawMatches(img1, keypoints1, img2, keypoints2, match, None, None, None,
```



Planar Homography

- Example) **2D video stabilization** [video_stabilization.py]

```
# Open a video and get the reference image and feature points
video = cv.VideoCapture('../data/traffic.avi')

_, gray_ref = video.read()
if gray_ref.ndim >= 3:
    gray_ref = cv.cvtColor(gray_ref, cv.COLOR_BGR2GRAY)
pts_ref = cv.goodFeaturesToTrack(gray_ref, 2000, 0.01, 10)

# Run and show video stabilization
while True:
    # Read an image from `video`
    valid, img = video.read()
    if not valid:
        break
    if img.ndim >= 3:
        gray = cv.cvtColor(img, cv.COLOR_BGR2GRAY)
    else:
        gray = img.copy()

    # Extract optical flow and calculate planar homography
    pts, status, err = cv.calcOpticalFlowPyrLK(gray_ref, gray, pts_ref, None)
    H, inlier_mask = cv.findHomography(pts, pts_ref, cv.RANSAC)

    # Synthesize a stabilized image
    warp = cv.warpPerspective(img, H, (img.shape[1], img.shape[0]))
```

A shaking CCTV video

Planar Homography

- Assumption) **A <u>plane</u> is observed by two views.**
  - Perspective distortion correction: A complete plane
  - Planar image stitching: An approximated plane (← distance ≫ depth variation)
  - 2D video stabilization: An approximated plane (← small motion)

Epipolar Geometry

- **Epipolar constraint**

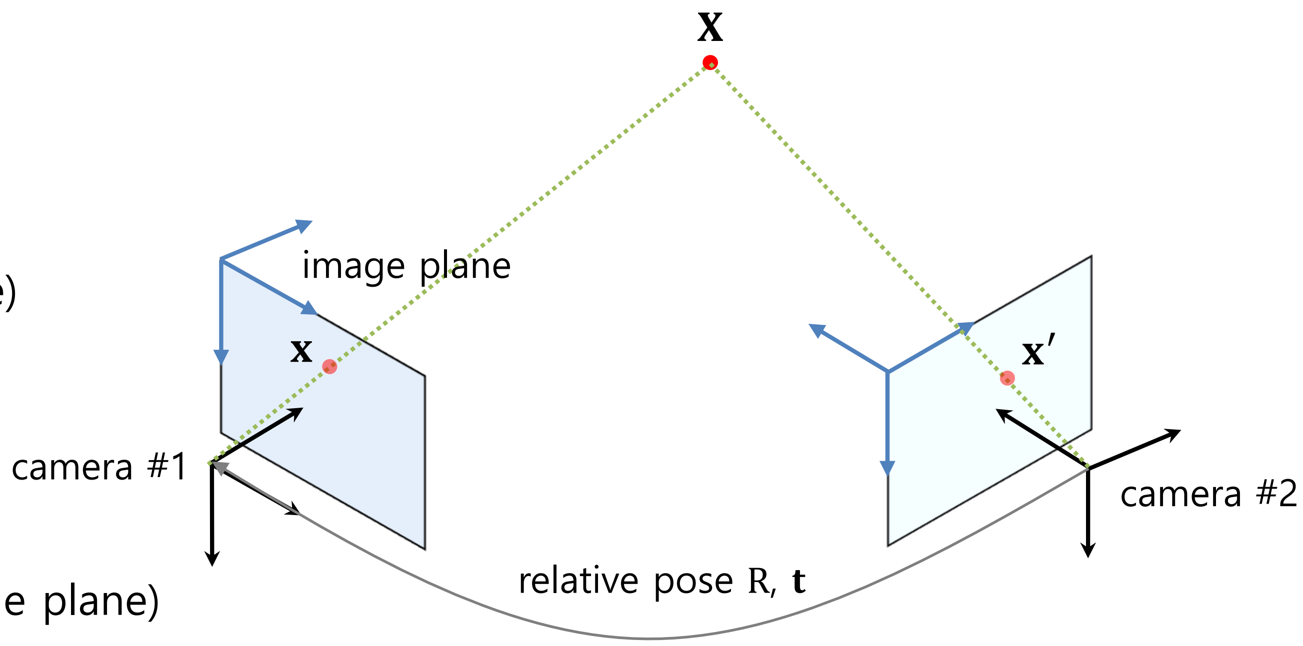
$\mathbf{x}'^{\top}\mathrm{F}\mathbf{x} = 0$ (F: Fundamental matrix on the image plane)

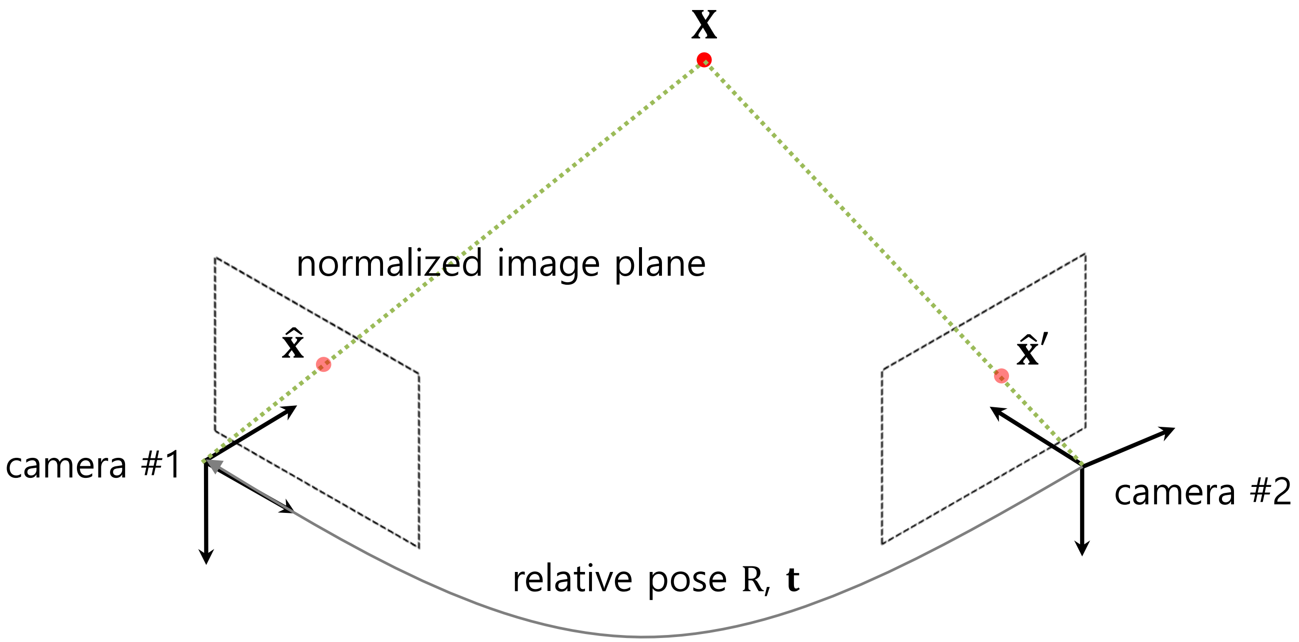
$\updownarrow$ $(\mathbf{x} = \mathrm{K}\hat{\mathbf{x}})$

$\hat{\mathbf{x}}'^{\top}\mathrm{K}'^{\top}\mathrm{F}\mathrm{K}\hat{\mathbf{x}} = 0$

$\updownarrow$ $(\mathrm{E} = \mathrm{K}'^{\top}\mathrm{F}\mathrm{K})$

$\hat{\mathbf{x}}'^{\top}\mathrm{E}\hat{\mathbf{x}} = 0$ (E: Essential matrix on the normalized image plane)





Epipolar Geometry

- **Epipolar constraint: Derivation**

$\mathbf{X}' = \mathrm{R}\mathbf{X} + \mathbf{t}$

$\downarrow \ (\mathbf{X} = \lambda\hat{\mathbf{x}})$

$\lambda'\hat{\mathbf{x}}' = \lambda\mathrm{R}\hat{\mathbf{x}} + \mathbf{t}$

$\mathbf{t} \times \downarrow$

$\lambda'\mathbf{t} \times \hat{\mathbf{x}}' = \lambda\mathbf{t} \times \mathrm{R}\hat{\mathbf{x}}$

$\hat{\mathbf{x}}' \cdot \downarrow$

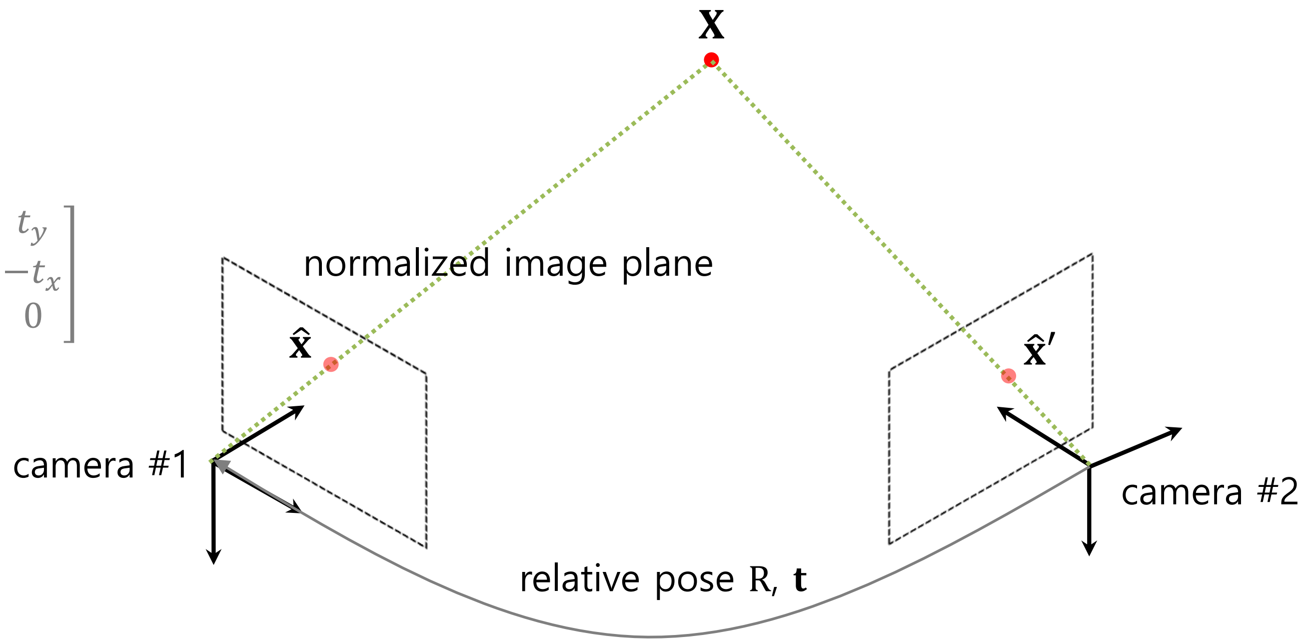
$\lambda'\hat{\mathbf{x}}' \cdot (\mathbf{t} \times \hat{\mathbf{x}}') = \lambda\hat{\mathbf{x}}' \cdot \mathbf{t} \times \mathrm{R}\hat{\mathbf{x}}$

$\downarrow \ (\hat{\mathbf{x}}' \cdot (\mathbf{t} \times \hat{\mathbf{x}}') = 0)$

$\lambda\hat{\mathbf{x}}' \cdot \mathbf{t} \times \mathrm{R}\hat{\mathbf{x}} = 0$

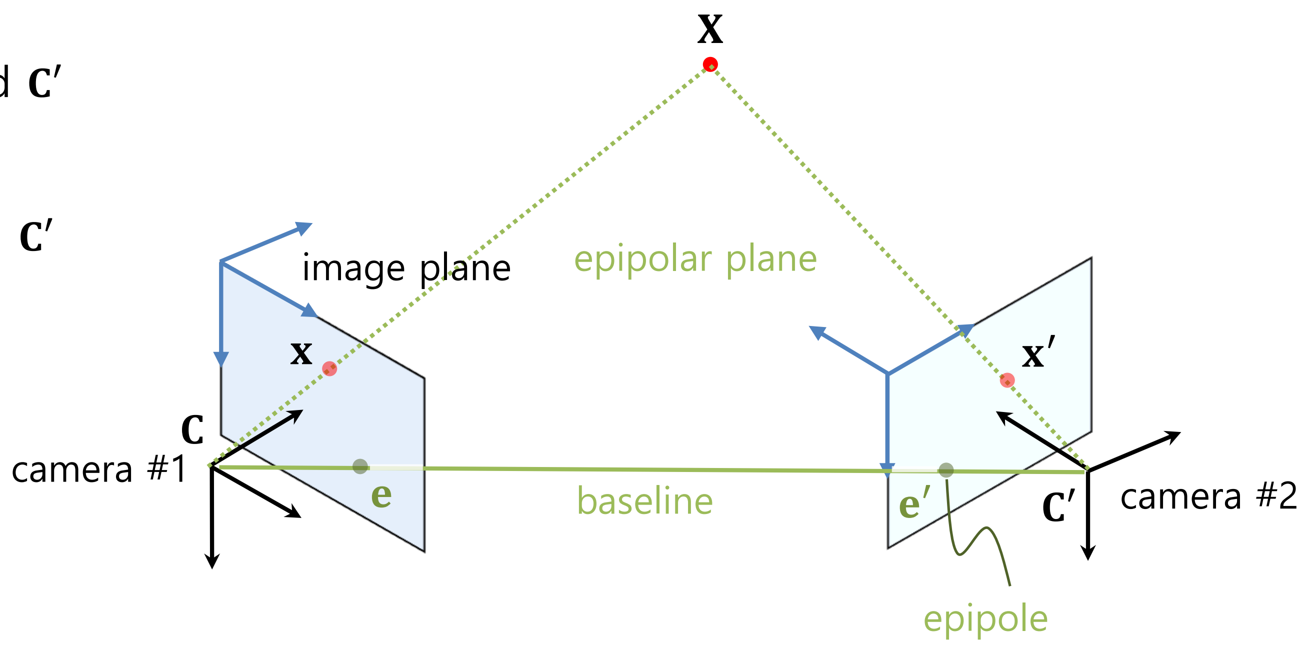
$\downarrow \ (\mathrm{E} = \mathbf{t} \times \mathrm{R}$ or $\mathrm{E} = [\mathbf{t}]_\times \mathrm{R})$ Note) $[\mathbf{t}]_\times = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$

$\hat{\mathbf{x}}'\mathrm{E}\hat{\mathbf{x}} = 0$



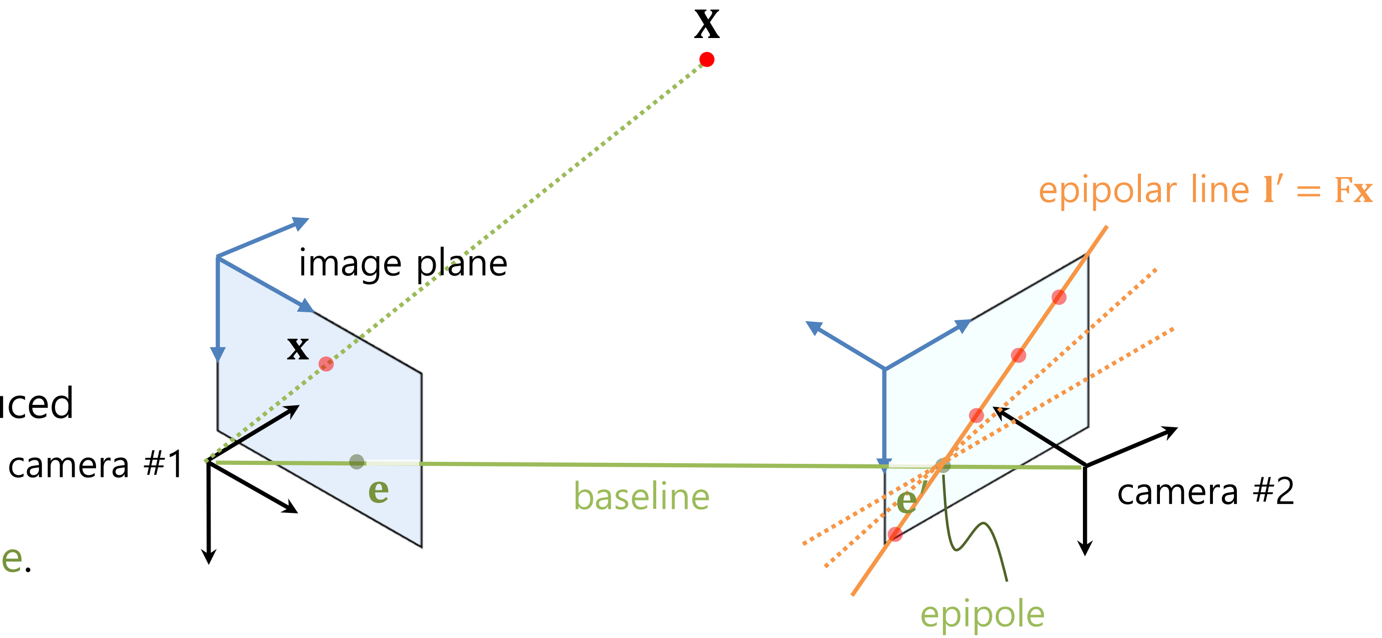
Epipolar Geometry

- Epipolar geometry
  - Baseline
    - Distance between two camera centers, $\mathbf{C}$ and $\mathbf{C}'$
  - Epipolar plane
    - Plane generated from three points, $\mathbf{X}$, $\mathbf{C}$, and $\mathbf{C}'$
  - Epipole (a.k.a. epipolar point)
    - Projection of other camera centers
    - $\mathbf{e} = \mathrm{P}\mathbf{C}'$ and $\mathbf{e}' = \mathrm{P}'\mathbf{C}$
      - e.g. $\mathrm{P} = \mathrm{K}[\mathrm{I}|0]$ and $\mathrm{P}' = \mathrm{K}'[\mathrm{R}|\mathbf{t}]$
        - $\mathbf{e} = -\mathrm{K}\mathrm{R}^{\mathsf{T}}\mathbf{t}$ and $\mathbf{e}' = \mathrm{K}'\mathbf{t}$

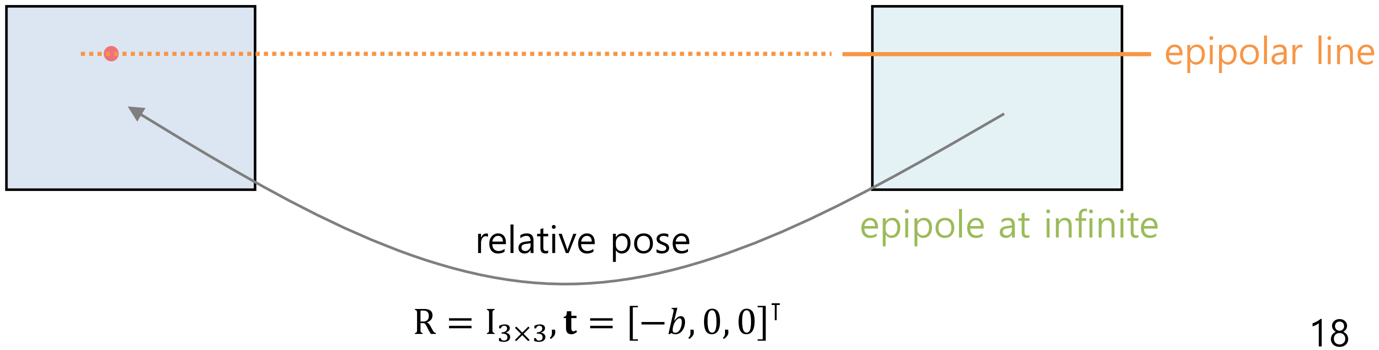


Epipolar Geometry

- **Epipolar geometry**
  - Epipolar line
    - $\mathbf{x}'^{\top}\mathrm{F}\mathbf{x} = 0 \rightarrow \mathbf{x}'^{\top}\mathbf{l}' = 0$ where $\mathbf{l}' = \mathrm{F}\mathbf{x}$
      - $\mathbf{x}'$ will lie on the line $\mathbf{l}'$.
    - $\mathbf{x}'^{\top}\mathrm{F}\mathbf{x} = 0 \rightarrow \mathbf{l}^{\top}\mathbf{x} = 0$ where $\mathbf{l} = \mathrm{F}^{\top}\mathbf{x}'$
      - $\mathbf{x}$ will lie on the line $\mathbf{l}$.
    - The search space of feature matching is reduced from a image plane to the epipolar line.
  - Note) Every epipolar line intersects at the epipole.
    - $\mathrm{F}\mathbf{e} = 0$
    - $\mathbf{e}$ is the null space of F.

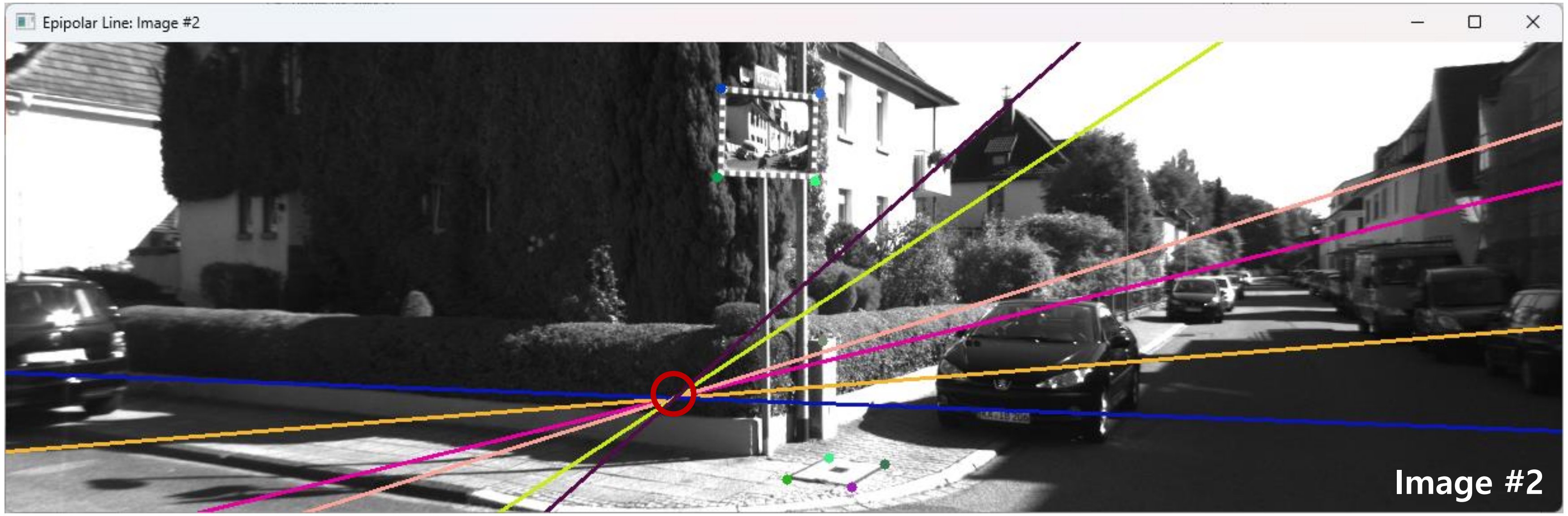


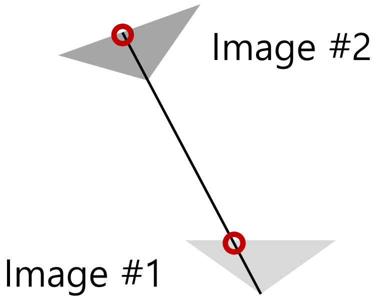
Special case) Stereo cameras



Epipolar Geometry

- Example) **Epipolar line visualization** `[epipolar_line_visualization.py]`

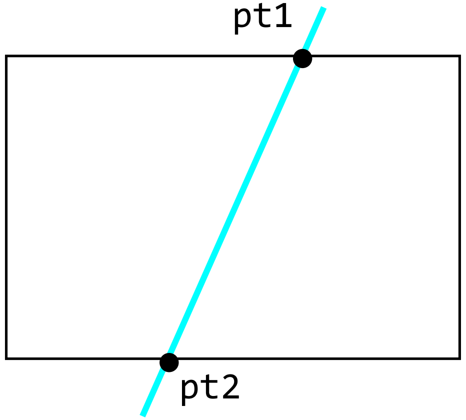




Epipolar Geometry

- Example) **Epipolar line visualization** [epipolar_line_visualization.py]

```python
def mouse_event_handler(event, x, y, flags, param):
    if event == cv.EVENT_LBUTTONDOWN:
        param.append((x, y))

def draw_straight_line(img, line, color, thickness=1):
    h, w, *_ = img.shape
    a, b, c = line # Line: ax + by + c = 0
    if abs(a) > abs(b):
        pt1 = (int(c / -a), 0)
        pt2 = (int((b*h + c) / -a), h)
    else:
        ...
    cv.line(img, pt1, pt2, color, thickness)

if __name__ == '__main__':
    # Load two images
    img1 = cv.imread('../data/KITTI07/image_0/000000.png', cv.IMREAD_COLOR)
    img2 = cv.imread('../data/KITTI07/image_0/000023.png', cv.IMREAD_COLOR)
    # Note) `F` is derived from `fundamental_mat_estimation.py`.
    F = np.array([[ 3.34638533e-07,  7.58547151e-06, -2.04147752e-03], ...])

    # Register event handlers and show images
    wnd1_name, wnd2_name = 'Epipolar Line: Image #1', 'Epipolar Line: Image #2'
    img1_pts, img2_pts = [], []
    cv.namedWindow(wnd1_name)
    cv.setMouseCallback(wnd1_name, mouse_event_handler, img1_pts)
```



Epipolar Geometry

- Example) **Epipolar line visualization** [epipolar_line_visualization.py]

```
if __name__ == '__main__':
    # Load two images
    img1 = cv.imread('../data/KITTI07/image_0/000000.png', cv.IMREAD_COLOR)
    img2 = cv.imread('../data/KITTI07/image_0/000023.png', cv.IMREAD_COLOR)
    F = np.array([[ 3.34638533e-07,  7.58547151e-06, -2.04147752e-03], ...])

    # Register event handlers and show images
    ...

    # Get a point from a image and draw its correponding epipolar line on the other image
    while True:
        if len(img1_pts) > 0:
            for x, y in img1_pts:
                color = (random.randrange(256), random.randrange(256), random.randrange(256))
                cv.circle(img1, (x, y), 4, color, -1)
                epipolar_line = F @ [[x], [y], [1]]
                draw_straight_line(img2, epipolar_line, color, 2)
            img1_pts.clear()
        if len(img2_pts) > 0:
            for x, y in img2_pts:
                color = (random.randrange(256), random.randrange(256), random.randrange(256))
                cv.circle(img2, (x, y), 4, color, -1)
                epipolar_line = F.T @ [[x], [y], [1]]
                draw_straight_line(img1, epipolar_line, color, 2)
            img2_pts.clear()
        cv.imshow(wnd2_name, img2)
```

$$\mathbf{l}' = \mathrm{F}\mathbf{x}$$

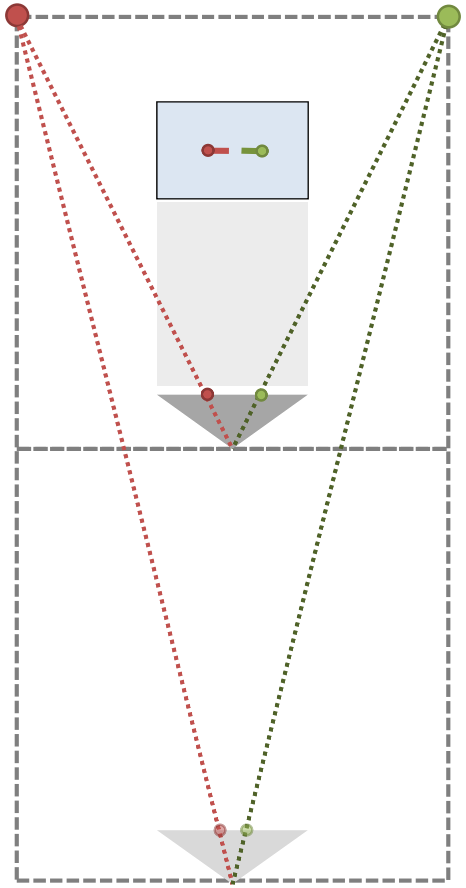
$$\mathbf{l} = \mathrm{F}^{\mathsf{T}}\mathbf{x}'$$

Relative Camera Pose Estimation

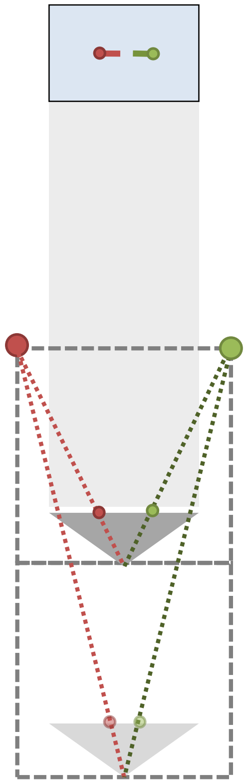
- **Relative camera pose estimation** (~ fundamental/essential matrix estimation)
  - Unknown: Rotation and translation R, $\mathbf{t}$ (**5 DOF**; up-to scale **"scale ambiguity"**)
  - Given: Point correspondence $(\mathbf{x}_1, \mathbf{x}'_1)$, …, $(\mathbf{x}_n, \mathbf{x}'_n)$ and camera matrices K, K'
  - Constraints: n x epipolar constraint ($\mathbf{x}'^\top \mathrm{F} \mathbf{x} = 0$ or $\hat{\mathbf{x}}'^\top \mathrm{E} \hat{\mathbf{x}} = 0$)
  - Solutions) **Fundamental matrix**: 7/8-point algorithm (**7 DOF**; intrinsic+extrinsic)
    - **Properties**: $\det(\mathrm{F}) = 0$ (or $\mathrm{rank}(\mathrm{F}) = 2$)
    - **Estimation**: `cv.findFundamentalMat()` → 1 solution
    - **Conversion to E**: $\mathrm{E} = \mathrm{K}'^\top \mathrm{F} \mathrm{K}$
  - Solutions) **Essential matrix**: 5-point algorithm (**5 DOF**; extrinsic)
    - **Properties**: $\det(\mathrm{E}) = 0$ and $2\mathrm{E}\mathrm{E}^\top\mathrm{E} - \mathrm{tr}(\mathrm{E}\mathrm{E}^\top)\,\mathrm{E} = 0$
    - **Estimation**: `cv.findEssentialMat()` → k solutions
    - **Decomposition**: `cv.decomposeEssentialMat()` → 4 solutions **"relative pose ambiguity"**
    - **Decomposition with positive-depth check**: `cv.recoverPose()` → 1 solution

R, $\mathbf{t}$
$\updownarrow$ ($\mathrm{E} = [\mathbf{t}]_\times \mathrm{R}$)
E
$\updownarrow$ ($\mathrm{E} = \mathrm{K}'^\top \mathrm{F} \mathrm{K}$)
F

Relative Camera Pose Estimation: Scale Ambiguity



(a) 2-meter-wide tunnel



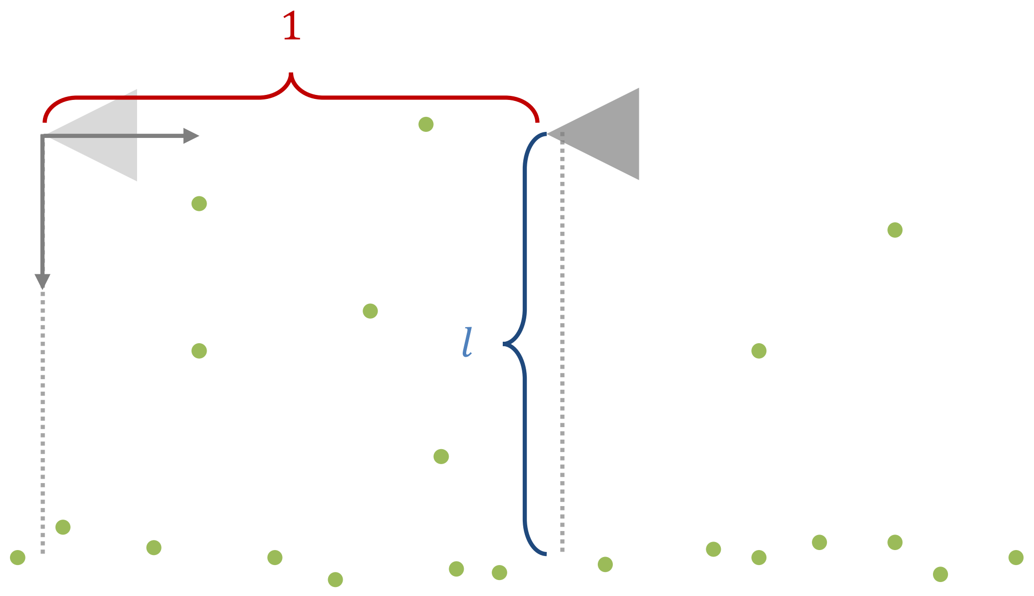
(b) 1-meter-wide tunnel

The Sandcrawler @ Star Wars IV: A New Hope (1977)

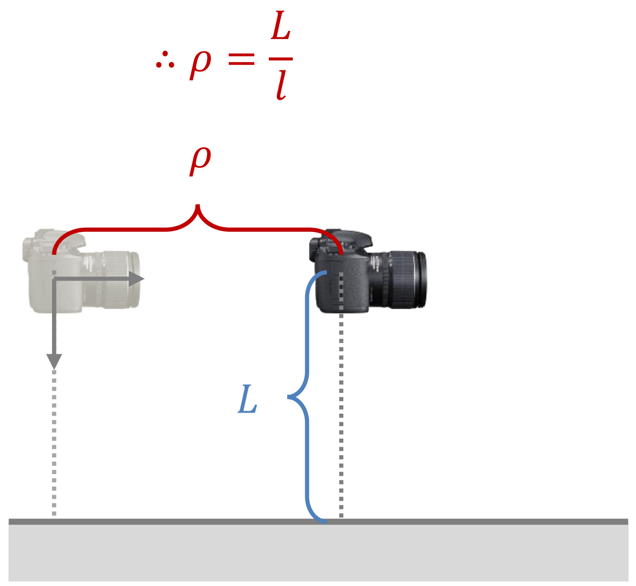
Reference: Choi et al., **Resolving Scale Ambiguity in Monocular Visual Odometry**, URAI, 2013

Relative Camera Pose Estimation: Scale Ambiguity

- **How to resolve scale ambiguity**
  - **Additional sensors**: Speedometers (odometers), IMUs, GPSs, depth/distance (stereo, RGB-D, LiDAR, …)
  - **Motion constraints**: Known initial translation, Ackerman's steering kinematics
  - **Observation constraints**: Known size of objects, known and constant height of camera



(a) The reconstructed world



(b) The real world

Reference: Choi et al., **Resolving Scale Ambiguity in Monocular Visual Odometry**, URAI, 2013

Relative Camera Pose Estimation: Relative Pose Ambiguity

- **How to resolve** **pose ambiguity**
  - Positive depth constraint

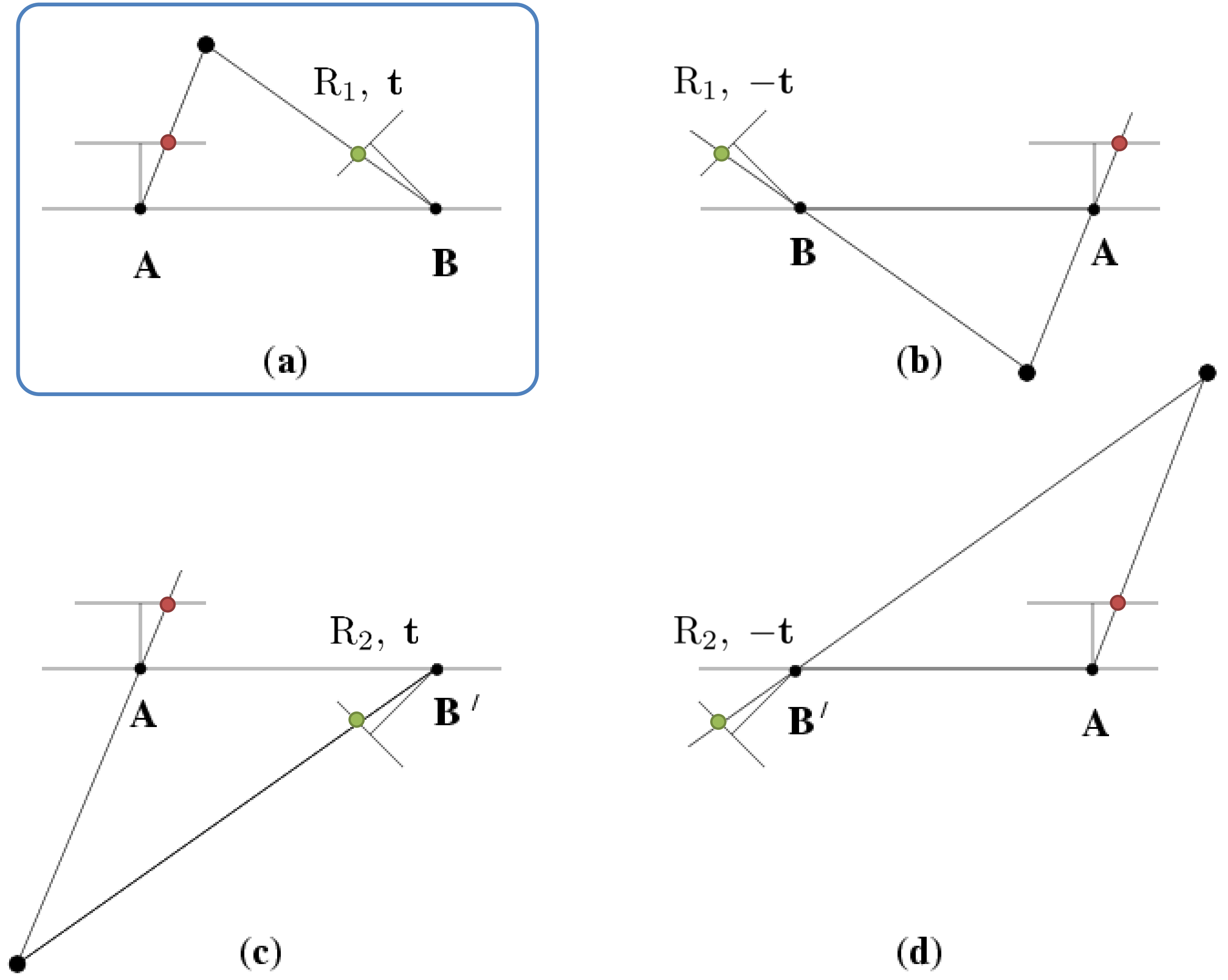


Image: Hartley and Zisserman, **Multiple View Geometry in Computer Vision** (2nd Edition), Cambridge University Press, 2004

Relative Camera Pose Estimation

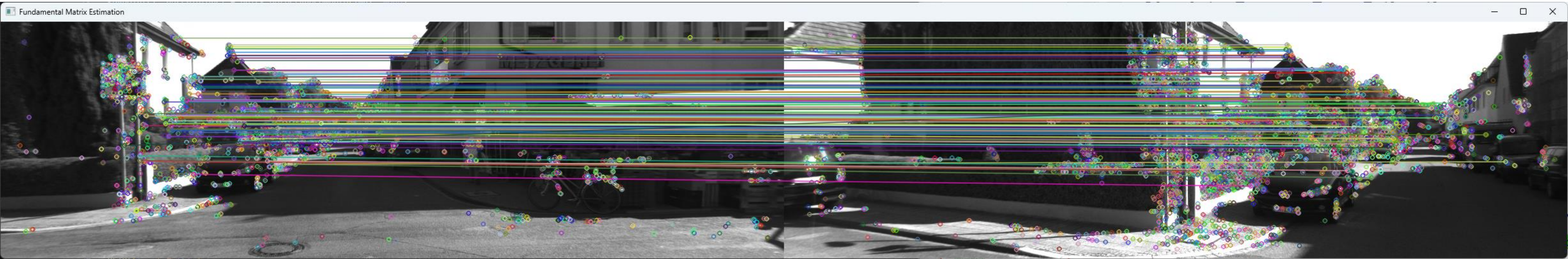
Example) **Fundamental matrix estimation** [fundamental_mat_estimation.py]

```python
# Load two images
img1 = cv.imread('../data/KITTI07/image_0/000000.png')
img2 = cv.imread('../data/KITTI07/image_0/000023.png')
f, cx, cy = 707.0912, 601.8873, 183.1104 # From the KITTI dataset
K = np.array([[f, 0, cx], [0, f, cy], [0, 0, 1]])

# Retrieve matching points
brisk = cv.BRISK_create()
keypoints1, descriptors1 = brisk.detectAndCompute(img1, None)
keypoints2, descriptors2 = brisk.detectAndCompute(img2, None)

fmatcher = cv.DescriptorMatcher_create('BruteForce-Hamming')
match = fmatcher.match(descriptors1, descriptors2)
```

Relative Camera Pose Estimation

Example) **Fundamental matrix estimation** [fundamental_mat_estimation.py]

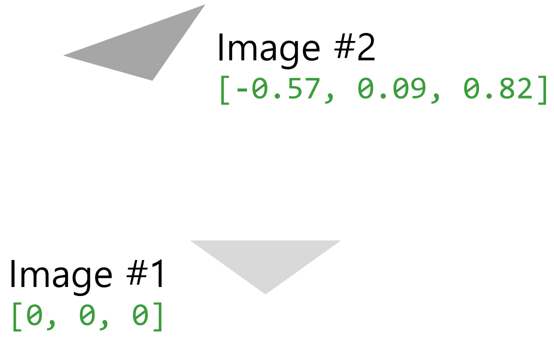
```
# Load two images
...

# Retrieve matching points
...

# Calculate the fundamental matrix
pts1, pts2 = [], []
for i in range(len(match)):
    pts1.append(keypoints1[match[i].queryIdx].pt)
    pts2.append(keypoints2[match[i].trainIdx].pt)
pts1 = np.array(pts1, dtype=np.float32)
pts2 = np.array(pts2, dtype=np.float32)
F, inlier_mask = cv.findFundamentalMat(pts1, pts2, cv.FM_RANSAC, 0.5, 0.999)
```



Relative Camera Pose Estimation

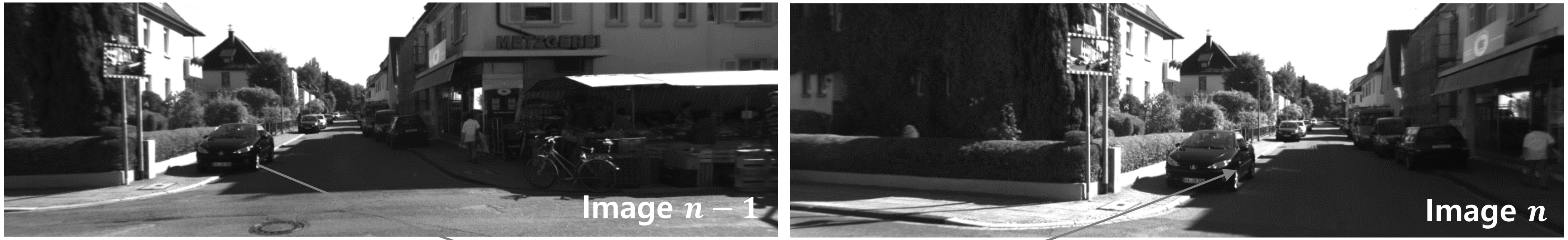
Example) **Fundamental matrix estimation** [fundamental_mat_estimation.py]

```python
# Load two images
...
f, cx, cy = 707.0912, 601.8873, 183.1104 # From the KITTI dataset
K = np.array([[f, 0, cx], [0, f, cy], [0, 0, 1]])

# Retrieve matching points
...

# Calculate the fundamental matrix
...
F, inlier_mask = cv.findFundamentalMat(pts1, pts2, cv.FM_RANSAC, 0.5, 0.999)

# Extract relative camera pose between two images
E = K.T @ F @ K
positive_num, R, t, positive_mask = cv.recoverPose(E, pts1, pts2, K, mask=inlier_mask)
...
print(f'* The position of Image #2 = {-R.T @ t}') # [-0.57, 0.09, 0.82]
```



Relative Camera Pose Estimation

- Example) **Monocular Visual Odometry (Epipolar Version)** `[vo_epipolar.cpp]`



Relative pose R, $\mathbf{t}$

Given K

`Use_5pt = true`

[Tentative Matching]
- Extract optical flow between adjacent images

Pass tentative matches

[Outlier Rejection]
- Remove wrong matches using `cv::findEssentialMat()`

Pass the essential matrix E

[Relative Pose Estimation]
- Decompose camera pose using `cv::recoverPose()`

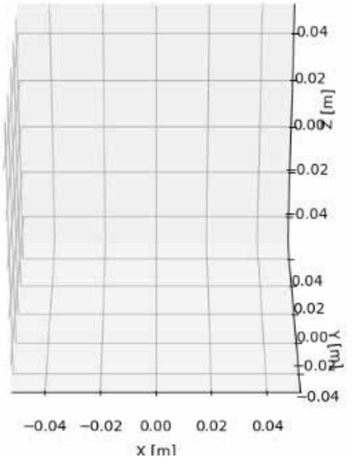
Pass the relative pose R, $\mathbf{t}$ (Note: $\|\mathbf{t}\| = 1$)

[Relative Pose Accumulation]

$$\mathrm{T}_n^0 = \mathrm{T}_{n-1}^0 \Delta_n^{n-1} = \mathrm{T}_{n-1}^0 \begin{bmatrix} \mathrm{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}^{-1}$$

Relative Camera Pose Estimation

- Example) **Monocular Visual Odometry (Epipolar Version)** `[vo_epipolar.cpp]`



Given K `Use_5pt = true`

[Tentative Matching]
- Extract optical flow between adjacent images

→ Pass tentative matches →

[Outlier Rejection]
- Remove wrong matches using `cv::findEssentialMat()`

→ Pass the essential matrix E →

[Relative Pose Estimation]
- Decompose camera pose using `cv::recoverPose()`

→ Pass the relative pose R, $\mathbf{t}$ (Note: $\| \mathbf{t} \| = 1$) →

[Relative Pose Accumulation]

$$\mathrm{T}_n^0 = \mathrm{T}_{n-1}^0 \Delta_n^{n-1} = \mathrm{T}_{n-1}^0 \begin{bmatrix} \mathrm{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}^{-1}$$

Relative Camera Pose Estimation

- Example) **Monocular Visual Odometry (Epipolar Version)** [vo_epipolar.cpp]

```
video_file = '../data/KITTI07/image_0/%06d.png'
f, cx, cy = 707.0912, 601.8873, 183.1104
use_5pt = True
min_inlier_num = 100
min_inlier_ratio = 0.2
traj_file = '../data/vo_epipolar.xyz'

# Open a video and get an initial image
video = cv.VideoCapture(video_file)

_, gray_prev = video.read()
if gray_prev.ndim >= 3 and gray_prev.shape[2] > 1:
    gray_prev = cv.cvtColor(gray_prev, cv.COLOR_BGR2GRAY)

# Prepare a plot to visualize the camera trajectory
...

# Run the monocular visual odometry
K = np.array([[f, 0, cx], [0, f, cy], [0, 0, 1]])
camera_traj = np.zeros((1, 3))
camera_pose = np.eye(4)
while True:
    # Grab an image from the video
    valid, img = video.read()
    if not valid:
        break
```

Relative Camera Pose Estimation

- Example) **Monocular Visual Odometry (Epipolar Version)** [vo_epipolar.cpp]

```
# Run the monocular visual odometry
K = np.array([[f, 0, cx], [0, f, cy], [0, 0, 1]])
camera_traj = np.zeros((1, 3))
camera_pose = np.eye(4)
while True:
    # Grab an image from the video
    valid, img = video.read()
    if not valid:
        break
    if img.ndim >= 3 and img.shape[2] > 1:
        gray = cv.cvtColor(img, cv.COLOR_BGR2GRAY)
    else:
        gray = img.copy()

    # Extract optical flow
    pts_prev = cv.goodFeaturesToTrack(gray_prev, 2000, 0.01, 10)
    pts, status, err = cv.calcOpticalFlowPyrLK(gray_prev, gray, pts_prev, None)
    gray_prev = gray

    # Calculate relative pose
    if use_5pt:
        E, inlier_mask = cv.findEssentialMat(pts_prev, pts, f, (cx, cy), cv.FM_RANSAC, 0.99, 1)
    else:
        F, inlier_mask = cv.findFundamentalMat(pts_prev, pts, cv.FM_RANSAC, 1, 0.99)
        E = K.T @ F @ K
    inlier_num, R, t, inlier_mask = cv.recoverPose(E, pts_prev, pts, focal=f, pp=(cx, cy), mask=inlier_mask)
```

Relative Camera Pose Estimation

- Example) **Monocular Visual Odometry (Epipolar Version)** [vo_epipolar.cpp]

```
# Run the monocular visual odometry
while True:
    # Grab an image from the video
    ...

    # Extract optical flow
    ...

    # Calculate relative pose
    if use_5pt:
        E, inlier_mask = cv.findEssentialMat(pts_prev, pts, f, (cx, cy), cv.FM_RANSAC, 0.99, 1)
    else:
        F, inlier_mask = cv.findFundamentalMat(pts_prev, pts, cv.FM_RANSAC, 1, 0.99)
        E = K.T @ F @ K
    inlier_num, R, t, inlier_mask = cv.recoverPose(E, pts_prev, pts, focal=f, pp=(cx, cy), mask=inlier_mask)
    inlier_ratio = inlier_num / len(pts)

    # Accumulate relative pose if result is reliable
    info_color = (0, 255, 0)
    if inlier_num > min_inlier_num and inlier_ratio > min_inlier_ratio:
        T = np.eye(4)
        T[:3, :3] = R
        T[:3, 3] = t.flatten()
        camera_pose = camera_pose @ np.linalg.inv(T)
        info_color = (0, 0, 255)
```

$$\mathrm{T}_n^0 = \mathrm{T}_{n-1}^0 \Delta_n^{n-1} = \mathrm{T}_{n-1}^0 \begin{bmatrix} \mathrm{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}^{-1}$$

Relative Camera Pose Estimation

- **Relative camera pose estimation** (~ fundamental/essential matrix estimation)
  - Unknown: Rotation and translation R, $\mathbf{t}$ (**5 DOF**; up-to scale **"scale ambiguity"**)
  - Given: Point correspondence $(\mathbf{x}_1, \mathbf{x}'_1)$, …, $(\mathbf{x}_n, \mathbf{x}'_n)$ and camera matrices K, K'
  - Constraints: n x epipolar constraint ($\mathbf{x}'^\top \mathrm{F} \mathbf{x} = 0$ or $\hat{\mathbf{x}}'^\top \mathrm{E} \hat{\mathbf{x}} = 0$)
  - Solutions) **Fundamental matrix**: 7/8-point algorithm (**7 DOF**; intrinsic+extrinsic)
    - **Properties**: $\det(\mathrm{F}) = 0$ (or $\mathrm{rank}(\mathrm{F}) = 2$)
    - **Estimation**: `cv.findFundamentalMat()` → 1 solution
    - **Conversion to E**: $\mathrm{E} = \mathrm{K}'^\top \mathrm{F} \mathrm{K}$
    - **Degenerate cases**: No translation, correspondence from a single plane
  - Solutions) **Essential matrix**: 5-point algorithm (**5 DOF**; extrinsic)
    - **Properties**: $\det(\mathrm{E}) = 0$ and $2\mathrm{E}\mathrm{E}^\top\mathrm{E} - \mathrm{tr}(\mathrm{E}\mathrm{E}^\top)\,\mathrm{E} = 0$
    - **Estimation**: `cv.findEssentialMat()` → k solutions
    - **Decomposition**: `cv.decomposeEssentialMat()` → 4 solutions **"relative pose ambiguity"**
    - **Decomposition with positive-depth check**: `cv.recoverPose()` → 1 solution
    - **Degenerate cases**: No translation ($\because \mathrm{E} = [\mathbf{t}]_\times \mathrm{R}$)

R, $\mathbf{t}$
$\updownarrow$ ($\mathrm{E} = [\mathbf{t}]_\times \mathrm{R}$)
E
$\updownarrow$ ($\mathrm{E} = \mathrm{K}'^\top \mathrm{F} \mathrm{K}$)
F

Relative Camera Pose Estimation

- **Relative camera pose estimation**

complementary

- Solutions) **Fundamental matrix**: 7/8-point algorithm (**7 DOF**; intrinsic+extrinsic)
  - **Estimation**: `cv.findFundamentalMat()` → 1 solution
  - **Conversion to E**: $\mathrm{E} = \mathrm{K}'^{\top}\mathrm{FK}$
  - **Degenerate cases**: No translation, correspondence from a single plane
- Solutions) **Essential matrix**: 5-point algorithm (**5 DOF**; extrinsic)
  - **Estimation**: `cv.findEssentialMat()` → k solutions
  - **Decomposition**: `cv.decomposeEssentialMat()` → 4 solutions **"relative pose ambiguity"**
  - **Decomposition with positive-depth check**: `cv.recoverPose()` → 1 solution
  - **Degenerate cases**: No translation ($\because \mathrm{E} = [\mathbf{t}]_\times \mathrm{R}$)
- Solutions) **Planar homography**: 4-point algorithm (**8 DOF**; up-to scale **"scale ambiguity"**)
  - **Estimation**: `cv.findHomography()` → 1 solutions
  - **Conversion to calibrated H**: $\widehat{\mathrm{H}} = \mathrm{K}'^{-1}\mathrm{HK}$
  - **Decomposition**: `cv.decomposeHomographyMat()` → 4 solutions **"relative pose ambiguity"**
  - **Degenerate cases**: Correspondence **not** from a single plane

Relative Camera Pose Estimation

- **Relative camera pose estimation**
  - Solutions) **Planar homography**: 4-point algorithm (**8 DOF**; up-to scale **"scale ambiguity"**)
    - **Estimation**: `cv.findHomography()` → 1 solutions
    - **Derivation**

$$\lambda' \hat{\mathbf{x}}' = \lambda \mathrm{R} \hat{\mathbf{x}} + \mathbf{t}$$

$$\downarrow \quad \frac{1}{d} \mathbf{n}^\intercal \hat{\mathbf{x}} = 1 \ \ (\because n_x \hat{x} + n_y \hat{y} + n_z \hat{z} - d = 0)$$

$$\hat{\mathbf{x}}' = \lambda'' \left(\mathrm{R} + \frac{1}{d'} \mathbf{t}\, \mathbf{n}^\intercal \right) \hat{\mathbf{x}}$$

$$\downarrow \quad \widehat{\mathrm{H}} = \mathrm{R} + \frac{1}{d'} \mathbf{t}\, \mathbf{n}^\intercal$$

$$\hat{\mathbf{x}}' = \widehat{\mathrm{H}}\, \hat{\mathbf{x}}$$

$$\downarrow \quad \hat{\mathbf{x}} = \mathrm{K}^{-1} \mathbf{x} \ \text{ and } \ \hat{\mathbf{x}}' = \mathrm{K}'^{-1} \mathbf{x}'$$

$$\mathbf{x}' = \mathrm{H}\, \mathbf{x}$$

    - **Conversion to calibrated H**: $\widehat{\mathrm{H}} = \mathrm{K}'^{-1} \mathrm{H} \mathrm{K}$
    - **Decomposition**: `cv.decomposeHomographyMat()` → 4 solutions **"relative pose ambiguity"**
    - **Degenerate cases**: Correspondence **not** from a single plane

Relative Camera Pose Estimation: Overview

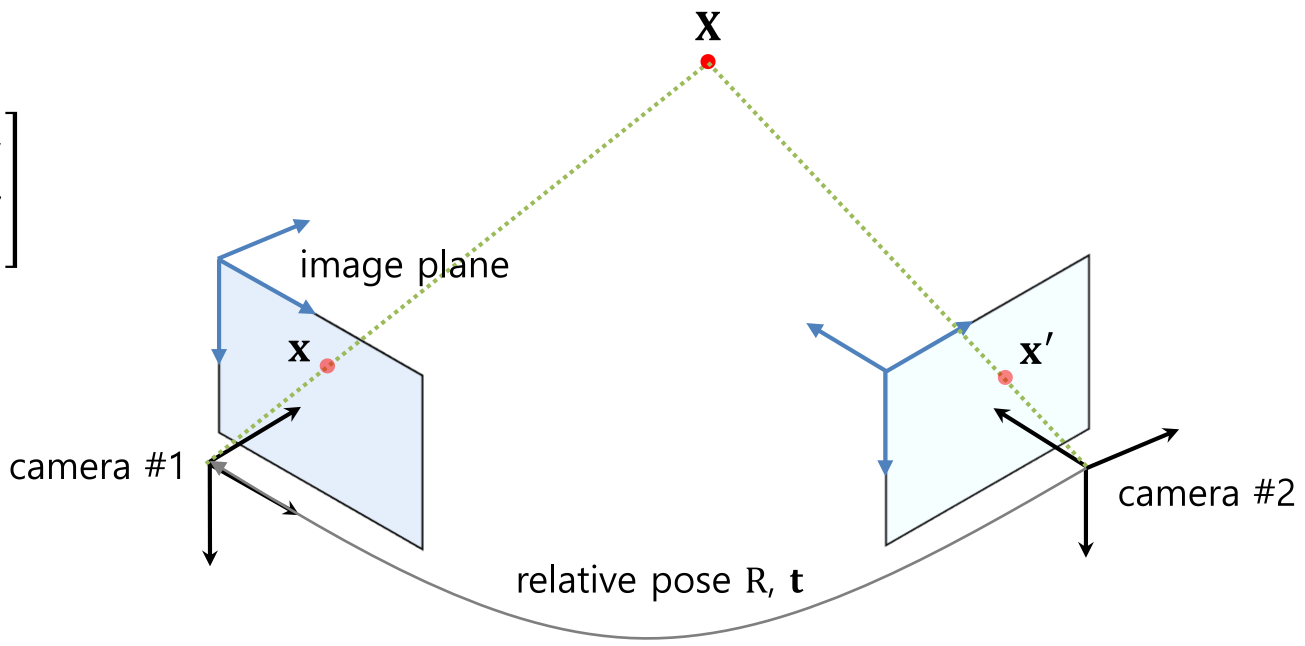
	Items	General 2D-2D Geometry	Planar 2D-2D Geometry
On Image Planes	Model	**Fundamental Matrix (7 DOF)**	**Planar Homography (8 DOF)**
	Formulation	$\mathrm{F} = \mathrm{K}'^{-\intercal} \mathrm{E} \mathrm{K}^{-1}$ $\quad \mathrm{E} = \mathrm{K}'^{\intercal} \mathrm{F} \mathrm{K}$	$\mathrm{H} = \mathrm{K}' \hat{\mathrm{H}} \mathrm{K}^{-1}$ $\quad \hat{\mathrm{H}} = \mathrm{K}'^{-1} \mathrm{H} \mathrm{K}$
	Estimation	- **7-point algorithm** ($n \geq 7$) → k solution - **(normalized) 8-point algorithm** → 1 solution - `cv.findFundamentalMat()`	- **4-point algorithm** ($n \geq 4$) → 1 solution - `cv::findHomography()`
	Input	- $(\mathbf{x}_i, \mathbf{x}'_i)$ [px] on the image plane	- $(\mathbf{x}_i, \mathbf{x}'_i)$ [px] on a plane in the image plane
	Degenerate Cases	- No translational motion - Correspondence from a single plane	- Correspondence not from a single plane
	Decomposition to R and **t**	- Convert to an essential matrix and decompose it	- `cv.decomposeHomographyMat()`
On Normalized Image Planes	Model	**Essentials Matrix (5 DOF)**	**(Calibrated) Planar Homgraphy (8 DOF)**
	Formulation	$\mathrm{E} = [\mathbf{t}]_\times \mathrm{R}$	$\hat{\mathrm{H}} = \mathrm{R} + \frac{1}{d} \mathbf{t} \mathbf{n}^\intercal$
	Estimation	- **5-point algorithm** ($n \geq 5$) → k solution - `cv.findEssentialMat()`	- **4-point algorithm** ($n \geq 4$) → 1 solution - `cv::findHomography()`
	Input	- $(\hat{\mathbf{x}}_i, \hat{\mathbf{x}}'_i)$ [m] on the normalized image plane	- $(\hat{\mathbf{x}}_i, \hat{\mathbf{x}}'_i)$ [m] on a plane in the normalized image plane
	Degenerate Cases	- No translational motion	- Correspondence not from a single plane
	Decomposition to R and **t**	- `cv.decomposeEssentialMat()` - `cv.recoverPose()`	- `cv.decomposeHomographyMat()` with $\mathrm{K} = \mathrm{I}_{3\times3}$

Triangulation

- **Triangulation** (point localization)
  - Unknown: Position of a 3D point $\mathbf{X}$ (3 DOF)
  - Given: Point correspondence $(\mathbf{x}, \mathbf{x}')$, camera matrices $(\mathrm{K}, \mathrm{K}')$, and relative pose $(\mathrm{R}, \mathbf{t})$
  - Constraints: $\mathbf{x} = \mathrm{K}\,[\,\mathrm{I}\,|\,\mathbf{0}\,]\,\mathbf{X}$, $\mathbf{x}' = \mathrm{K}'\,[\,\mathrm{R}\,|\,\mathbf{t}\,]\,\mathbf{X}$
  - Solution
    - OpenCV `cv.triangulatePoints()`
    - Special case) Stereo cameras

$$\mathrm{R} = \mathrm{I}_{3\times3},\ \mathbf{t} = \begin{bmatrix} -b \\ 0 \\ 0 \end{bmatrix}, \text{ and } \mathrm{K} = \mathrm{K}' = \begin{bmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore Z = \frac{f}{x - x'} b$$



Triangulation

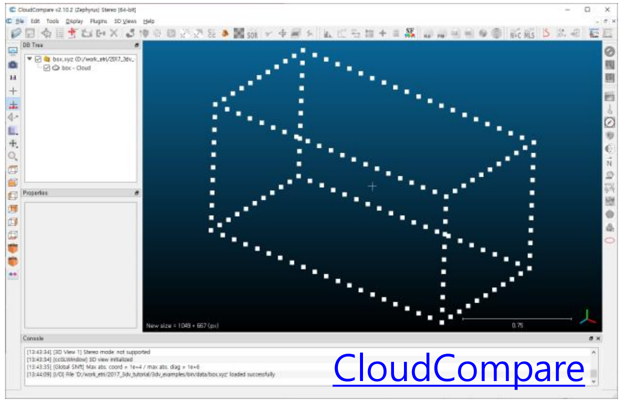
- Example) Triangulation

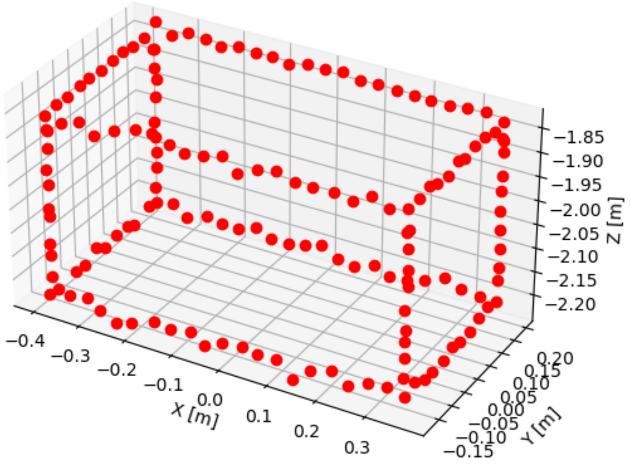
```python
f, cx, cy = 1000., 320., 240.
pts0 = np.loadtxt('../data/image_formation0.xyz')[:,:2]
pts1 = np.loadtxt('../data/image_formation1.xyz')[:,:2]
output_file = '../data/triangulation.xyz'

# Estimate relative pose of two view
F, _ = cv.findFundamentalMat(pts0, pts1, cv.FM_8POINT)
K = np.array([[f, 0, cx], [0, f, cy], [0, 0, 1]])
E = K.T @ F @ K
_, R, t, _ = cv.recoverPose(E, pts0, pts1)

# Reconstruct 3D points (triangulation)
P0 = K @ np.eye(3, 4, dtype=np.float32)
Rt = np.hstack((R, t))
P1 = K @ Rt
X = cv.triangulatePoints(P0, P1, pts0.T, pts1.T)
X /= X[3]
X = X.T

# Write the reconstructed 3D points
np.savetxt(output_file, X)
```

$$\mathbf{x} = \mathbf{K}\,[\,\mathbf{I} \mid \mathbf{0}\,]\,\mathbf{X}$$

$$\mathbf{x}' = \mathbf{K}'\,[\,\mathbf{R} \mid \mathbf{t}\,]\,\mathbf{X}$$

A point cloud: data/box.xyz





output_file: data/triangulation.xyz

Summary

- **Planar Homography**: $\mathbf{x}'_i = \mathrm{H}\mathbf{x}_i$
  - Example) Perspective distortion correction
  - Example) Planar image stitching
  - Example) 2D video stabilization
- **Epipolar Geometry**: $\mathbf{x}'^{\top}\mathrm{F}\mathbf{x} = 0$ (on the image plane), $\hat{\mathbf{x}}'^{\top}\mathrm{E}\hat{\mathbf{x}} = 0$ (on the *normalized* image plane)
  - Example) Epipolar line visualization
- **Relative Camera Pose Estimation**: Finding R and $\mathbf{t}$ (5 DOF)
  - Solutions) **Fundamental matrix**: 7/8-point algorithm (7 DOF)
  - Solutions) **Essential matrix**: 5-point algorithm (5 DOF)
  - Solutions) **Planar homography**: 4-point algorithm (8 DOF)
  - Example) Fundamental matrix estimation
  - Example) Monocular Visual Odometry (Epipolar Version)
- **Triangulation**: Finding $\mathbf{X}$ (3 DOF)
  - Example) Triangulation

$$
\begin{array}{ll}
\mathrm{R}, \mathbf{t} & \\
\updownarrow & (\mathrm{E} = [\mathbf{t}]_{\times}\mathrm{R}) \\
\mathrm{E} & \\
\updownarrow & (\mathrm{E} = \mathrm{K}'^{\top}\mathrm{F}\mathrm{K}) \\
\mathrm{F} &
\end{array}
$$