Supplementary Material: R^{2}LIVE: A Robust, Real-time, LiDAR-Inertial-Visual tightly-coupled state Estimator and mapping

A. *Perturbation on $SO(3)$*

In this appendix, we will use the following approximation of perturbation $\delta\mathbf{r} \to \mathbf{0}$ on $SO(3)$ [25, 26]:

$$\begin{aligned}
\mathtt{Exp}(\mathbf{r}+\delta\mathbf{r}) &\approx \mathtt{Exp}(\mathbf{r})\mathtt{Exp}(\mathbf{J}_r(\mathbf{r})\delta\mathbf{r}) \\
\mathtt{Exp}(\mathbf{r})\mathtt{Exp}(\delta\mathbf{r}) &\approx \mathtt{Exp}(\mathbf{r}+\mathbf{J}_r^{-1}(\mathbf{r})\delta\mathbf{r}) \\
\mathbf{R}\cdot\mathtt{Exp}(\delta\mathbf{r})\cdot\mathbf{u} &\approx \mathbf{R}\left(\mathbf{I}+[\delta\mathbf{r}]_\times\right)\mathbf{u} = \mathbf{R}\mathbf{u} - \mathbf{R}[\mathbf{u}]_\times\delta\mathbf{r}
\end{aligned}$$

where $\mathbf{u} \in \mathbb{R}^3$ and we use $[\cdot]_\times$ denote the skew-symmetric matrix of vector $(\cdot)$; $\mathbf{J}_r(\mathbf{r})$ and $\mathbf{J}_r^{-1}(\mathbf{r})$ are called the *right Jacobian* and the *inverse right Jacobian* of $SO(3)$, respectively.

$$\begin{aligned}
\mathbf{J}_r(\mathbf{r}) =& \mathbf{I} - \frac{1-\cos||\mathbf{r}||}{||\mathbf{r}||^2}[\mathbf{r}]_\times + \frac{||\mathbf{r}||-\sin(||\mathbf{r}||)}{||\mathbf{r}||^3}[\mathbf{r}]_\times^2 \\
\mathbf{J}_r^{-1}(\mathbf{r}) =& \mathbf{I} + \frac{1}{2}[\mathbf{r}]_\times + \left(\frac{1}{||\mathbf{r}||^2} - \frac{1+\cos(||\mathbf{r}||)}{2||\mathbf{r}||\sin(||\mathbf{r}||)}\right)[\mathbf{r}]_\times^2
\end{aligned}$$

B. *Computation of $\mathbf{F}_{\delta\mathbf{x}}$ and $\mathbf{F}_{\mathbf{w}}$*

Combing (4) and (6) , we have:

$$\begin{aligned}
&\delta\hat{\mathbf{x}}_{i+1} = \mathbf{x}_{i+1} \boxminus \hat{\mathbf{x}}_{i+1} \\
&= \left(\mathbf{x}_i \boxplus \left(\Delta t\cdot\mathbf{f}(\mathbf{x}_i,\mathbf{u}_i,\mathbf{w}_i)\right)\right) \boxminus \left(\hat{\mathbf{x}}_i \boxplus \left(\Delta t\cdot\mathbf{f}(\hat{\mathbf{x}}_i,\mathbf{u}_i,\mathbf{0})\right)\right) \\
&= \begin{bmatrix}
\mathbf{Log}\left(\left({}^G\hat{\mathbf{R}}_{I_i}\mathtt{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)\right)^T \cdot \left({}^G\hat{\mathbf{R}}_{I_i}\mathtt{Exp}\left({}^G\delta\mathbf{r}_{I_i}\right)\mathtt{Exp}\left(\boldsymbol{\omega}_i\Delta t\right)\right)\right) \\
{}^G\delta\mathbf{p}_{I_i} + {}^G\delta\mathbf{v}_i\Delta t + \frac{1}{2}\mathbf{a}_i\Delta t^2 - \frac{1}{2}\hat{\mathbf{a}}_i\Delta t^2 \\
{}^I\delta\mathbf{r}_{C_i} \\
{}^I\delta\mathbf{p}_{C_i} \\
{}^G\delta\mathbf{v}_i + \left({}^G\hat{\mathbf{R}}_{I_i}\mathtt{Exp}\left({}^G\delta\mathbf{r}_{I_i}\right)\right)\mathbf{a}_i\Delta t - {}^G\hat{\mathbf{R}}_{I_i}\hat{\mathbf{a}}_i\Delta t \\
\delta\mathbf{b}_{g_i} + \mathbf{n}_{\mathbf{bg}_i} \\
\delta\mathbf{a}_{g_i} + \mathbf{n}_{\mathbf{ba}_i}
\end{bmatrix}
\end{aligned}$$

with:

$$\hat{\boldsymbol{\omega}}_i = \boldsymbol{\omega}_{m_i} - \mathbf{b}_{\mathbf{g}_i},\ \ \boldsymbol{\omega}_i = \hat{\boldsymbol{\omega}}_i - \delta\mathbf{b}_{\mathbf{g}_i} - \mathbf{n}_{\mathbf{g}_i} \tag{S1}$$

$$\hat{\mathbf{a}}_i = \mathbf{a}_{m_i} - \mathbf{b}_{\mathbf{a}_i},\ \ \mathbf{a}_i = \hat{\mathbf{a}}_i - \delta\mathbf{b}_{\mathbf{a}_i} - \mathbf{n}_{\mathbf{a}_i} \tag{S2}$$

And we have the following simplification and approximation from Section. A.

$$\begin{aligned}
&\mathtt{Log}\left(\left({}^G\hat{\mathbf{R}}_{I_i}\mathtt{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)\right)^T \cdot \left({}^G\hat{\mathbf{R}}_{I_i}\mathtt{Exp}\left({}^G\delta\mathbf{r}_{I_i}\right)\mathtt{Exp}\left(\boldsymbol{\omega}_i\Delta t\right)\right)\right) \\
=&\mathtt{Log}\left(\mathtt{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)^T \cdot \left(\mathtt{Exp}\left({}^G\delta\mathbf{r}_{I_i}\right)\cdot\mathtt{Exp}\left(\boldsymbol{\omega}_i\Delta t\right)\right)\right) \\
\approx&\mathtt{Log}\left(\mathtt{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)^T\mathtt{Exp}\left({}^G\delta\mathbf{r}_{I_i}\right)\mathtt{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)\cdot\right. \\
&\quad \mathtt{Exp}\left(-\mathbf{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)\left(\delta\mathbf{b}_{g_i}+\mathbf{n}_{\mathbf{g}_i}\right)\right)) \\
\approx&\mathtt{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)\cdot{}^G\delta\mathbf{r}_{I_i} - \mathbf{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)^T\delta\mathbf{b}_{\mathbf{g}_i} - \mathbf{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)^T\mathbf{n}_{\mathbf{g}_i} \\
&\left({}^G\mathbf{R}_{I_i}\mathtt{Exp}\left({}^G\delta\mathbf{r}_{I_i}\right)\right)\mathbf{a}_i\Delta t \\
\approx&\left({}^G\mathbf{R}_{I_i}\left(\mathbf{I}+[{}^G\delta\mathbf{r}_{I_i}]_\times\right)\right)\left(\hat{\mathbf{a}}_i - \delta\mathbf{b}_{\mathbf{a}_i} - \mathbf{n}_{\mathbf{a}_i}\right)\Delta t \\
\approx&{}^G\mathbf{R}_{I_i}\hat{\mathbf{a}}_i\Delta t - {}^G\mathbf{R}_{I_i}\delta\mathbf{b}_{\mathbf{a}_i}\Delta t - {}^G\mathbf{R}_{I_i}\mathbf{n}_{\mathbf{a}_i}\Delta t - {}^G\mathbf{R}_{I_i}[\hat{\mathbf{a}}_i]_\times{}^G\delta\mathbf{r}_{I_i}
\end{aligned}$$

To conclude, we have the computation of $\mathbf{F}_{\delta\mathbf{x}}$ and $\mathbf{F}_{\mathbf{w}}$ as follow:

$$\begin{aligned}
\mathbf{F}_{\delta\hat{\mathbf{x}}} =& \left.\frac{\partial\left(\delta\hat{\mathbf{x}}_{i+1}\right)}{\partial\delta\hat{\mathbf{x}}_i}\right|_{\delta\hat{\mathbf{x}}_i=\mathbf{0},\mathbf{w}_i=\mathbf{0}} \\
=& \begin{bmatrix}
\mathtt{Exp}(-\hat{\boldsymbol{\omega}}_i\Delta t) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)^T & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{I}\Delta t & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-{}^G\hat{\mathbf{R}}_{I_i}[\hat{\mathbf{a}}_i]_\times\Delta t & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & -{}^G\hat{\mathbf{R}}_{I_i}\Delta t \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}
\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}_{\mathbf{w}} =& \left.\frac{\partial\left(\delta\hat{\mathbf{x}}_{i+1}\right)}{\partial\mathbf{w}_i}\right|_{\delta\hat{\mathbf{x}}_i=\mathbf{0},\mathbf{w}_i=\mathbf{0}} \\
=& \begin{bmatrix}
-\mathbf{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -{}^G\hat{\mathbf{R}}_{I_i}\Delta t & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}\Delta t & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}\Delta t
\end{bmatrix}
\end{aligned}$$

C. *The computation of $\boldsymbol{\mathcal{H}}$*

Recalling (15), we have:

$$\begin{aligned}
\boldsymbol{\mathcal{H}} =& \frac{\left(\check{\mathbf{x}}_{k+1}\boxplus\delta\check{\mathbf{x}}_{k+1}\right)\boxminus\hat{\mathbf{x}}_{k+1}}{\partial\delta\check{\mathbf{x}}_{k+1}}|_{\delta\check{\mathbf{x}}_{k+1}=\mathbf{0}} \\
=& \begin{bmatrix}
\mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{3\times 9} \\
\mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{3\times 9} \\
\mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{0}_{3\times 9} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}_{3\times 9} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{9\times 9}
\end{bmatrix}
\end{aligned}$$

with the 3×3 matrix $\mathbf{A} = \mathbf{J}_r^{-1}(\mathtt{Log}({}^G\hat{\mathbf{R}}_{I_{k+1}}{}^T\,{}^G\check{\mathbf{R}}_{I_{k+1}}))$ and $\mathbf{B} = \mathbf{J}_r^{-1}(\mathtt{Log}({}^I\hat{\mathbf{R}}_{C_{k+1}}{}^T\,{}^I\check{\mathbf{R}}_{C_{k+1}}))$.

D. *The computation of $\mathbf{H}_j^l$*

Recalling (12) and (15), we have:

$$\begin{aligned}
&\mathbf{r}_l(\check{\mathbf{x}}_{k+1}\boxplus\delta\check{\mathbf{x}}_{k+1},{}^L\mathbf{p}_j) = \mathbf{u}_j^T\left({}^G\check{\mathbf{p}}_{I_{k+1}} + {}^G\delta\check{\mathbf{p}}_{I_{k+1}} - \right. \\
&\mathbf{q}_j + {}^G\check{\mathbf{R}}_{I_{k+1}}\mathtt{Exp}({}^G\delta\check{\mathbf{r}}_{I_{k+1}})\left({}^I\mathbf{R}_L{}^L\mathbf{p}_j + {}^I\mathbf{p}_L\right)\Big)
\end{aligned} \tag{S3}$$

And with the small perturbation approximation, we get:

$$\begin{aligned}
&{}^G\check{\mathbf{R}}_{I_{k+1}}\mathtt{Exp}({}^G\delta\check{\mathbf{r}}_{I_{k+1}})\mathbf{P_a} \\
\approx&{}^G\check{\mathbf{R}}_{I_{k+1}}\left(\mathbf{I}+\left[{}^G\delta\check{\mathbf{r}}_{I_{k+1}}\right]_\times\right)\mathbf{P_a} \\
=&{}^G\check{\mathbf{R}}_{I_{k+1}}\mathbf{P_a} - {}^G\check{\mathbf{R}}_{I_{k+1}}[\mathbf{P_a}]_\times{}^G\delta\check{\mathbf{r}}_{I_{k+1}}
\end{aligned} \tag{S4}$$

where $\mathbf{P_a} = {}^I\mathbf{R}_L{}^L\mathbf{p}_j + {}^I\mathbf{p}_L$. Combining (S3) and (S4) together we can obtain:

$$\mathbf{H}_j^l = \mathbf{u}_j^T\begin{bmatrix}-{}^G\check{\mathbf{R}}_{I_{k+1}}[\mathbf{P_a}]_\times & \mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 15}\end{bmatrix}$$

E. *The computation of $\mathbf{H}_s^c$ and $\mathbf{F}_{\mathbf{P}_s}$*

Recalling (16), we have:

$${}^C\mathbf{P}_s = \mathbf{P_C}(\check{\mathbf{x}}_{k+1},{}^G\mathbf{P}_s) = \begin{bmatrix}{}^CP_{sx} & {}^CP_{sy} & {}^CP_{sz}\end{bmatrix}^T$$

where the function $\mathbf{P_C}(\check{\mathbf{x}}_{k+1},{}^G\mathbf{P}_s)$ is:

$$\mathbf{P_C}(\check{\mathbf{x}}_{k+1},{}^G\mathbf{P}_s) = \left({}^G\check{\mathbf{R}}_{I_{k+1}}{}^I\check{\mathbf{R}}_{C_{k+1}}\right)^T{}^G\mathbf{P}_s \tag{S5}$$

$$-\left({}^I\check{\mathbf{R}}_{C_{k+1}}\right)^T{}^G\check{\mathbf{p}}_{I_{k+1}} - {}^I\check{\mathbf{p}}_{C_{k+1}} \tag{S6}$$

From (20), we have:

$$\mathbf{r}_c\left(\check{\mathbf{x}}_{k+1},{}^C\mathbf{p}_s,{}^G\mathbf{P}_s\right) = {}^C\mathbf{p}_s - \boldsymbol{\pi}({}^C\mathbf{P}_s)$$

$$\boldsymbol{\pi}({}^C\mathbf{P}_s) = \begin{bmatrix} f_x\frac{{}^CP_{sx}}{{}^CP_{sz}} + c_x & f_y\frac{{}^CP_{sy}}{{}^CP_{sz}} + c_y\end{bmatrix}^T \tag{S7}$$

where f_x and f_y are the focal length, c_x and c_y are the principal point offsets in image plane.

For conveniently, we omit the $(\cdot)|_{\delta\check{\mathbf{x}}_{k+1}^i=\mathbf{0}}$ in the following derivation, and we have:

$$\mathbf{H}_s^c = -\frac{\partial\boldsymbol{\pi}({}^C\mathbf{P}_s)}{\partial{}^C\mathbf{P}_s}\cdot\frac{\partial\mathbf{P_C}(\check{\mathbf{x}}_{k+1}\boxplus\delta\check{\mathbf{x}}_{k+1},{}^G\mathbf{P}_s)}{\partial\delta\check{\mathbf{x}}_{k+1}} \tag{S8}$$

$$\mathbf{F}_{\mathbf{P}_s} = -\frac{\partial\boldsymbol{\pi}({}^C\mathbf{P}_s)}{\partial{}^C\mathbf{P}_s}\cdot\frac{\partial\mathbf{P_C}(\check{\mathbf{x}}_{k+1},{}^G\mathbf{P}_s)}{\partial{}^G\mathbf{P}_s} \tag{S9}$$

where:

$$\frac{\partial\boldsymbol{\pi}({}^C\mathbf{P}_s)}{\partial{}^C\mathbf{P}_s} = \frac{1}{{}^CP_{s_z}}\begin{bmatrix} f_x & 0 & -f_x\frac{{}^CP_{s_x}}{{}^CP_{s_z}} \\ 0 & f_y & -f_y\frac{{}^CP_{s_y}}{{}^CP_{s_z}}\end{bmatrix} \tag{S10}$$

$$\frac{\partial\mathbf{P_b}(\check{\mathbf{x}}_{k+1},{}^G\mathbf{P}_s)}{\partial{}^G\mathbf{P}_s} = \left({}^G\check{\mathbf{R}}_{I_{k+1}}{}^I\check{\mathbf{R}}_C\right)^T \tag{S11}$$

According to Section. A, we have the following approximation of $\mathbf{P_C}(\check{\mathbf{x}}_{k+1}\boxplus\delta\check{\mathbf{x}}_{k+1},{}^G\mathbf{P}_s)$:

$$\begin{aligned}
&\mathbf{P_C}(\check{\mathbf{x}}_{k+1}\boxplus\delta\check{\mathbf{x}}_{k+1},{}^G\mathbf{P}_s)\\
&= \left({}^G\check{\mathbf{R}}_{I_{k+1}}\mathtt{Exp}\left({}^G\delta\check{\mathbf{r}}_{I_{k+1}}\right){}^I\check{\mathbf{R}}_{C_{k+1}}\mathtt{Exp}\left({}^I\delta\check{\mathbf{r}}_{C_{k+1}}\right)\right)^T{}^G\mathbf{P}_s - {}^I\check{\mathbf{p}}_C\\
&- {}^I\delta\check{\mathbf{p}}_C - \left({}^I\check{\mathbf{R}}_C\mathtt{Exp}\left({}^I\delta\check{\mathbf{r}}_C\right)\right)^T\left({}^G\check{\mathbf{p}}_{I_{k+1}} + {}^G\delta\check{\mathbf{p}}_{I_{k+1}}\right)\\
&\approx \mathbf{P_b}(\check{\mathbf{x}}_{k+1}^i,{}^G\mathbf{P}_s) + \left[\left({}^G\check{\mathbf{R}}_{I_{k+1}}{}^I\check{\mathbf{R}}_C\right)^T{}^G\mathbf{P}_s\right]_\times{}^I\delta\check{\mathbf{r}}_C\\
&+ \left({}^I\check{\mathbf{R}}_C\right)^T\left[\left({}^G\check{\mathbf{R}}_{I_{k+1}}\right)^T{}^G\mathbf{P}_s\right]_\times{}^G\delta\check{\mathbf{r}}_{I_{k+1}} - \left({}^I\check{\mathbf{R}}_C\right)^T{}^G\delta\check{\mathbf{p}}_{I_{k+1}}\\
&- \left[\left({}^I\check{\mathbf{R}}_C\right)^T{}^G\check{\mathbf{p}}_{I_{k+1}}\right]_\times{}^I\delta\check{\mathbf{r}}_C - {}^I\delta\check{\mathbf{p}}_C
\end{aligned}$$

With this, we can derive:

$$\frac{\partial\mathbf{P_C}(\check{\mathbf{x}}_{k+1}\boxplus\delta\check{\mathbf{x}}_{k+1},{}^G\mathbf{P}_s)}{\partial\delta\check{\mathbf{x}}_{k+1}} = \begin{bmatrix}\mathbf{M_A} & \mathbf{M_B} & \mathbf{M_C} & -\mathbf{I} & \mathbf{0}_{3\times12}\end{bmatrix} \tag{S12}$$

$$\begin{aligned}
\mathbf{M_A} &= \left({}^I\check{\mathbf{R}}_C\right)^T\left[\left({}^G\hat{\mathbf{R}}_{I_{k+1}}\right)^T{}^G\mathbf{P}_s\right]_\times\\
\mathbf{M_B} &= -\left({}^I\hat{\mathbf{R}}_C\right)^T\\
\mathbf{M_C} &= \left[\left({}^G\hat{\mathbf{R}}_{I_{k+1}}{}^I\hat{\mathbf{R}}_C\right)^T{}^G\mathbf{P}_s\right]_\times - \left[\left({}^I\hat{\mathbf{R}}_C\right)^T{}^G\hat{\mathbf{p}}_{I_{k+1}}^i\right]_\times
\end{aligned}$$

Substituting (S10), (S11) and (S12) into (S8) and (S9), we finish the computation of $\mathbf{H}_s^c$ and $\mathbf{F}_{\mathbf{P}_s}$.