International Journal of Forecasting 7 (1991) 339–347
North-Holland

ARIMA forecasts with restrictions derived from a structural change

Victor M. Guerrero *
Instituto Tecnológico Autónomo de México (ITAM), México 01000, DF, Mexico

Abstract: Some time series models, which account for a structural change either in the deterministic or in the stochastic part of an ARIMA model are presented. The structural change is assumed to occur during the forecast horizon of the series and the only available information about this change, besides the time point of its occurrence, is provided by only one or two linear restrictions imposed on the forecasts. Formulas for calculating the variance of the restricted forecasts as well as some other statistics are derived. The methods here suggested are illustrated by means of empirical examples.

Keywords: Deterministic change, Intervention, Minimum mean-square error, Stochastic change, Time series models.

1. Introduction

It sometimes happens that a time series analyst is provided with additional information, besides the historical record of a time series. For instance, one such situation occurred when the Mexican Government wanted to evaluate the recovery of the Mexican economy, as measured by the annual rate of growth of the Index of Manufacturing Production (IMP). There was an understanding between the Mexican authorities and the International Monetary Fund of reaching a specified rate of growth of IMP at the end of a certain year. Since some new economic policies were to be implemented, a structural change on the behavior of IMP was expected and a higher than usual rate of growth of IMP was agreed upon. Then, a future monthly path, consistent with the annual target and with the avaiable historical record, as well as tolerance limits for that path, were needed to determine whether the observed behavior of IMP during that year should be considered adequate.

This paper considers the case in which the monthly figures are to be forecasted with the aid of an autoregressive-integrated-moving-average (ARIMA) model. Evidently, the probability of achieving the annual target with the conventional ARIMA forecasts is zero, unless the additional information is properly taken into consideration as imposing some (linear) restrictions on the forecasts. We propose here some models which assume that the structural change will affect specifically either the deterministic or the stochastic part of the ARIMA model. It is also assumed that enough historical data exist for building such a model by means of the Box and Jenkins (1970) strategy. While the additional information consists only of one or two restrictions to be satisfied by the forecasts.

Some papers that have previously considered the problem of incorporating external information to the ARIMA forecasts are those of Cholette (1982), Guerrero (1989) and Trabelsi and Hillmer (1989). Although the Trabelsi–Hillmer's setting is more general, all three solutions can be shown to be equivalent under

* The author would like to express his appreciation to the associate editor and two anonymous referees for all their comments and constructive criticisms.

appropriate conditions. However, none of those papers considered the possibility of a structural change during the forecast horizon. On the other hand, the work of Thompson and Miller (1986) addressed specifically that possibility through a 'forecast intervention' approach, in which a bundle of many possible realizations is used to project the future in several ways, including a change in the level of the series. So, the idea of the present paper is not entirely new and its aim is mainly to help the decision makers decide which of the possible scenarios portrayed is more realistic. Therefore the methods here suggested may also prove to be valuable tools for what Thompson and Miller call 'what if analysis'.

Derivation of the restricted forecast is carried out by Lagrangian minimization within the framework of a general model which allows for changes both in its deterministic as well as in its stochastic structure. The deterministic change situation is approached through an ex-ante intervention model, of the kind proposed by Box and Tiao (1975), with a first order structure which affects only the local level of the series. The corresponding parameters of this model are determined entirely by the restrictions imposed on the forecasts. When the change is assumed to be of a stochastic nature, the original ARIMA model is augmented by adding a white noise process. Formulas for the restricted forecasts in this situation involve the variance of the white noise process. Again this parameter has to be determined from the information provided by the restrictions and a statistic is derived as an aid for performing this task.

It is worth noticing that perhaps in some practical applications we may not be sure that the additional information implies a structural change in the historical behavior of the series. In that case we should test for compatibility between the historical and the additional information through a statistical test as the one provided by Guerrero (1989). Then if the test rejects the null hypothesis of compatibility, we could assume the existence of a structural change at a certain time point during the forecast horizon and proceed to apply either the technique of Thompson and Miller (1986) or ours. If on the other hand, the test fails to reject the hypothesis, there is no reason to anticipate a structural change. Anyhow, the additional information should be incorporated into the forecasts to improve both their accuracy and precision, and this can be done as illustrated by Trabelsi and Hillmer (1989) or by Guerrero (1989).

2. Modeling a change in the ARIMA structure

Let $\{Z_t\}$ denote a time series observed during the time period $t = 1, \ldots, N$, which can be modeled as an ARIMA process. Its minimum mean square error (MMSE) forecast, given the column vector of historical data $\boldsymbol{Z}_0 = (Z_1, \ldots, Z_N)'$ and expressed in terms of the pure moving average representation coefficients, produces for $t = N+1, N+2, \ldots$, the forecast error

$$Z_t - E(Z_t \mid \boldsymbol{Z}_0) = \sum_{j=0}^{t-N-1} \psi_j a_{t-j}, \qquad \psi_0 = 1. \tag{2.1}$$

Here $\{a_t\}$ represents a Gaussian zero-mean, white noise process with variance σ_a^2, and the weights ψ_1, $\psi_2, \ldots$ are assumed to be known. This expression can be validated in the stationary case by way of Wold's Decomposition Theorem and in the nonstationary case by the results of Bell (1984, section 2).

A structural change affecting the deterministic structure of an ARIMA process, namely the local level, can be thought of as being due to a (deterministic) intervention, as it was suggested by Box and Tiao (1975). If such an intervention is justified and the time point of its occurrence is precisely known, then the original ARIMA model can be augmented by including the dynamic function D_t^{τ}, which will be assumed to be given by

$$(1 - \delta B) D_t^{\tau} = \omega S_t^{\tau}, \tag{2.2}$$

where B denote the backshift operator such that $BZ_t = Z_{t-1}$, while δ and ω are the intervention parameters. This function includes as special cases an immediate change of level of size ω (when $\delta = 0$), a gradual change of level with eventual gain $\omega/(1-\delta)$ (when $|\delta| \leq 1$) and an unlimited change (when $|\delta| > 1$). The applicability of this particular function in empirical work has been demonstrated by several authors [e.g. Box and Tiao (1975), Thompson and Miller (1986) or Tsay (1987)]. Expression (2.2) involves the step function S_t^τ which takes on the value 1 when $t \geq \tau$ and is zero otherwise, with the value τ denoting the moment of the intervention. Of course, a more general form of the dynamic model (2.2) could have been postulated, but that would involve also more parameters, which to be specified would require information from more than two (linearly independent) restrictions. However, in practice it seldom happens that more than one or two restrictions are available, so there is no real need of considering a more general case.

On the other hand, it is well known [e.g. Granger and Morris (1976)] that adding a white noise process to an ARIMA model, produces another ARIMA model with different stochastic structure. Then, if the stochastic structure of the original model is deemed to be affected by the structural change, we shall consider adding a white noise process $\{v_t\}$ to the ARIMA model for taking account of that change. Thus, given $\boldsymbol{Z}_0$, $\{v_t\}$ will be assumed to be a Gaussian zero-mean white noise process independent of $\{a_t\}$, with variance σ_v^2. This is a very simple but practical approach, which can be considered as a particular case of contaminating the original ARIMA model with the addition of another independent ARIMA model. This general situation is practically untractable, since it would require full knowledge of the contaminating ARIMA model, so again it does not make much sense to consider a formulation of this kind.

Thus, a model which includes both deterministic (D) and stochastic (V) effects of the form previously described, is

$$Z_{t,\mathrm{D,V}} = Z_t + \left(\frac{\omega}{1-\delta B} + v_t\right)S_t^\tau, \qquad t = 1,\ldots, N+H, \tag{2.3}$$

with H the forecast horizon. Without loss of generality, let us suppose the intervention occurs at the time point $\tau = N+1$ (otherwise $\boldsymbol{Z}_0$ would be augmented with the conventional ARIMA forecasts $\hat{Z}_N(1),\ldots, \hat{Z}_N(\tau - N - 1)$, in Box–Jenkins notation, and the origin for the forecasts obtained with model (2.3) would be $\tau - 1$). Then by defining the future observations of the series as the column vector $\boldsymbol{Z}_\mathrm{F} = (Z_{N+1},\ldots, Z_{N+H})'$, with similar definitions for $\mathrm{D_F}$, $\mathrm{V_F}$ and $\boldsymbol{Z}_{\mathrm{F,D,V}}$, we can express the values obtained with (2.3) for $t = N+1,\ldots, N+H$ as

$$\boldsymbol{Z}_{\mathrm{F,D,V}} = \boldsymbol{Z}_\mathrm{F} + \mathrm{D_F} + \mathrm{V_F}. \tag{2.4}$$

So, to obtain the MMSE forecast under formulation (2.3) we could simply take conditional expectation of (2.4), given $\boldsymbol{Z}_O$. This yields the forecast

$$E(\boldsymbol{Z}_{\mathrm{F,D,V}} \mid \boldsymbol{Z}_0) = E(\boldsymbol{Z}_\mathrm{F} \mid \boldsymbol{Z}_0) + \mathrm{D_F}, \tag{2.5}$$

whose error has covariance matrix

$$\mathrm{Cov}\big[\boldsymbol{Z}_{\mathrm{F,D,V}} - E(\boldsymbol{Z}_{\mathrm{F,D,V}} \mid \boldsymbol{Z}_0) \mid \boldsymbol{Z}_0\big] = \sigma_a^2\psi\psi' + \sigma_v^2 I, \tag{2.6}$$

with ψ the $H \times H$ lower triangular matrix with values $1, \psi_1,\ldots, \psi_{H-1}$ in the first column, the values $0, 1, \psi_1,\ldots, \psi_{H-2}$ in the second column, and so on.

If ω, δ and σ_v^2 were known, expressions (2.5)–(2.6) would provide a solution to the problem of obtaining forecasts which account for structural changes in the forecast horizon. However, in practice those parameters are unknown and we should rely on some additional information besides $\boldsymbol{Z}_0$, to determine their values. Here we are concerned with obtaining the vector of forecasts when additional information about the future values of the series is given in the form of linear restrictions, that is

$$\mathrm{Y} = C\boldsymbol{Z}_{\mathrm{F,D,V}}, \tag{2.7}$$

where Y is an m-dimensional vector of known values and C is an $m \times H$ constant matrix. In Subsections 2.1 and 2.2 we shall link (2.7) with specifying the parameters which appear in (2.3). At this point, let us notice that (2.1), (2.4) and (2.5) imply

$$\boldsymbol{Z}_{\mathrm{F,D,V}} = E(\boldsymbol{Z}_{\mathrm{F,D,V}} \mid \boldsymbol{Z}_0) + \psi \mathrm{a_F} + \mathrm{V_F}, \tag{2.8}$$

with $a_F = (a_{N+1}, \ldots, a_{N+H})'$. Therefore, since $\mathrm{Cov}(\psi a_F + V_F \mid Z_0) = \sigma_a^2 \psi\psi' + \sigma_v^2 I$, we pose the problem as one of Lagrangian minimization of $Z_{F,D,V} - E(Z_{F,D,V} \mid Z_0)$ subject to the restriction (2.7). To solve this problem let us consider the following function, which involves knowledge of both Z_0 and Y:

$$Q = \left[Z_{F,D,V} - E(Z_{F,D,V} \mid Z_0)\right]' \left(\sigma_a^2 \psi\psi' + \sigma_v^2 I\right)^{-1} \left[Z_{F,D,V} - E(Z_{F,D,V} \mid Z_0)\right] + 2L'(Y - CZ_{F,D,V}), \tag{2.9}$$

in which L is a vector of Lagrange multipliers. By solving the equation $O = \partial Q / \partial Z_{F,D,V}$ we obtain

$$\hat{Z}_{F,D,V} = E(Z_{F,D,V} \mid Z_0, Y) = E(Z_{F,D,V} \mid Z_0) + A\left[Y - CE(Z_{F,D,V} \mid Z_0)\right], \tag{2.10}$$

with

$$A = \left(\sigma_a^2 \psi\psi' C' + \sigma_v^2 C'\right)\left(\sigma_a^2 C\psi\psi' C' + \sigma_v^2 CC'\right)^{-1}. \tag{2.11}$$

Moreover, the covariance matrix of the restricted forecast errors can be shown to be

$$\mathrm{Cov}\left[\left(Z_{F,D,V} - \hat{Z}_{F,D,V}\right) \mid Z_0, Y\right] = (I - AC)\left(\sigma_a^2 \psi\psi' + \sigma_v^2 I\right). \tag{2.12}$$

Thus, $\hat{Z}_{F,D,V}$ as given by (2.10)–(2.11), satisfies the restrictions imposed by (2.7) and is a theoretically sound solution, but we still need to be able to provide the values of ω, δ and σ_v^2 in order to apply this result in practice. When (2.7) imposes at most two restrictions, it is impossible to determine the values of three parameters, so in order to do that we specialize the results to the two different types of changes so far considered, separately. Thus in (2.4) we will have $Z_{F,D} = Z_F + D_F$ for a deterministic change, while $Z_{F,V} = Z_F + V_F$ will be used when the change is stochastic. The best way of choosing between these models is based on subject matter knowledge of the phenomenon under study, but we can also discriminate between them empirically by looking at their corresponding restricted forecasts, as is illustrated in Section 3.

2.1. Change in the deterministic structure

If only a deterministic change is expected to occur, we take $\sigma_v^2 = 0$ in the previous formulation and obtain the corresponding restricted forecast, which will be denoted $\hat{Z}_{F,D}$. To solve the problem of determining ω and δ we should make use of the explicit form of D_t^T [see (2.2)] which is 0 if $t \le N$ and is $\omega(1 - \delta^{t-N})/(1 - \delta)$ if $t > N$. That is, we know that the vector of future values associated with the intervention is given by

$$D_F = \left(\omega,\ \omega(1+\delta), \ldots, \omega(1 + \delta + \delta^2 + \cdots + \delta^{H-1})\right)'.$$

On the other hand, we also know that all the information about the behavior of the series during the forecast horizon, is provided by the unrestricted ARIMA forecasts $E(Z_F \mid Z_0)$ and the additional information Y. Therefore D_F and hence the values of ω and δ are specified by solving

$$C\hat{D}_F = Y - CE(Z_F \mid Z_0), \tag{2.13}$$

which we assume is a system of consistent equations (i.e. any linear relationship existing among the rows of C also exists among the corresponding elements of $Y - CE(Z_F \mid Z_0)$). The solution of that system is

$$\hat{D}_F = C^-\left[Y - CE(Z_F \mid Z_0)\right] + (I - C^- C)w, \tag{2.14}$$

with C^- a generalized inverse of C and w an arbitrary H-dimensional constant vector. Then the restricted forecasts which incorporate the deterministic effects to the ARIMA forecasts are given by

$$\hat{Z}_{F,D} = E(Z_F \mid Z_0) + \hat{D}_F, \tag{2.15}$$

with

$$\mathrm{Cov}\left[\left(Z_{\mathrm{F,D}}-\hat{Z}_{\mathrm{F,D}}\right) \mid Z_0, \mathrm{Y}\right]=\sigma_a^2\left(I-A_{\mathrm{D}} C\right) \psi \psi' \tag{2.16}$$

and

$$A_{\mathrm{D}}=\psi \psi' C'\left(C \psi \psi' C'\right)^{-1}. \tag{2.17}$$

Since the covariance matrix of the unrestricted forecast errors, given by $\sigma_a^2 \psi \psi'$, exceeds the covariance matrix (2.16) by the matrix $\sigma_a^2 A_{\mathrm{D}} C \psi \psi'$, which is positive semidefinite, it follows that the restricted forecast $\hat{Z}_{\mathrm{F,D}}$ is at least as precise as the unrestricted one. In fact, the covariance matrix (2.16) is identical to that obtained by Guerrero (1989) for the forecast error $Z_{\mathrm{F}}-E(Z_{\mathrm{F}} \mid Z_0, \mathrm{Y})$ which was derived on the assumption of no structural change.

2.2. Change in the stochastic structure

When only the stochastic structure is deemed to change due to the intervention, we take $\omega=0$ in the general model and make use of the fact that, given Z_0,

$$\mathrm{Y}-C E\left(Z_{\mathrm{F}} \mid Z_0\right)=C \psi a_{\mathrm{F}}+C \mathrm{V}_{\mathrm{F}} \sim N\left(\mathbf{0}, \sigma_a^2 C \psi \psi' C'+\sigma_v^2 C C'\right). \tag{2.18}$$

Then we propose to employ the statistic

$$K_V=\left[\mathrm{Y}-C E\left(Z_{\mathrm{F}} \mid Z_0\right)\right]'\left(\sigma_a^2 C \psi \psi' C'+\sigma_v^2 C C'\right)^{-1}\left[\mathrm{Y}-C E\left(Z_{\mathrm{F}} \mid Z_0\right)\right], \tag{2.19}$$

which, for N large enough, is approximately distributed as a Chi-square variable with m degrees of freedom, to test the validity of the restriction $\mathrm{Y}=C Z_{\mathrm{F,V}}$ which is equivalent to $\mathrm{Y}=C E(Z_{\mathrm{F}} \mid Z_0)+C \psi a_{\mathrm{F}} + C \mathrm{V}_{\mathrm{F}}$. This restriction will be valid when σ_v^2 is reasonably chosen. Thus we consider now the restriction as a null hypothesis, which will not be rejected when σ_v^2 is selected appropriately. Besides, this appropriateness criterion will be satisfied in accordance with a predetermined significance level. That is, we do not reject the hypothesis when $K_{\mathrm{V}}<\chi_m^2(\alpha)$, where $\chi_m^2(\alpha)$ denotes the upper α percentage point of the Chi-square distribution.

Hence, we suggest to select the value $\hat{\sigma}_v^2$ by trial and error until K_{V} reaches a value less than the specified percentage point. Evidently this procedure does not produce a unique $\hat{\sigma}_v^2$ value, but we know from (2.12) that the higher this value the lower the precision of the restricted forecast, so we should choose it as the minimum value (or close to it) for which $K_{\mathrm{V}}<\chi_m^2(\alpha)$, with α given beforehand. Another approach, suggested by a referee, consists of applying a Bayesian approach for estimating σ_v^2, that would consist of specifically incorporating a prior distribution for this parameter, but that idea will not be pursued any further here.

From (2.10)–(2.12) we get

$$\hat{Z}_{\mathrm{F,V}}=E\left(Z_{\mathrm{F}} \mid Z_0\right)+A_{\mathrm{V}}\left[\mathrm{Y}-C E\left(Z_{\mathrm{F}} \mid Z_0\right)\right], \tag{2.20}$$

with

$$A_{\mathrm{V}}=\left(\sigma_a^2 \psi \psi' C'+\hat{\sigma}_v^2 C'\right)\left(\sigma_a^2 C \psi \psi' C'+\hat{\sigma}_v^2 C C'\right)^{-1} \tag{2.21}$$

and

$$\mathrm{Cov}\left[\left(Z_{\mathrm{F,V}}-\hat{Z}_{\mathrm{F,V}}\right) \mid Z_0, \mathrm{Y}\right]=\left(I-A_v C\right)\left(\sigma_a^2 \psi \psi'+\hat{\sigma}_v^2 I\right). \tag{2.22}$$

Thus, in this case the precision of the restricted forecasts can be higher or lower than that of the unrestricted ones, depending on whether the matrix $A_{\mathrm{V}} C(\sigma_a^2 \psi \psi'+\hat{\sigma}_v^2 I)-\hat{\sigma}_v^2 I$ is positive or negative semidefinite.

3. Empirical illustration of the procedures

An ARIMA model for the annual percent rate of growth of the Index of Manufacturing Production (IMP) for México $\{R_t^{\mathrm{IMP}}\}$ is now used for illustrative purposes. Such a model is

$$\nabla R_t^{\mathrm{IMP}} = (1-\theta B)(1-\Theta B^{12})a_t,$$

with estimated parameters (calculated by maximum likelihood from monthly data ranging from 01, 1975 to 09, 1986) $\hat{\theta}=0.1031$, $\hat{\Theta}=0.8111$ and $\hat{\sigma}_a^2=0.9313$.

3.1. Forecasts subjected to one linear restriction

Here we shall assume that the interest lies in obtaining a monthly path for R_t^{IMP} such that at the end of 1987 the rate of growth is 7%. It is also assumed that a structural change is to take place in october of 1986.

First we will consider the possibility of a change in the deterministic structure (the local level) of the series. Here we have $m=1$, $Y=7$, $C=(0, 0, \ldots, 0, 1)$ and, from the original forecasts we know $\hat{Z}_N(15) = -0.05$, so the equation to be solved for determining $\hat{\omega}$ and $\hat{\delta}$ becomes

$$\hat{\omega}(1+\hat{\delta}+\cdots+\hat{\delta}^{14}) = Y - \hat{Z}_N(15) = 7.05.$$

As an extra condition which will allow us to get a unique solution, it will be supposed that the eventual gain of the intervention function is, for all practical purposes, already attained in December of 1987. This means that $\hat{\omega}/(1-\hat{\delta}) \doteq \hat{\omega}(1-\hat{\delta}^{15})/(1-\hat{\delta})$, thus by choosing $\hat{\delta}^{15}=0.0001 \doteq 0$ the solution became $\hat{\delta}=0.55$, $\hat{\omega}=3.1725$. The restricted forecasts obtained with these values are shown in Table 1, under the heading of procedure A_{D}.

Now, if we fear the change will occur in the stochastic structure, we could alternatively postulate a model contaminated with a white noise process. Here the problem lies in selecting σ_v^2 appropriately. To this end we used the statistic K_{V} and fixed the significance level at $\alpha=0.05$, so we had to solve the inequality

$$\left[Y-\hat{Z}_N(15)\right]^2/\chi_1^2(0.05) - \sigma_a^2 C\psi\psi'C' < \hat{\sigma}_v^2,$$

where $C\psi\psi'C' = 9.9131$. Then for $\hat{\sigma}_v^2 = 4.5$ we obtained the restricted forecasts which appear in Exhibit 1, under the heading of procedure A_{V}.

Exhibit 1 also shows the original ARIMA forecasts $E(\boldsymbol{Z}_{\mathrm{F}} \mid \boldsymbol{Z}_0)$ and the corresponding standard errors of the different forecasts. Were we asked to decide whether procedure A_{D} or A_{V} is preferable, we could do that naively by looking at the standard errors of the forecasts, and procedure A_{D} would be the choice. Of course, a better way to decide between a deterministic or a stochastic change should be based on subject matter knowledge about the structural change. For instance, in this particular illustration it would be hard to believe that the IMP could change its observed historical pattern, reflected in the conventional ARIMA forecasts, as drastically as the A_{D} procedure suggests. Hence, the path shown by the forecasts obtained with procedure A_{V} could be thought as more realistic. However, by imposing the restrictions we force the series to do something strange and we should pay for it with more uncertainty, which shows up through larger standard errors.

It is worth emphasizing the fact that the three sets of forecasts shown are drastically different, because the corresponding assumptions leading to each of them were also considerably different. In the first place, $E(\boldsymbol{Z}_{\mathrm{F}} \mid \boldsymbol{Z}_0)$ differs from the other sets of forecasts in that only the historical information was taken into account in that case, while the restricted forecasts involve more information. Secondly, the dramatic difference between the two sets of restricted forecasts, is entirely due to the assumption of either a

Exhibit 1
Forecasts of the rate of growth of IMP: $Y=7$.

Month	$E(Z_F \mid Z_0)$		Procedure A_D		Procedure A_V	
	Fore-cast	Std. error	Fore-cast	Std. error	Fore-cast	Std. error
Oct. 86	−3.72	0.931	−0.55	0.930	−3.64	2.316
Nov.	−3.66	1.251	1.26	1.247	−3.51	2.462
Dec.	−3.55	1.504	2.33	1.500	−3.37	2.599
Jan. 87	−1.31	1.721	5.09	1.679	−0.71	2.714
Feb.	−2.04	1.913	4.66	1.811	−1.07	2.812
Mar.	−2.94	2.087	3.91	1.904	−1.59	2.894
Apr.	−2.23	2.248	4.71	1.965	−0.51	2.962
May	−1.82	2.398	5.17	1.996	0.28	3.015
Jun.	−1.07	2.539	5.92	1.998	1.40	3.055
Jul.	−0.71	2.673	6.32	1.971	2.14	3.083
Aug.	−0.19	2.801	6.85	1.915	3.04	3.099
Sep.	−0.03	2.923	7.01	1.827	3.57	3.102
Oct.	−0.05	2.924	7.00	1.551	3.86	3.003
Nov.	−0.05	2.928	7.00	1.179	4.18	2.891
Dec.	−0.05	2.932	7.00	0.000	7.00	0.000

deterministic or a stochastic intervention. To visualize the behavior of these forecasts, we show them in Exhibit 2 together with some historical data. In this exhibit we also include the restricted forecasts provided by Guerrero (1989) with his procedure A which assumes no structural change (even though that assumption was rejected by a significance test at the 5% level, as reported in that paper). Here we notice that procedures A_V and A produce similar results, which also agree with the historical record, however in this case procedure A_V is more honest by showing higher uncertainty and assuming explicitly the existence of a structural change.

We should also notice that procedures A, A_D and A_V all assume implicitly that the restrictions are not subject to uncertainty. In practice it would perhaps be more reasonable to assume the presence of random error in the restrictions, but this would imply the need of estimating the variability of such an error. Guerrero (1989) proposes a method to do that when no structural change is feared during the forecast horizon. Trabelsi and Hillmer (1989) deal with such a variability by assuming that, instead of restrictions, we have access to forecasts obtained from alternative models and to their corresponding covariance matrix. In this paper we prefer not to attack that problem explicitly, because that requires more information than the few restrictions so far assumed known and we rather view the present methodology as a tool for performing scenario or 'what if' analysis, as Thompson and Miller (1986) call it.

3.2. Two linear restrictions on the forecasts

The following illustration considers the case in which two restrictions are imposed on the forecasts, namely: (1) the rate of growth at the end of 1987 is 7% and (2) the average rate of growth during that year is 3%. In this situation we have $m=2$, $H=15$,

$$\mathrm{Y}=\begin{pmatrix}7\\3\end{pmatrix},\quad C=\begin{pmatrix}0&0&0&0&\dots&0&1\\0&0&0&1/12&\dots&1/12&1/12\end{pmatrix}.$$

The equations associated with procedure A_D became

$$\frac{\hat{\omega}}{1-\hat{\delta}}(1-\hat{\delta}^{15})=7.05,\quad \frac{\hat{\omega}}{1-\hat{\delta}}\left[1-\frac{\hat{\delta}^{4}-\hat{\delta}^{16}}{12(1-\hat{\delta})}\right]=4.0408,$$

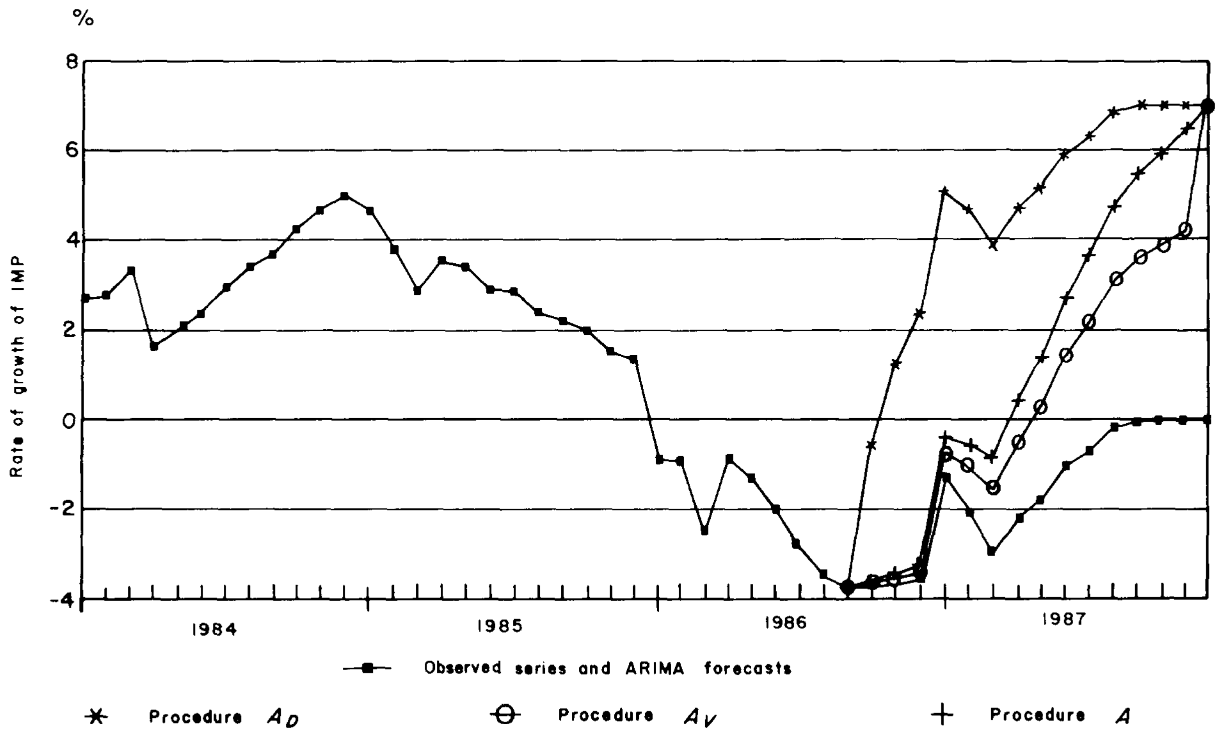

Exhibit 2. Restricted and unrestricted forecasts.

so that $\hat{\delta} = 1.0456$ and $\hat{\omega} = 0.3377$. For applying procedure A_V, we proceeded by trial and error and found that $\hat{\sigma}_v^2 = 3$ was an appropriate choice, according to the rule $K_V < \chi_2^2(\alpha)$ with $\alpha = 0.10$.

The results, in terms of the calculated forecasts and their standard errors are summarized in Exhibit 3. From the viewpoint of a practitioner, the two sets of monthly forecasts may be equally good and indeed

Exhibit 3
Forecasts for the rate of growth of IMP: Y = (7, 3)′.

Month	Procedure A_D Forecast	Std. error	Procedure A_V Forecast	Std. error
Oct. 86	−3.38	0.850	−3.45	1.938
Nov.	−2.97	0.996	−3.13	2.030
Dec.	−2.49	0.915	−2.77	2.044
Jan. 87	0.14	0.953	0.14	1.967
Feb.	−0.19	0.931	−0.06	1.945
Mar.	−0.67	0.890	−0.44	1.915
Apr.	0.48	0.848	0.76	1.884
May	1.35	0.818	1.65	1.858
Jun.	2.59	0.808	2.85	1.845
Jul.	3.45	0.821	3.65	1.846
Aug.	4.50	0.850	4.59	1.864
Sep.	5.21	0.883	5.15	1.898
Oct.	5.77	1.056	5.29	2.051
Nov.	6.37	0.995	5.43	2.143
Dec.	7.00	0.000	7.00	0.000
Average 87	3.00	–	3.00	–

their paths are very similar. However, the underlying assumptions (the models) for each one are different and we should keep this in mind when selecting one particular procedure. In the present situation, Exhibit 3 allows us to appreciate the forecast precision which is undoubtedly better for procedure A_D, nonetheless this procedure could be discarded on the grounds that the intervention function involved implies an infinite eventual gain (since $\hat{\delta} > 1$). Therefore procedure A_V would be recommended as the most appropriate in this application.

4. Summary and conclusions

This paper presents two basic models which may account for a structural change to occur during the forecast horizon of the time series under study. The information about this change was assumed to be provided only by some linear restrictions on the future values of the series. These models are the basis from which procedures A_D and A_V are developed for obtaining restricted forecasts.

The illustrations presented allow us to appreciate the potential usefulness of the methods. Even though in practical applications we should rely on subject matter considerations for discriminating among procedures A_D or A_V, an extension of this work would consider working with the general augmented model to include the possibility of changes in both the deterministic and the stochastic structure of the original ARIMA model. Of course, that would require more information to be provided by the linear restrictions than it was assumed here.

References

Bell, W., 1984, "Signal extraction for nonstationary time series", *The Annals of Statistics*, 12, 646–664.

Box, G.E.P. and G.M. Jenkins, 1970, *Time Series Analysis, Forecasting and Control* (Holden-Day, San Francisco).

Box, G.E.P. and G.C. Tiao, 1975, "Intervention analysis with applications to economic and environmental problems", *Journal of the American Statistical Association*, 70, 70–79.

Cholette, P.A., 1982, "Prior information and ARIMA forecasting", *Journal of Forecasting*, 1, 375–383.

Granger, C.W.J. and M.J. Morris, 1976, "Time series modelling and interpretation", *Journal of the Royal Statistical Society*, A-139, 246–257.

Guerrero, V.M., 1989, "Optimal conditional ARIMA forecasts", *Journal of Forecasting*, 8, 215–229.

Thompson, P.A. and R.B. Miller, 1986, "Sampling the future: A Bayesian approach to forecasting from univariate time series", *Journal of Business and Economic Statistics*, 4, 427–436.

Trabelsi, A. and S.C. Hillmer, 1989, "A benchmarking approach to forecast combination", *Journal of Business and Economic Statistics*, 7, 353–362.

Tsay, R.S., 1987, "Detecting and modeling changes in time series. Comment on 'Non-Gaussian state-space modeling of nonstationary time series' by Kitagawa, G.", *Journal of the American Statistical Association*, 82, 1056–1059.

Biography: Víctor M. GUERRERO is Professor of Statistics at the Instituto Tecnológico Autónomo de México (ITAM). He received a Ph.D. degree in statistics from the University of Wisconsin–Madison in 1979. His research interests include time series analysis, econometrics and forecasting techniques. He has published articles in such journals as Biometrika, Insurance: Mathematics and Economics, Letters in Probability and Statistics, Economics Letters, Journal of Forecasting, International Statistical Review and has some other publications in Spanish, including two textbooks.