Multivariate Analysis for the Behavioral Sciences, Second Edition (Chapman and Hall/CRC, 2019)

Examples of Chapter 15: Exploratory Factor Analysis

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12 December 2018

Contents

Examples

Table 15.1: Maximum Likelihood Two-Factor Solution for Correlations of Six School Subjects

```
R <- c(  1.00, 0.44, 0.41, 0.29, 0.33, 0.25,
         0.44, 1.00, 0.35, 0.35, 0.32, 0.33,
         0.41, 0.35, 1.00, 0.16, 0.19, 0.18,
         0.29, 0.35, 0.16, 1.00, 0.59, 0.47,
         0.33, 0.32, 0.19, 0.59, 1.00, 0.46,
         0.25, 0.33, 0.18, 0.47, 0.46, 1.00)

school_subjects <- c("French", "English", "History", "Arithmetic", "Algebra", "Geometry")
R <- matrix(R, ncol = 6, byrow = TRUE, dimnames = list(school_subjects, school_subjects))
R
```

```
##            French English History Arithmetic Algebra Geometry
## French       1.00    0.44    0.41       0.29    0.33     0.25
## English      0.44    1.00    0.35       0.35    0.32     0.33
## History      0.41    0.35    1.00       0.16    0.19     0.18
## Arithmetic   0.29    0.35    0.16       1.00    0.59     0.47
## Algebra      0.33    0.32    0.19       0.59    1.00     0.46
## Geometry     0.25    0.33    0.18       0.47    0.46     1.00
```

```
Rfa1 <- factanal(covmat = R, factors = 1, n.obs = 220, rotation = "none")
Rfa2 <- factanal(covmat = R, factors = 2, n.obs = 220, rotation = "none")
Rfa3 <- factanal(covmat = R, factors = 3, n.obs = 220, rotation = "none")
Rfa1
```

```
##
## Call:
## factanal(factors = 1, covmat = R, n.obs = 220, rotation = "none")
##
## Uniquenesses:
##     French    English    History Arithmetic    Algebra   Geometry
##      0.750      0.710      0.878      0.473      0.468      0.621
##
## Loadings:
##            Factor1
## French     0.500
## English    0.539
## History    0.349
## Arithmetic 0.726
## Algebra    0.729
## Geometry   0.615
##
##                Factor1
## SS loadings       2.10
## Proportion Var    0.35
##
## Test of the hypothesis that 1 factor is sufficient.
## The chi square statistic is 51.6 on 9 degrees of freedom.
## The p-value is 5.37e-08
```

```
Rfa2
```

```
##
## Call:
## factanal(factors = 2, covmat = R, n.obs = 220, rotation = "none")
##
## Uniquenesses:
##     French    English    History Arithmetic    Algebra   Geometry
##      0.508      0.595      0.644      0.377      0.440      0.628
##
## Loadings:
##            Factor1 Factor2
## French      0.558   0.425
## English     0.569   0.286
## History     0.392   0.450
## Arithmetic  0.738  -0.279
## Algebra     0.718  -0.209
## Geometry    0.595  -0.133
##
##                Factor1 Factor2
## SS loadings      2.204   0.603
## Proportion Var   0.367   0.101
## Cumulative Var   0.367   0.468
##
## Test of the hypothesis that 2 factors are sufficient.
## The chi square statistic is 2.18 on 4 degrees of freedom.
## The p-value is 0.703
```

```
Rfa3
```

```
##
## Call:
## factanal(factors = 3, covmat = R, n.obs = 220, rotation = "none")
##
## Uniquenesses:
##     French    English    History Arithmetic    Algebra   Geometry
##      0.448      0.497      0.679      0.411      0.376      0.611
##
## Loadings:
##            Factor1 Factor2 Factor3
## French      0.582   0.432  -0.162
## English     0.594   0.306   0.238
## History     0.398   0.402
## Arithmetic  0.713  -0.281
## Algebra     0.731  -0.269  -0.130
## Geometry    0.591  -0.158   0.123
##
##                Factor1 Factor2 Factor3
## SS loadings      2.241   0.617   0.118
## Proportion Var   0.373   0.103   0.020
## Cumulative Var   0.373   0.476   0.496
##
## The degrees of freedom for the model is 0 and the fit was 0.001
```

```r
#install.packages("psych")
library(psych)

RfaNone <- fa(R, nfactors = 2, fm = "ml", rotate = "none")
print.psych(RfaNone)
```

```
## Factor Analysis using method =  ml
## Call: fa(r = R, nfactors = 2, rotate = "none", fm = "ml")
## Standardized loadings (pattern matrix) based upon correlation matrix
##             ML1   ML2   h2   u2 com
## French     0.56  0.42 0.49 0.51 1.9
## English    0.57  0.29 0.41 0.59 1.5
## History    0.39  0.45 0.36 0.64 2.0
## Arithmetic 0.74 -0.28 0.62 0.38 1.3
## Algebra    0.72 -0.21 0.56 0.44 1.2
## Geometry   0.59 -0.13 0.37 0.63 1.1
##
##                        ML1  ML2
## SS loadings           2.20 0.60
## Proportion Var        0.37 0.10
## Cumulative Var        0.37 0.47
## Proportion Explained  0.79 0.21
## Cumulative Proportion 0.79 1.00
##
## Mean item complexity =  1.5
## Test of the hypothesis that 2 factors are sufficient.
##
## The degrees of freedom for the null model are  15  and the objective function was  1.43
## The degrees of freedom for the model are 4  and the objective function was  0.01
##
## The root mean square of the residuals (RMSR) is  0.01
## The df corrected root mean square of the residuals is  0.03
##
## Fit based upon off diagonal values = 1
## Measures of factor score adequacy
##                                                 ML1  ML2
## Correlation of (regression) scores with factors   0.91 0.73
## Multiple R square of scores with factors          0.82 0.53
## Minimum correlation of possible factor scores     0.64 0.07
```

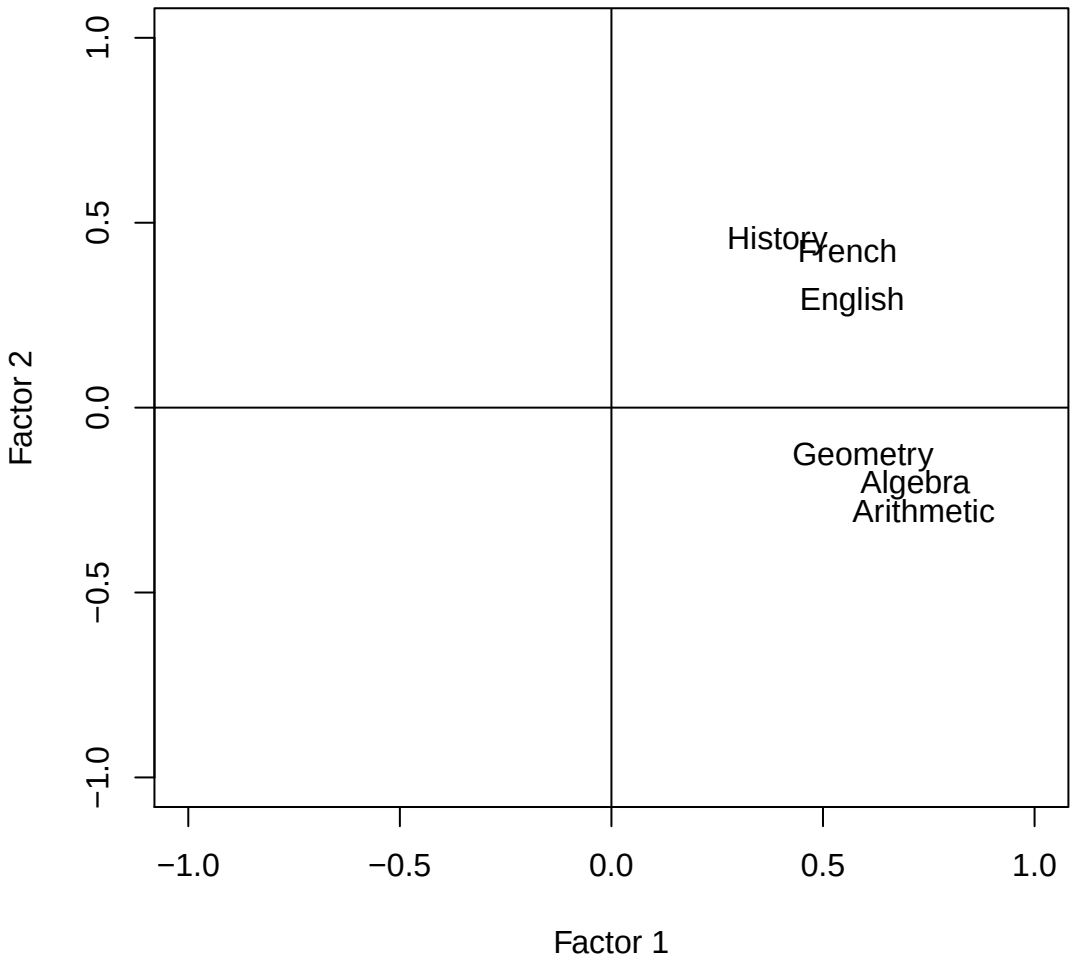
Correlation matrix of the six school subjects implied by the fitted two-factor model:

```r
lambdaHat <- as.matrix(RfaNone$loadings)
Rhat <- lambdaHat %*% t(lambdaHat) + RfaNone$residual
Rhat
```

```
##            French English History Arithmetic Algebra Geometry
## French       1.00    0.44    0.41       0.29    0.33     0.25
## English      0.44    1.00    0.35       0.35    0.32     0.33
## History      0.41    0.35    1.00       0.16    0.19     0.18
## Arithmetic   0.29    0.35    0.16       1.00    0.59     0.47
## Algebra      0.33    0.32    0.19       0.59    1.00     0.46
## Geometry     0.25    0.33    0.18       0.47    0.46     1.00
```

Figure 15.1

```
plot(RfaNone$loadings, type = "n",
     xlim = c(-1, +1), ylim = c(-1, +1),
     xlab = "Factor 1", ylab = "Factor 2")
text(RfaNone$loadings, labels = school_subjects)
abline(h = 0, v = 0)
```



```
# same figure with ggplot (overkill, perhaps) - just testing:
library(ggplot2)
```

```
##
## Attaching package: 'ggplot2'
```

```
## The following objects are masked from 'package:psych':
##
##     %+%, alpha
```

```
RfaNoneLoadings <- as.matrix(RfaNone$loadings)
class(RfaNoneLoadings) <- "matrix"
RfaNoneLoadings <- as.data.frame(RfaNoneLoadings)
RfaNoneLoadings$Subject <- row.names(RfaNoneLoadings)

p1 <- ggplot(RfaNoneLoadings, aes(x = ML1, y = ML2)) + geom_point()
p2 <- p1 + geom_text(aes(x = ML1+0.03, y = ML2+0.00,
                         label = RfaNoneLoadings$Subject), size=4, hjust=0)
p3 <- p2 + theme_bw() + theme(panel.grid.major = element_blank(),
                              panel.grid.minor = element_blank())
p4 <- p3 + scale_x_continuous(limits = c(-1, +1), name = "Factor 1")
p5 <- p4 + scale_y_continuous(limits = c(-1, +1), name = "Factor 2")
p6 <- p5 + geom_vline(xintercept=0) + geom_hline(yintercept=0)
p6
```

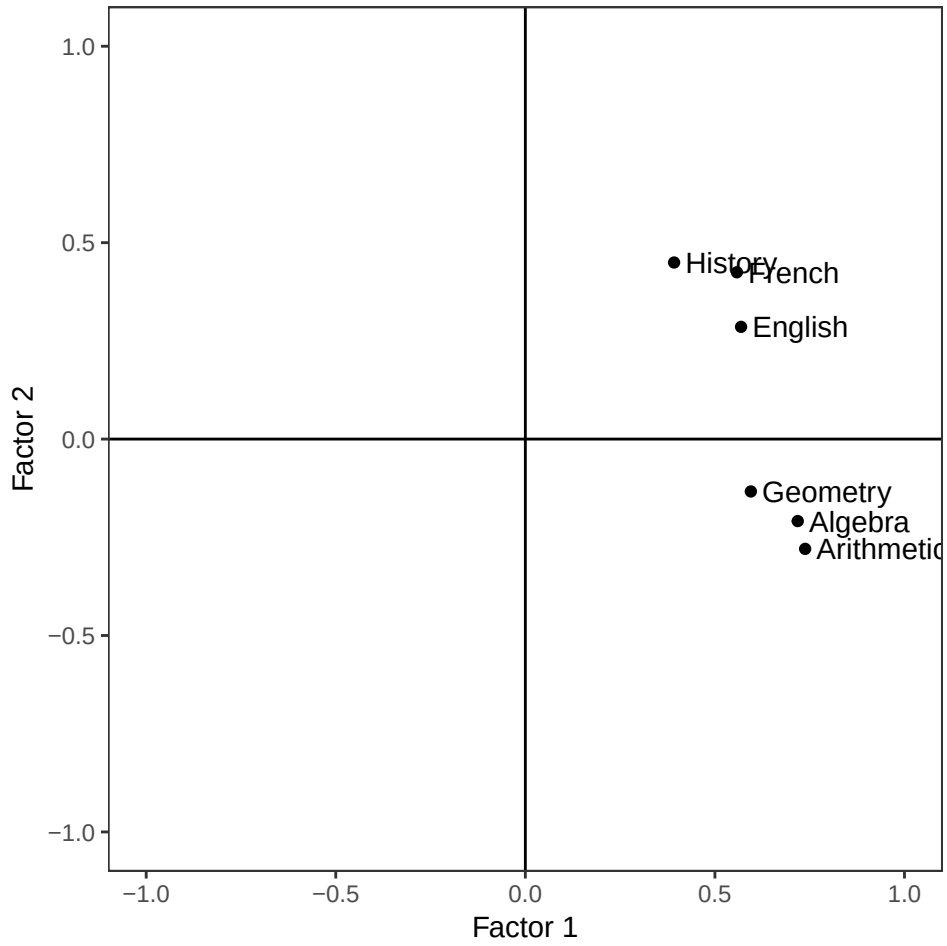


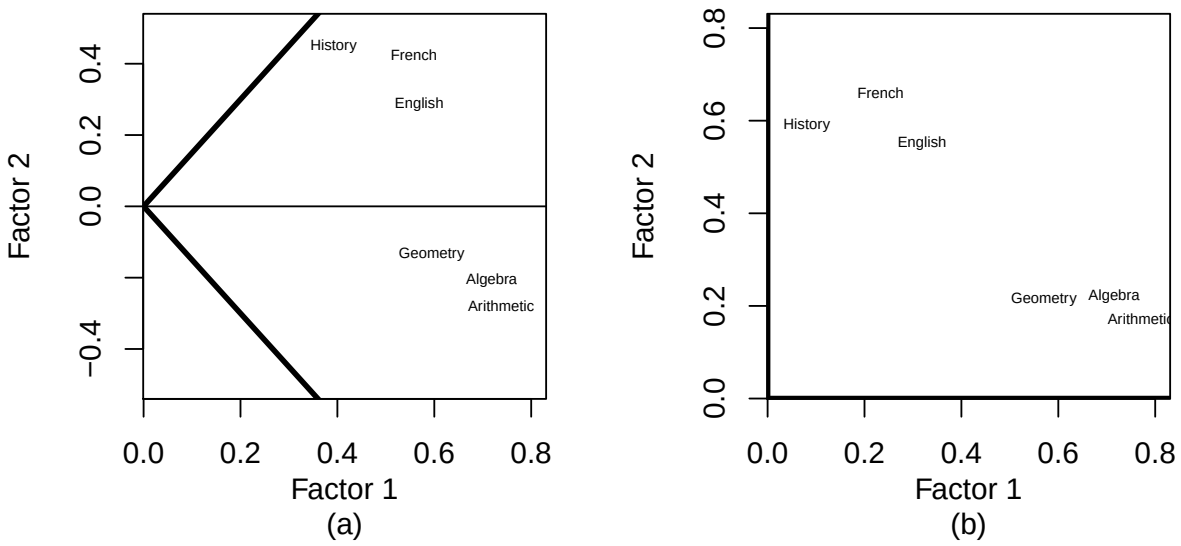
Figure 15.2

```
par(mfrow=c(1,2))

plot(RfaNone$loadings, type = "n", xlim = c(0.03, 0.8), ylim = c(-0.5, +0.5),
     xlab = "Factor 1\n(a)", ylab = "Factor 2")
text(RfaNone$loadings, labels = school_subjects, cex=0.5)
abline(h = 0)
abline(0, +1.5, lwd = 3)
abline(0, -1.5, lwd = 3)

RfaVarimax <- fa(R, nfactors = 2, fm = "ml", rotate = "varimax") # See the note below!

plot(RfaVarimax$loadings, type = "n", xlim = c(0.03, 0.8), ylim = c(0.03, 0.8),
     xlab = "Factor 1\n(b)", ylab = "Factor 2")
text(RfaVarimax$loadings, labels = school_subjects, cex=0.5)
abline(h = 0, v = 0, lwd = 3)
```



Above, the rotation was achieved using the varimax, because, as far as we know, no function for graphical rotation exists in R. For an inspiration to program such a function, you should take a good look at **Survo**: *an environment for creative processing of text and numerical data*, created by professor **Seppo Mustonen** (see www.survo.fi/presentation/history.html), cf. the example of *classical music composers* in **Chapter 14**.

Interactive graphical rotation has been available in various generations of Survo **since the 1970s**:

- A short demo on YouTube shows how an **oblique graphical rotation** is applied in **SURVO MM**: https://youtu.be/-yZRUR6gpAg (see www.survo.fi/demos/#ex38 for more details and links).
- A historical example of using the graphical rotation in **SURVO 76** appears on pp. 29-30 of the report: Mustonen, S. (1977). SURVO 76, a statistical data processing system. Research Report No. 6. Dept of Statistics, University of Helsinki. www.survo.fi/publications/Research_Report_6_Mustonen_1977.pdf

Publications related to Survo since 1963: www.survo.fi/publications

Table 15.2

```
RfaVarimax <- fa(R, nfactors = 2, fm = "ml", rotate = "varimax")
print.psych(RfaVarimax, sort = TRUE)
```

```
## Factor Analysis using method =  ml
## Call: fa(r = R, nfactors = 2, rotate = "varimax", fm = "ml")
## Standardized loadings (pattern matrix) based upon correlation matrix
##            item  ML1  ML2   h2   u2 com
## Arithmetic    4 0.77 0.17 0.62 0.38 1.1
## Algebra       5 0.72 0.22 0.56 0.44 1.2
## Geometry      6 0.57 0.22 0.37 0.63 1.3
## French        1 0.23 0.66 0.49 0.51 1.2
## History       3 0.08 0.59 0.36 0.64 1.0
## English       2 0.32 0.55 0.41 0.59 1.6
##
##                        ML1  ML2
## SS loadings           1.59 1.21
## Proportion Var        0.27 0.20
## Cumulative Var        0.27 0.47
## Proportion Explained  0.57 0.43
## Cumulative Proportion 0.57 1.00
##
## Mean item complexity =  1.2
## Test of the hypothesis that 2 factors are sufficient.
##
## The degrees of freedom for the null model are  15  and the objective function was  1.43
## The degrees of freedom for the model are 4  and the objective function was  0.01
##
## The root mean square of the residuals (RMSR) is  0.01
## The df corrected root mean square of the residuals is  0.03
##
## Fit based upon off diagonal values = 1
## Measures of factor score adequacy
##                                                 ML1  ML2
## Correlation of (regression) scores with factors   0.86 0.79
## Multiple R square of scores with factors          0.73 0.62
## Minimum correlation of possible factor scores     0.47 0.24
```

Table 15.3

```
#install.packages("GPArotation")
library(GPArotation)
RfaOblimin <- fa(R, nfactors = 2, fm = "ml", rotate = "oblimin")
print.psych(RfaOblimin, sort = TRUE)
```

```
## Factor Analysis using method =  ml
## Call: fa(r = R, nfactors = 2, rotate = "oblimin", fm = "ml")
## Standardized loadings (pattern matrix) based upon correlation matrix
##            item   ML1   ML2   h2   u2 com
## Arithmetic    4  0.81 -0.04 0.62 0.38 1.0
## Algebra       5  0.73  0.03 0.56 0.44 1.0
## Geometry      6  0.57  0.07 0.37 0.63 1.0
## French        1  0.04  0.68 0.49 0.51 1.0
## History       3 -0.11  0.65 0.36 0.64 1.1
## English       2  0.17  0.53 0.41 0.59 1.2
##
##                        ML1  ML2
## SS loadings           1.60 1.21
## Proportion Var        0.27 0.20
## Cumulative Var        0.27 0.47
## Proportion Explained  0.57 0.43
## Cumulative Proportion 0.57 1.00
##
##  With factor correlations of
##      ML1  ML2
## ML1 1.00 0.52
## ML2 0.52 1.00
##
## Mean item complexity =  1.1
## Test of the hypothesis that 2 factors are sufficient.
##
## The degrees of freedom for the null model are  15  and the objective function was  1.43
## The degrees of freedom for the model are 4  and the objective function was  0.01
##
## The root mean square of the residuals (RMSR) is  0.01
## The df corrected root mean square of the residuals is  0.03
##
## Fit based upon off diagonal values = 1
## Measures of factor score adequacy
##                                                  ML1  ML2
## Correlation of (regression) scores with factors   0.89 0.84
## Multiple R square of scores with factors          0.79 0.70
## Minimum correlation of possible factor scores     0.58 0.40
```

Table 15.4: Estimated Parameters for the Three-Factor Model Fitted to the Crime Rate Data by Maximum Likelihood

```
# see Chapter 13, where this data set was used for the first time:
crime <- read.table("data/crime.txt", sep = '\t')

crime_fa1 <- factanal(crime, factors = 1, rotation = "none")
crime_fa2 <- factanal(crime, factors = 2, rotation = "none")
crime_fa3 <- factanal(crime, factors = 3, rotation = "none")

crime_fa1
```

```
##
## Call:
## factanal(x = crime, factors = 1, rotation = "none")
##
## Uniquenesses:
##   Murder     Rape  Robbery  Assault Burglary    Theft  Vehicle
##    0.306    0.440    0.290    0.207    0.383    0.639    0.424
##
## Loadings:
##          Factor1
## Murder   0.833
## Rape     0.749
## Robbery  0.843
## Assault  0.890
## Burglary 0.785
## Theft    0.601
## Vehicle  0.759
##
##                Factor1
## SS loadings      4.311
## Proportion Var   0.616
##
## Test of the hypothesis that 1 factor is sufficient.
## The chi square statistic is 64.63 on 14 degrees of freedom.
## The p-value is 1.78e-08
```

```
crime_fa2
```

```
##
## Call:
## factanal(x = crime, factors = 2, rotation = "none")
##
## Uniquenesses:
##   Murder     Rape  Robbery  Assault Burglary    Theft  Vehicle
##    0.295    0.380    0.056    0.276    0.103    0.353    0.349
##
## Loadings:
##          Factor1 Factor2
## Murder    0.829  -0.133
## Rape      0.709   0.343
## Robbery   0.919  -0.317
## Assault   0.847
```

```
## Burglary  0.785   0.531
## Theft     0.601   0.535
## Vehicle   0.799  -0.114
##
##                Factor1 Factor2
## SS loadings      4.365   0.824
## Proportion Var   0.624   0.118
## Cumulative Var   0.624   0.741
##
## Test of the hypothesis that 2 factors are sufficient.
## The chi square statistic is 20.02 on 8 degrees of freedom.
## The p-value is 0.0102
```

```
crime_fa3
```

```
##
## Call:
## factanal(x = crime, factors = 3, rotation = "none")
##
## Uniquenesses:
##   Murder     Rape  Robbery  Assault Burglary    Theft  Vehicle
##    0.030    0.360    0.185    0.253    0.136    0.279    0.005
##
## Loadings:
##          Factor1 Factor2 Factor3
## Murder    0.654   0.727  -0.115
## Rape      0.611   0.307   0.415
## Robbery   0.828   0.344  -0.103
## Assault   0.697   0.479   0.181
## Burglary  0.625   0.330   0.604
## Theft     0.422   0.231   0.700
## Vehicle   0.992  -0.106
##
##                Factor1 Factor2 Factor3
## SS loadings      3.523   1.145   1.083
## Proportion Var   0.503   0.164   0.155
## Cumulative Var   0.503   0.667   0.822
##
## Test of the hypothesis that 3 factors are sufficient.
## The chi square statistic is 4.9 on 3 degrees of freedom.
## The p-value is 0.179
```

```r
crime_fa3None <- fa(crime, nfactors = 3, fm = "ml", rotate = "none")
print.psych(crime_fa3None, digits = 3)
```

```
## Factor Analysis using method =  ml
## Call: fa(r = crime, nfactors = 3, rotate = "none", fm = "ml")
## Standardized loadings (pattern matrix) based upon correlation matrix
##            ML1    ML2    ML3    h2     u2  com
## Murder   0.654  0.727 -0.114 0.970 0.0301 2.03
## Rape     0.611  0.307  0.415 0.640 0.3601 2.30
## Robbery  0.828  0.344 -0.103 0.815 0.1854 1.37
## Assault  0.697  0.479  0.181 0.747 0.2526 1.93
## Burglary 0.625  0.330  0.604 0.864 0.1364 2.51
## Theft    0.422  0.231  0.700 0.721 0.2787 1.90
## Vehicle  0.992 -0.106 -0.010 0.995 0.0050 1.02
##
##                          ML1   ML2   ML3
## SS loadings            3.523 1.145 1.083
## Proportion Var         0.503 0.164 0.155
## Cumulative Var         0.503 0.667 0.822
## Proportion Explained   0.613 0.199 0.188
## Cumulative Proportion  0.613 0.812 1.000
##
## Mean item complexity =  1.9
## Test of the hypothesis that 3 factors are sufficient.
##
## The degrees of freedom for the null model are  21  and the objective function was  5.818 with Chi Sq
## The degrees of freedom for the model are 3  and the objective function was  0.109
##
## The root mean square of the residuals (RMSR) is  0.015
## The df corrected root mean square of the residuals is  0.04
##
## The harmonic number of observations is  51 with the empirical chi square  0.5  with prob <  0.919
## The total number of observations was  51  with Likelihood Chi Square =  4.898  with prob <  0.179
##
## Tucker Lewis Index of factoring reliability =  0.9446
## RMSEA index =  0.1281  and the 90 % confidence intervals are  0 0.2852
## BIC =  -6.897
## Fit based upon off diagonal values = 0.999
## Measures of factor score adequacy
##                                                 ML1   ML2   ML3
## Correlation of (regression) scores with factors   0.998 0.979 0.921
## Multiple R square of scores with factors          0.995 0.958 0.847
## Minimum correlation of possible factor scores     0.991 0.915 0.695
```

Table 15.5

```
crime_fa3varimax <- fa(crime, nfactors = 3, fm = "ml", rotate = "varimax")
print.psych(crime_fa3varimax, digits = 3, sort = TRUE)
```

```
## Factor Analysis using method =  ml
## Call: fa(r = crime, nfactors = 3, rotate = "varimax", fm = "ml")
## Standardized loadings (pattern matrix) based upon correlation matrix
##          item   ML3   ML2   ML1    h2     u2  com
## Theft       6 0.831 0.132 0.120 0.721 0.2787 1.09
## Burglary    5 0.828 0.330 0.263 0.864 0.1364 1.53
## Rape        2 0.645 0.369 0.297 0.640 0.3601 2.06
## Murder      1 0.259 0.922 0.229 0.970 0.0301 1.29
## Robbery     3 0.243 0.664 0.561 0.815 0.1854 2.24
## Assault     4 0.492 0.629 0.331 0.747 0.2526 2.46
## Vehicle     7 0.285 0.317 0.902 0.995 0.0050 1.46
##
##                        ML3   ML2   ML1
## SS loadings           2.24 2.049 1.463
## Proportion Var        0.32 0.293 0.209
## Cumulative Var        0.32 0.613 0.822
## Proportion Explained  0.39 0.356 0.254
## Cumulative Proportion 0.39 0.746 1.000
##
## Mean item complexity =  1.7
## Test of the hypothesis that 3 factors are sufficient.
##
## The degrees of freedom for the null model are  21  and the objective function was  5.818 with Chi Sq
## The degrees of freedom for the model are 3  and the objective function was  0.109
##
## The root mean square of the residuals (RMSR) is  0.015
## The df corrected root mean square of the residuals is  0.04
##
## The harmonic number of observations is  51 with the empirical chi square  0.5  with prob <  0.919
## The total number of observations was  51  with Likelihood Chi Square =  4.898  with prob <  0.179
##
## Tucker Lewis Index of factoring reliability =  0.9446
## RMSEA index =  0.1281  and the 90 % confidence intervals are  0 0.2852
## BIC =  -6.897
## Fit based upon off diagonal values = 0.999
## Measures of factor score adequacy
##                                                 ML3   ML2   ML1
## Correlation of (regression) scores with factors   0.931 0.975 0.991
## Multiple R square of scores with factors          0.868 0.951 0.982
## Minimum correlation of possible factor scores     0.735 0.903 0.963
```

Figure 15.3

```
crime_fa3 <- factanal(crime, factors = 3, method = "mle", rotation = "varimax",
                      scores = "regression")
pairs(crime_fa3$scores,
      panel = function(x,y) text(x, y, labels = row.names(crime), cex=0.8))
```