Modelling of Climate

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Problem Statement

To make a basic model for the Temperature of the Earth.

Assumptions :

1. Earth receives incoming insolation (solar energy)
2. Earth radiates some of the energy back into space.

Global Mean Temperature

The global mean temperature T can be modeled by the energy balance equation (EBM) [Kaper and Engler 2013, 16]

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4.$$

The first term on the right is **incoming heat absorbed by the Earth** and its atmosphere system.

The second term is heat radiating out as if the **Earth were a blackbody.**

- **T** (K, kelvins) is the average temperature in the Earth's photosphere where the energy balance occurs)

- **t** (years) is time;

- **R** is the averaged heat capacity of the Earth system ; **R** = 2.912 W-yr/m^2 K)

- **Q** is the annual global mean incoming solar radiation (or insolation) per square meter of the Earth's surface ; **Q** = 342 W/m^2

- $\boldsymbol{\alpha}$ is planetary albedo (reflectivity) ; $\boldsymbol{\alpha}$ = 0.30 (dimensionless)

- $\boldsymbol{\sigma}$ is the Stefan-Boltzmann constant. ; $\boldsymbol{\sigma}$ = 5.67 ⇥ 10-8 W/m K^4

Analytic Solution of EBM

Taking initial condition T(0) = 0

```
In[=]:= ebm = {T'[t] == (1/R) * (Q * (1 - alpha) - sigma * (T[t])^4), T[0] == 0};
        soln = DSolve[ebm, T, t]
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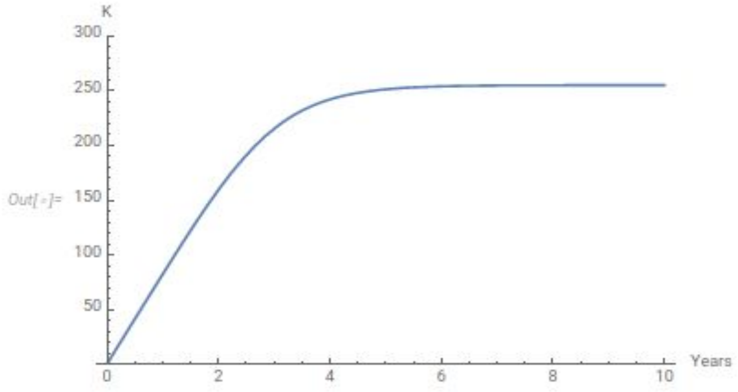
Out[=]=

$$\Big\{\Big\{T \to \mathrm{Function}\Big[\{t\},\ \mathrm{InverseFunction}\Big[\frac{-2\,\mathrm{ArcTan}\Big[1-\frac{\sqrt{2}\,\mathrm{sigma}^{1/4}\,\#1}{(-1+\mathrm{alpha})^{1/4}Q^{1/4}}\Big]+2\,\mathrm{ArcTan}\Big[1+\frac{\sqrt{2}\,\mathrm{sigma}^{1/4}\,\#1}{(-1+\mathrm{alpha})^{1/4}Q^{1/4}}\Big]-\mathrm{Log}\Big[\sqrt{-1+\mathrm{alpha}}\,\sqrt{Q}-\sqrt{2}\,(-1+\mathrm{alpha})^{1/4}Q^{1/4}\,\mathrm{sigma}^{1/4}\,\#1+\sqrt{\mathrm{sigma}}\,\#\ldots}{4\sqrt{2}\,(-1+\mathrm{alpha})^{3/4}Q^{3/4}\,\mathrm{sigma}^{1/4}}$$

Numerical Solution of EBM

Taking initial condition T(0) = 0 over the time interval of first 10 years and value of parameters as mentioned in above slides, we solve the EBM using NDSolve function in Mathematica.



Basic Analysis

Equilibrium Temperature (T^*)

Equating above EBM to zero i.e $Q(1-\alpha) - \sigma T^4 = 0$

$$T = \{ (1-\alpha)Q/\sigma)^{1/4} \} = 255 \text{ K} = -18°C$$

Now, considering emissivity factor ε so that we can see the effect of atmosphere on temperature.

$$R\frac{dT}{dt} = Q(1-\alpha) - \epsilon\sigma T^4.$$

The value ε = 1 yields an atmosphere completely transparent to OLR.

Ideal value of ε

Taking other parameters same, what value of ε gives an equilibrium global mean temperature T = 288.4 (= 15.4° C) which is our current annual global average temperature?

$$R\frac{dT}{dt} = Q(1-\alpha) - \epsilon\sigma T^4.$$

$\varepsilon = Q(1-\alpha)/(\sigma * T^4) = 342(1-0.3)/(5.67 * 10^{-8})*(288.4^4) = 0.61$ (approx)

Therefore, 61% of the Outgoing longwave radiation (OLR) would have to escape the Earth's atmosphere and radiate to space to maintain a temperature of 288.4 K (15.4°C)

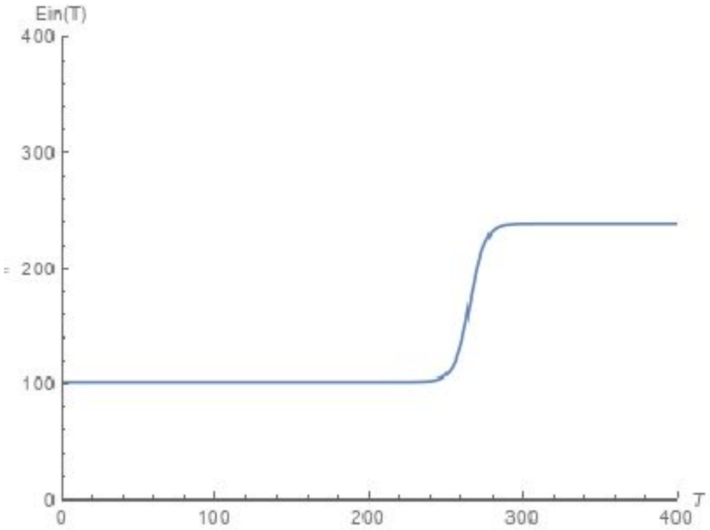
Continued...

$$\frac{dT}{dt} = E_{\text{in}}(T) - E_{\text{out}}(T)$$

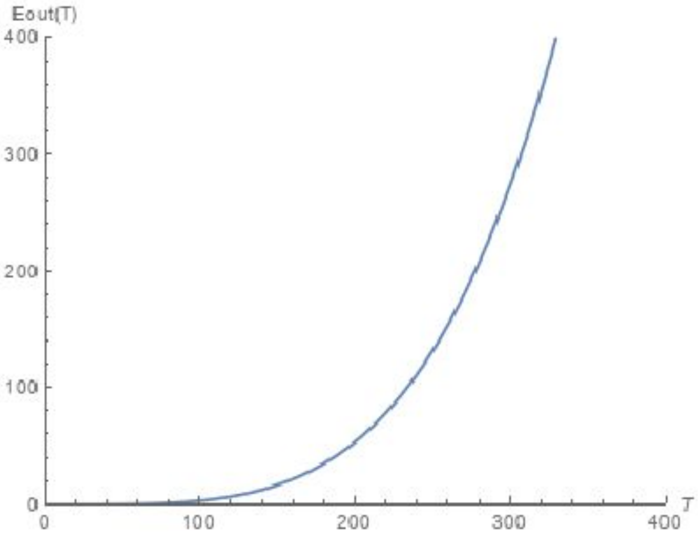
ε is an emissivity factor, allowing us to incorporate the effect of the atmosphere on climate. For example, as the concentration of an atmospheric greenhouse gas such as CO_2 increases, less outgoing longwave radiation escapes to space, corresponding to a decrease in ε and vice versa.

Therefore, ε acts as a proxy for greenhouse gases. Ein (T) would be low for smaller T -values because if the temperature was very cold then much of the planet's surface would be covered with ice which implies low value of albedo and vice versa for a warm planet with little ice cover.

Variation with Temperature



Variation : Energy absorbed with Temperature (Q= 342 W/m^2)

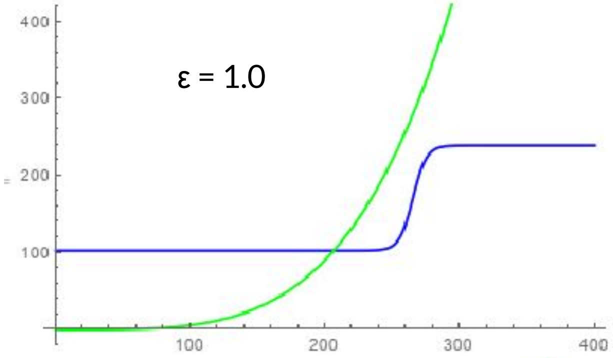


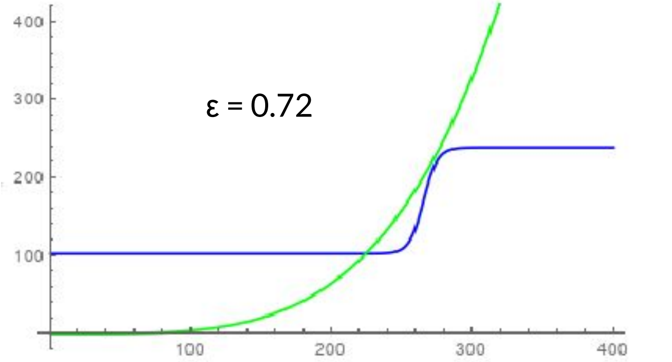
Variation : Energy emitted with Temperature (ε = 0.6)

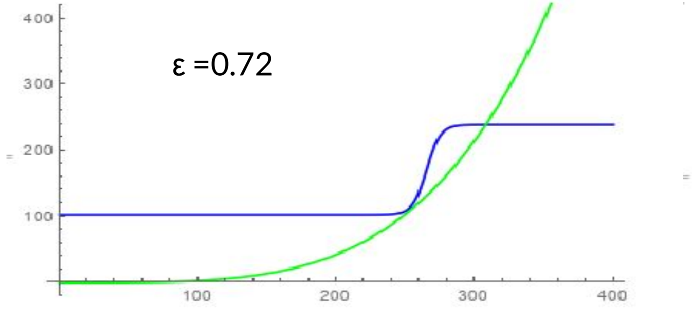
Effect of ε

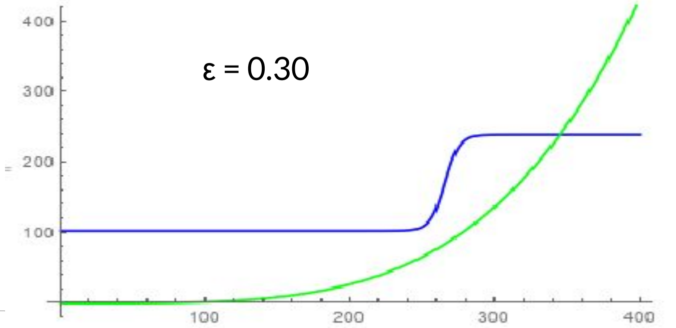
Green Line : Energy Emitted

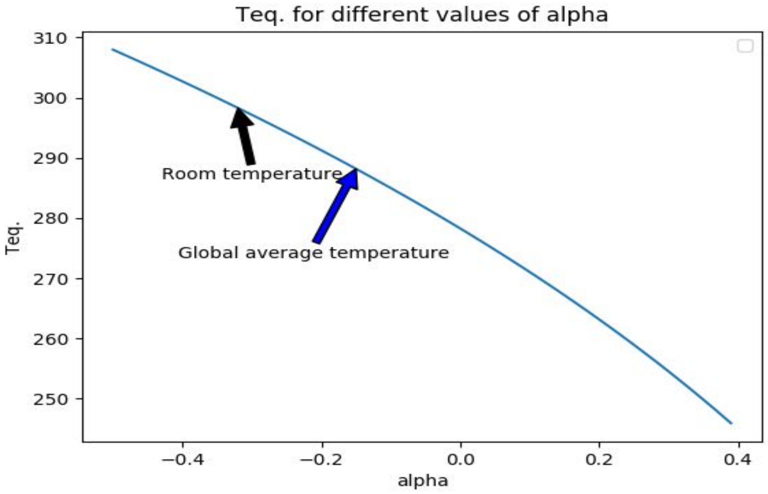
Blue Line : Energy Absorbed



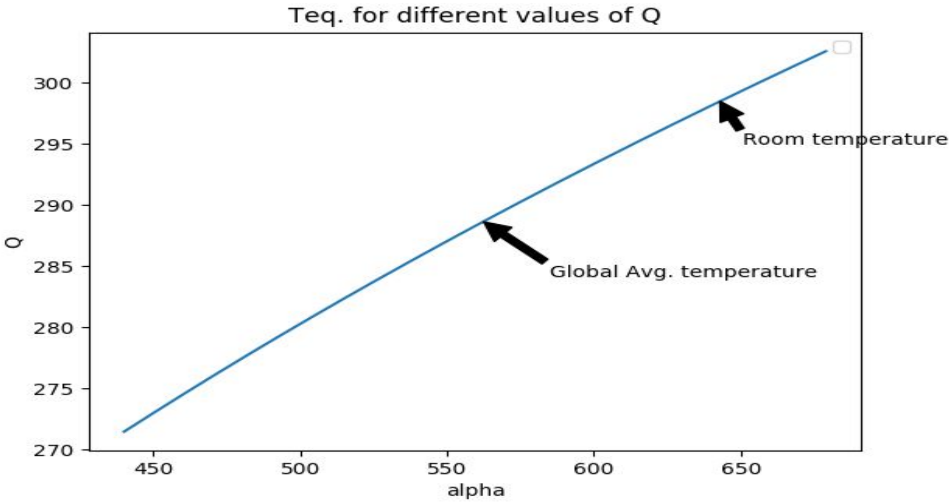








On changing the value of α, we got the room temperature 298 K at -0.31 and Global avg. temperature 288K at -0.15



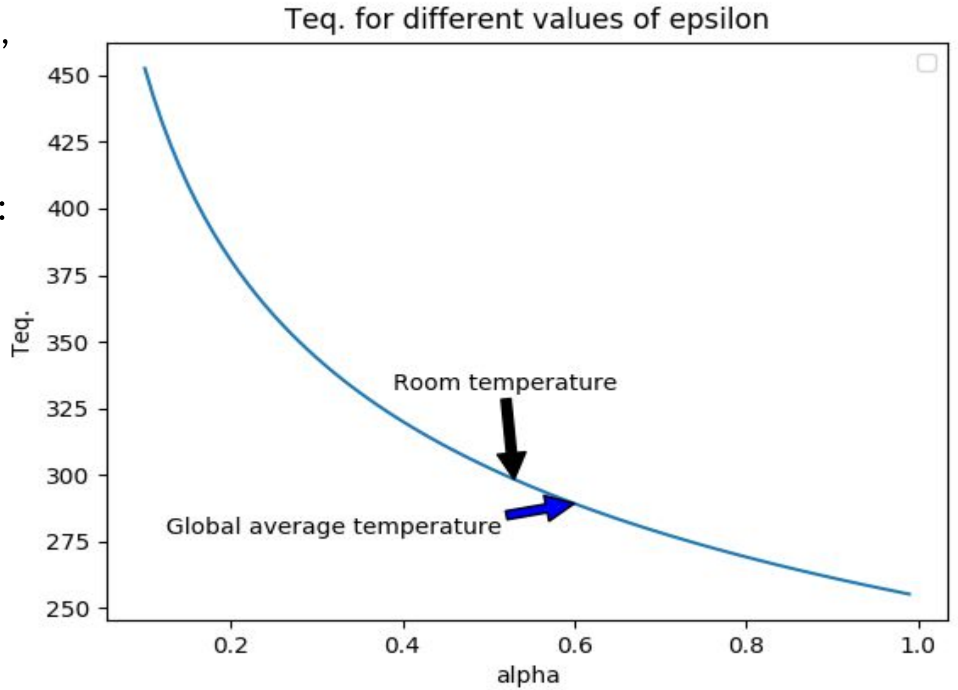
On changing the value of Q, we got the room temperature 298 K at 643 W/m^2 and Global avg. temperature 288K at 562 W/m^2

So in order to obtain Room temperature : α=0.3and Q=640 W/m^2
And in order to obtain Global avg. temperature:α =0.3 and Q=560 W/m^2

So in order to obtain Room temperature : ε=0.3, Q=340 W/m^2 and epsilon=0.53

And in order to obtain Global avg. temperature: α=0.3, Q=340 W/m^2 and ε =0.61

On changing the value of ε, we got the room temperature 298 K at 0.53 and Global avg. temperature 288K at 0.61



Modified Modelling

Modelling the Outgoing longwave radiation (OLR) in a global surface temperature model via a linear term of the form A+B*T where B > 0 : [as in Graves et al. [1993, 20]]

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT).$$

(A + BT) models the radiation that the Earth emits to space.

Earth’s average surface temperature T*at equilibrium :

$$T^* = (1/B)*(Q(1-\alpha) - A)$$

General Solution of Modified Equation

The equation of the slightly modified model would be:

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

which can also be rewritten as:

$$\frac{dT}{dt} + \frac{B}{R}T = \frac{Q}{R}(1-\alpha) - \frac{A}{R}$$

Let's take $k = \frac{Q}{R}(1-\alpha) - \frac{A}{R}$. This turns our equation to:

$$\frac{dT}{dt} + \frac{B}{R}T = k$$

The integrating factor for this equation is $e^{\int \frac{B}{R}dt} = e^{\frac{B}{R}t}$.
Using this integration factor gives us:

$$T(t)e^{\frac{B}{R}t} = k\frac{R}{B}e^{\frac{B}{R}t} + c$$

which can be written as:

$$T(t) = k\frac{R}{B} + ce^{-\frac{B}{R}t}$$

or, if we replace $k = \frac{Q}{R}(1-\alpha) - \frac{A}{R}$, then

$$T(t) = \frac{Q(1-\alpha) - A}{B} + ce^{-\frac{B}{R}t}$$

Average Temperature at Equilibrium

Oh! By the way, this is considering there's no initial conditions on the system.

Q. Variation in magnitude of T^* with the parameters A and B ?

The slightly modified model yields:

$$T(t) = \frac{Q(1-\alpha) - A}{B} + ce^{-\frac{B}{R}t}$$

which, at equilibrium, ends up being:

$$T^* = \frac{Q(1-\alpha) - A}{B}$$

which, by the way, is also the asymptotic value of the temperature. From Graves et al. [1993], we have the best current estimates of A and B. $A = 202 W/m^2$ and $B = 1.90 W/m^2 K$.

Now, if we take the values of Q and α from previous slides, then:

$$T^* = \frac{343x(1-0.3) - 202}{1.9}$$

which then yields $T^* = 19.7^o C$ or $290.7K$. This is pretty close to the actual average of $15.4^o C$. Makes sense? Yeah. This model attempts to take into consideration the role our atmosphere plays in determining surface temperature.

Interpretations

Therefore, A can be interpreted as a proxy for atmospheric greenhouse gas concentrations.

Finding T* in different cases :

1 . Case when the Earth is ice-free (Albedo of any ice-free surface is around σ_w = 0.32)

$$T^* = (1/B)*(Q(1-\alpha) - A) = 1/(1.9)*(342(1-0.32)-202) = 16°C$$

2. Case when the Earth is in snowball state (Albedo of any icy surface is around σ_w = 0.62)

$$T^* = (1/B)*(Q(1-\alpha) - A) = 1/(1.9)*(342(1-0.62)-202) = -37.9°C$$

Q. What happens after a long time ?

Bifurcation

Bifurcation occurs when a small smooth change made to the parameter values of a system causes a sudden 'qualitative change in its behavior.

We will be using bifurcation diagrams to study how our ODE depends on Q (insolation)

Albedo function used by Kaper and Engler [2013, 18]:

$$\alpha(T) = 0.5 + 0.2 \tanh\{0.1(265 - T)\}$$

Q. What will be the albedo of the hottest and coldest surfaces ?

Effect of Insolation on Equilibrium Solutions

Insolation Q has changed significantly over the Earth's history.For example, 3.5 billion years ago insolation was less than 80% of its current value.

$$f(T) = R\frac{dT}{dt} = Q(1-\alpha) - \epsilon\sigma T^4.$$

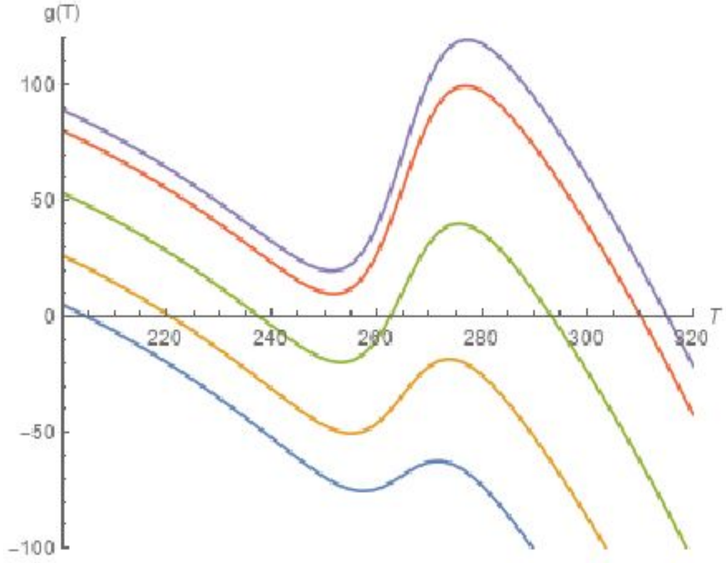
Plotting f(T) for different values of insolation :

From bottom to top, are are plots of f (T) for increasing

Q-values i.e Q = 200 , 270, 360, 450 and 480 W/m^2.

Note : For higher Q values, we are not getting any

equilibrium points.



Bifurcation of Equilibrium Points

We know, bifurcations of equilibrium points occur at f(T) = 0 and f'(T) = 0

1. f(T) = 0 gives :

$$Q = Q(T) = \frac{\epsilon\sigma T^4}{1 - \alpha(T)}.$$

2. f'(T) = 0 gives :

$$-\epsilon\sigma T^3 \left(\frac{T}{1 - \alpha(T)} \alpha'(T) + 4 \right) = 0.$$

On Solving for T , we get 2 values : T1 = 252.06K and T2 = 274.23K

and Q(T1) = 418.7 and Q(T2) = 298.1K

Conclusion

For Q slightly larger than Q1,there will be only one equilibrium temperature T = T3 > 305 K, corresponding to an extremely hot planet. This makes sense in terms of the model,given that Q1 is very large in this case. Note that f'(T3) < 0, implying that the equilibrium point T = T3 is a sink.

For Q slightly below Q1 or slightly above Q2 (as with the green or gold middle curves in Figure S2), there are three equilibrium points T 1 < T2 < T3 .We see that T = T1 and T = T3 are sinks while T = T2 is a source.

Conclusion

Hence, as Q decreases through Q1 ,a second stable equilibrium solution T1 appears, with T1 between roughly T = 240 K and T = 260 K. Thus, the temperature of the planet can tend to either the warmer T = T3 world or to the much colder T = T1 world.This phenomenon is know as bistability.

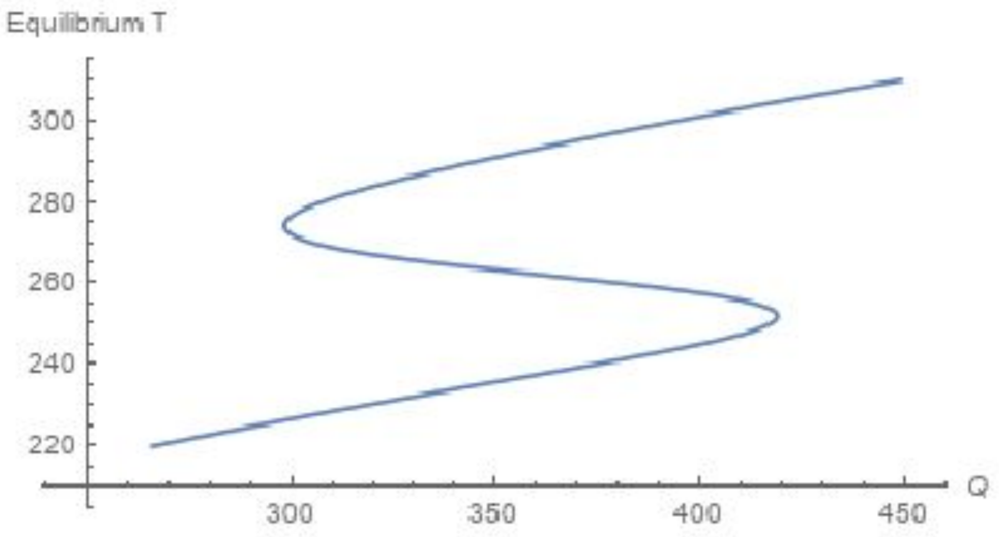
For Q just below Q2(as with the bottom),there is one equilibrium temperature T = T1 < 220 K, corresponding to a snowball Earth.

Note that T = T1 is a sink. Thus, as Q decreases sufficiently,the planet's temperature inexorably tends to T = T1 . This makes sense (with current atmospheric conditions given by ε = 0.6),since Q2 = 298.1 is far below today's insolation value of Q = 342 W/m^2.

Bifurcation Diagram

Bifurcation diagram of equilibrium

solution versus the parameter Q

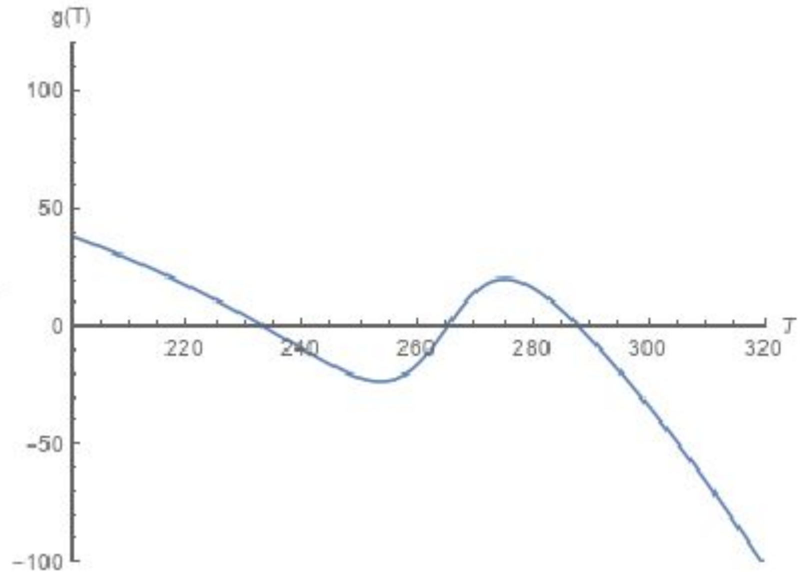


Faint young Sun Paradox

The equilibrium surface temperature when Q = 280 is T = 223 K = 50 C. Hence, for this value of Q and the atmospheric emissivity factor ε = 0.6, the surface of the Earth would be covered with ice.

This conclusion contradicts the fact that 3.5 billion years ago,at a time when Q was less than 280 W/m2 , liquid water is known to have existed on the Earth's surface. This contradiction, known as the faint young Sun paradox.

It has has been explained by positing that the atmosphere contained more greenhouse gases early in Earth's history.



A Latitude-Dependent Model

The temperature profile denoted by a $T = T(t, y)(°C)$ is the annual average surface temperature at latitude y

Assumption: T(t, y) is symmetric across the Equator

The global annual average temperature is given simply by

$$\overline{T} = \overline{T}(t) = \int_0^1 T(t, y)\, dy.$$

Glaciers

Glaciers are incorporated into this model via an adjustment to the albedo function i.e. the parameter η s referred to as the ice line, with the albedo $\alpha_\eta(y) = \alpha(y, \eta)$ depending on y and the position of the ice line η

$$\alpha(y,\eta) = \begin{cases} \alpha_1, & y < \eta; \\ \alpha_2, & y > \eta; \\ \frac{1}{2}(\alpha_1 + \alpha_2), & y = \eta, \end{cases}$$

where $\alpha_1 < \alpha_2$.

Distribution of Insolation

A second adjustment concerns the distribution of insolation. The tropics receive more energy from the sun on an annual basis than do the polar regions. This difference is taken into account by modeling the energy absorbed by the surface via the term

$$Qs(y)(1-\alpha(y,\eta)),$$

where s(y) is the distribution of insolation over latitude

$$\begin{aligned} s(y) &= 1.241 - 0.723y^2 \\ &= 1 \cdot 1 + (-0.482)\left(\tfrac{1}{2}(3y^2-1)\right) \\ &= s_0p_0(y) + s_2p_2(y), \end{aligned}$$

where we have expressed s(y) as a linear combination of the first two even Legendre polynomials, with s_0 = 1 and s_2 = 0.482

Note that s(y) is largest at the Equator and decreases monotonically to a minimum at the North Pole

Meridional Heat Transport

A final adjustment concerns meridional heat transport, encompassing physical processes such as the heat flux carried by the circulation of the ocean and the fluxes of water vapor and heat transported via atmospheric currents.

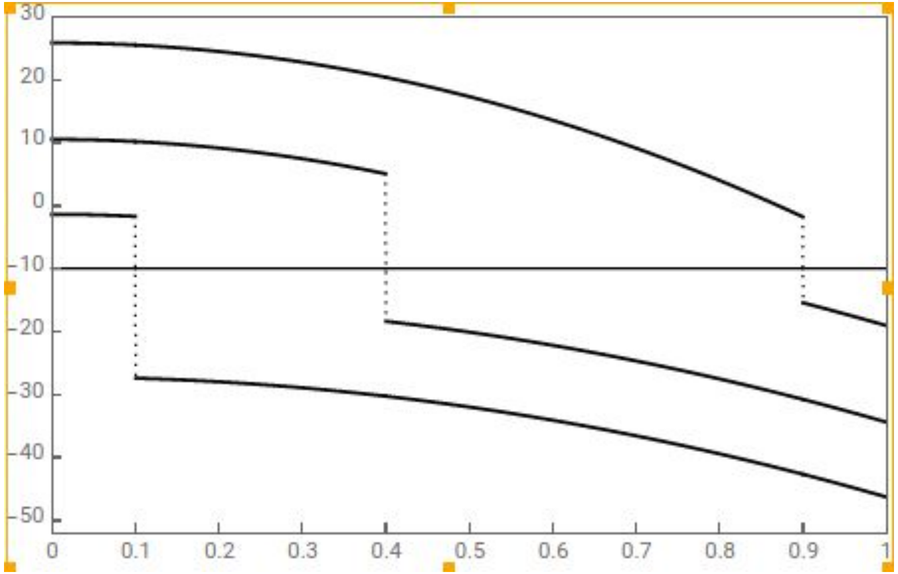
$$R\frac{\partial T(t,y)}{\partial t} = Qs(y)\left(1-\alpha(y,\eta)\right) - (A+BT) - C\left(T-\overline{T}\right)$$

The final term models the simple idea that warm latitudes (relative to the global mean temperature) lose heat energy through transport, while cooler latitudes gain heat energy.

Assumption; each of A, B, C,and Q is strictly positive

Equilibrium solutions T*(y) for the model

Top: η = 0.1. Middle: η = 0.4. Bottom: η = 0.9



$Qs(y)[1-\alpha(y,n)] - (A+B^*T(y))-C[T^*(y) - \underline{T}^*(y)] = 0$

Note : $\frac{\partial}{\partial t}T^*(y) = 0$, that is, T^* is a function of only y.)

Note that T (t, y) is an even function of y , due to the assumption of symmetry across the Equator. Thus, an expansion for T in the variable y requires only the even Legendre polynomial

$$T(t,y) = \begin{cases} U(t,y), & y < \eta; \\ V(t,y), & y > \eta; \\ (U(t,\eta) + V(t,\eta))/2, & y = \eta, \end{cases}$$

where

$$\begin{aligned} U(t,y) &= u_0(t)p_0(y) + u_2(t)p_2(y), \\ V(t,y) &= v_0(t)p_0(y) + v_2(t)p_2(y). \end{aligned}$$

The coefficients of p_0 and p_2 are now functions of t , rather than constants, since U and V are functions of both t and y.

Please view this link to see all calulations regarding nullclines

Nullclines and vector fields

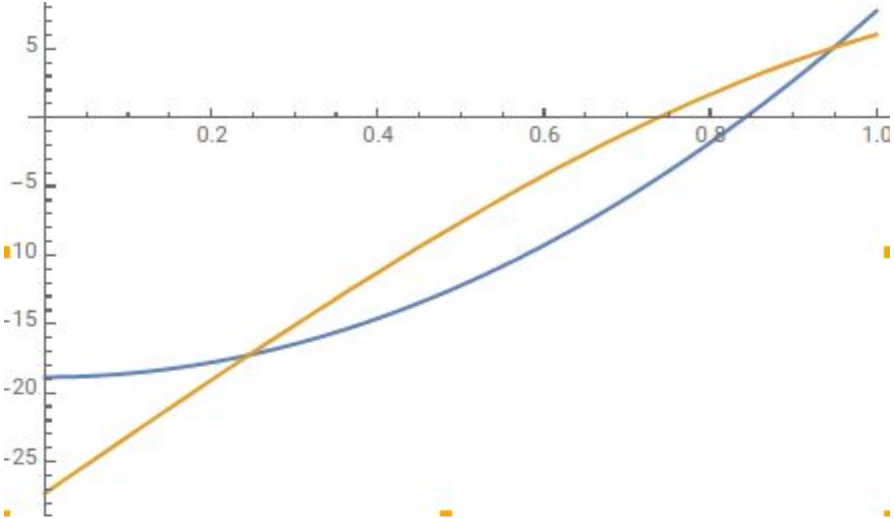
Plot of η-nullcline and *w*-nullcline for a coupled temperature–ice line model where T(η) is

$$T(\eta) = \frac{1}{2}\left(\lim_{y\to\eta^-} T(t,y) + \lim_{y\to\eta^+} T(t,y)\right)$$

$$\dot{\eta} = \epsilon\left(w + \frac{Qs_2(1-\alpha_0)}{B+C}p_2(\eta) - T_c\right) = \epsilon(w - G(\eta)),$$

$$\dot{w} = \frac{1}{R}\left\{Q(1-\alpha_0) - A - Bw + \frac{CQ(\alpha_2-\alpha_1)}{B+C}\left[\eta - \frac{1}{2} + s_2P_2(\eta)\right]\right\}$$

where $\alpha_0 = \frac{1}{2}(\alpha_1 + \alpha_2)$.

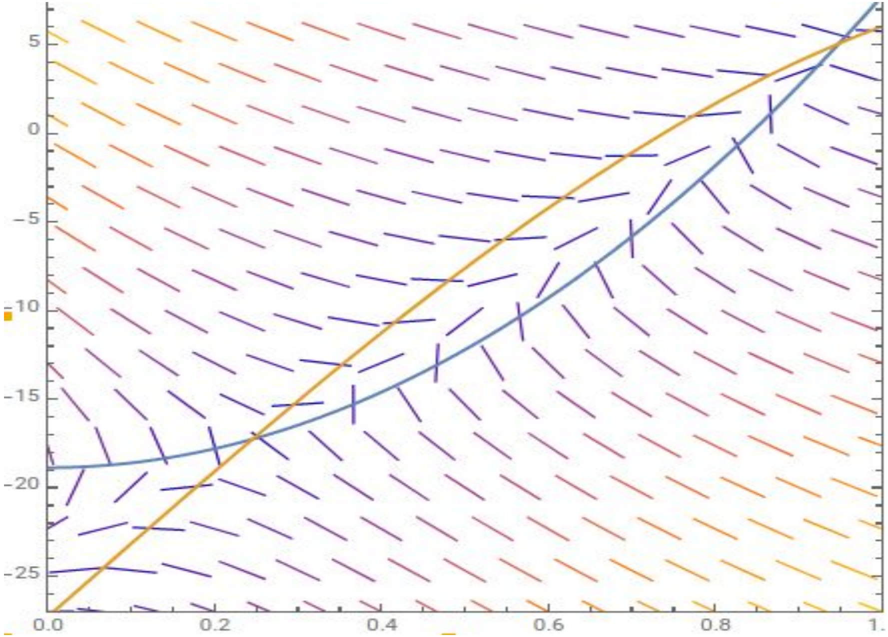


Concave upward curve on right (red): η-nullcline
Concave-downward curve (yellow): *w*-nullcline

Continued...

For ε = 1, the nullclines and vector fields for the system are shown The η-nullcline is the concave-upward (blue) curve, while the *w*-nullcline is the concave-downward (red) curve. There are two equilibrium points, (η_1, w_1) and (η_2, w_2), $\eta_1 < \eta_2$ given by the two points of intersection of the nullclines . Mathematica gives $(\eta_1, w_1) \approx (0.246, 17.265)$ and $(\eta_2, w_2) \approx (0.949, 5.080)$.

$$\dot{\eta} = \epsilon(T(\eta) - T_c)$$



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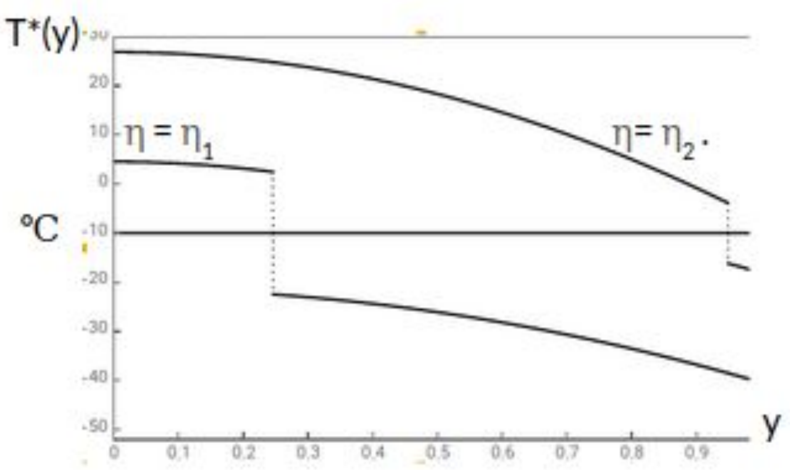
Current data indicate that ice forms at the ice line (so η decreases) if $T(\eta) < T_c$, and ice melts at the ice line (so η increases) if $T(\eta) > T_c$, where the critical temperature $T_c = 10°C$.

We plot $T^*(y)$ when $\eta = \eta_1$ and when $\eta = \eta_2$. We see that for i = 1 and i = 2, we get $T^*(\eta_i) = 10°C$, and the system is at equilibrium. Thus, there are but two ice line positions that satisfy for which the temperature at the ice line at equilibrium equals the critical temperature. One has a very large ice cap ($\eta = \eta_1$), while the other has a small ice cap ($\eta = \eta_2$)



Chaotic Behaviour

For any dynamical system to show chaotic behaviour, it must fulfill the following conditions:

1) It must be **sensitive to initial conditions.** (***BUTTERFLY EFFECT***….You get it….)
2) It must be **topologically transitive**, which means that for any f : X -> X and A,B in powerset of X P(X), there exists an integer 'n' for which $f^n(A) \cap B \neq \varnothing$.
3) It must have **dense periodic orbits.**

Because it's not possible to get an accurate fix on the initial atmospheric conditions of our planet, the initial conditions have to be assumed and that gives us different models of temperature. There you go, the Butterfly effect of weather.

Results

From our multiple trials, we can see that:

1) The Kaper-Engler iteration of the EBM is able to model temperature as a function of time using blackbody radiation principles but is flawed because of the assumption of uniform radiation spread which is not practically the case.

2) The Graves iteration of EBM is able to ease out calculations and also help us model temperature in terms of latitude and time. This is important as it keeps aside the earlier assumption of uniform radiation, and instead says that while the average is Q, only part of the Q is what reaches a latitude.

3) Using a piecewise continuous function for albedo instead of a constant, the Graves EBM is able to come closer to reality.

Conclusion

Thus, we saw two models in this project and tweaked with some of the parameters.

The first model used basic blackbody radiation principles to give us an energy-balance based model of how temperature varies with time. We made modifications to that by taking a sigmoid albedo as opposed to a constant albedo.

The second one used the outgoing radiation as a linear function of temperature and that gave us another model of temperature as a function of time. Then we varied both insolation and temperature with latitude and obtained a more reliable model.

We can say that energy balance comes pretty close to modelling temperature, which is usually the highlight of modelling climate. It still has a long way to go though...

Discussion

This model does not take into account the Earth's atmosphere.

E.g Effect of Global Warming on the temperature of Earth

We know,there will be different insolation at different latitudes. In this model,they have neglected this view point and assumed uniform insolation over all latitudes. Therefore, it will be incorrect to predict the same temperature at different latitudes on the Earth. I mean, how is it cold in the UK and Canada whilst it’s warm in our country if insolation were the same throughout the surface of the earth?

Also, they have neglected the variation of insolation with changes in Earth’s orbital parameters.

E.g Effect of obliquity of the spin axis, precession of the spin axis, and the eccentricity of the orbit.

Codes and References

All programs were written in Wolfram Mathematica (courtesy Wolfram Research of Champaign, Illinois) and python.

Please use **this** link to access all codes.

Please view **this** file to see all references.


Thanks