Nudllines-calc.

$T(t,y) = U(t,y)$ for $y < \eta$

and $T(t,y) = V(t,y)$ for $y > \eta$

$$R\frac{d}{dt}U(t,y) = Qs(y)(1-\alpha_1) - (A+BU) - C(U-\bar{T}) \quad 0 \le y < \eta$$

$$R\frac{d}{dt}V(t,y) = Qs(y)(1-\alpha_2) - (A+BV) - C(V-\bar{T}) \quad \eta < y < 1$$

Given: $P_2(\eta) = \int_0^{\eta} p_2(y)\,dy = \frac{1}{2}(\eta^3-\eta)$

$$\bar{T} = \int_0^1 T(t,y)\,dy = \int_0^{\eta} U(t,y)\,dy + \int_{\eta}^{1} V(t,y)\,dy$$

$$= \int_0^{\eta}(u_0 + u_2 p_2(y))\,dy + \int_{\eta}^{1}(v_0 + v_2 p_2(y))\,dy$$

$$= (u_0\eta + u_2 P_2(\eta)) + (v_0(1-\eta) + v_2(P_2(1) - P_2(\eta)))$$

$$= \eta u_0 + (1-\eta)v_0 + P_2(\eta)(u_2 - v_2)$$

$$T(t,\eta) = \frac{1}{2}(U(\eta) + V(\eta)) = \frac{1}{2}(u_0 + v_0) + \frac{1}{2}(u_2 + v_2)p_2(\eta)$$

Substituting value of T in the above differential eqn

$$R(\dot{u}_0 p_0 + \dot{u}_2 p_2) = Qs(y)(1-\alpha_1) - (A + B(u_0 p_0 + u_2 p_2)) - C((u_0 p_0 + u_2 p_2) - \bar{T})$$

Similarly

$$R(\dot{v}_0 p_0 + \dot{v}_2 p_2) = Qs(y)(1-\alpha_2) - (A + B(v_0 p_0 + v_2 p_2)) - C((v_0 p_0 + v_2 p_2) - \bar{T})$$

$S(y) = 1\,p_0 + s_2 p_2 \qquad A = Ap_0 \quad$ and $\bar{T} = \bar{T}p_0$

So $R\dot{u}_0 = Q(1-\alpha_1) - A - (B+C)u_0 + C\bar{T}$

$R\dot{v}_0 = Q(1-\alpha_2) - A - (B+C)v_0 + C\bar{T}$

$R\dot{u}_2 = QS_2(1-\alpha_1) - (B+C)u_2$

$R\dot{v}_2 = QS_2(1-\alpha_2) - (B+C)v_2$

Now considering $w = \frac{1}{2}(u_0+v_0)$ and $z = u_0 - v_0$ as change of variables

$R\dot{w} = \frac{1}{2}(R\dot{u}_0 + R\dot{v}_0)$

$= \frac{1}{2}\big(Q(1-\alpha_1) - A - (B+C)u_0 + C\bar{T} + Q(1-\alpha_2) - A - (B+C)v_0 + C\bar{T}\big)$

$= Q(1-\alpha_0) - A - (B+C)w + C\bar{T}$

where $\alpha_0 = \frac{1}{2}(\alpha_1+\alpha_2)$

Similarly

$R\dot{z} = R\dot{u}_0 - R\dot{v}_0$

$= \big(Q(1-\alpha_1) - A - (B+C)u_0 + C\bar{T}\big) - \big(Q(1-\alpha_2) - A - (B+C)v_0 + C\bar{T}\big)$

$= Q(\alpha_2-\alpha_1) - (B+C)z$

Also as $w = \frac{1}{2}(u_0+v_0)$ and $z = u_0 - v_0$

$v_0 = w - \frac{1}{2}z$ $\quad u_0 = w + \frac{1}{2}z$

So $\bar{T} = n u_0 - n v_0 + P_2(n)(v_2 - v_2)$

$$= n z + \left(\omega - \frac{1}{2} z\right) + P_2(n)(v_2 - v_2)$$

$$= \omega + \left(n - \frac{1}{2}\right) z + P_2(n)(u_2 - v_2)$$

$$T(t, \eta) = w + \frac{u_2 + v_2}{2} p_2(n)$$

Also $\bar{T} = \omega + \left(n - \frac{1}{2}\right) \frac{Q(\alpha_2 - \alpha_1)}{B+C} + P_2(n)\left(\frac{Q S_2 (1-\alpha_1)}{B+C} - \frac{Q S_2(1-\alpha_2)}{B+C}\right)$

$$= \omega + \left(n - \frac{1}{2}\right) \frac{Q(\alpha_2 - \alpha_1)}{B+C} + P_2(n)\left(\frac{Q S_2(\alpha_2 - \alpha_1)}{B+C}\right)$$

$$= \omega + \frac{Q(\alpha_2 - \alpha_1)}{B+C}\left[n - \frac{1}{2} + S_2 P_2(n)\right]$$

Plugging this into $\dot{\omega}$ eq^n

$$\dot{\omega} = \frac{1}{R}\left\{Q(1-\alpha_0) - A - B\omega + CQ\frac{(\alpha_2 - \alpha_1)}{B+C}\left(\eta - \frac{1}{2} + S_2 P_2(\eta)\right)\right\}$$

$$= \frac{-B}{R}(\omega - F(\eta))$$

where $F(\eta)$ is the cubic polynomial

$$F(\eta) = \frac{1}{B}\left[Q(1-\alpha_0) - A + \frac{CQ(\alpha_2 - \alpha_1)}{B+C}\left(\eta - \frac{1}{2} + S_2 P_2(\eta)\right)\right]$$

Now we consider dynamic equation lay ice line

$$\dot{x} = \epsilon\left(T(t,x) - T_c\right)$$

~~Also~~ now $u_{2eq} = \dfrac{Q s_2 (1-\alpha_1)}{B+C}$ (given)

$$v_{2eq} = \frac{Q s_2 (1-\alpha_2)}{B+C}$$

So $T(t,\eta) = \omega + \dfrac{1}{2}\left(\dfrac{Q s_2 (1-\alpha_1)}{B+C} + \dfrac{Q s_2 (1-\alpha_2)}{B+C}\right) p_2(x)$

$$= \omega + \frac{Q s_2 (1-\alpha_0)}{B+C} p_2(x)$$

where $\alpha_0 = \dfrac{1}{2}(\alpha_1 + \alpha_2)$

So $\dot{x} = \epsilon\left(\omega + \dfrac{Q s_2 (1-\alpha_0)}{B+C} p_2(x) - T\right) = \epsilon(\omega - G(\eta))$

where $G(x) = -\dfrac{Q}{B+C} s_2 (1-\alpha_0) p_2(x) + T_c$

So $\dot{x} = f(x,\omega) = \epsilon(\omega - G(x))$

and $\dot{\omega} = g(x,\omega) = -\dfrac{B}{R}(\omega - F(x))$