Advanced Theory Of Communication

University of Tehran

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Homework 2

Due : 1402/12/27

Problem 1

The random variable Y is defined as

$$Y = \sum_{i=1}^{n} X_i$$

where the $X_i, i = 1, 2, \cdots, n$ are statistically independent random variables with

$$X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

1. Determine the characteristic function of Y .
2. From the characteristic function, determine the moments $E(Y)$ and $E(Y^2)$.

Problem 2

This problem provides some useful bounds on $Q(x)$.

1. By integrating $e^{-\frac{u^2+v^2}{2}}$ on the region $u > x$ and $v > x$ in $\mathbb{R}^2$, where $x > 0$, then changing to polar coordinates and upper bounding the integration region by the region $r > \sqrt{2}x$ in the first quadrant, show that $Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}$ for all $x \geq 0$.
2. Apply integration by parts to

$$\int_x^\infty e^{\frac{-y^2}{2}} \frac{dy}{y^2}$$

and show that

$$\frac{x}{\sqrt{2\pi}(1+x^2)} e^{\frac{-x^2}{2}} < Q(x) < \frac{1}{\sqrt{2\pi}x} e^{\frac{-x^2}{2}}$$

3. Based on the result of part 2 show that, for large x,

$$Q(x) \approx \frac{1}{x\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

Problem 3

$X_1, X_2, X_3, \cdots$ are i.i.d random variables each uniformly distributed on $[0, A]$, where $A > 0$. Let $Y_n = \min\{X_1, X_2, \cdots, X_n\}$.

a. What is the PDF of Y_n?

b. Show that if both A and n tend to infinity such that $\frac{n}{A} = \lambda$, where $\lambda > 0$ is a constant,the density function of Y_n tends to an exponential density function. Specify this density function.

Problem 4

The four random variables X_1, X_2, X_3, X_4 are zero-mean jointly Gaussian random variables with covariance $C_{ij} = E(X_i X_j)$. Show that

$$E(X_1 X_2 X_3 X_4) = C_{12}C_{34} + C_{13}C_{24} + C_{14}C_{23}$$

Problem 5

Let R_0 denote a Rayleigh random variable with PDF

$$f_{R_0}(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} & x > 0 \\ 0 & otherwise \end{cases}$$

and R_1 be Ricean with PDF

$$f_{R_1}(x) = \begin{cases} \frac{x}{\sigma^2} I_0(\frac{\mu x}{\sigma^2}) e^{-\frac{x^2+\mu^2}{2\sigma^2}} & x > 0 \\ 0 & otherwise \end{cases}$$

Furthermore, assume that R_0 and R_1 are independent. Show that

$$P(R_0 > R_1) = \frac{1}{2} e^{-\frac{\mu^2}{4\sigma^2}}$$

Problem 6

Suppose that we have a complex-valued Gaussian random variable $Z = X + jY$, where (X, Y) are statistically independent variables with zero mean and variance $E[X^2] = E[Y^2] = \sigma^2$ Let $R = Z+m$, where $m = m_r+jm_i$ and define R as $R = A+jB$. Clearly, $A = X+m_r$ and $B = Y+m_i$. Determine the following probability density functions:

a. $P_{A,B}(a, b)$.

b. $P_{U,\Phi}(u, \phi)$,where $U = \sqrt{A^2 + B^2}$ and $\Phi = \tan^{-1}(B/A)$.

c. $P_U(u)$.

Problem 7

Show that if $\mathbf{Z}$ is a proper complex vector, then any transform of the form $\mathbf{W} = \mathbf{AZ} + \mathbf{b}$ is also a proper complex vector.

Problem 8

Assume that real random processes $X(t)$ and $Y(t)$ are jointly stationary.

1. Determine the auto-correlation function of $Z(t) = X(t) + jY(t)$.
2. Determine the auto-correlation function of $Z(t)$ when $X(t)$ and $Y(t)$ are uncorrelated.
3. Determine the auto-correlation function of $Z(t)$ when $X(t)$ and $Y(t)$ are uncorrelated and have zero means.

Problem 9

The auto-correlation function of a stochastic process $X(t)$ is,

$$R_X(\tau) = \frac{1}{2}N_0\delta(\tau)$$

Suppose $x(t)$ is the input to an ideal band-pass filter having the frequency response characteristic shown in Figure 1figure.1. Determine the total noise power at the output of the filter.

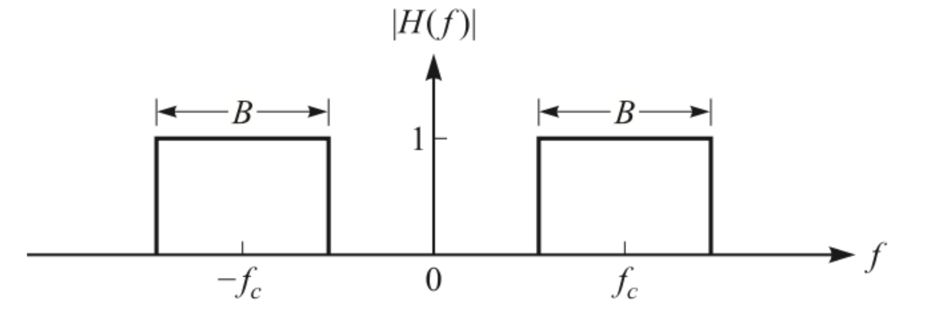

Figure 1:

Problem 10

The covariance matrix of three random variables X_1, X_2, and X_3 is:

$$\begin{bmatrix} C_{11} & 0 & C_{13} \\ 0 & C_{22} & 0 \\ C_{31} & 0 & C_{33} \end{bmatrix}$$

The random vector $\mathbf{Y}$ is defined as $\mathbf{Y} = \mathbf{AX}$ where,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Determine the covariance matrix of $\mathbf{Y}$.

Problem 11

The random process $V(t)$ is defined as:

$$V(t) = X\cos(2\pi f_c t) - Y\sin(2\pi f_c t)$$

where X and Y are random variables. Show that $V(t)$ is wide-sense stationary if and only if $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2)$, and $E(XY) = 0$.

Problem 12

Determine the auto-correlation function of the stochastic process:

$$X(t) = A\sin(2\pi f_c t + \Theta)$$

where f_c is a constant and Θ is a uniformly distributed phase, i.e.,

$$p(\theta) = \frac{1}{2\pi},\ 0 \le \theta \le 2\pi$$

Problem 13

Suppose $x(t)$ is Proper process and input to a **LTI** system with impulse response $h(t)$. If we represent output process with $y(t)$, is $y(t)$ a Proper process? why?

Problem 14

Suppose $x(t)$ is Gaussian and Circular process and input to a **LTI** system with impulse response $h(t)$. If we represent output process with $y(t)$, is $y(t)$ a Circular process? why?
