Introduction:

- **Routh-Hurwitz**
  - + powerful technique to determine how many roots of a given polynomial have real parts larger than a specified value
  - + useful to find conditions on unknown parameters that must be met in order to ensure system stability and/or a specified settling time
  - + can be used with any number of unknown parameters
  - – **does not provide actual roots themselves**
  - – **does not tell us anything about imaginary parts of roots**
- **Root Locus**
  - + Shows the **actual values** of roots for a range of unknown parameter
  - – Is limited to a **single** unknown parameter

Problem Set-up

Consider the n^{th}-order characteristic equation:

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 + K * (b_m\lambda^m + b_{m-1}\lambda^{m-1} + ... + b_1\lambda + b_0) = 0 \tag{1}$$

where

$$\begin{aligned} \lambda_1, \lambda_2, ...\lambda_n \equiv& \text{ are the roots} \\ a_0\ ,\ a_1\ ,\ ...\ a_n \equiv& \text{ known, real constants} \\ b_0\ ,\ b_1\ ,\ ...\ b_m \equiv& \text{ known, real constants} \\ m \le& n \\ K \equiv& \text{ variable real parameter} \end{aligned}$$

We want to select K in order to obtain the best system (in some sense), measured in terms of the roots. Note:

- There are n roots for each value of K
- Each *set* of roots is a numerical solution for a specific K; individual roots are not continuous functions of K
- We could utilize a *do-loop* or *for-loop* or other trial-and-error method, but may miss important information

The Root Locus plot

- root locus plots are drawn on the complex plane (since roots may be real, or complex conjugates)
- displays all of the roots of Eq. (1) for some range of K, usually $0 \le K < \infty$
- each specific value of K corresponds to n specific points (roots) on the plot
- since values of K may be infinitesimally close to each other, the corresponding sets of n roots appear as n continuous lines in the complex plane

A quick closed-form example: Consider the system with characteristic equation

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

Suppose we know that $\omega_n = 1$, but the value of ζ has not yet been determined. This could represent two different but equally important scenarios:

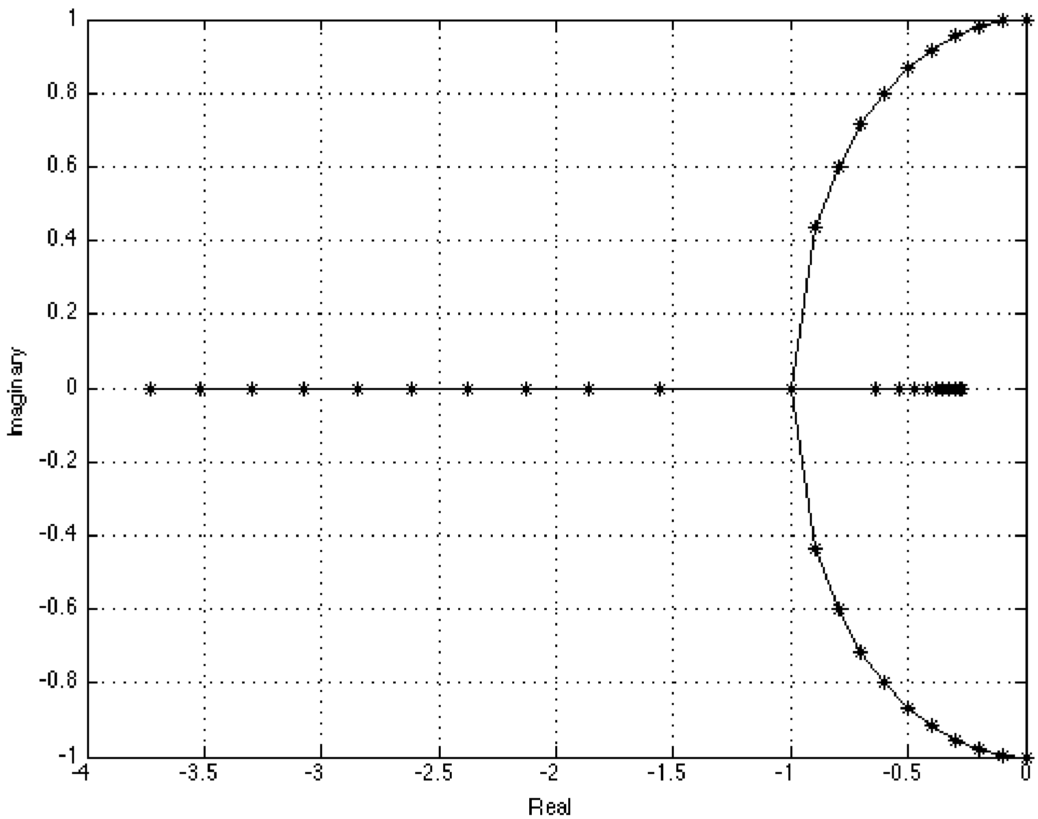
1. It could be that we have no control over ζ, and we're worried about what could happen for different values
2. It could be that we are to select ζ as part of a design process

Since roots of the CE are so important, we want to see how they are affected by ζ. We can write

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

We proceed to calculate the roots for various values of ζ:

ζ	λ_1	λ_2
0	i	-i
0.1	-0.1+0.9995 i	-0.1 - 0.9995 i
0.2	-0.2+0.9798 i	-0.2 - 0.9798 i
0.3	-0.3+0.9539 i	-0.3 - 0.9539 i
0.4	-0.4+0.9165 i	-0.4 - 0.9165 i
0.5	-0.5+0.8660 i	-0.5 - 0.8660 i
0.6	-0.6+0.8000 i	-0.6 - 0.8000 i
0.7	-0.7+0.7141 i	-0.7 - 0.7141 i
0.8	-0.8+0.6000 i	-0.8 - 0.6000 i
0.9	-0.9+0.4359 i	-0.9 - 0.4359 i
1.0	-1.0000	-1.0000
1.1	-0.6417	-1.5583
1.2	-0.5367	-1.8633
1.3	-0.4693	-2.1307
1.4	-0.4202	-2.3798
1.5	-0.3820	-2.6180
...	...	...
100	-0.0050	-199.995



A plot of the roots in a complex conjugate plane is called a **Root Locus** plot, which graphically shows all of the roots which result from different values of an unknown parameter (in this case, ζ) in a characteristic equation.

The graph enables us to see very important information about the effect of the unknown parameter on the system behavior, including:

- Stability: We can see whether or not any roots are in the RHP
- Speed: We can see how far the roots are into the LHP
- Oscillatory behavior: We can see whether or not all roots are real
- Oscillatory behavior: We can read the frequencies of any complex pairs directly from the graph, and we can measure the corresponding damping ratios

It's usually quite tedious to vary our unknown parameter, calculate tables of roots, and then draw the corresponding plots as shown above.

Moreover, we may miss critical transition points (e.g., transition from LHP to RHP) if we don't happen to use the exact value of the unknown parameter.

In the following pages, we learn how to draw root locus plots without explicitly calculating sets of roots. Moreover, we learn how to find the exact values for critical points on the plot (e.g., transition from LHP to RHP).

How to draw a Root Locus plot :

A root locus plot has incredibly useful information, but in order to understand how to read this information, it's important to understand exactly how root locus plots are drawn and what they represent. We will discuss how to read and interpret the plots after we learn how to draw them.

For convenience, we re-write Eq. (1) in the form:

$$D(\lambda) + K * N(\lambda) = 0 \tag{2}$$

where $D(\lambda)$ is an n^{th}-order polynomial, and $N(\lambda)$ is an m_{th}-order polynomial.

The Root Locus plot for $0 \leq K \leq \infty$ - often called the *positive root locus* - is then drawn using the following 8 guidelines:

1. The root locus plot has n branches and is symmetric about the Re axis
2. The n branches begin at the roots of $D(\lambda)$, marked by an X on the plot
3. m of the branches end at the roots of $N(\lambda)$; the remaining $n - m$ branches go off to different versions of ∞
4. The branches going to ∞ do so along asymptotes that are symmetric about the $-Re$ axis
5. The asymptotes from guide 4 originate at the point
$$\lambda = \frac{\sum X's - \sum O's}{n - m}$$
6. The root locus exists on the Re axis if, and only if, there is an odd number of (X+O) points to the right along the Re axis
7. If break-in and break-out points exist, they must satisfy
$$N(\lambda) * (\frac{d}{d\lambda}(D(\lambda))) - D(\lambda) * (\frac{d}{d\lambda}(N(\lambda))) = 0$$
8. To find any branches crossing the Im axis, substitute $\lambda = i\omega$ into the equation, then set the real and imaginary parts of the equation to zero (independently of each other) to solve for ω and K
9. The sum of the roots of the equation
$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$
must equal $-a_{n-1}$

   When using Guide 9, be sure that the coefficient of $\lambda_n = 1$!

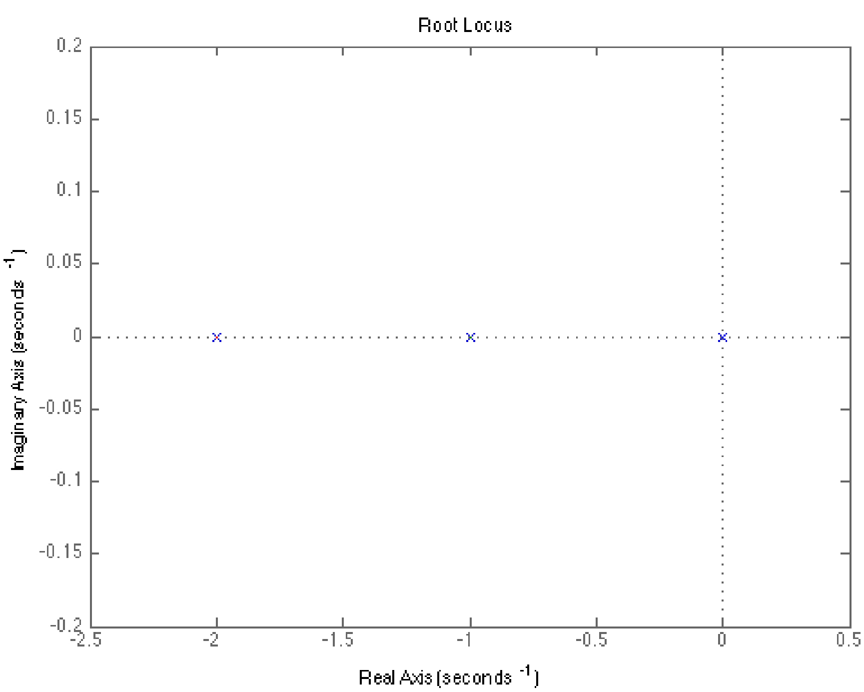
1. **The number of *branches* of the root locus plot is n.**

Example
The polynomial:
$$\lambda^3 + 3\lambda^2 + 2\lambda + K = 0 \equiv D(\lambda) + K * N(\lambda) \tag{3}$$
has 3 roots for any value of K. The root locus shows all possible values of these roots for the range $0 \leq K \leq \infty$. The root locus will appear as 3 continuous lines (called *branches*) in a complex plane.

Example
The polynomial:
$$\lambda^2 + (K-1)\lambda + 2K = \lambda^2 - \lambda + K(\lambda + 2) = 0 \equiv D(\lambda) + K * N(\lambda) \tag{4}$$
has 2 roots for any value of K. The root locus shows all possible values of these roots for the range $0 \leq K \leq \infty$. The root locus will appear as 2 continuous lines (called *branches*) in a complex plane.

2. **The n branches begin at the roots of $D(\lambda)$**
We are plotting the root locus for $0 \leq K \leq \infty$, so one extreme of this range is $K = 0$. When $K = 0$, the n roots of the polynomial are equal to the n roots of $D(\lambda)$. It is customary to mark these points in the complex plane with an X; they are the beginning points of the n branches of the root locus.

Example
For the polynomial:
$$\lambda^3 + 3\lambda^2 + 2\lambda + K = 0 \tag{5}$$
we have $D(\lambda) = \lambda^3 + 3\lambda^2 + 2\lambda$. The three branches of the root locus plot begin at the roots of D, which are $\lambda = 0$, $\lambda = -1$, and $\lambda = -2$, which are the roots of the polynomial when $K = 0$.



3. **The *ends* of the branches correspond to the roots when $K \to \infty$.**

   (a) In Eq. (2), if $K \to \infty$, then we must have $N(s) \to 0$. Thus, a total of m of the branches must end at the m roots of $N(s)$. These points are marked with an 0 in the root locus plot.

   (b) But if $K = \infty$ and $N(\lambda) \neq 0$, then $D(\lambda) \to \infty$ to offset $KN(\lambda)$ so that the polynomial is 0. Therefore, the remaining $n - m$ branches end at various different versions of $\lambda = \infty$, as defined in guideline 4. These $n - m$ branches, each heading off towards a different ∞, disappear off the edges of the plot, in different directions, as described in guideline 4.

Example
The polynomial:

$$\lambda^3 + 3\lambda^2 + 2\lambda + K = 0 \equiv D(\lambda) + K * N(\lambda) \tag{6}$$

has 3 branches, all of which end at some ∞ since N has no roots

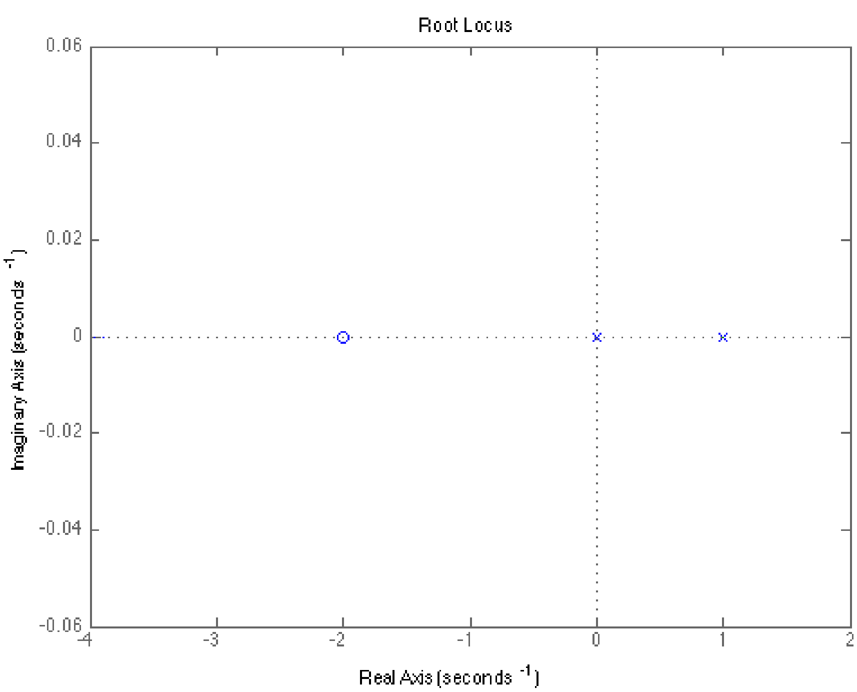
Example
The polynomial:

$$\lambda^2 + (K-1)\lambda + 2K = \lambda^2 - \lambda + K(\lambda + 2) = 0 \equiv D(\lambda) + K * N(\lambda) \tag{7}$$

has 2 branches.

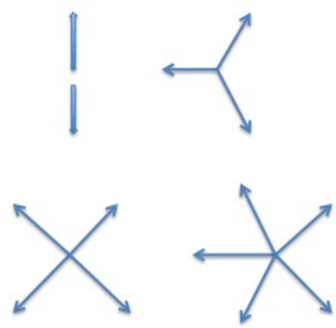
The branches begin at the roots of D, i.e., $\lambda = 1, 0$

The branches end at the roots of N or at ∞; for this example, the 2 branches end at $\lambda = -2$ and at ∞



4. **Branches ending at ∞ do so along asymptotes that are symmetric about the negative *real* axis.**
   (a) If $n-m$ is odd, one of the asymptotes is the negative real axis.
   (b) The asymptotes (if more than one) are always separated by equal angles. Thus,

   If $n-m=1$, the single asymptote goes to $-\infty$ along the negative real axis
   If $n-m=2$, the two asymptotes go to $\pm i\infty$, i.e., the two asymptotes are parallel to the *imaginary* axis
   If $n-m=3$, the three asymptotes are the negative *real* axis, and $\pm 120\,\mathrm{deg}$ to the negative real axis
   If $n-m=4$, the four asymptotes lie at angles of $\pm 45\,\mathrm{deg}$ and $\pm 135\,\mathrm{deg}$ to the negative real axis
   If $n-m=5$, the five asymptotes are the negative *real* axis, and $\pm 72\,\mathrm{deg}$, $\pm 144\,\mathrm{deg}$ to the negative *real* axis



Upper left: 2 asymptotes pattern
$\pm 90^o$ to $-Re$

Upper right: 3 asymptotes pattern
$-Re$, and $\pm 120^o$ to $-Re$

Lower left: 4 asymptotes pattern
$\pm 45^o$, $\pm 135^o$ to $-Re$

Lower right: 5 asymptotes pattern
$-Re$, and $\pm 72^o$, $\pm 144^o$ to $-Re$

Example
For the polynomial:

$$\lambda^3 + 3\lambda^2 + 2\lambda + K = 0 \tag{8}$$

all three branches of the root locus plot will go to ∞ as $K \to \infty$, along asymptotes relative to the $-Re$ axis defined by 0^0 , $\pm 120^0$ (the pattern shown in the upper right in the figure above)

5. (Goes with guideline 4): **If asymptotes exist, they intersect on the real axis at the point**

$$\lambda = \frac{\sum X's - \sum O's}{n - m} \tag{9}$$

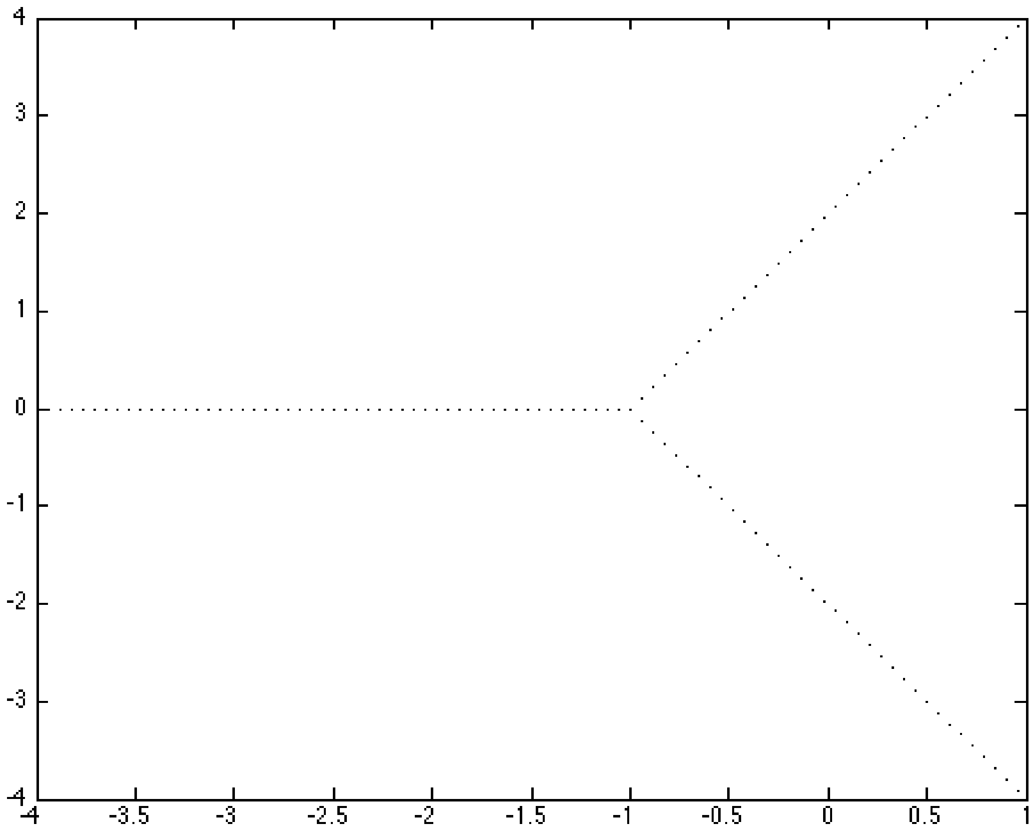
Example
For the polynomial:

$$\lambda^3 + 3\lambda^2 + 2\lambda + K = 0 \tag{10}$$

as we have seen, the three branches end at ∞ along asymptotes at 0^0 , $\pm 120^0$.

These three asymptotes intersect at the point

$$\lambda = \frac{(0 - 1 - 2) - (0)}{3} = -1$$

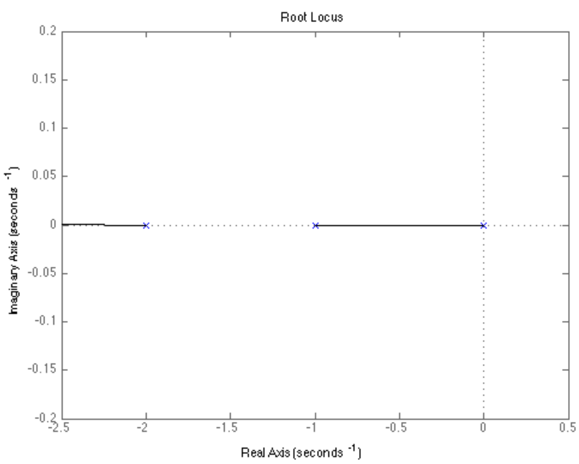


6. **Existence on the *Re* axis:**
The root locus exists on the *Re* axis if, and only if, there is an odd number of (X+O) points to the right along the *Re* axis

Example
For the polynomial:

$$\lambda^3 + 3\lambda^2 + 2\lambda + K = 0 \tag{11}$$

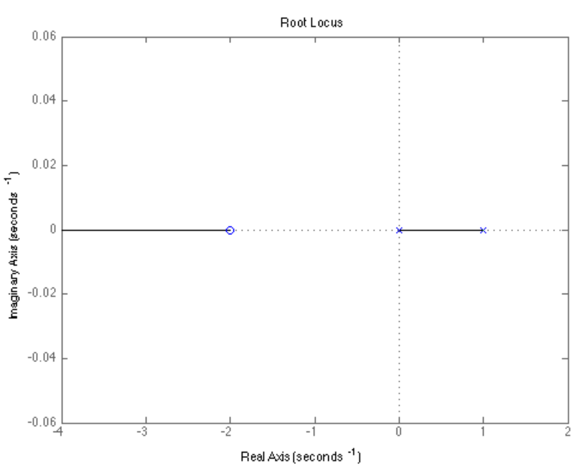
has 3 branches with X's at $0, -1, and - 2$, and no O's. Therefore, a branch of the root locus must exist on the real axis between $-1 \leq \lambda \leq 0$, because in this range, there is a total of 1 (an odd number) of O's and X's to the right (positive direction). The root locus cannot exist on the real axis in the range $-2 \leq \lambda \leq -1$ because the total number of O's and X's to the right is two (even). The root locus must exist on the real axis in the range $-\infty \leq \lambda \leq\ -2$ because the total number of O's and X's to the right is three (odd).



Example
The polynomial:

$$\lambda^2 + (K-1)\lambda + 2K = \lambda^2 - \lambda + K(\lambda + 2) = 0 \equiv D(\lambda) + K * N(\lambda) \tag{12}$$

has 2 X's at $\lambda = 1, 0$ and one O at $\lambda = -2$. Therefore, the root locus must exist on the *Re* axis between $0 \leq \lambda \leq 1$ and $-\infty \leq \lambda \leq -2$



7. **Break-in and break-out points, if they exist, are located where**

$$N(\lambda) * (\frac{d}{d\lambda}(D(\lambda))) - D(\lambda) * (\frac{d}{d\lambda}(N(\lambda))) = 0$$

The concept of break-in and break-out points arises because multiple branches of the root locus *cannot* co-exist on the real axis except at *single* (discrete) points. Two branches cannot occupy the same stretch of the *Re* axis, except at a single point. Since branches can't coexist along the *Re* axis,

(a) if two branches are already on the *Re* axis, without end points (0's) between them on *Re*, they must *break out* from the *Re* axis in order to head off to their end points.

(b) if two branches must end on the real axis, but are not yet on the *Re* axis, they must *break in* to the *Re* axis

Break-in and break-out points mark the spots (if any) where repeated real roots are found.

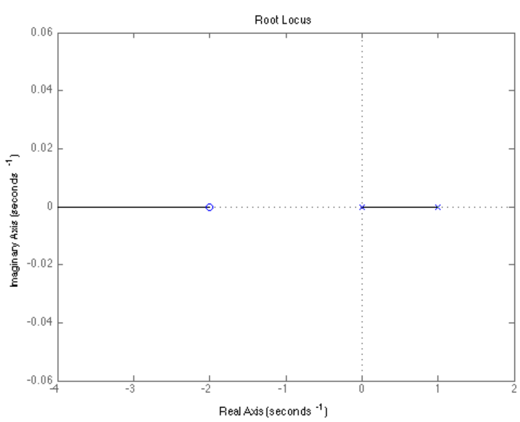
Example
As we've seen, the polynomial:

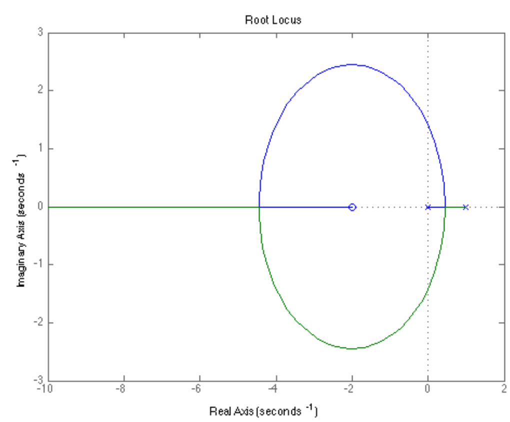
$$\lambda^2 + (K-1)\lambda + 2K = \lambda^2 - \lambda + K(\lambda+2) = 0 \equiv D(\lambda) + K * N(\lambda) \tag{13}$$

has 2 X's at $\lambda = 1, 0$ and one O at $\lambda = -2$. Therefore, the root locus must exist on the *Re* axis between $0 \le \lambda \le 1$ and $-\infty \le \lambda \le -2$

The two branches begin at $\lambda = 1$ and $\lambda = 0$ when $K = 0$. As K is increased, the root locus must cover the *Re* axis between 0 and 1. However, the two branches cannot co-exist except at a single point; therefore, when they meet each other between 0 and 1, they must *break out* from *Re*.

As $K \to \infty$, the two branches end at $\lambda = -2$ and $\lambda = -\infty$ along the $-Re$ axis. The root locus must cover the *Re* axis from $-\infty \le \lambda \le -2$. But the branches can only coexist at a point. Therefore, they must *break in* to *Re* somewhere to the left of $\lambda = -2$. After the two branches break in, one moves to the right (to end at $\lambda = -2$) and the other one moves to the left to end at $\lambda \to -\infty$





By inspection of the figure on the left, we must have a break-out between $0 \le \lambda \le 1$ and a break-in between $-\infty \le \lambda \le -2$. In the figure on the right, we see the break-in and break-out points. They are located where

$$N(\lambda)*(\frac{d}{d\lambda}(D(\lambda)))-D(\lambda)*(\frac{d}{d\lambda}(N(\lambda))) = 0 = (s+2)(2s-1)-(s^2-1) = s^2+4s-2 \;\Rightarrow\; \lambda = 0.4495\ ,\ -4.4495$$

Thus, there is a value of K that results in repeated roots $\lambda_{1,2} = 0.4495$, and another value of K that results in repeated roots $\lambda_{1,2} = -4.4495$. Those values are $K = 0.101$ and $K = 9.899$, easily found by substitution of λ

The repeated roots at -4.4495 constitute the fastest possible settling time = 4/4.4495 = 0.899 seconds

Example
As we've seen, the polynomial:

$$\lambda^3 + 3\lambda^2 + 2\lambda + K = 0 \tag{14}$$

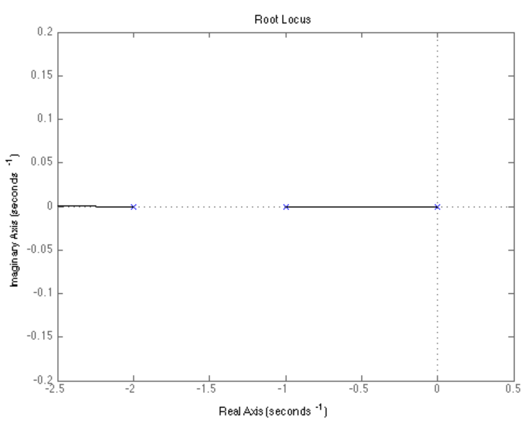
has 3 branches that begin on the real axis at $\lambda = 0$, $\lambda = -1$, and $\lambda = -2$. Also, the root locus must exist on the real axis between $-1 \leq \lambda \leq 0$ and between $-\infty \leq \lambda \ -2$. Also, 2 of the 3 branches go to ∞ along asymptotes that lie at an angle of $\pm 120\,\text{deg}$ to the negative real axis; since all branches begin on the *Re* axis, two of them must break out in order to go off along these asymptotes. Applying the condition from guide 7 to find break-in and break-out points,

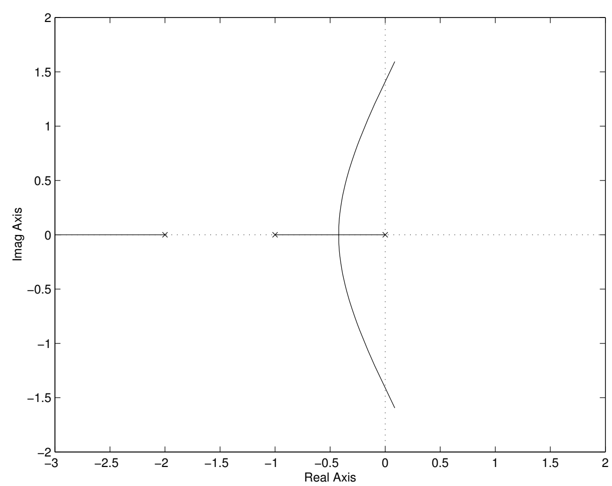
$$1 * \frac{d}{d\lambda}(\lambda^3 + 3\lambda^2 + 2\lambda) - (\lambda^3 + 3\lambda^2 + 2\lambda) * \frac{d}{d\lambda}(1) = 3\lambda^2 + 6\lambda + 2 = 0 \quad \rightarrow \quad \lambda = -0.4226\ ,\ -1.5774$$

These two points *may* be break-in or break-out points, but are not necessarily either. The point $\lambda = -1.5774$ cannot be a break-in or break-out point, *because the root locus doesn't exist on the Re axis in the range* $-2 \leq \lambda \leq -1$! On the other hand, the point $\lambda = -0.4226$ must be a break-out point, because the two branches that begin at 0 and -1 must leave the real axis somewhere between 0 and -1.

We now know that a repeated root exists at $\lambda = -0.4226$, and we can find the value of K that produces this repeated pair of roots by substituting into the original equation:

$$(-0.4226)^3 + 3 * (-0.4226)^2 + 2 * (-0.4226) + K = 0 \ \rightarrow \ K = 0.3849$$





From the figure on the left, we see that a break-out point must exist between $-1 \leq \lambda \leq 0$; it is shown in the figure on the right. We calculate its value to be $\lambda = -0.4226$, and the value of K that produces these repeated real roots is calculated to be $K = 0.3849$

8. Branches cross the imaginary axis at $\lambda = \pm i\omega$. Substituting these value, the characteristic equation will then contain two unknowns, ω and K. By setting both the real and imaginary parts of the characteristic equation to zero (independently of each other), we can solve two equations to obtain the values for ω and K.

Example
We've seen that the root locus for the polynomial:

$$\lambda^3 + 3\lambda^2 + 2\lambda + K = 0$$

has three branches, two of which break out at $\lambda = -0.4226$, and then go off to ∞ along asymptotes at angles $\pm 120^o$ to the $-Re$ axis. Therefore, these two branches must cross the imaginary axes. To find the crossing points, we substitute $\lambda = i\omega$ and collect terms to find

$$-i\omega^3 - 3\omega^2 + i2\omega + K = 0$$

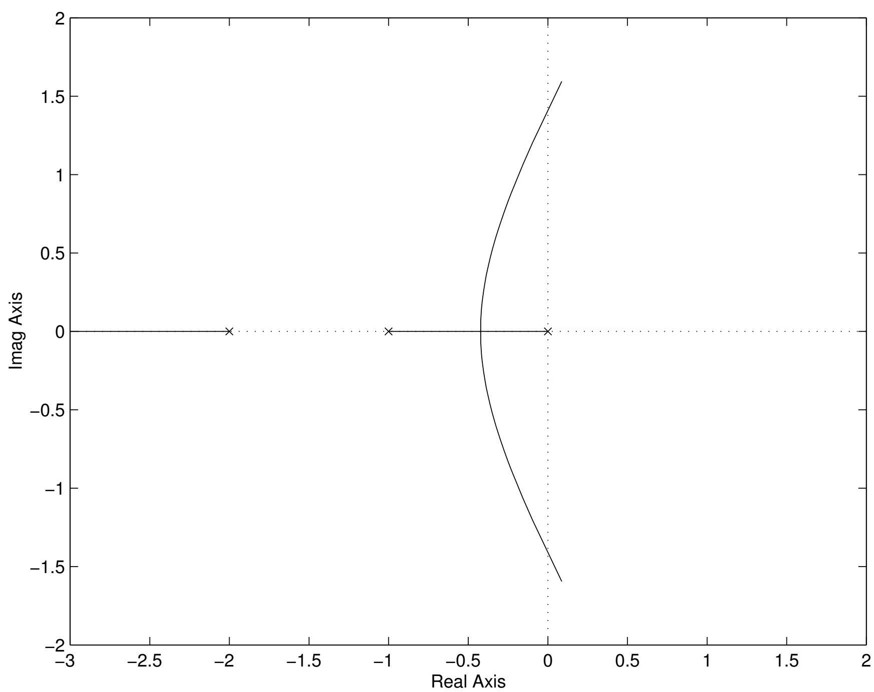
There are two unknowns, ω and K. We solve for them by setting both the real and imaginary parts of this equation to zero:

$$\text{Imaginary part: } 2\omega - \omega^3 = 0 \;\rightarrow\; \omega = 0 \;\;,\;\; \omega = \sqrt{2} = 1.414 \quad (15)$$

$$\text{Real part: } K - 3\omega^2 = 0 \;\rightarrow\; K = 0 \;\;,\;\; K = 3\omega^2 = 6 \quad (16)$$

Thus, we have found that the value $K = 0$ produces a real root at 0 (which we already knew, from the beginning point of one of the branches). We have also found that the value $K = 6$ produces a pair of roots at $\lambda_{1,2} = \pm i1.414$

- Crossings of the imaginary axis are important, because they often represent boundaries between stable and unstable behavior (not always - if there are additional roots already in the RHP, the system's stability does not change for the roots whose crossing we calculate)
- In this example, we've found that $0 < K < 6$ produces a stable system; $K = 0$ and $K = 6$ produce marginal stability; and any other value of K produces an unstable system. We knew that there would be a limited range of stability from simply looking at the plot, which shows two branches going off to ∞ along asymptotes at $\pm 120^o$; inevitably, these two branches must cross into the RHP sooner or later



Example
As we've seen, the polynomial:

$$\lambda^2 + (K-1)\lambda + 2K = \lambda^2 - \lambda + K(\lambda+2) = 0 \equiv D(\lambda) + K * N(\lambda) \tag{17}$$

has 2 branches which begin at $\lambda = 1$ and $\lambda = 0$ when $K = 0$. As K is increased, they break out at $\lambda = 0.73$, then break back in to Re at $\lambda = -2.73$. Where do they cross from the RHP into the LHP?

Substituting $\lambda = \pm i\omega$, we find

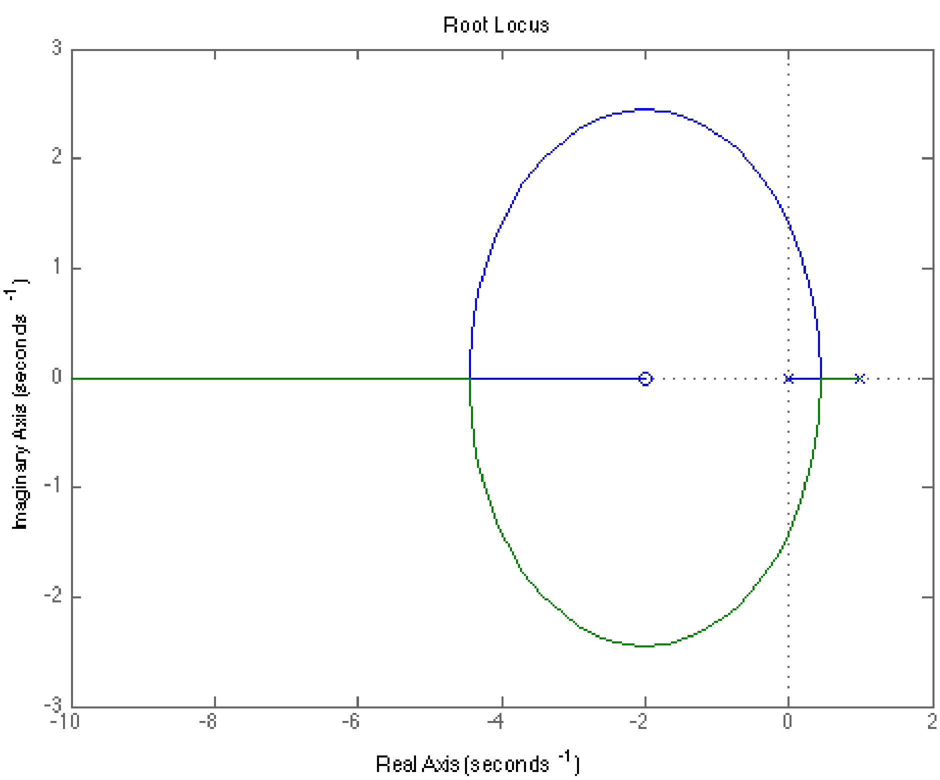
$$-\omega^2 - i\omega + K(i\omega + 2) = 0$$

Setting the real and imaginary components equal to zero independently, we have

$$-\omega^2 + 2K = 0$$
$$-\omega + K\omega = 0$$

The two solutions are ($\omega = 0$, $K = 0$) and ($K = 1$, $\omega = \sqrt{2}$).

In this case, we've found that the range $0 < K < 1$ produces roots in the RHP (unstable system). $K = 0$ and $K = 1$ produce a marginally-stable system, and all $K > 1$ produces a stable system.



9. In the polynomial equation

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

the sum of the n roots must equal $-a_{n-1}$

This guide is often useful for finding the values of the remaining roots after we've found specific roots such as break-in, break-out, or imaginary axis crossings.

Example
We've seen that the root locus for the polynomial:

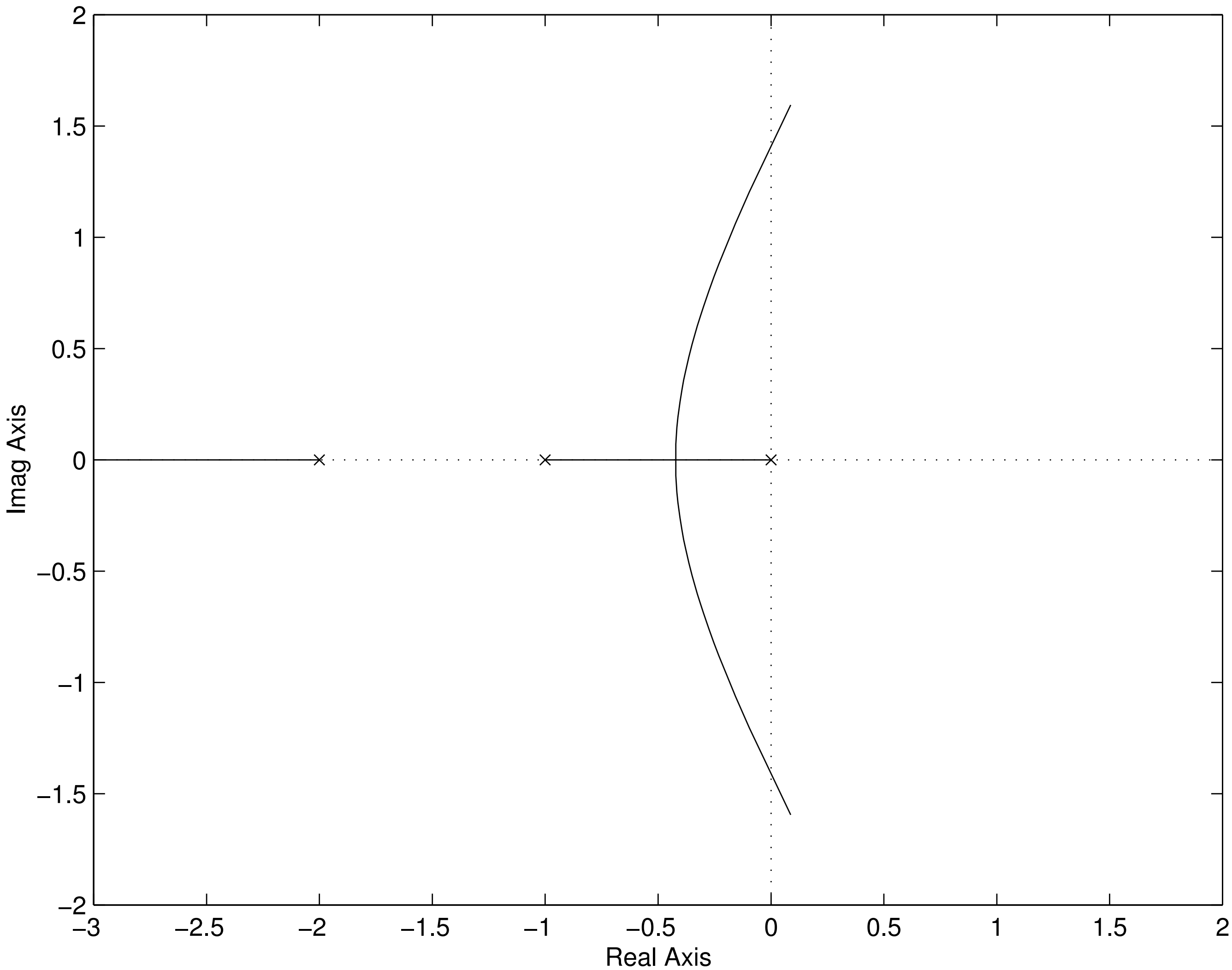
$$\lambda^3 + 3\lambda^2 + 2\lambda + K = 0$$

has three branches, two of which break out at $\lambda_{1,2} = -0.4226$, corresponding to a value of $K = 0.3849$. Thus, for this value of K, the third root of the CE is

$$\lambda_3 = -a_{n-1} - 2 * (-0.4266) = -3 - 2 * (-0.4226) = -2.16$$

We also found that for $K = 6$, two branches cross the Im axis. Since these two roots add up to zero, i.e., $i\omega - i\omega = 0$, the third root for $K = 6$ is $\lambda_3 = -a_{n-1} = -3$

This guide is also sometimes very helpful in determining the fastest possible speed of response in a system with multiple branches heading in opposite directions relative to the Re axis. We will see examples of this later.



Example
For the polynomial:

$$\lambda^3 + 3\lambda^2 + 2\lambda + K = 0$$

we used guidelines 1-8 to sketch the complete root locus.

1. 3 branches
2. Branches begin at $\lambda = 0$, $\lambda = -1$, and $\lambda = -2$
3. Branches end at ∞
4. Branches go off to ∞ along asymptotes on negative real axis and ± 120 deg from negative real axis
5. Asymptotes intersect at $\lambda = -1$
6. Branches are on real axis between $0 \leq \lambda \leq -1$ and $-2 \leq \lambda \;\; -\infty$
7. Break-out point exists at $\lambda = -0.4226$
8. Two branches cross the imaginary axis at $\lambda = i \pm 1.414$

Interpretation of the Root Locus

The root locus provides critical information that enables us to see how a system's behavior is affected by the value of a variable (K) that appears in the coefficients of the characteristic polynomial.

- **Sensitivity**: Depending on the application, K itself may have markedly different meanings. It may represent a design variable that can be selected in order to optimize the desired behavior of they system. But it may also represent some level of uncertainty in the model of a system, and by examining the root locus, we can learn important information about the *sensitivity* of a system to this uncertainty. We may discover that in the range of possible values for the unknown K, the behavior is always acceptable, and thus, the uncertainty is not a problem. Or we may discover that within the range of K, there is potential behavior that is unacceptable, and we must do something to eliminate that possibility.

- **Stability**: We need all roots to be in the LHP. Therefore, branch crossings of the Im axis are usually crucial. There may be multiple crossings within a single root locus plot, and we need to find them all and evaluate them for their effect on the system's stability.

  In the example above, by applying guideline 8, we found that the system is only stable for the range $0 < K < 6$.

- **Fastest settling time**: The settling time for a stable system is controlled by the settling time of the root whose negative real part is the least negative among all the roots - i.e., the *dominant* root. For any specific value of K, there is a specific dominant root (if real), or pair (if a complex conjugate pair). In graphical terms, the dominant root is the root that is *farthest to the right* (least negative) in the complex plane.

  Thus, if we seek to find K such that the system has the fastest settling time, then we seek the dominant root which is farthest to the left. Since the root locus plot shows the sets of roots that correspond to all K within some range, and since each specific value of K has its own dominant root, the root locus plot contains an essentially infinite number of possible dominant roots. Therefore, it is important to recognize the location of the dominant root as it moves among the branches of the root locus.

  In the example above, for $K = 0$, the three roots are at $\lambda = 0, -1, -2$, marked by X's in the plot. Let's call these $\lambda_1, \lambda_2, \lambda_3$. Thus, for $K = 0$, the dominant root is $\lambda_1 = 0$.

  We know that different values of K produce different roots. As K increases, initially (for small K), looking at the plot, we see that λ_1 moves left along the real axis, λ_2 moves to the right along the real axis, and λ_3 moves to the left along the real axis. Therefore, in this range of K, the dominant root is always λ_1. For small values of K, as K increases, the dominant root moves to the left (becoming more negative), so the system settling time is getting faster!

  When K gets large enough, λ_1 and λ_2 meet each other, break out, and become a complex conjugate pair. The break-out occurs at the value $\lambda_{1,2} = -0.4226$, when $K = 0.3849$. For $K > 0.3849$, these two roots are a complex conjugate pair that then moves back to the right as K increases. Thus, for $K > 0.3849$, the dominant root is the complex conjugate pair represented by these branches. Since they move to the right as K increases, the settling time increases for $K > 0.3849$.

  Thus $K = 0.3849$ provides the fastest settling time - it is the value of K which has the most negative dominant root, the repeated root $\lambda = -0.4226$. The settling time for this K is $4\tau = 4\frac{1}{0.4226} = 9.465$ seconds.

- **Oscillatory behavior**: If one or more complex conjugate pairs of roots exist, then the system may have sinusoidal (oscillatory) behavior. Since this behavior is unacceptable in certain applications, we may wish to consider only values of K that produce no complex conjugate roots (i.e., all roots of the system must be real). From guidelines 6 and 7 in the example above, we learned that the range of K for which the system has only real roots is $0 < K < 0.3849$