General Form of the Particular Solution

Consider again the general form of a single n^{th}-order LTI ODE:

$$a_n \frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + ... + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t) \tag{1}$$

We have previously seen that the complete solution may be written as the sum of a homogeneous solution and a particular solution, and that the homogeneous solution may be written in the general form $x_h(t) = Ae^{\lambda t}$.

The general form of the particular solution may be written

$$x_p(t) = C_1 f(t) + C_2 \dot{f}(t) + C_3 \ddot{f}(t) + \cdots + C_{n+1} \frac{d^n f(t)}{dt^n} \tag{2}$$

where $C_1, C_2, \ldots, C_{n+1}$ are constants which are determined by substituting Eq. (2) into Eq. (1).

Although the general form for x_p is given in Eq. (2), in practice it is rarely necessary to include all of the terms as shown, because:

1. in some cases, the derivatives of $f(t)$ are zero
2. in other cases, $f(t)$ and/or its derivatives contain the same functions of time, and any repeated functional forms on the RHS of Eq. (2) can be combined into a single term with an appropriate constant

Example

Suppose

$$f(t) = t^2 + 2t - 7 \tag{3}$$

Applying Eq. (2) exactly, we would write

$$\begin{aligned} x_p(t) &= C_1 * (t^2 + 2t - 7) + C_2 * (2t + 2) + C_3 * 2 + C_4 * (0) + C_5 * (0) + ... \\ &= C_1 t^2 + (2C_1 + 2C_2)t + (-7C_1 + 2C_2 + 2C_3) \end{aligned} \tag{4}$$

Following the two points above,

1. all derivatives of $f(t)$ higher than 2nd are zero, so of course we don't need to actually write them (i.e., we don't need to write any term beyond $C_3 * 2$)
2. since C_1, C_2, and C_3 are constants anyway, we can simplify to

   $$x_p(t) = C_4 t^2 + C_5 t + C_6 \tag{5}$$

   Although we could solve for x_p using Eq. (4), there's no need to do so. We simply solve for x_p using the form of Eq. (5).

We substitute $x_p(t)$ from Eq. (2) (simplifying using the two guides shown above, if appropriate) into the LTI ODE:

$$a_n \frac{d^n x_p}{dt^n} + a_{n-1} \frac{d^{n-1} x_p}{dt^{n-1}} + \cdots + a_1 \frac{dx_p}{dt} + a_0 x_p = f(t)$$

which enables us to solve for the $C_i's$ to obtain the exact $x_p(t)$.

The Complete Solution and Initial Conditions

The general form of the particular solution contains constants C_i which are obtained by solving

$$a_n \frac{d^n x_p}{dt^n} + a_{n-1} \frac{d^{n-1} x_p}{dt^{n-1}} + \cdots + a_1 \frac{dx_p}{dt} + a_0 x_p = f(t) \tag{6}$$

As we have seen, the general form of the homogeneous solution is

$$x_h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \cdots + A_n e^{\lambda_n t}$$

The general form of the complete solution of the LTI ODE is the sum of the homogeneous and particular solutions:

$$x(t) = x_h(t) + x_p(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \cdots + A_n e^{\lambda_n t} + C_1 f(t) + C_2 \dot{f}(t) + C_3 \ddot{f}(t) + \cdots + C_n + 1 \frac{d^n f(t)}{dt^n} \tag{7}$$

The constants C_i are found from Eq. (6). The final step to obtain the exact complete solution is to use the n initial conditions with Eq. (7) to find the values of the n $A_i' s$.

The particular solution affects the homogeneous response through the initial conditions

- The $A_i' s$ that result from Eq. (7) are not all equal to those that result from a zero particular solution
- Of particular interest is the case with IC's =0 . We have seen that: $x_h(t) = 0$ if the IC's=0 and there is no $f(t)$

  But from Eq. (7), if $f(0) \neq 0$, then $x_h(t) \neq 0$ even for IC's=0 ! The forcing produces **both** nonzero x_p **and** nonzero x_h

Example

Consider the LTI ODE

$$\ddot{x} + 3\dot{x} + 2x = 0$$

with initial conditions $x(0) = 0$, $\dot{x}(0) = 0$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0$$

which has roots at $\lambda_{1,2} = -1\ ,\ -2$. The homogeneous solution is

$$x_h(t) = A_1 e^{-t} + A_2 e^{-2t}$$

Applying the IC's to find A_1 and A_2, we obtain

$$x(0) = 0 = A_1 + A_2 \quad , \quad \dot{x}(0) = 0 = -A_1 - 2A_2 \qquad \Rightarrow A_1 = A_2 = 0 \qquad \Rightarrow \underline{x_h(t) = 0}$$

Now suppose the system is subjected to an input $f(t) = 1$. Applying Eq. (2), we see that $x_p = C_1$, and we substitute into the LTI ODE to find

$$\ddot{C}_1 + 3\dot{C}_1 + 2C_1 = 1 \qquad \Rightarrow x_p(t) = C_1 = 1/2$$

The complete solution is

$$x(t) = A_1 e^{-t} + A_2 e^{-2t} + 1/2$$

Applying the IC's to find A_1 and A_2, we obtain

$$x(0) = 0 = A_1 + A_2 + 1/2 \ , \ \dot{x}(0) = 0 = -A_1 - 2A_2 \quad \Rightarrow A_1 = -1\ ,\ A_2 = 1/2 \quad \Rightarrow \underline{x_h(t) = -e^{-t} + 0.5e^{-2t}}$$

Linearity

Linearity is a crucial property of LTI ODE's !!

Suppose that we have an LTI ODE, called, say, 'model'.

Suppose that an input $f_1(t)$, applied by itself to the LTI ODE 'model', produces a particular solution $x_{p1}(t)$:

$$f_1(t) \rightarrow \text{ model } \rightarrow x_{p1}(t)$$

Suppose that another input $f_2(t)$, applied by itself to the same system, produces the particular solution $x_{p2}(t)$.

$$f_2(t) \rightarrow \text{ model } \rightarrow x_{p2}(t)$$

The property of *linearity* allows us to find the response of the system to a linear combination of the two inputs as the same linear combination of the two outputs!

$$[c_1 f_1(t) + c_2 f_2(t)] \rightarrow \text{ model } \rightarrow [c_1 x_{p1}(t) + c_2 x_{p2}(t)]$$

where c_1 and c_2 may be any real constants.

Linearity actually applies to any linear combination of any number of inputs:

$$[c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) + c_4 f_4(t) + \ldots] \rightarrow \text{ model } \rightarrow [c_1 x_{p1}(t) + c_2 x_{p2}(t) + c_3 x_{p3}(t) + c_4 x_{p4}(t) + \ldots]$$

Several important consequences of linearity in LTI ODE models:

1. The effect of any input is *independent* of the effect of any other input
2. We can calculate and then store particular solutions for common generic input forms, then simply scale and/or combine these stored solutions to obtain the actual response to any linear combination of those inputs

Three Important Forcing Functions and Their Corresponding Particular Solutions

1. $f(t)$= constant= $F \quad \Rightarrow \quad x_p(t)$ = constant = C

   If the input is constant, the particular solution is also constant

   Constant inputs are very common in practice; many real-world systems experience them.

   Because of linearity, of particular interest is the *step* input, defined as

$$f(t) = \begin{pmatrix} 0 & t < 0 \\ 1 & t \geq 0 \end{pmatrix} \equiv \text{ Step Input} \tag{8}$$

   The complete solution for a step input, applied to a system with zero initial conditions, is called the *step response*. The step response is one of the most commonly-used descriptions of dynamic systems. Using linearity, the step response gives us the response to any constant input F applied at $t = 0$ to a system with zero initial conditions, i.e., $x_p(t) = F*$(step response)

2. $f(t) \equiv$ ramp input $= k_1 t + k_2 \quad \Rightarrow \quad x_p(t) =$ ramp output $= C_1 t + C_2$

   If the input is a *ramp*, the particular solution is also a ramp

   Ramp inputs are often used to model the transition between two constant-value inputs in systems where the actual physical input value cannot be changed instantaneously. A common example is a change in the throttle position for a transportation vehicle. The engine power output (which is the input to the vehicle) does not actually change instantaneously, due to the inertia of the engine itself. A ramp may be used to model the transition from one constant power setting to another.

   Real-world inputs rarely increase without bound, so a single ramp input is rarely applied to a system for $t \to \infty$. We will return to this topic shortly .

   Although ramp inputs are the most common form of input polynomials, the particular solution for any input polynomial is a polynomial of the same order:

$$f(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \ldots \qquad \Rightarrow \qquad x_p(t) = C_0 + C_1 t + C_2 t^2 + C_3 t^3 + \ldots$$

   The $C_i's$ must be determined from substitution into the LTI ODE, and usually are not equal to the $b_i's$

3. $f(t) = Asin(\omega t) \quad \Rightarrow \quad x_p(t) = C_1 sin(\omega t) + C_2 \omega cos(\omega t) = C sin(\omega t + \phi)$

   If the input is a sinusoid, the particular solution is a sinusoid with the same frequency

   A sinusoidal input produces a sinusoidal output with the *identical frequency.* In general, the amplitude of the output is different from the input (i.e., $C \neq A$), and in addition, the output is shifted along the time axis by a *phase* angle, ϕ.

   We have already seen that *each pair* of complex conjugate roots in the homogeneous solution leads to a sinusoidal term with its own frequency, phase angle, and exponentially-changing amplitude. By contrast, for a sinusoidal input, the particular solution for the whole system (regardless of order) has one frequency (equal to the input frequency), with constant amplitude (not exponentially-changing), and there is only one phase angle.

   The ratio C/A and the phase angle ϕ are generally different for different values of input frequency ω. As we will see, it is often valuable to analyze this kind of forcing (and the resulting particular solution) in terms of ω rather than t. Since ω is the frequency of the input, we call this *frequency domain* analysis. An entire section of the course is devoted to frequency-domain analysis.

   In many applications, sinusoidal responses are the single most important type of particular solution.

Example:

(a) Find the particular solution of the system with LTI ODE model

$$\dddot{x} + 5\ddot{x} + 9\dot{x} + 5x = f(t)$$

to the following input:

$$f(t) = \begin{pmatrix} 0 & t < 0 \\ 1 & t \geq 0 \end{pmatrix}$$

Solution: Using Eq. (2), the form of the particular solution is

$$x_p(t) = constant = C$$

Substituting into the ODE,

$$\dddot{x}_p + 5\ddot{x}_p + 9\dot{x}_p + 5x_p = C = f(t) = \begin{pmatrix} 0 & t < 0 \\ 1 & t \geq 0 \end{pmatrix} \quad \Rightarrow x_p = \begin{pmatrix} 0 & t < 0 \\ 0.2 & t \geq 0 \end{pmatrix}$$

(b) Find the complete response of the system to the input given above, for initial conditions

$$x(0) = 0 \quad , \quad \dot{x}(0) = 0 \quad , \quad \ddot{x}(0) = 0$$

To find the homogeneous solution, we need the roots of the characteristic equation:

$$\lambda^3 + 5\lambda^2 + 9\lambda + 5 = 0 \quad \Rightarrow \quad \lambda_{1,2,3} = -1 \ , \ -2 \pm i$$

Thus, the general form of the homogeneous solution is

$$x_h(t) = A_1 e^{-t} + A_2 e^{-2t} sin(t + \phi)$$

The general form of the complete solution for $t \geq 0$ is

$$x(t) = x_h(t) + x_p(t) = A_1 e^{-t} + A_2 e^{-2t} sin(t + \phi) + 0.2$$

We use the IC's to find the constants in the homogeneous part of the solution:

$$\begin{aligned} x(0) &= 0 = A_1 + A_2 sin\phi + 0.2 \\ \dot{x}(0) &= 0 = -A_1 - 2A_2 sin\phi + A_2 cos\phi \\ \ddot{x}(0) &= 0 = A_1 + 3A_2 sin\phi - 4A_2 cos\phi \end{aligned}$$

Solving simultaneously using MATLAB function **inv**, we obtain

$$\begin{pmatrix} A_1 \\ A_2 sin\phi \\ A_2 cos\phi \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.3 \\ 0.1 \end{pmatrix} \quad \Rightarrow \ A_2 = 0.3162 \ , \ \phi = 1.249$$

The complete solution is

$$\underline{x(t) = -0.5e^{-t} + 0.3162e^{-2t} sin(t + 1.249) + 0.2}$$

Note that if the input $f(t) = 0$, then $x_p(t) = 0$; solving for the corresponding $x_h(t)$ for these IC's, we find

$$\begin{pmatrix} A_1 \\ A_2 sin\phi \\ A_2 cos\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \ x_h(t) = 0$$

Thus, the presence of a nonzero input produces **both** nonzero $x_p(t)$ and nonzero $x_h(t)$!

Example:
Find the particular solution of the system with LTI ODE model

$$5\dddot{x} + 3\ddot{x} + 12\dot{x} + x = f(t)$$

to the following input:

$$f(t) = 24t + 8$$

Solution:
Using Eq. (2), the form of the particular solution is

$$x_p(t) = C_1 t + C_2$$

Substituting into the ODE,

$$5\dddot{x}_p + 3\ddot{x}_p + 12\dot{x}_p + x_p = 12C_1 + C_1 t + C_2 = 24t + 8$$

Equating the coefficients on like powers of t, we find:

$$\begin{aligned} t: &\quad 2C_1 = 24 \rightarrow C_1 = 12 \\ 1: &\quad C_1 + C_2 = 8 \rightarrow C_2 = -4 \end{aligned}$$

Thus, the particular solution is

$$x_p(t) = 12t - 4$$

Example:
Find the particular solution of the system with LTI ODE model

$$5\dddot{x} + 3\ddot{x} + 12\dot{x} + x = f(t)$$

to the following input:

$$f(t) = 4sin(3t)$$

Solution:
Using Eq. (2), the form of the particular solution is

$$x_p(t) = C_1 sin(3t) + C_2 cos(3t) = C sin(3t + \phi)$$

As with homogeneous solutions, it's often algebraically easier to obtain the constants C_1, C_2 than C, ϕ :

$$\begin{aligned} 5\dddot{x}_p + 3\ddot{x}_p + 12\dot{x}_p + x_p = {} & 135(C_2 sin(3t) - C_1 cos(3t)) \\ & - 27(C_1 sin(3t) + C_2 cos(3t)) \\ & + 36(C_1 cos(3t) - C_2 sin(3t)) \\ & + C_1 sin(3t) + C_2 cos(3t) \\ & \qquad = f(t) = 4sin(3t) \end{aligned}$$

The *sin* and *cos* terms must be equal independently:

$$\begin{aligned} sin(3t): & \quad (135 - 36)C_2 + (-27 + 1)C_1 = 4 \\ cos(3t): & \quad (-135 + 36)C_1 + (-27 + 1)C_2 = 0 \end{aligned}$$

Solving simultaneously, $C_1 = -0.0099$ and $C_2 = 0.0378$, so we have

$$x_p(t) = 0.0378cos(3t) - 0.0099sin(3t)$$

For practice, convert this back into the form $x_p = Csin(3t + \phi)$

Sequential Forcing

Many applications consist of multiple inputs that may start and/or stop at times other than $t = 0$. For such cases, the general solution strategy is

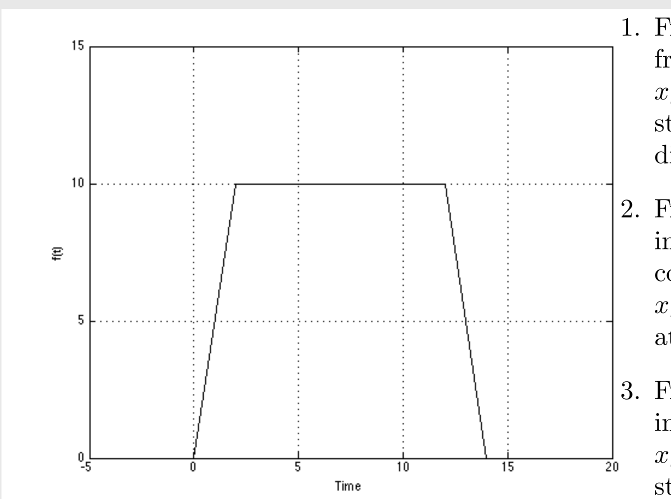
1. Evaluate the complete solution $x(t)$, beginning from $t = 0$, using initial conditions
2. Use the current complete solution until a change occurs in the RHS of the LTI ODE (i.e., a new forcing function begins and/or an existing forcing function ends)
3. Calculate the particular solution corresponding to the new RHS of the LTI ODE
4. Use the values of $x(t)$, $\dot{x}(t)$, etc., from step (2), as the *initial conditions*, and construct a new complete solution (homogeneous plus particular) using the particular solution from (3)
5. Repeat steps (2)-(4) for every change of RHS

Example
A common real-world input model is shown in the figure, representing a constant input that needs some time after it is switched on to reach its constant value, and then needs some time to return to zero after it is switched off.

We use a ramp input function to represent the start-up period, followed by a constant input during the operational period, followed by another ramp to represent the powering-down period.

Suppose the constant input is 10; the start-up period is $0 \leq t \leq 2$ seconds; the operational period is $2 \leq t \leq 12$ seconds; and the shut-down period is $12 \leq t \leq 14$ seconds. The system starts from rest.

The complete solution $x(t)$ can be constructed from the following components:



1. From $0 \leq t \leq 2$: Particular solution resulting from the ramp $f(t) = 5t$, plus general form of $x_h(t)$, combined, and then the unknown constants in $x_h(t)$ are evaluated using initial conditions = 0 (start from rest)
2. From $2 \leq t \leq 12$: Particular solution resulting from $f(t) = 10$, plus general form of $x_h(t)$, combined, and then the unknown constants in $x_h(t)$ are evaluated using the solution for $x(t)$ at $t = 2$ from step (1)
3. From $12 \leq t \leq 14$: Particular solution resulting from $f(t) = 70 - 5t$, plus general form of $x_h(t)$, combined, and then the unknown constants in $x_h(t)$ are evaluated using the solution for $x(t)$ at $t = 12$ from step (2)
4. From $14 \leq t \leq (14 +$ the settling time$)$: the general form of the homogeneous solution, with unknown constants evaluated using the solution for $x(t)$ at $t = 14$ from step (3)