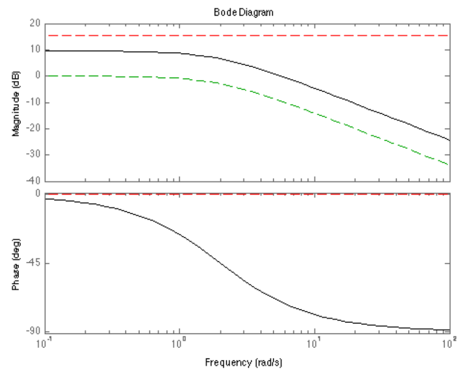
Find state-space models for each of the following systems:

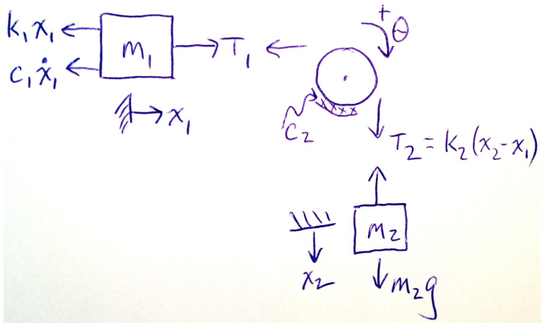
1.



Mass m_1 is connected to the wall through spring k_1 and damper c_1. It slides on the floor without friction. An inextensible cord connects mass 1, over the pulley with inertia J, to spring k_2. The cord does NOT slide relative to the pulley. The hub of the pulley has rotational damping c_2. Gravity acts downward as usual.

The system's input is gravity; the outputs are (i) the tension in the cord connected to m_1, and (ii) the force in spring k_2.

Solution:



The FBD's are shown at left. For m_1, with $x_1 = 0$ at the unstretched position of spring k_1, we have

$$m_1\ddot{x}_1 = T_1 - c_1\dot{x}_1 - k_1x_1 \tag{1}$$

For the pulley,

$$J\ddot{\theta} = R(T_2 - T_1) - C_2\dot{\theta} \tag{2}$$

For m_2,

$$m_2\ddot{x}_2 = m_2g - T_2 \tag{3}$$

Since the cord is inextensible, we have $x_1 = R\theta$, so we don't need both x_1 and θ. I'll choose to keep x_1 and eliminate θ. Also, note that $T_2 = k_2(x_2 - x_1)$.

Now solve Eq. (2) for T_1:

$$T_1 = \frac{1}{R}\big(RT_2 - J\ddot{\theta} - C_2\dot{\theta}\big) = k_2(x_2 - x_1) - \frac{J}{R^2}\ddot{x}_1 - \frac{C_2}{R^2}\dot{x}_1 \tag{4}$$

Now substitute for T_1 in Eq. (1), and collect terms:

$$\ddot{x}_1 = (\frac{R^2}{m_1R^2 + J})(-c_1 - \frac{C_2}{R^2})\dot{x}_1 - (k_1 + k_2)x_1 + k_2x_2 \tag{5}$$

The state-space model may be written from equations (3) and (5) as follows:

$$\underline{z}(t) \equiv \begin{pmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \quad , \quad \underline{u}(t) = (g) \quad , \quad \underline{y}(t) = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

Expanding the outputs,

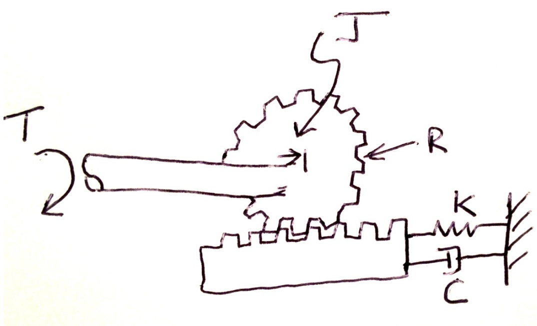
$$\begin{pmatrix} T_1 & = \left[-k_2 + \frac{k_1J}{m_1R^2+J}\right]x_1 + \left[k_2 - \frac{2k_2J}{m_1R^2+J}\right]x_2 + \left[\frac{(c_1+C_2/R^2)J}{m_1R^2+J}\right]\dot{x}_1 \\ \\ T_2 & = -k_2x_1 + k_2x_2 \end{pmatrix}$$

****Definitely double-check my algebra! **** Assuming that it's correct,

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2) & k_2 & (\frac{R^2}{m_1R^2+J})(-c_1-\frac{C_2}{R^2}) & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \left[-k_2+\frac{k_1J}{m_1R^2+J}\right] & \left[k_2-\frac{2k_2J}{m_1R^2+J}\right] & \left[\frac{(c_1+C_2/R^2)J}{m_1R^2+J}\right] & 0 \\ -k_2 & k_2 & 0 & 0 \end{pmatrix} \quad , \quad D = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

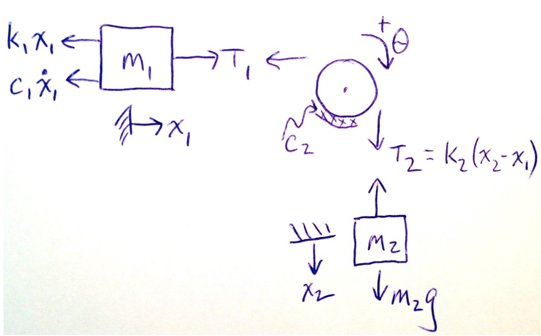
2.



The diagram shows an idealized rack-and-pinion steering mechanism. The driver creates input torque (moment) T on the shaft of the pinion gear, which has inertia J and radius R. The rotation of the pinion gear slides the rack, which has mass m. Assume that all motion of the pinion gear is pure rotation, and all motion of the rack is pure translation. The resistance of the tires is modeled by a spring k and dashpot c, shown connected to ground.

The system's input is T; the output is the force on the tires (through k and c).

__Solution:__



The FBD's are shown at left. For the rack, with $x = 0$ at the unstretched position of the spring k, we have

$$m\ddot{x} = F - c\dot{x} - kx \tag{6}$$

For the pinion,

$$J\ddot{\theta} = T - RF \tag{7}$$

From geometry, we have $x = R\theta$, so we don't need both x and θ. I'll choose to keep x and eliminate θ.

Now solve Eq. (7) for F:

$$F = \frac{T - J\ddot{\theta}}{R} = \frac{1}{R}T - \frac{J}{R^2}\ddot{x} \tag{8}$$

Now substitute for F in Eq. (6), and collect terms:

$$\ddot{x} = (\frac{R^2}{mR^2+J})(\frac{1}{R}T - c\dot{x} - kx) \tag{9}$$

The state-space model may be written from equation (9) as follows:

$$\underline{z}(t) \equiv \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \quad , \quad \underline{u}(t) = (T) \quad , \quad \underline{y}(t) = (kx + c\dot{x})$$

Thus,

$$A = \begin{pmatrix} 0 & 1 \\ \frac{-kR^2}{mR^2+J} & \frac{-cR^2}{mR^2+J} \end{pmatrix} \quad , \quad B = (\frac{R}{mR^2+J}) \quad , \quad C = \begin{pmatrix} k & c \end{pmatrix} \quad , \quad D = (0)$$