From our knowledge of particular solutions, we know that:

$$\textbf{If input} = Asin(\omega t)$$

$$\textbf{then particular solution} = Bsin(\omega t + \phi)$$

$\Rightarrow$ **The response to a sinusoidal input is a sinusoidal output with the same frequency** $\Leftarrow$

Magnitude may be different (usually is), and may have a phase shift (usually do)

We are interested in (i) ratio $\frac{B}{A}$, and (ii) ϕ – both of which are functions of ω

We define the Frequency Response Function:

$$\text{FRF} \equiv \text{TF}(s = i\omega) = \text{ complex function of } \omega$$
$$\equiv Re(\omega) + i * Im(\omega)$$

$$\text{FRF} = \text{ function of } \omega \quad \Rightarrow \quad \omega \text{ is the independent variable in the Frequency Domain}$$

It can be shown that

$$\text{Magnitude(FRF)} \equiv \sqrt{Re^2 + Im^2} = \frac{B}{A}$$

and

$$\text{Phase(FRF)} \equiv tan^{-1}(\frac{Im}{Re}) = \phi$$

Therefore:

1. Find the system TF's
2. Evaluate them at $s = i\omega \qquad \Rightarrow \; FRF = Re(\omega) + iIm(\omega)$
3. For any input $u = Asin(\omega t)$, the steady-state output (particular solution) is

$$y(t) = Bsin(\omega t + \phi) = A * Mag(FRF) * sin(\omega t + Phase(FRF))$$

Example

Find the frequency response functions for the system modeled by

$$\ddot{x}(t) + 5\dot{x}(t) + 6 = f(t)$$

where the outputs are

$$y_1(t) = 7x$$
$$y_2(t) = 5\dot{x}$$
$$y_3(t) = 3\ddot{x}$$

First, we find the system TF's:

$$\mathcal{L}\left[\ddot{x}(t) + 5\dot{x}(t) + 6x(t) = f(t)\right] = s^2X(s) - sx(0) - \dot{x}(0) - 5X(s) - 5x(0) + 6X(s) = F(s)$$

For all $TF's$, set initial conditions to zero, collect terms to find:

$$(s^2 + 5s + 6)X(s) = F(s)$$

The 3 $TF's$ are:

$$\frac{Y_1(s)}{F(s)} = \frac{7X(s)}{F(s)} = \frac{7}{s^2 + 5s + 6} \qquad FRF_{Y_1} = \frac{7}{(6 - \omega^2) + i5\omega}$$

$$\frac{Y_2(s)}{F(s)} = \frac{5sX(s)}{F(s)} = \frac{5s}{s^2 + 5s + 6} \qquad FRF_{Y_2} = \frac{i5\omega}{(6 - \omega^2) + i5\omega}$$

$$\frac{Y_3(s)}{F(s)} = \frac{3s^2X(s)}{F(s)} = \frac{3s^2}{s^2 + 5s + 6} \qquad FRF_{Y_3} = \frac{-3\omega^2}{(6 - \omega^2) + i5\omega}$$

To find the Re and Im parts of FRF, and thus the Mag and Phase, it's convenient to convert the FRF denominator to a real function.

Example

We previously found the frequency response functions for the system modeled by

$$\ddot{x}(t) + 5\dot{x}(t) + 6 = f(t)$$

where the outputs are

$$y_1(t) = 7x$$
$$y_2(t) = 5\dot{x}$$
$$y_3(t) = 3\ddot{x}$$

The first FRF was found to be:

$$FRF_{Y_1} = \frac{7}{(6-\omega^2) + i5\omega}$$

In this expression, we can't easily distinguish between the Re and Im parts because of the complex denominator. Therefore, we multiply both numerator and denominator by the conjugate of the denominator (multiplication by 1 doesn't change the value of the function):

$$FRF_{Y_1} = \frac{7}{(6-\omega^2) + i5\omega} * \frac{(6-\omega^2) - i5\omega}{(6-\omega^2) - i5\omega} = \frac{7(6-\omega^2) - i35\omega}{(6-\omega^2)^2 + 25\omega^2}$$

From RHS, we now easily see that:

$$Re(\omega) \equiv \frac{7(6-\omega^2)}{(6-\omega^2)^2 + 25\omega^2}$$

$$Im(\omega) \equiv \frac{-35\omega}{(6-\omega^2)^2 + 25\omega^2}$$

$$Mag(\omega) \equiv \sqrt{Re^2 + Im^2} = \frac{\sqrt{\left[7(6-\omega^2)\right]^2 + (35\omega)^2}}{(6-\omega^2)^2 + 25\omega^2}$$

$$Phase \equiv tan^{-1}\frac{Im}{Re} = tan^{-1}\frac{-35\omega}{7(6-\omega^2)}$$

In the example above, if $f(t) = 5sin(t)$, then

$$y_1(t) = Mag(\omega = 1) * 5 * sin(t + Phase(\omega = 1)) = 4.95sin(t - 0.785)$$

Similarly, if $f(t) = 3sin(2t)$, then

$$y_1(t) = Mag(\omega = 2) * 3 * sin(2t + Phase(\omega = 2)) = 2.06sin(2t - 1.379)$$

Bode Plots

$\Rightarrow$ Plots of mag(FRF) and phase(FRF) vs ω

THUS, Bode plots show the system response to every possible sinusoidal input!!!!

$\Rightarrow$ Normally draw mag(FRF) and phase(FRF) on separate plots, but for the same range of ω

$\Rightarrow$ These are semi-log plots, with ω axis = $\log_{10}$ scale = horizontal

1. **Frequency:** The horizontal axis in the Bode plots is ω, plotted on $\log_{10}$ axis
   (a) Frequency units may be *Hertz* (Hz) $\equiv$ cycles per second , or rad/sec, or even degrees/sec
   (b) Regardless of units, we define a **Decade** $\equiv$ any factor of 10 range of ω

Example

$$\begin{aligned} \textbf{One decade} &= 1 \leq \omega \leq 10 \\ \textbf{One decade} &= 23 \leq \omega \leq 230 \\ \textbf{Two decades} &= 0.18 \leq \omega \leq 18 \\ \textbf{Four decades} &= 0.0053 \leq \omega \leq 53 \end{aligned}$$

On semi-log scale, every decade is the exact same distance along the horizontal axis!

   (c) Typical Bode plots have 3-4 decades, but can be more or less (depends on actual system)

Examples:

Human hearing	$\approx 20 \leq \omega \leq 20,000Hz$	Three decades
Sound amplification	$\approx 20 \leq \omega \leq 20,000Hz$	Three decades
Aircraft control	$\approx 0.01 \leq \omega \leq 100rad/sec$	Four decades
Human Voice	$\approx 300 \leq \omega \leq 3,400Hz$	2^+ decades

   (d) Actual range of ω used in Bode plots is normally limited to a few decades where changes in magnitude and phase are most important (next sections); no point in plotting entire range of ω

2. Magnitude is plotted in units of *decibels* $\equiv 20\log_{10}(\text{Magnitude})$

Example

Magnitude	**Magnitude in db**
0	$20\log_{10}(0) = -\infty \; db$
1	$20\log_{10}(1) \;= 0 \; db$
10	$20\log_{10}(10) \;= 20 \; db$
100	$20\log_{10}(100) \;= 40 \; db$
1000	$20\log_{10}(1000) \;= 60 \; db$
...	
0.1	$20\log_{10}(0.1) \;= -20 \; db$
0.01	$20\log_{10}(0.01) \;= -40 \; db$
0.001	$20\log_{10}(0.001) \;= -60 \; db$
...	
$0 < mag < 1$	$-\infty < \cdots < 0 \; db$
$1 < mag < \infty$	$0 < \cdots < \infty \; db$

We define an **Amplifier** when $output > input$; Thus, an amplifier has $0 <$ [mag(FRF)] db $< \infty$

We define an **Attenuator** when $output < input$; Thus, an attenuator has $0 >$ [mag(FRF)] db $> -\infty$

Note that a **10-X increase in output = 20 db ; a 10-X decrease in output = -20 db**

$$e.g., \; \frac{output}{input} = 5 \;\rightarrow\; 13.98 \; db \qquad \frac{output}{input} = 50 \;\rightarrow\; 33.98 \; db$$

$$\frac{output}{input} = 0.5 \;\rightarrow\; -6.02 \; db \qquad \frac{output}{input} = 0.05 \;\rightarrow\; -26.02 \; db$$

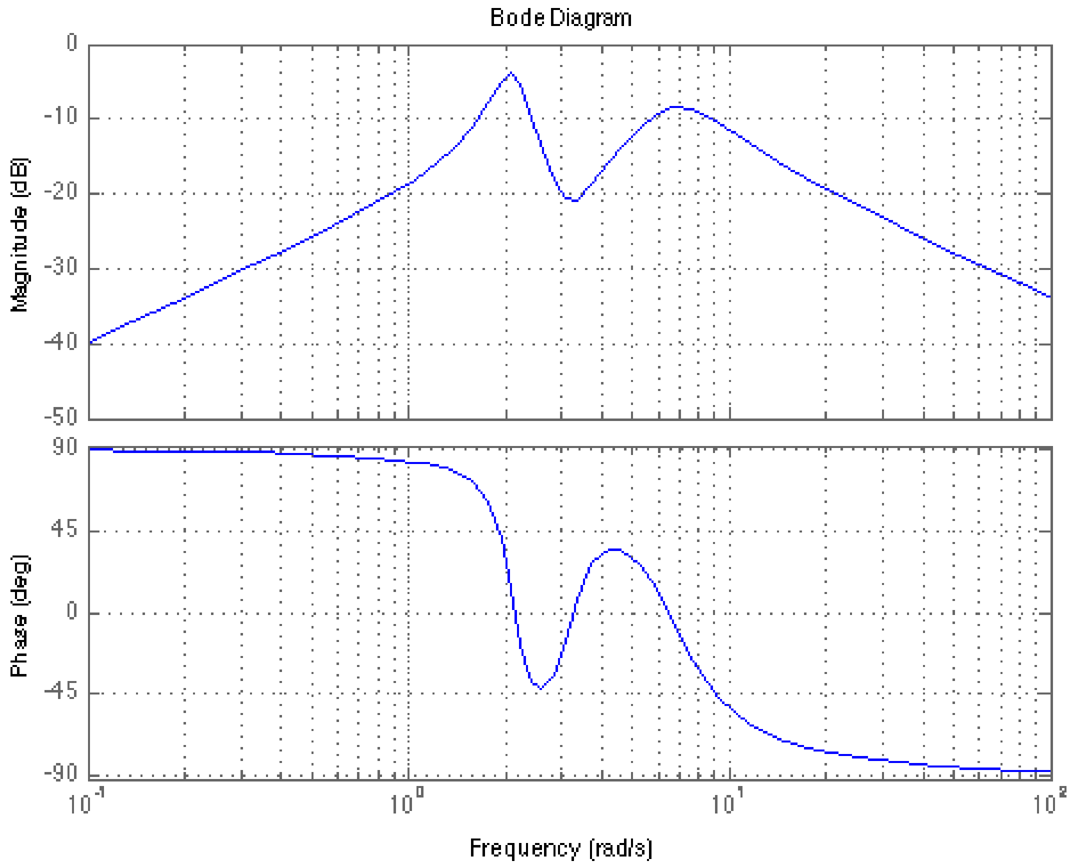
3. Phase is usually plotted in degrees

   (a) Use same range of ω as magnitude plot, but on separate set of axes

   (b) From a purely mathematical standpoint, phase is not unique; it repeats every $360°$. A phase angle of $+180°$ is mathematically identical to phase of $-180°$; phase of $+60°$ is identical to phase of $-300°$ or a phase of $420°$; etc. Any phase angle ϕ is equivalent mathematically to a phase of $\phi \pm 360°$, $\phi \pm 720°$, etc.

   (c) For our *physical* interpretation, we normally use $(-)$ phase angle $0 \leq \phi \leq 360°$. We use negative phase because it corresponds to *lag*, i.e., if phase is negative, the system output comes *after* the input, which mimics the physical reality that the system can only respond after the input has been applied. If we have $(+)$ phase angle (called *lead*), it implies that the output comes before the input, which is not correct physically even though it is equivalent mathematically.

   (d) Although we don't cover *control system design* in this course, many controllers utilize *lead* phase angles, i.e., the control system is designed to deliver an input to the system in advance of an expected behavior in order to counteract (control) the response of the system. The physical response of a real system always lags the input, but the control input in these cases is calculated to precede (*lead*) other undesirable inputs, and creates a *leading* response in the system that counteracts the undesirable response to the other inputs.



The Bode plots shown above correspond to an LTI ODE system. The plots tell us all of the following:

(a) The system never acts as an amplifier

(b) The input sinusoidal frequency that produces the highest output/input amplitude is approximately $\omega = 2$; at this frequency, the output amplitude is approximately 70% of the input amplitude

(c) For a fixed input magnitude, the magnitude of the response for input frequency $\omega = 0.1\ rad/s$ is approximately 1/10 the magnitude of the response for input frequency $\omega = 0.9\ rad/s$

(d) Similarly, the output magnitude for input frequencies $\omega \approx 1.6\ rad/s$, $2.4\ rad/s$, $4.7\ rad/s$, $9\ rad/s$ are all about 10 times the output magnitude for frequencies $\omega = 0.3\ rad/s$, $63\ rad/s$

(e) There are four input frequencies for which the output magnitude is approximately 1/10 the input magnitude

(f) There is one input frequency for which the output magnitude is approximately 1/100 the input magnitude

(g) There are three input frequencies for which the output is completely in phase with the input

(h) If the system input is $u(t) = 8sin(0.88t) + 15sin(21t)$, the steady-state output is approximately

$$y(t) = 0.8sin(0.88t + 80/360 * 2\pi) + 1.5sin(21t - 70/360 * 2\pi)$$

4. Construction of Bode plots

Suppose that A and B are two complex functions:

Converting to (db),

$$20\log_{10}\left[Mag(A*B)\right] = 20\log_{10}\left[Mag(A)\right] + 20\log_{10}\left[Mag(B)\right]$$
$$20\log_{10}\left[Mag(\frac{A}{B})\right] = 20\log_{10}\left[Mag(A)\right] - 20\log_{10}\left[Mag(B)\right] \tag{1}$$

Similarly,

$$phase(A*B) = phase(A) + phase(B) \quad , \quad phase(\frac{A}{B}) = phase(A) - phase(B) \tag{2}$$

Example: Eqs. (1) and (2)

Suppose $A = 5 + i3$ and $B = 4 - i2$.
Thus,

$$A*B = (5+i3)*(4-i2) = 26 + i2$$
$$\frac{A}{B} = \frac{5+i3}{4-i2} * \frac{4+i2}{4+i2} = \frac{14+i22}{20} = 0.7 + i1.1$$

Find the magnitudes in (db) of both A and B:

$$20\log_{10}\left[Mag(A)\right] = 20\log_{10}(\sqrt{5^2+3^2}) \approx 15.315\ db$$
$$20\log_{10}\left[Mag(B)\right] = 20log_{10}(\sqrt{4^2+2^2}) \approx 13.01\ db$$

Now illustrate the use of Eqs. (1):

$$20\log_{10}\left[Mag(A*B)\right] = 20\log_{10}\left[Mag(26+i2)\right] \approx 28.325\ db \approx (15.315 + 13.01)db$$
$$20\log_{10}\left[Mag(\frac{A}{B})\right] = 20\log_{10}\left[Mag(0.7+i1.1)\right] \approx 2.305\ db \approx (15.315 - 13.01)db$$

Similarly, find the $phase(A)$ and $phase(B)$ in degrees:

$$phase(A) = \tan^{-1}\frac{Im(A)}{Re(A)} = \tan_{-1}\frac{3}{5} \approx 31^\circ$$

$$phase(B) = \tan^{-1}\frac{Im(B)}{Re(B)} = \tan_{-1}\frac{-2}{4} \approx -26.6^\circ$$

Now illustrate the use of Eqs. (2):

$$phase(A*B) = \tan^{-1}\frac{Im(A*B)}{Re(A*B)} = \tan_{-1}\frac{2}{26} \approx 4.4^\circ \approx (31 - 26.6)^\circ$$

$$phase(\frac{A}{B}) = \tan^{-1}\frac{Im(\frac{A}{B})}{Re(\frac{A}{B})} = \tan_{-1}\frac{1.1}{0.7} \approx 57.6^\circ \approx (31 + 26.6)^\circ$$

We now proceed to use Eqs. (1) and (2) to construct and interpret Bode plots, piece by piece, by factoring the numerator and denominator of the FRF into independent complex functions corresponding to the poles and the zeros:

$$FRF = TF_{(s=i\omega)} = \frac{Num(s=iw)}{Den(s=iw)} = \frac{K_N(i\omega - z_1)(i\omega - z_2)(\dots)(i\omega - z_m)}{K_D(i\omega - p_1)(i\omega - p_2)(\dots)(i\omega - p_n)} \tag{3}$$

where z_i are the zeros, and p_i are the poles, of the TF. K_N and K_D are constants that correspond to the coefficients of the highest powers of s in the numerator and denominator, respectively.

We plot magnitude and phase of Eq. (3) using Eqs. (1) and (2)

In other words, rather than trying to calculate the overall magnitude and phase of Eq. (3), we use Eqs. (1) and (2) to get magnitude and phase in Eq. (3) *piece-by-piece.*

We already know that there are only four types of terms in either the numerator or denominator of the FRF:

(a) Constant (i.e., K_N or K_D)

(b) Zero or pole = 0

(c) Zero or pole = Re number

(d) Zero or pole = complex conjugate pair

Next, we will see that these four pieces are easy to plot individually, and thus it is simple to construct the Bode plots of any FRF

Piece-by-piece construction of Bode plots

We consider each possible piece of the factored Eq. (3) separately:

(a) Constant (K_N or K_D):

These constants must be real, since our original LTI ODE has only real coefficients. If our system is stable, they must also be positive. Thus, the magnitude is given by

$$20\log_{10}(K)\ db = \textit{constant} \text{ for all } \omega = \begin{pmatrix} < 0 & K_N < 1 \text{ or } K_D > 1 \\ > 0 & K_N > 1 \text{ or } K_D < 1 \end{pmatrix}$$

Similarly, the phase is given by

$$\textit{phase}(K) = \tan^{-1}\frac{0}{K} = 0^\circ$$

Notes:

i. Looking at Eq. (1), note that K_N terms add to *mag*, and K_D terms subtract from *mag*

ii. If $K = 1$, $mag(K) = 0\ db$, i.e., it has no effect on the overall $mag(FRF)$ plot for Eq. (3)

iii. K terms have no effect on the *phase* plot of Eq. (3)

iv. If $K > 1$, then $mag(K) > 0\ db$. Alternatively, if $K < 1$, then $mag(K) < 0\ db$.
Thus, $K_N > 1$ or $K_D < 1$ produces a straight, horizontal line above 0 *db*; in the *mag* plot for Eq. (3), the effect is to raise the entire plot.
Alternatively, $K_N < 1$ or $K_D > 1$ produces a straight, horizontal line below 0 *db*; in the *mag* plot for Eq. (3), the effect is to lower the entire plot.

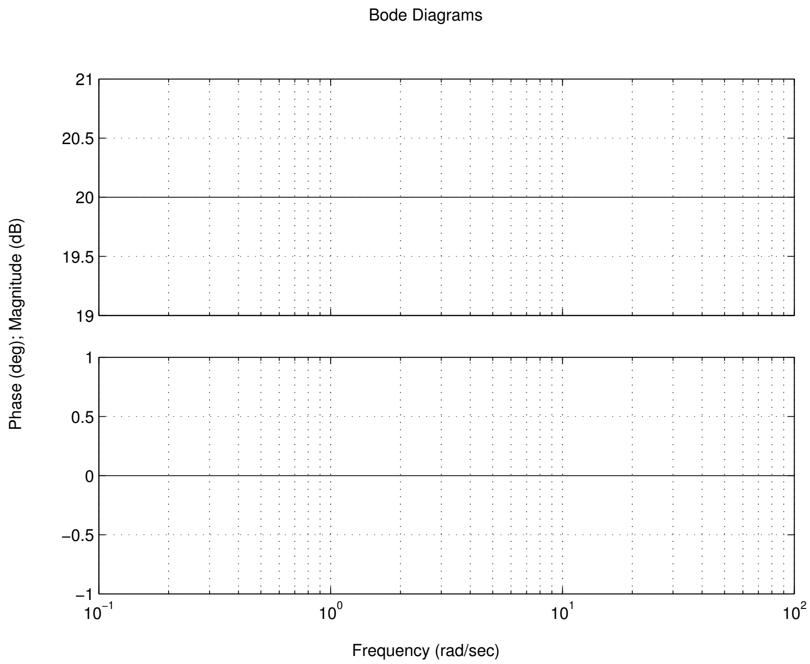


Figure 1:Bode plot for $K_N = 10$

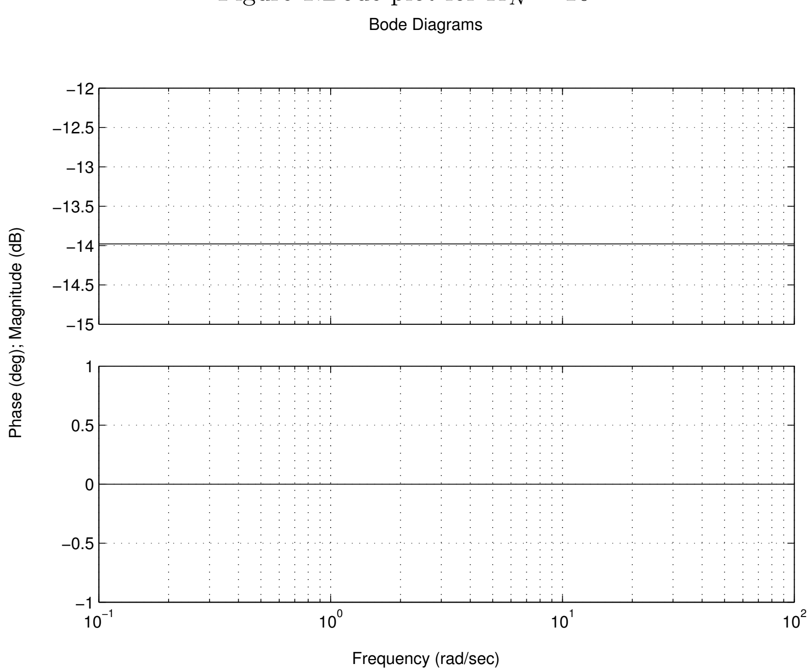


Figure 2: Bode plot for $K_D = 5$ (note: $20\log_{10}(5) = 13.98$)

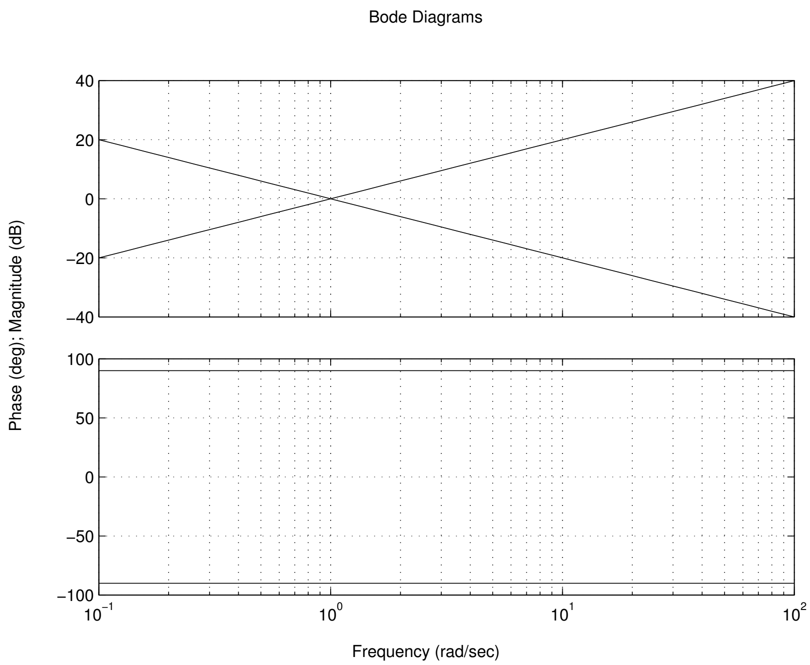


Figure 3:Bode plot for $z = 0$ (positive slope, $phase = 90°$) and $p = 0$ (negative slope, $phase = -90°$)

(b) $z_i = 0$ or $p_i = 0$:

These terms appear in the factorization of Eq. (3) as $i\omega$. The magnitude is

$$20\log_{10}\left[Mag(i\omega)\right]\ db = 20\log_{10}(\omega)\ db = \begin{pmatrix} 0\ db & \text{at } \omega = 1 \\ 20\ db/decade & \text{slope of straight line} \end{pmatrix}$$

Since $\log_{10}(10) = 1$, we see that this term produces a change of 20 db in mag for every decade change in ω - i.e., the mag changes by 20 db as ω changes by a multiple of 10. For $\omega = 1$, $mag = 0\ db$. Thus, the magnitude plot is a straight line with a slope of 20 $db/decade$ and an intercept of $mag = 0\ db$ at $\omega = 1$.

The *phase* is given by

$$\phi = \tan^{-1}(\frac{\omega}{0}) = 90° \qquad \text{for all } \omega$$

(c) z_i or $p_i =$ real $= \lambda$

Drawing the plot is easier if we re-write these terms as:

$$(i\omega - \lambda) = \lambda * (\frac{i\omega}{\lambda} - 1)$$

The magnitude in *db* is given by

$$20\log_{10}\left[Mag(\lambda * (\frac{i\omega}{\lambda} - 1))\right] = 20log_{10}\lambda + 20\log_{10}\left[Mag(\frac{i\omega}{\lambda} - 1)\right] \tag{4}$$

The first term on the right-hand side of Eq. (4) is a constant for all ω.

The second term varies with ω, as

$$20\log_{10}\left[Mag(\frac{i\omega}{\lambda} - 1)\right] = \begin{pmatrix} \approx 20\log_{10}(1) = 0 \ db & \omega << \lambda \\ 20log_{10}(\sqrt{2}) \approx 3 \ db & \omega = \lambda \\ \approx 20\log_{10}(\frac{\omega}{\lambda}) = 20 \ db/decade & \omega >> \lambda \end{pmatrix}$$

Putting the two terms together on the right-hand side of Eq. (4), we see

$$\begin{pmatrix} \text{For } \omega << \lambda & Mag \approx 20log_{10}\lambda = \text{constant} \\ \text{For } \omega >> \lambda & Mag \text{ changes by } 20db/decade \\ \text{For } \omega = \lambda & Mag \approx 3 + 20log_{10}\lambda \end{pmatrix} \tag{5}$$

Thus, the *mag* plot is

horizontal for $\omega << \lambda$

has a slope of 20 $db/decade$ for $\omega >> \lambda$

The value $\omega = \lambda$ is called the *corner frequency* because it is in the center of the transition of the *mag* from horizontal to a slope of 20 $db/decade$.

A close approximation to the *mag* plot may be constructed by hand, using two straight-line *asymptotes*:

(1) for $0 \leq \omega \leq \lambda$, the horizontal line at $20log_{10}\lambda$

(2) beyond $\omega = \lambda$, a line with slope 20 $db/decade$

The actual *mag* plot curves smoothly from (1) to (2) for about 1 decade centered around the corner frequency (it does not have the sharp corner defined by the intersection of lines (1) and (2). This curving transition can be approximated by hand, noting that it should pass 3 *db* above the intersection of these two asymptotes.

The phase is given by

$$\begin{pmatrix} \text{For } \omega << \lambda & \phi \approx \tan^{-1}(\frac{0}{1}) \approx 0^\circ \\ \text{For } \omega = \lambda & \phi = \tan^{-1}(\frac{1}{1}) = 45^\circ \\ \text{For } \omega >> \lambda & \phi \approx \tan^{-1}(\frac{1}{0}) \approx 90^\circ \end{pmatrix} \tag{6}$$

The transition from 0° to 90° occurs across approximately 2 decades. Although the actual *phase* plot is curved, an approximation can be drawn by the straight line that connects 0° to 90° from $\omega = \frac{\lambda}{10}$ to $\omega = 10\lambda$. The actual plot will have smaller slopes near the two ends of this transition range, and a steeper slope in the center around $\omega = \lambda$.

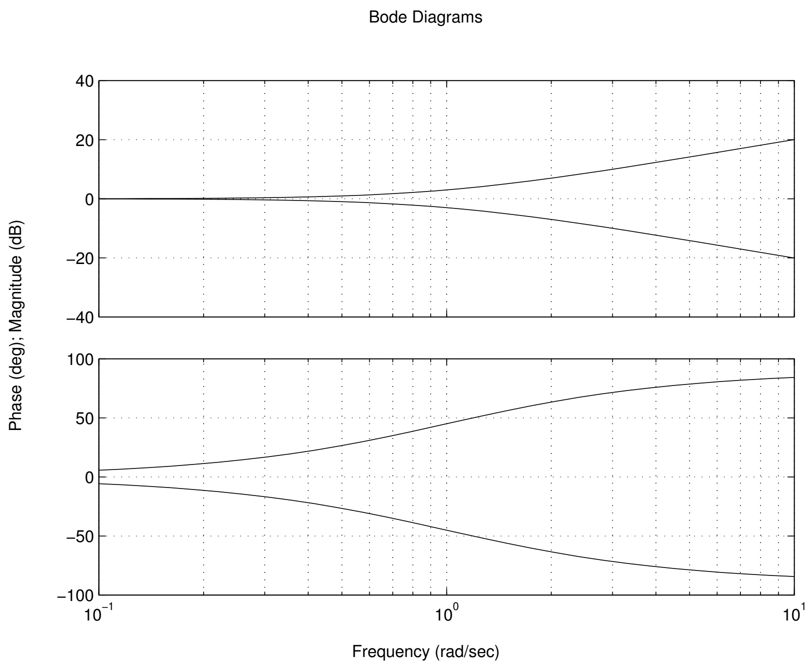


Figure 4:Bode plots for $TF = \frac{s+1}{1}$ and $TF = \frac{1}{s+1}$

(d) In the case of complex-conjugate zeros or poles, we combine the corresponding two terms into a single 2^{nd}-order factor in the numerator or denominator.

Let s_1 and s_2 be two complex conjugate roots given by $\sigma \pm i\gamma$:

$$(s - (\sigma + i\gamma))\,(s - (\sigma - i\gamma)) = s^2 - 2\sigma s + \sigma^2 + \gamma^2 = s^2 + 2\zeta\omega_n + \omega_n^2$$

$$= \omega_n^2 * (\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1)$$

In the FRF, with $s = i\omega$, this becomes

$$\omega_n^2 * \left[1 - (\frac{\omega}{\omega_n})^2 + i\frac{2\zeta\omega}{\omega_n}\right]$$

The *mag* is given by

$$20\log_{10}(\omega_n)^2 + 20\log_{10}\left[1 - (\frac{\omega}{\omega_n})^2 + i\frac{2\zeta\omega}{\omega_n}\right] \tag{7}$$

The first term in Eq. (7) is a constant. The second term behaves as follows:

$$\begin{aligned} \text{For } \omega << \omega_n \qquad & mag \approx 20\log_{10}(1) = 0\ db \\ \text{For } \omega >> \omega_n \qquad & mag \approx 20\log_{10}(\frac{\omega}{\omega_n})^2 \approx 40\ db/decade \\ \text{For } \omega = \omega_n \qquad & mag \approx 20\log_{10}(2\zeta)\ db \end{aligned} \tag{8}$$

Thus, the *mag* is a horizontal constant line at $20log_{10}(\omega_n^2)$ for $\omega << \omega_n$; it is a straight line with a slope of 40 $db/decade$ for $\omega >> \omega_n$; and in between, it transitions according the value of ζ.

Here, we define the *corner frequency* as $\omega = \omega_n$. The decade centered around this point contains the transition of the *mag* plot from a horizontal line (constant) to a sloped line (slope of 40 $db/decade$). Conceptually, this is very similar to the case of the single, real, nonzero root (shown in the previous section, (c)).

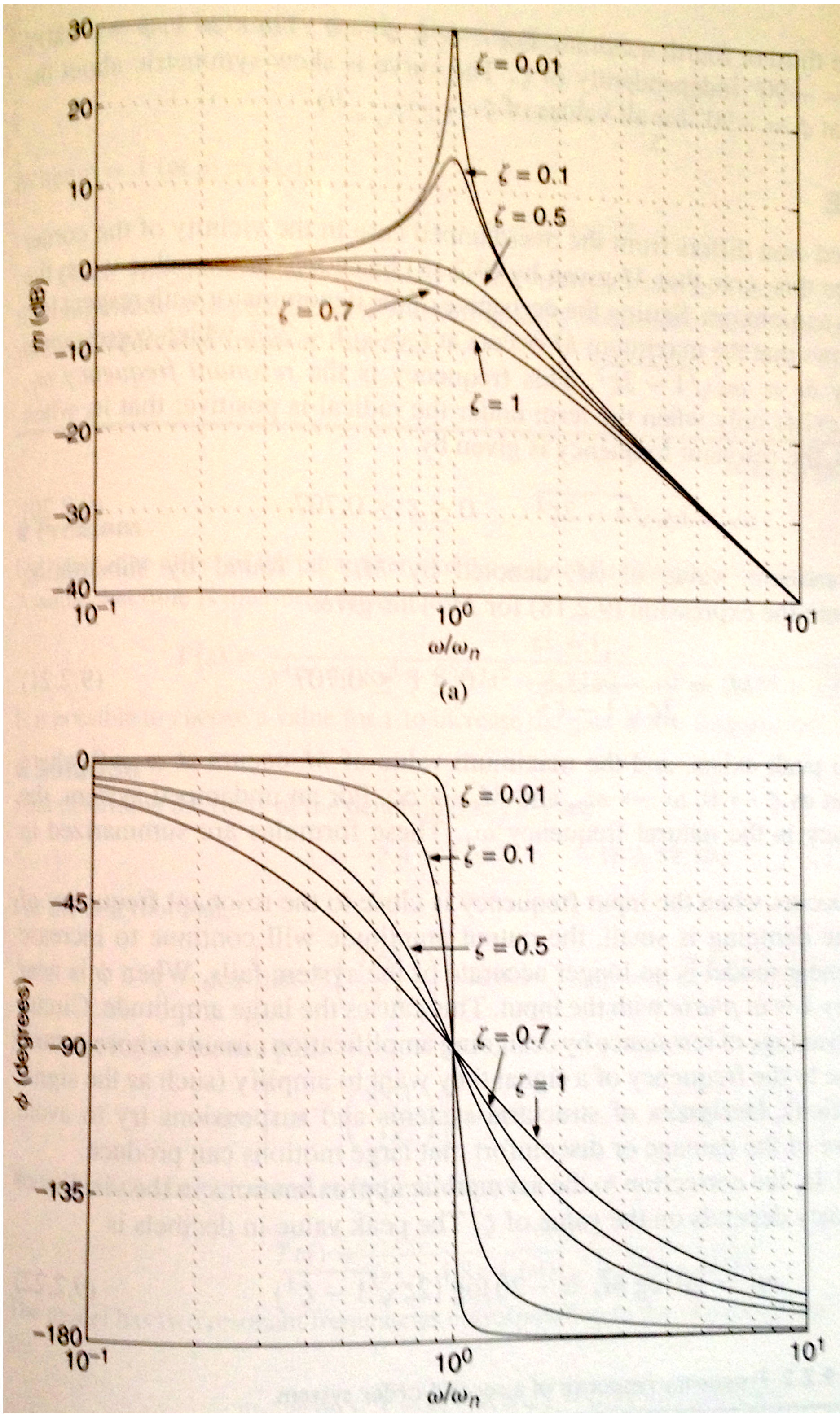
However, there is potentially an enormous difference when complex-conjugate roots are present. Figs. (5) and (6) show magnitude plots for complex-conjugate pairs having several different values of ζ (Fig. 5

is for zeros, Fig. 6 is for poles). Looking at, for example, Fig. (5) for zeros, we see that the *mag* may not simply transition from horizontal to an upward slope of 40 *db/decade*; instead, it may first *decrease* below the horizontal line, then curve back upwards to meet the 40 *db/decade* slope. Similarly, in Fig. (6) for poles, the *mag* may first increase above the horizontal line, then curve back downward to meet the slope of -40 *db/decade*.

It turns out that this behavior appears for any value of $0 < \zeta < \frac{\sqrt(2)}{2}$.

The *phase* is given by

$$
\begin{aligned}
\text{For } \omega << \omega_n & \qquad \phi \approx \tan^{-1}(\frac{0}{1}) \approx 0^\circ \\
\text{For } \omega >> \omega_n & \qquad \phi \approx \tan^{-1}(\frac{0}{-1}) \approx 180^\circ \\
\text{For } \omega = \omega_n & \qquad \phi \approx \tan^{-1}(\frac{1}{0}) \approx 90^\circ \; db
\end{aligned} \tag{9}
$$

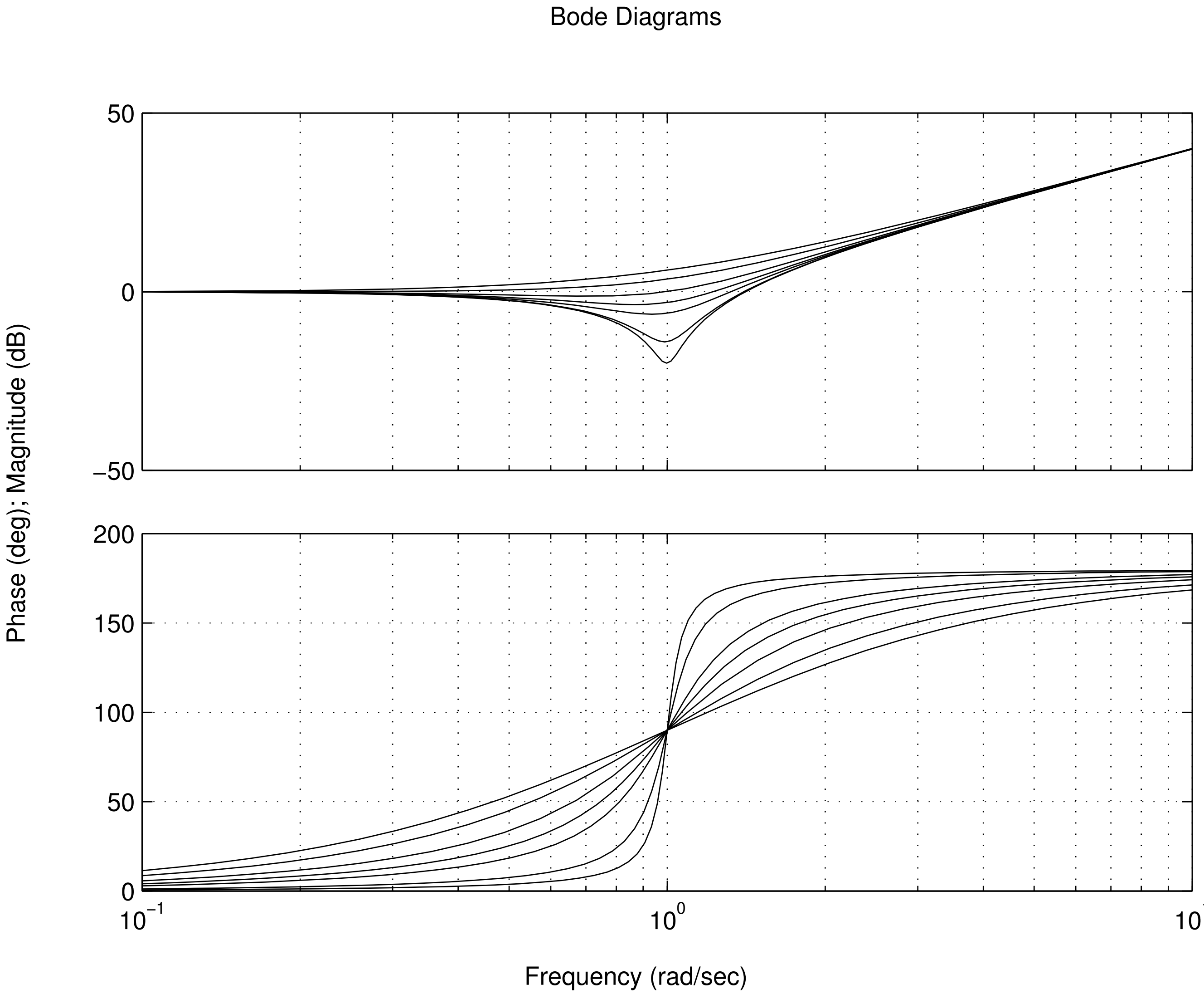


Bode plot for complex-conjugate pair of poles:

$$TF = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Note that magnitude transition $\approx$ one decade; phase transition $\approx$ two decades.

In magnitude plot, the 'hump' occurs for $0 \leq \zeta < 0.707$



Bode plot for complex-conjugate pair of zeroes:

$$TF = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{1}$$

This is the mirror-image of the pole plot.

(e) Sketching Bode plots:

i. Determine the zeros and the poles of the transfer function, then sketch the corresponding *piece* of the overall FRF using the guides shown in parts (a)-(d) above. The sketch of the complete FRF is the graphical combination of the sketches of each piece.

ii. Pieces due to zeros are *added* ; pieces due to poles are *subtracted* (see Eqs. (1) and (2)).

iii. For convenience, all constant terms can be combined into a single constant multiplier (they don't have to be calculated and sketched independently), i.e., the FRF from Eq. (3) can be re-written

$$FRF = TF_{(s=i\omega)} = \frac{Num(s=iw)}{Den(s=iw)} = \frac{K_N(i\omega - z_1)(i\omega - z_2)(\dots)(i\omega - z_m)}{K_D(i\omega - p_1)(i\omega - p_2)(\dots)(i\omega - p_n)} \qquad (3)$$

$$\equiv K * \frac{(\frac{i\omega}{z_1} - 1)(\frac{i\omega}{z_2} - 1)(\dots)(\frac{i\omega}{z_m} - 1)}{(\frac{i\omega}{p_1} - 1)(\frac{i\omega}{p_2} - 1)(\dots)(\frac{i\omega}{p_n} - 1)}$$

$$\text{where } K = \frac{K_n * z_1 * z_2 * \cdots * z_m}{K_d * p_1 * p_2 * \cdots * p_n}$$

iv. If this is done, then all of the individual zero and pole piece plots (for nonzero poles and zeros) have $mag = 0$ for low ω below their corner frequencies, making them easier to draw and easier to sum

v. Since every zero results in a slope of $+20$ *db/decade* beyond its corner frequency, and every pole results in a slope of -20 *db/decade* beyond its corner frequency, the slope of any Bode plot beyond the highest corner frequency is equal to 20 *db/decade* $*$ (number of zeros)-(number of poles).

In other words, the high-frequency slope of any Bode plot is an integer multiple of 20 *db/decade*, where the integer is the difference between the number of zeros and the number of poles.

vi. Similarly, since every zero results in a *phase* of $+90°$ beyond its corner frequency, and every pole results in a *phase* of $-90°$ beyond its corner frequency, the final *phase* angle of any Bode plot is equal to $+90° *$ (number of zeros)-(number of poles).

vii. When sketching the curving transition around corner frequencies, a range of about 1 decade is sufficient for *mag*, and a range of about 2 decades is required for *phase*

Some comments

i. Bode plots theoretically give solution for ANY input
   A. Fourier Transform (or Fast Fourier Transform) converts $f(t)$ into sum of sinusoidal terms
   B. Bode plots contain solution for any sinusoidal input within frequency range
   C. Linearity enables calculation of response to any input independent of any other input
   D. $\Rightarrow$ Use Fourier to convert $f(t)$ into sinusoids; use Bode to get solution for each sinusoid; sum these solutions to get solution to $f(t)$

ii. Since many $f(t)$'s have at least some randomness, combination of Fourier and Bode enables prediction of strengths/vulnerabilities in system response

iii. From design standpoint, simplistically,
   A. Bode plots for response to uncontrollable inputs (*the world*) should have low amplitudes around expected frequencies
   B. Bode plots for response to command inputs should have high amplitudes around expected frequencies

iv. For many systems, Bode plots can be created experimentally via *sin sweep* testing
   A. Devise a method for creating sinusoidal input at constant frequency
   B. Run until steady-state achieved, then measure amplitude and phase shift of the outputs
   C. This creates one point of each Bode plot
   D. Repeat at additional frequencies until sufficient points on Bode plot to draw smooth curves

Example Test Questions from a given set of Bode plots:

i. If the input is $7sin(3t)$, what is the output?
ii. If the magnitude of the output is 12 at input frequency $\omega = 6rad/sec$, what is the input?
iii. For what frequencies (if any) is the phase angle $90°$?
iv. For what input frequency is the output magnitude greatest?
v. For what range of frequencies (if any) does the system behave like an amplifier?
vi. For what input frequency(ies) is the output amplitude equal to $\frac{1}{2}$ the input amplitude?
vii. For what range of phase angles (if any) does the system behave like an amplifier?
viii. What is the (number of poles)-(number of zeros)?
ix. What are the approximate values of any poles and/or zeros?
x. Are there any poles or zeros at zero?