I Introduction

Real World ⇒ Mathematical Model

Engineers almost always create mathematical models to represent physical systems, and then, they may perform a very wide variety of calculations based on these mathematical models, in order to analyze the behavior of the actual physical system *without the need to build a prototype or experiment.*

Cheaper, Easier, Faster, Unlimited Prototypes

The engineering motivation for using mathematical models is quite simple: they reduce (or, in some cases, entirely eliminate) the need to build actual physical prototypes and experiments. An accurate mathematical model, properly evaluated, can enable analysis of physical behavior across a wide variety of conditions that may be difficult or impossible to duplicate with an experiment. They also may allow design ideas and concepts to be tested and/or optimized in a far more thorough, and far more efficient, manner than is possible using physical prototypes and experiments.

MUST HAVE: Accurate Model + Accurate and Useful Analysis

There are two critically-important steps to this approach: (1) creating a sufficiently accurate mathematical model of the physical system, and (2) accurately performing meaningful calculations using the model.

MAE 340 is devoted to both of these two critical steps

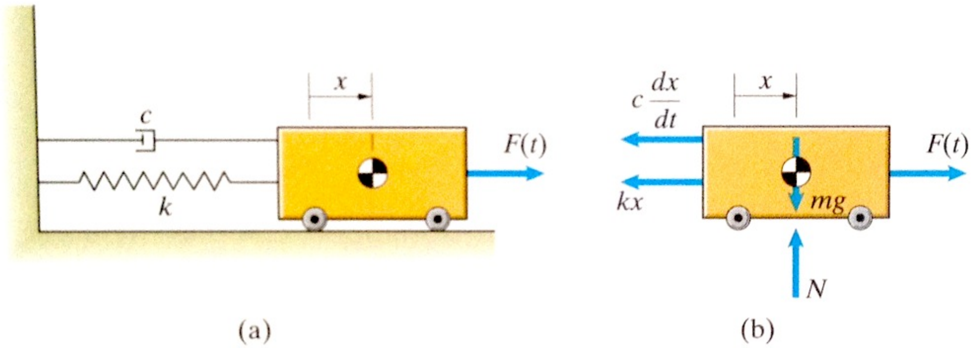

Figure 1:
(a) a simple vibrating system consisting of a cart with mass m, rolling on a frictionless surface, connected to a stationary wall by massless spring k and massless dashpot c, subject to the external forcing $F(t)$
(b) the free-body diagram of the system

Writing $\sum \vec{F} = m\vec{a}$ in the horizontal direction, we obtain: $m\ddot{x} + c\dot{x} + kx = F(t)$

Thus, the *mathematical model* of the system's *physical* behavior, using fundamental principles of dynamics, is a 2nd-order, linear, time-invariant, ordinary differential equation (LTI ODE).

The solution of this LTI ODE may be used to determine the actual physical behavior of the system. Moreover, we may quickly evaluate any changes to m, c, k, and/or $F(t)$, without need to build and test a new physical system, thus greatly enhancing our ability to design a better system, and/or to evaluate how the system will perform under different $F(t)$.

II Types of Models

(1) In engineering practice, there is generally a trade-off between the overall physical accuracy of a model and our ability to perform meaningful calculations based on it. Most of the time, the more complicated the model, the less accurate and/or less efficient is the analysis.

(2) Most physical systems are capable of exhibiting very complex behavior, especially under extreme conditions of operation. This complex behavior may require a complicated mathematical model that is difficult to solve and/or analyze accurately.

(3) On the other hand, simpler models may often be analyzed very accurately and efficiently.

(4) **Engineers often must make a choice: Either use a simple (imperfect) model that lends itself to perfect and efficient analysis, or use a complex (more accurate) model that is subject to inefficient and/or inaccurate analysis.**

(5) In this section of the course, we focus on linear, time-invariant, ordinary differential equations (LTI ODE's). LTI ODE's are the simplest form of dynamic model, and as mechanical and aerospace engineers, virtually every system we design and/or analyze exhibits dynamic behavior. In the next several weeks, you will learn to very accurately and efficiently analyze an LTI ODE model. Later, we will learn methods for constructing LTI ODE's for complex engineering systems.

(6) LTI ODE's are widely used throughout engineering practice to model systems of many different physical behaviors and components. Ability to understand the information available from accurate LTI ODE's is critical in modern engineering design and analysis.

III General Form of LTI ODE

The general form of a single-variable **linear, time-invariant, ordinary** differential equation (LTI ODE) may be written

$$a_n \frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + ... + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t) \tag{1}$$

where

$$t = \text{the } \textit{independent} \text{ variable, typically, time} \tag{2}$$
$$x = \text{the } \textit{dependent} \text{ variable, typically, a physical quantity} \tag{3}$$
$$a_0, a_1, ..., a_n = \text{constants} \tag{4}$$
$$f(t) = \text{"external" or "applied" forcing, independent of } x \tag{5}$$
$$n = \text{the } \textit{order} \text{ of the system} \tag{6}$$

For example, applying the definitions in Eqs (1-6) to the system shown in Figure (1), we find the following:

$$\begin{aligned} t &= \text{time} \\ x &= \text{the horizontal position of the cart} \\ a_0 &= \text{k} \\ a_1 &= c \\ a_2 &= \text{m} \\ f(t) &= F(t) \\ n &= 2 \end{aligned}$$

It is extremely important to understand the physical implications of this class of models, since we know that using a simpler model is generally less physically accurate. Consider the physical and mathematical importance of each adjective in the LTI ODE name:

(1) *Linear* refers to the fact that the dependent variable, and all of its derivatives, appear in the LTI ODE in a linear form, i.e., they do not appear as multiplicative products of each other, exponents, trigonometric functions, etc. Most real physical systems are, at best, linear only for limited ranges of operation. For example, in the system of Figure (1), large enough motions of the cart would exceed the physical dimensions of the spring and/or the dashpot, which would completely change the behavior of the system. Generally speaking, all physical systems are nonlinear if the range of operation is large enough, and some physical systems are always nonlinear even for the smallest ranges of operation. Therefore, a linear model should never be considered adequate for all ranges of operation of any physical system. From a purely physical point of view, linear models are not strictly accurate. But linear models may be perfectly analyzed mathematically in many crucial ways that are impossible or difficult with nonlinear models. Moreover, all linear models are represented by a single general form, so all of the analysis and design tools based on that model are immediately applicable to any physical system modeled linearly, whereas there are an infinity of different ways in which a model may be nonlinear, thus eliminating our ability to generalize the analysis of such models. Therefore, from the engineer's mathematical standpoint, linear models are far preferable. And, it turns out that there are many practical situations in which the behavior of a system may be adequately well-represented by a linear model. This is especially true in many well-designed engineered systems, in which the *design itself* maintains an operational range which is limited to linear behavior. For example, if the system shown in Figure (1) is properly designed, it may only operate in a range well within the physical limits of the spring and dashpot, and therefore a linear model may be extremely accurate. Note that there is an inherent *chicken vs egg* (which comes first?) aspect to many engineering design approaches. The designer proceeds to approach the problem by assuming linear behavior, although at the beginning of the process, there's no way to know; then, by creating a good design (using the linear model), the resulting system does indeed exhibit linear behavior. The design is based on an assumption of linearity; but actual linearity is ensured only by the result of the design.

(2) *Time-invariant* means that the system model equation doesn't change with time. In practical terms, this means that the values of the coefficients a_0, a_1, ... , a_n are constant, and that the terms representing dependent variables and/or external forces continue to be represented in the original equation. It does NOT mean that the dependent variable or external forcing is constant; it only means that the form of the equation is constant. In the system of Figure (1), the model is time-invariant if m, c, and k are constants. $F(t)$ is, by definition, a function of time, but its representation in the model never changes. Similarly, the cart position x is also a function of time, but the representation of x, $\dot{x}$, and $\ddot{x}$ are fixed in the equation itself. Thus, the form of the equation does not change with time. As with linearity, physical time-invariance of a given system may be only true within a range of operation, or not at all - for example, many mechanical and aerospace vehicles burn large quantities of fuel, resulting in substantial changes to m over time. But even systems which are not strictly time-invariant may often be modeled accurately within specific windows of time by a time-invariant model. Time-invariant models lend themselves mathematically to a far greater range of accurate and efficient solutions than do time-varying models.

(3) *Ordinary* refers to the presence of only one *independent* variable (usually, time; in MAE 340, it will always be time, and thus, we will always be solving for the behavior of our systems as a function of time). In practice, most physical quantities also vary spatially, so that spatial coordinates become additional independent variables. Thus, partial derivatives with respect to both time and space then appear in the differential equation model of the system. But ordinary differential equation models lend themselves to a far greater number of accurate and efficient solution techniques than do partial differential equations. From a physical standpoint, using an ordinary differential equation model implies that we may "lump" the dependent-variable physical quantities into single specific values that do not vary with a spatial coordinate - for example, in Figure (1), all of the mass is located precisely in the cart (treated as a single point located at x); all of the stiffness is represented by a single spring which undergoes a single deformation (also x); all of the damping is represented by a single dashpot that undergoes a single velocity ($\dot{x}$). There is no spatial distribution of the physical parameters. For this reason, ODE models are also often referred to as "lumped-parameter" models.

(4) *Order* of a single-variable LTI ODE model refers to the greatest number of derivatives of the dependent variable. In Eq. (1), and most other LTI ODE's, the order is n. But there are systems whose physical properties do not depend on $x(t)$ - for example, if the spring is removed in Figure (1), the corresponding model is $m\ddot{x} + c\dot{x} = F(t)$. This model is 1^{st}-order, easily seen by defining a new variable $z(t) \equiv \dot{x}(t)$. Therefore, although we often choose to use a physical quantity like x in our model, it's important to recognize that the mathematical order of the model does not necessarily correspond to the highest derivative of x; it refers to the highest number of derivatives necessary mathematically.

In each simplification noted above (linear vs nonlinear, time-invariant vs time-varying, and ordinary vs partial differential equations), we are choosing the simpler mathematical form of the model over one which is physically more accurate. The reason we do this is that analysis and design of the simpler mathematical forms is much quicker and easier, and also, more accurate. Thus, we utilize a model that sacrifices some physical perfection, but offers a perfect solution. The alternative is to start with a better physical model, but since these are generally (not always) much harder to solve perfectly, we then must accept an approximate solution to the better model. Moreover, obtaining this approximate solution may be much more time-consuming and potentially less accurate. Which approach is better? In fact, this question remains open-ended in many real-world applications. Certainly the last few decades have seen enormous advances in our ability to obtain accurate solutions of more complicated models, but there are still many physically accurate models that are very difficult to solve. Perhaps more important in the engineering sense, many very powerful design techniques are based on LTI ODE models.

IV Creating an LTI ODE

In courses such as dynamics, thermodynamics, fluid mechanics, mechanics of materials, etc, you learned so-called "first principles" of physics. For example, in dynamics, the fundamental first principle is $\sum \vec{F} = m\vec{a}$. But in dynamics and other introductory courses, you considered only idealized, simple examples, based only on the physical topic of the course. Real-world engineering systems typically contain many different types of physical subsystems, each of which is far more complex than the simple examples studied in introductory courses. In addition, these subsystems are often inter-dependent, and cannot be modeled and analyzed independently of each other. For example, pumps are found as components of many other systems. The power source for a pump may be electrical or mechanical; the output may be fluid flow; and the internal workings of the pump itself is a dynamic mechanical system. Later in the course, we will study methods for combining multiple, physically-different subsystems, into single, large-order models. For now, we proceed to analyze such models.

V General Form of Solution of LTI ODE

The solution of an LTI ODE consists of two distinct parts, namely, the *homogeneous* solution and the *particular* solution. Thus, in general, we write that the solution of Eq. (1) is

$$x(t) = x_h(t) + x_p(t) \tag{7}$$

There are critically important distinctions between these two components, in both a mathematical and a physical sense:

(a) The *homogeneous* solution, denoted by $x_h(t)$, is found by ignoring $f(t)$ (i.e., by setting $f(t) = 0$ in Eq. (1), even when it may not actually be zero). Considering Eq. (1) with $f(t) = 0$, we see that any solution for this case is equivalent (mathematically) to the seemingly-trivial condition that $0 = 0$. Since $0 = 0$ can be added to Eq. (1) without affecting the equivalency of the two sides of the equation, even when $f(t) \neq 0$, we see that, mathematically, the homogeneous solution can be a permanent component of the solution of Eq. (1) regardless of the value of $f(t)$.
This result is extremely important, because in Eq. (1), $f(t)$ represents forces or disturbances that are independent of the system - meaning forces that do not arise from the system itself - e.g., $f(t)$ can represent the input of the world around the system, which may include natural forces, as well as man-made inputs intended to control the system for some useful purpose. In general, such independent forces change frequently. However, the homogeneous solution is always present, regardless of the actual current value of $f(t)$, because $0 = 0$ can always be added to any other solution of Eq. (1). The physical meaning of this mathematical observation is equally critical: namely, the homogeneous solution represents physical behavior that may always be present, regardless of whatever inputs (natural or man-made) may be applied to the system.
The homogeneous solution $x_h(t)$ results in $0 = 0$ in Eq. (1), but this does not mean that $x_h(t)$ itself must be zero. In fact, there may be many different non-zero mathematical functions for $x_h(t)$ that each result, independently, in $0 = 0$ in Eq. (1). In addition, since multiplication of zero by any nonzero number is still equal to zero, each of these functions may be multiplied by any arbitrary multiplier, and it will still satisfy $0 = 0$ in Eq. (1). Thus, we see that the complete homogeneous solution is composed of a linear combination (i.e., a simple sum) of all of the various functions, each multiplied by its own multiplier. And since the homogeneous solution represents physical behavior that may always be possible regardless of the actual, current input (natural or man-made), any understanding of the physical behavior of the system is impossible without the homogeneous solution.

(b) The *particular* solution, denoted by $x_p(t)$, is any function which satisfies Eq. (1) when $f(t) \neq 0$. Since $f(t)$ represents independent inputs to the system, natural and/or man-made, it may change frequently, resulting in changing $x_p(t)$. In theory and in practice, there is infinite variability in $f(t)$, and thus, infinite variability in $x_p(t)$. However, as we will see, there are certain specific mathematical functions for $f(t)$ that may be used to represent virtually all behavior that is of practical interest, and for those specific $f(t)$, there are powerful tools readily available for analyzing the particular solution of the LTI ODE.

VI Systems of coupled LTI ODE's

First, consider a set of algebraic equations in multiple unknowns, such as

$$z_1 + 3z_2 = 4 \tag{8}$$

$$z_1 - z_2 = 0 \tag{9}$$

These equations are *coupled*, by which we mean that we cannot solve independently for z_1 or z_2 using just one of these equations. We must instead use both equations simultaneously in order to obtain a solution. The general approach to solving such a set of equations is to use back substitution to eliminate all of the unknowns until a single equation in one unknown remains. That equation is then solved for the one unknown, and the substitutions are used to find the other unknowns. For example,

$$\text{From Eq. (9):} \quad z_1 = z_2 \tag{10}$$

$$\text{Substitute into Eq. (8):} \quad z_2 + 3z_2 = 4 \ \ ; \quad z_2 = 1 \tag{11}$$

$$\text{Substitute } z_2 \text{ back into Eq. (10):} \quad z_1 = z_2 = 1$$

Here we see that we needed two algebraic equations in order to obtain the values of the two algebraic unknowns, z_1 and z_2.

We now consider a system of *coupled* LTI ODE's. Similar to algebraic problems in which we have multiple algebraic unknowns, requiring multiple algebraic equations in order to obtain a solution, we will encounter many examples that contain multiple *dependent variables*, requiring multiple LTI ODE's for solution. Such equation systems commonly arise in applications where multiple connected subsystems form a single functional unit, and each subsystem is modeled by appropriate physical laws such as $f = ma$. We will see many such examples when we begin to model complex physical systems.

Similar to the idea of using substitution in algebraic equations to eliminate unknowns until we are left with a single equation in a single unknown, it is possible to use back substitution in a set of LTI ODE's to ultimately obtain a single LTI ODE in a single dependent variable. This equation would then have an identical form to Eq. (1). However, the amount of work required is much greater than in an algebraic problem, because it is not nearly as simple to eliminate the dependent variables in LTI ODE's as it is to eliminate the algebraic variables in algebraic equations. The extra complexity arises from the presence of not only the dependent variables themselves, but also, their derivatives.

For example, consider the two coupled LTI ODE's given by

$$\begin{aligned}
a_{1,n}\frac{d^n x_1(t)}{dt^n} + a_{1,n-1}\frac{d^{n-1}x_1(t)}{dt^{n-1}} + ... + a_{1,1}\frac{dx_1(t)}{dt} + a_{1,0}x_1(t) \\
+b_{1,n-1}\frac{d^{n-1}x_2(t)}{dt^{n-1}} + ... + b_{1,1}\frac{dx_2(t)}{dt} + b_{1,0}x_2(t) = f_1(t) \qquad (12)\\
b_{2,m}\frac{d^m x_2(t)}{dt^m} + b_{2,m-1}\frac{d^{m-1}x_2(t)}{dt^{m-1}} + ... + b_{2,1}\frac{dx_2(t)}{dt} + b_{2,0}x_2(t) \\
+a_{2,m-1}\frac{d^{m-1}x_1(t)}{dt^{m-1}} + ... + a_{2,1}\frac{dx_1(t)}{dt} + a_{2,0}x_1(t) = f_2(t) \qquad (13)
\end{aligned}$$

where

$$\begin{aligned}
x_1\ ,\ x_2 &= \text{ dependent variables}\\
n\ ,\ m &= \text{ integers representing the highest derivatives of } x_1, x_2 \text{ respectively}
\end{aligned}$$

Eqs. (12-13) have two dependent variables, $x_1(t)$ and $x_2(t)$. Following the algebraic example, we could begin by solving Eq. (13) for x_1 to obtain:

$$\begin{aligned}
x_1(t) = \frac{-1}{a_{2,0}}\{b_{2,m}\frac{d^m x_2(t)}{dt^m} + b_{2,m-1}\frac{d^{m-1}x_2(t)}{dt^{m-1}} + ... + b_{2,1}\frac{dx_2(t)}{dt} + b_{2,0}x_2(t) \\
+a_{2,m-1}\frac{d^{m-1}x_1(t)}{dt^{m-1}} + ... + a_{2,1}\frac{dx_1(t)}{dt} - f_2(t)\} \qquad (14)
\end{aligned}$$

When we performed this same logical step in the algebraic problem (Eq. (10)), we found z_1 solely in terms of z_2 so that we could then substitute back into Eq. (8) and eliminate z_1 entirely. If we mimic that step by substituting Eq. (14) into Eq. (12), we eliminate x_1 but do not eliminate any of its derivatives. Thus we see that, unlike the algebraic problem, the presence of the *derivatives* of the dependent variables in a coupled set of LTI ODE's means that we cannot easily get to a single equation in a single dependent variable.

However, it is not impossible to complete this process and ultimately obtain a single LTI ODE. An example is given in a Homework Assignment. The resulting single LTI ODE will be of order equal to the sum of the orders of all of the ODE's in the coupled set. Thus, in the example defined by Eqs. (12)-(13), the resulting single LTI ODE would have order $n + m$.

Although it is possible to convert a coupled set of LTI ODE's into a single, large-order LTI ODE in one dependent variable, eliminating the others, in practice, we now have much more efficient tools available for handling this situation. Because the situation arises in so many practical applications of coupled physical subsystems, those tools are covered in detail in the section of the course on *State Space*.