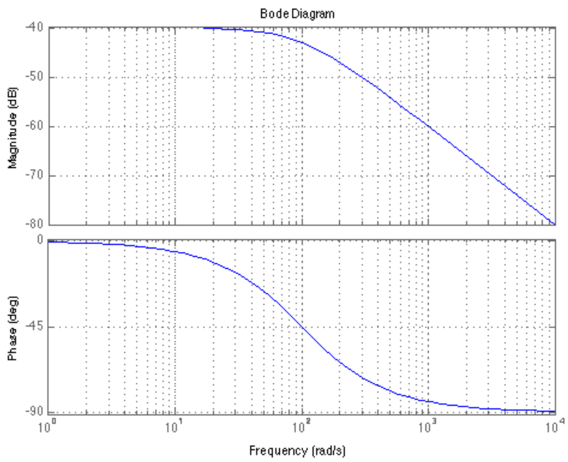
In each case below, find a transfer function that behaves as described, then use MATLAB to construct the Bode plots of your transfer function to verify that it is indeed as described

The solutions shown are not the only ones possible, but are typically the simplest TF format.

1. A low-pass filter, with bandwidth $0 \leq \omega \leq 100$ and acting as an amplifier for $0 \leq \omega \leq 1000$

   Solution: To achieve the shape of a low-pass filter, we need a plot with a constant magnitude for low ω, followed by a corner to turn the magnitude plot slope downward. A transfer function with a single real pole in the denominator has the necessary shape:

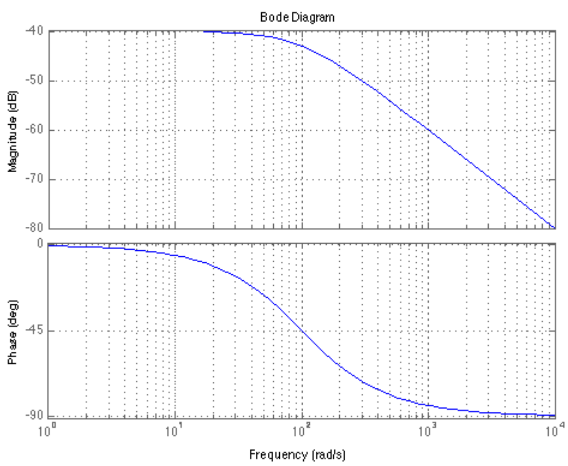


The plot shows the transfer function

$$TF = \frac{1}{s+100}$$

which has the correct bandwidth ($0 \leq \omega \leq 100$), since the drop in magnitude at the pole is 3 db as required. But this TF does not have the correct amplitude range. We need the system to act as an amplifier for $0 \leq \omega \leq 1000$, which means we need a higher magnitude across the range $0 \leq \omega \leq 1000$.

Specifically, from inspection of the amplitude at $\omega = 1000$, we see that we need to raise the magnitude by 60 db.



To raise the magnitude equally across the board, we use a constant multiplier (K) in the TF.

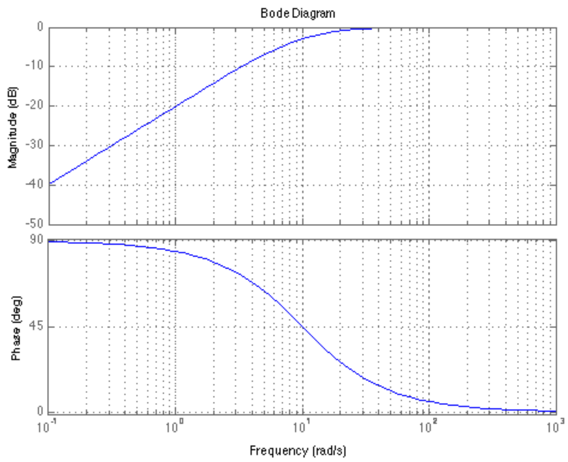
Since we need 60 db, and $20log_{10}(1000) = 60db$, we add a factor $K = 1000$ to the transfer function:

$$TF = \frac{1000}{s+100}$$

The Bode plots for this TF are shown at left, and satisfy both the bandwidth and amplifier requirements.

2. A high-pass filter, with bandwidth $10 \leq \omega \leq \infty$ and acting as an amplifier for $0.5 \leq \omega \leq \infty$

Solution: To achieve the shape of a high-pass filter, we need a magnitude sloping up from low frequencies, then flattening out to a constant magnitude for high frequencies greater than $\omega = 10$. A transfer function with a single real zero followed by a larger single real pole has the necessary shape.



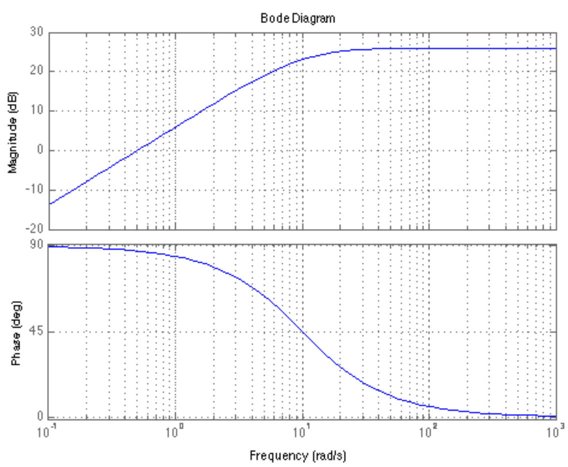
Since we want the high-pass range to begin at $\omega = 10$, we put the pole there:

$$TF = \frac{s+z}{s+10}$$

The location of the zero is somewhat arbitrary, but it does need to be located at a low-enough value of ω to allow the magnitude to rise sufficiently so that the low frequencies are not in the bandwidth. A simple choice is $z = 0$:

$$TF = \frac{s}{s+10}$$

which is shown in the plot to the left.



The plot has the correct bandwidth ($10 \leq \omega \leq \infty$), but does not have the correct amplitude range. We need the system to act as an amplifier for $0.5 \leq \omega \leq \infty$.

We can read the current amplitude at $\omega = 0.5$ as approximately (-26 db), which indicates that we should multiply our transfer function by $K = 10^{26/20} \approx 20$

$$TF = \frac{20s}{s+10}$$

The plot of this transfer function is shown to the left, and satisfies the amplification range.

Note that the amplitude correction can be calculated exactly (rather than estimating it from the plot, as done above). First, we find the FRF:

$$FRF = TF_{(s=i\omega)} = \frac{i\omega}{i\omega + 10} = \frac{\omega^2 + i10\omega}{100 + \omega^2}$$

We now evaluate the magnitude of the FRF for $\omega = 0.5$:

$$mag(FRF_{(\omega=0.5)}) = mag\left(\frac{\omega^2 + i10\omega}{100+\omega^2}\right)_{(\omega=0.5)} = \left.\frac{\sqrt{\omega^4 + 100\omega^2}}{(100+\omega^2)}\right|_{\omega=0.5} = 0.0499$$

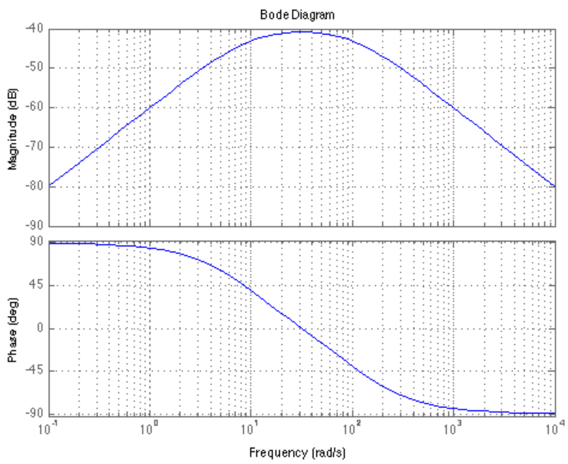
Since we want the actual magnitude to be 1 at $\omega = 0.5$, i.e., 0 db, we multiply the TF by $1/0.0499$:

$$TF = \frac{1}{0.0499} * \frac{s}{s+10} = \frac{20.025s}{s+10}$$

so the exact correction is 20.025, which is very close to what we estimated above by simply reading the plot.

3. A bandpass filter with BW $= 10 \le \omega \le 100$ and acting as an amplifier for $10 \le \omega \le 100$

Solution: To achieve the shape of a band-pass filter, we may use a transfer function with a single real zero, followed by a larger single real pole, followed by yet another single real pole.



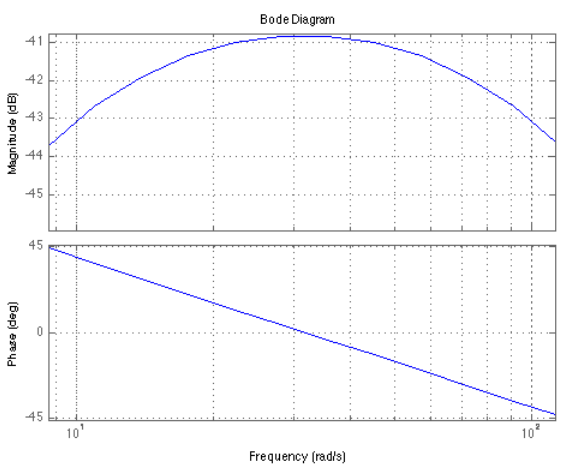
Since we want the band-pass range to begin at $\omega = 10$ and end at $\omega = 100$, we put the poles there:

$$TF = \frac{s+z}{(s+10)(s+100)}$$

The location of the zero is somewhat arbitrary, but it does need to be located at a low-enough value of ω to allow the magnitude to rise sufficiently so that the low frequencies are not in the bandwidth. A simple choice is $z = 0$:

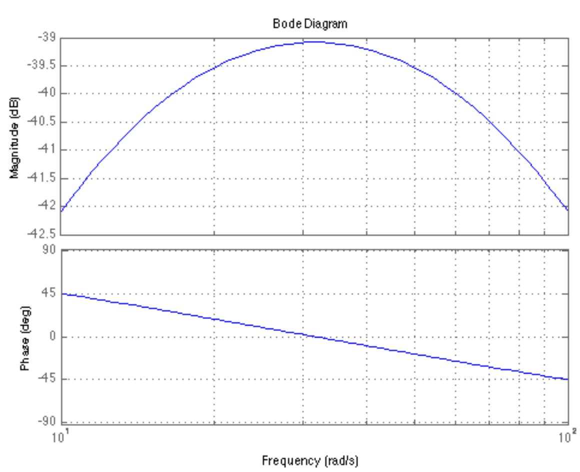
$$TF = \frac{s}{(s+10)(s+100)}$$

which is shown in the plot to the left.



A close-up of the intended bandwidth, shown at left, reveals that the transfer function shown above does not actually meet the bandwidth criterion. Here we can see that the peak magnitude is approximately -41 db, so the magnitude should be approximately -44 db at $\omega = 10$ and $\omega = 100$ (i.e., a -3 db change from the peak). We see that it is too high.

By moving the locations of the two poles toward the middle of the decade $10 \le \omega \le 100$, we also move their transition zones farther into the center of the decade, and thus we may steepen the slopes at the bandwidth limits, to lower values, such that we can meet the bandwidth criterion.



To keep the filter symmetric, we should move the poles equal distance along the semilog$_{10}$ ω-axis. This is accomplished by modifying the TF to

$$TF = \frac{s}{(s+a)(s+100*10/a)}$$

where $a =$ the location of the left-side pole. Through simple trial-and-error, $a = 13$ was found to work well:

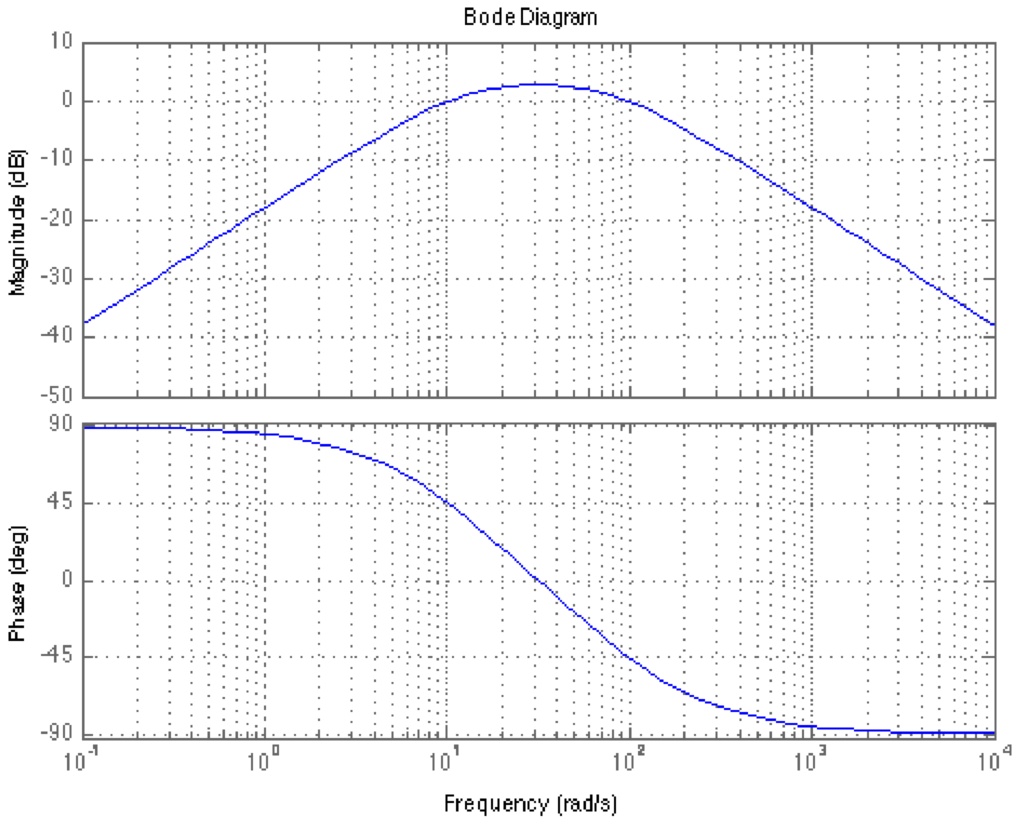
$$TF = \frac{s}{(s+13)(s+76.92)} \quad \text{(plotted at left)}$$

The peak magnitude is approximately -39 db and the magnitudes at $\omega = 10$ and $\omega = 100$ are approximately -42 db (i.e., -3 db from peak), so the bandwidth is now correct.

Since we need amplification from $\omega = 10$ to $\omega = 100$, and currently the magnitudes at those frequencies is approximately (-42 db), we need to multiply by $10^{42/20} = 125.9$:

$$TF = \frac{125.9s}{(s+13)(s+76.92)}$$

The bode plots are shown below and are seen to satisfy the bandwidth and amplification criteria.



4. A notch filter, with the filtered frequencies $1 \leq \omega \leq 10$

Solution: To achieve the shape of a notch filter, we may use a transfer function with a single real pole to begin the notch, followed by a larger single real zero to flatten the bottom of the notch, followed by another zero to send the magnitude back up into the bandwidth, followed by yet another single real pole to flatten the magnitude for high frequencies.

Since we want the notch range to begin at $\omega = 1$ and end at $\omega = 10$, we can try to put the poles there:

$$TF = \frac{(s+z_1)(s+z_2)}{(s+1)(s+10)}$$

Since the zeros will stop the downward slope of the magnitude, we don't want the first one to be too close to the first pole - otherwise, it may affect the bandwidth. Similarly, we don't want the second zero too close to the second pole. Let's try placing both of them exactly in the middle of the decade $1 \leq \omega \leq 10$ since this is as far away from the two poles as we can go.

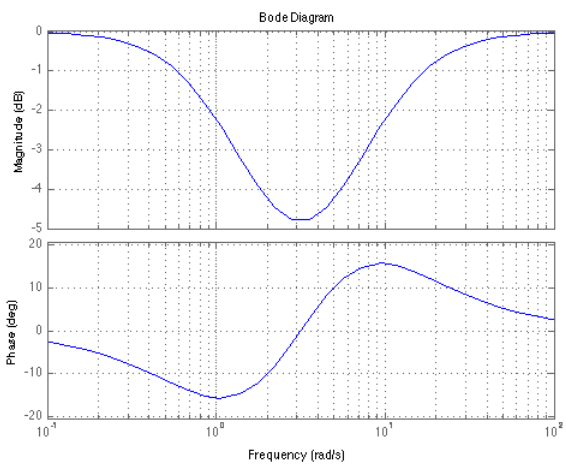
Note that the half-decade point (on the semilog$_{10}$ scale) in any decade $\omega_L \leq \omega \leq 10\omega_L$ is found as follows:

$$log_{10}(\omega_{half}) = \frac{log_{10}(\omega_L) + \log_{10}(10\omega_L)}{2} = \frac{2log_{10}(\omega_L) + 1}{2} = log_{10}(\omega_L) + \frac{1}{2}$$

$$\Rightarrow \omega_{half} = 10^{log_{10}(\omega_L)+0.5} = \sqrt{10}\omega_L$$

Thus, since our $\omega_L = 1$ (beginning of notch), the location of the half-decade is $\omega = \sqrt{10}$ and so we will define our notch filter as

$$TF = \frac{(s+\sqrt{10})^2}{(s+1)(s+10)} \qquad \text{(plotted just below and to the left)}$$

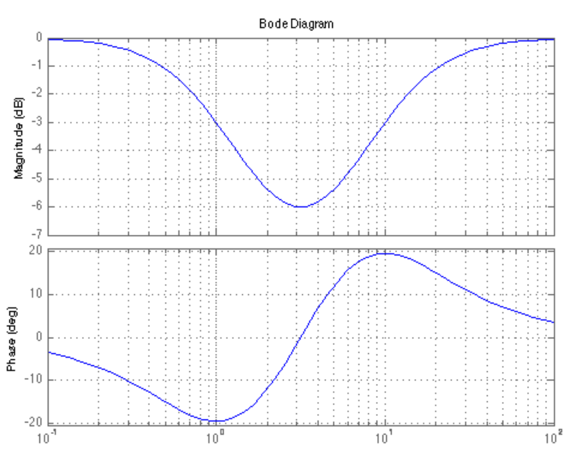


This plot does have the correct shape, but on closer inspection, we see that the bandwidth is too narrow. The zeroes are too close to the poles to allow the magnitude to drop to -3 db at $\omega = 1$, and to remain below (-3 db) at $\omega = 10$.

The simplest fix is to move the two poles out (i.e., spread them out from the zeroes a bit) so that the bandwidth is correct.

To keep the shape symmetric, we should move the two poles the same distance along the semilog$_{10}$ scale:

$$TF = \frac{(s+\sqrt{10})^2}{(s+a)(s+10/a)} \qquad a = \text{first pole}$$



After a little trial-and-error, a value of $a = 0.847$ appears to satisfy the bandwidth criterion (as seen in the plot at left), so our filter is defined as

$$TF = \frac{(s+\sqrt{10})^2}{(s+0.847)(s+10/0.847)}$$

$$= \frac{(s+\sqrt{10})^2}{(s+0.847)(s+11.806)}$$

The peak magnitude is 0 db and the bandwidth is defined by -3 db, so the notch is indeed $1 \leq \omega \leq 10$