Problem Statement

Find the unit step response for the 2nd-order system with the LTI ODE

$$\ddot{x} + 2\dot{x} + 12x = f(t)$$

Plot the step response, and use the plot to determine the following:

- Steady-state response
- Maximum overshoot
- Peak time
- 90% rise time
- 2% settling time

As the problem asks for the *unit* step response, it is assumed that the input $f(t)$ is the unit step input:

$$f_{step}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

The standard step response also assumes that all the initial conditions are zero: $x(0) = \dot{x}(0) = 0$.
This system has a natural frequency of $\omega_n = \sqrt{12} \approx 3.464$ rad/sec, and a damping ratio of $\zeta = 0.29$, so it is underdamped and we expect some amount of overshoot. The particular solution (also the steady-state response) is just

$$x_p(t) = x_{ss} = \frac{1}{\omega_n^2} = \frac{1}{12} \approx 0.08333$$

The solution procedure for the homogeneous solution will not be shown here, but it can be shown to be

$$x_h(t) = 0.08704\mathrm{e}^{-t} \sin\left(3.3166t + 4.4195\right)$$

Therefore the complete unit step response is given by

$$x(t) = 0.08704\mathrm{e}^{-t} \sin\left(3.3166t + 4.4195\right) + 0.08333$$

This solution is shown in Figure 1.

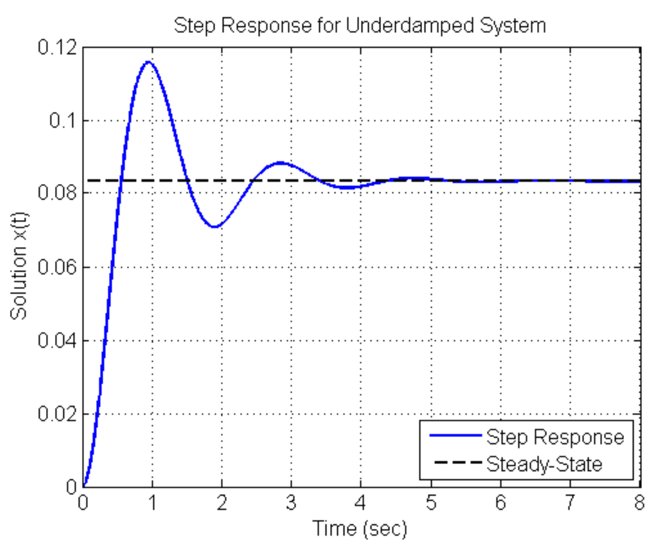

Figure 1: Unit Step Response

Steady-State Response

The steady-state response was already found analytically; it is equivalent to the particular solution for stable systems (where the homogeneous solution will eventually go to zero). We can verify from the plot, however, that the response oscillates around a value of approximately 0.083. This is indicated by a dashed horizontal line.

Overshoot and Peak Time

The maximum overshoot (or overshoot, or percent overshoot) for an underdamped system is the maximum amount by which the solution exceeds the steady-state value. This highest peak is always the first peak after the solution crosses the steady-state value. It is expressed as a percentage of the steady-state value:

$$\%\ \text{Overshoot} = \frac{x_{max} - x_{ss}}{x_{ss}} \times 100\%$$

In order to determine the maximum value as accurately as possible, it is useful to zoom in on the highest peak in Matlab using the "magnifying glass" tool at the top of the plot window, and then use data cursors to read values off of the plot. Figure 2 shows such a zoomed-in region.

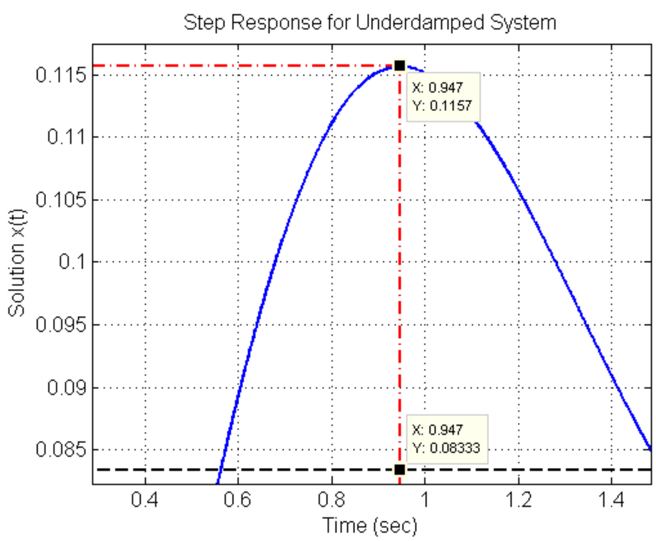

Figure 2: Step Response: Highest Peak Region

The maximum value is about $x_{max} = 0.1157$, so the overshoot is

$$\%\ \text{Overshoot} = \frac{0.1157 - 0.08333}{0.08333} \times 100\% = 38.8\%$$

The peak time t_p is just the time at which this maximum value occurs; from the plot we see that this is $t_p = 0.947$ seconds.

90% Rise Time

The 90% rise time is the time when the solution reaches 90% of the steady-state value (for the first time). As the steady-state value is 0.08333, the rise time is the time required to reach a level of $0.9x_{ss} = 0.075$. Figure 3 contains a zoomed-in version of the initial "rise" portion of the step response, which is useful for determining rise time.

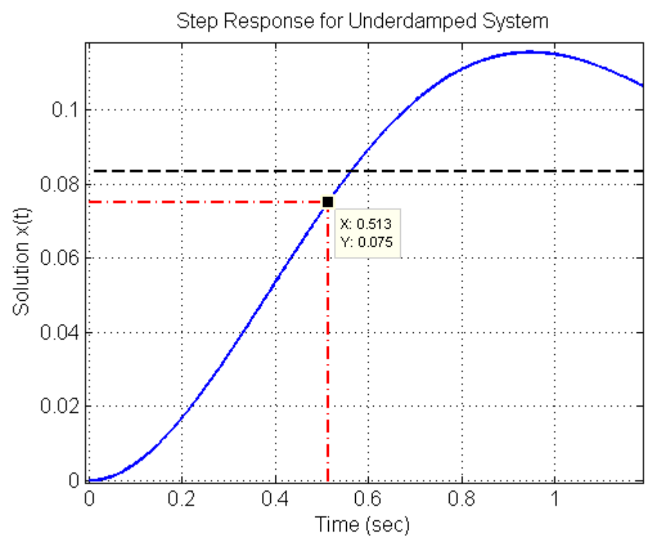

Figure 3: Step Response: Initial Rise Region

In the figure, we see that the response reaches a level of 0.075 at a time of $t_r = 0.513$ seconds.

2% Settling Time

The real part of the homogeneous solution roots is -1, so the settling time for the homogeneous solution is 4 seconds. For step response, however, we use a different definition which may result in slightly different values. The 2% settling time is the time after which the solution stays within ±2% of the steady-state response.

The ±2% boundaries for this problem are found by $0.98x_{ss} = 0.08167$ and $1.02x_{ss} = 0.085$. One can draw in (or just imagine) horizontal lines at these levels that the solution must stay between, as in Figure 4. After $t_r \approx 3.938$ seconds, the solution stays within ±2% of the steady-state, so this is the settling time. In this case it is close to the homogeneous solution settling time, but not identical.

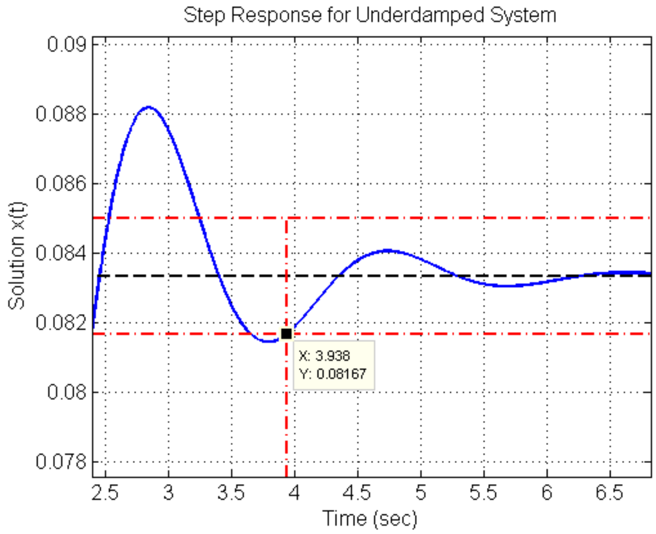

Figure 4: Step Response: Settling Region