Problem Statement

Find the complete solution for the system with the LTI ODE

$$\ddot{x} + 2\dot{x} + 10x = f(t), \qquad x(0) = \dot{x}(0) = 0$$

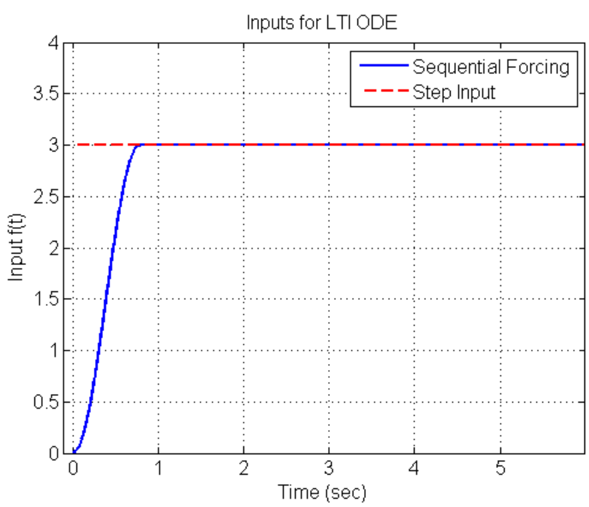
where the input $f(t)$ is given by the sequential forcing:

$$f(t) = \begin{cases} 0 & t < 0 \\ 1.5\left[1 - \cos(4t)\right] & 0 \le t < \frac{\pi}{4} \\ 3 & t \ge \frac{\pi}{4} \end{cases}$$

Compare the solution to the step response for a step input with an amplitude of 3:

$$f_{step}(t) = \begin{cases} 0 & t < 0 \\ 3 & t \ge 0 \end{cases}$$

Often it is undesirable (or just unrealistic) to have an input that jumps suddenly from 0 to some non-zero value without any elapsed time or transition region. One way to get around this is to have a ramp that starts at zero and goes to the desired value within some specified time. However, the ramp function will have sudden jumps (discontinuities) in its first derivative, which may also be unwanted. One possible alternative is to use a segment of a sinusoid to transition between two levels, as shown in this example. Figure 1 shows the input $f(t)$ and the corresponding step input $f_{step}(t)$ on the same axes for comparison, along with the MATLAB code used to create the inputs. To find the complete solution for the input $f(t)$, one must solve two initial value problems: first from $t = 0$




```
t1 = 0:0.005:pi/4;
t2 = pi/4:0.005:6;
t = [t1 t2];

f1 = 1.5*(1-cos(4*t1));
f2 = 3*ones(size(t2));
f = [f1 f2];
fstep = 3*ones(size(t));
```

Figure 1: Inputs for Sequential Forcing Example

to $t = \frac{\pi}{4}$, and then for $t \ge \frac{\pi}{4}$.

Part 1: $0 \le t < \frac{\pi}{4}$

The particular solution for this region is a linear combination of the input $f(t) = 1.5\left[1 - \cos(4t)\right]$ and its first two derivatives. Because this input is composed of a sum of two standard inputs (constant and sinusoidal), the particular solution will be a sum of the standard solutions for those two types of inputs:

$$x_{p1}(t) = C_0 + C_1 \sin(4t) + C_2 \cos(4t)$$

In order to find the constants C_0, C_1, and C_2, the first two derivatives of the $x_p(t)$ equation are required:

$$\dot{x}_{p1}(t) = 4C_1 \cos(4t) - 4C_2 \sin(4t)$$
$$\ddot{x}_{p1}(t) = -16C_1 \sin(4t) - 16C_2 \cos(4t)$$

Substitute these expressions into the characteristic equation and then divide the constant, sine, and cosine terms into separate equations:

$$[-16C_1\sin(4t) - 16C_2\cos(4t)] + 2\,[4C_1\cos(4t) - 4C_2\sin(4t)] + 10\,[C_0 + C_1\sin(4t) + C_2\cos(4t)] = 1.5 - 1.5\cos(4t)$$

$$\begin{aligned} 10C_0 &= 1.5 && (\text{constant, } 1) \\ -6C_1 - 8C_2 &= 0 && (\sin(4t)) \\ 8C_1 - 6C_2 &= -1.5 && (\cos(4t)) \end{aligned}$$

This system of equations can be written in matrix form and solved in MATLAB using the `inv` command.

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & -6 & -8 \\ 0 & 8 & -6 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \\ -1.5 \end{bmatrix}$$

The resulting values are $C_0 = 0.15$, $C_1 = -0.12$, and $C_2 = 0.09$. Therefore the particular solution for this segment (along with its first derivative, need later to apply initial conditions) is

$$\begin{aligned} x_{p1}(t) &= 0.15 - 0.12\sin(4t) + 0.09\cos(4t) \\ \dot{x}_{p1}(t) &= -0.48\cos(4t) - 0.36\sin(4t) \end{aligned}$$

Next, find the homogeneous solution for this segment. The characteristic equation for this system is

$$\lambda^2 + 2\lambda + 10 = 0$$

which has roots $\lambda_{1,2} = -1 \pm 3i$. Therefore the form of the homogeneous solution and its first derivative are given by

$$\begin{aligned} x_{h1}(t) &= A_1 \mathrm{e}^{-t}\sin(3t + \phi_1) \\ \dot{x}_{h1}(t) &= -A_1 \mathrm{e}^{-t}\sin(3t + \phi_1) + 3A_1\mathrm{e}^{-t}\cos(3t + \phi_1) \end{aligned}$$

Applying the initial conditions for this first segment gives

$$\begin{aligned} x_1(0) &= x_{h1}(0) + x_{p1}(0) \\ &= A_1\sin\phi_1 + 0.24 = 0 \\ \dot{x}_1(0) &= \dot{x}_{h1}(0) + \dot{x}_{p1}(0) \end{aligned}$$

$$\& = -A_1\sin\phi_1 + 3A_1\cos\phi_1 - 0.48 = 0$$

This equation can be written in matrix form and solved in MATLAB :

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} A_1\sin\phi_1 \\ A_1\cos\phi_1 \end{bmatrix} &= \begin{bmatrix} -0.24 \\ 0.48 \end{bmatrix} \\ \Rightarrow \qquad A_1\sin\phi_1 = -0.24, &\qquad A_1\cos\phi_1 = 0.08 \\ A_1 = \sqrt{(A_1\sin\phi_1)^2 + (A_1\cos\phi_1)} &= 0.25298 \\ \phi_1 = \tan^{-1}\left(\frac{A_1\sin\phi_1}{A_1\cos\phi_1}\right) &= -1.249 \text{ rad} \\ \phi_1 = \phi_1 + 2\pi &= 5.0341 \text{ rad} \end{aligned}$$

Note that 2π has been added to the phase without changing the solution because we prefer to have a positive phase value.

Given the calculated amplitude and phase, the homogeneous solution and complete solution for the first segment can now be written:

$$\begin{aligned} x_{h1}(t) &= 0.25298\mathrm{e}^{-t}\sin(3t + 5.0341) \\ x_1(t) &= x_{h1}(t) + x_{p1}(t) = 0.25298\mathrm{e}^{-t}\sin(3t + 5.0341) + 0.15 - 0.12\sin(4t) + 0.09\cos(4t) \end{aligned}$$

Part 2: $t > \frac{\pi}{4}$

The input $f(t)$ is a constant value of 3 for $t > \frac{\pi}{4}$, so the second part of the particular solution is given by

$$x_{p2}(t) = C_3$$

Substitution of this solution and its derivatives (all of which are zero) into the characteristic equation gives

$$\begin{aligned}(0) + 2(0) + 10\,(C_3) = 3\\ C_3 = \frac{3}{10} = 0.3\end{aligned}$$

Therefore the second part of the particular solution is

$$x_{p2}(t) = 0.3$$

The homogeneous solution for this segment has the same characteristic equation and same roots as for the previous segment, so the equation will also have the same form. However, the constants for the equation will be different.

$$\begin{aligned}x_{h2}(t) &= A_2 \mathrm{e}^{-t} \sin(3t + \phi_2)\\ \dot{x}_{h2}(t) &= -A_2 \mathrm{e}^{-t} \sin(3t + \phi_2) + 3A_2 \mathrm{e}^{-t} \cos(3t + \phi_2)\end{aligned}$$

For this segment of the solution, the initial conditions at $t = \frac{\pi}{4}$ are found by calculating the final values of the complete solution $x_1(t)$ and its derivative at this time. The complete solution and its derivative for the first segment are

$$\begin{aligned}x_1(t) &= 0.25298\mathrm{e}^{-t} \sin(3t + 5.0341) + 0.15 - 0.12 \sin(4t) + 0.09 \cos(4t)\\ \dot{x}_1(t) &= -0.25298\mathrm{e}^{-t} \sin(3t + 5.0341) + 0.75895\mathrm{e}^{-t} \cos(3t + 5.0341) - 0.48 \cos(4t) - 0.36 \sin(4t)\end{aligned}$$

At $t = \frac{\pi}{4}$, the new "initial" conditions are therefore

$$\begin{aligned}x_2\left(\frac{\pi}{4}\right) &= x_1\left(\frac{\pi}{4}\right)\\ &= 0.25298\mathrm{e}^{-\frac{\pi}{4}} \sin\left(\frac{3\pi}{4} + 5.0341\right) + 0.15 - 0.12 \sin(\pi) + 0.09 \cos(\pi) = 0.16317\\ \dot{x}_2\left(\frac{\pi}{4}\right) &= \dot{x}_1\left(\frac{\pi}{4}\right)\\ &= -0.25298\mathrm{e}^{-\frac{\pi}{4}} \sin\left(\frac{3\pi}{4} + 5.0341\right) + 0.75895\mathrm{e}^{-\frac{\pi}{4}} \cos\left(\frac{3\pi}{4} + 5.0341\right) - 0.48 \cos(\pi) - 0.36 \sin(\pi) = 0.53158\end{aligned}$$

To apply the initial conditions, one must evaluate $x_{h2}(t)$, $x_{p2}(t)$, and their derivatives at $t = \frac{\pi}{4}$ rather than at $t = 0$:

$$\begin{aligned}x_2\left(\frac{\pi}{4}\right) &= x_{h2}\left(\frac{\pi}{4}\right) + x_{p2}\left(\frac{\pi}{4}\right)\\ &= A_2 \mathrm{e}^{-\frac{\pi}{4}} \sin\left(\frac{3\pi}{4} + \phi_2\right) + 0.3 = 0.16317\\ \dot{x}_2\left(\frac{\pi}{4}\right) &= \dot{x}_{h2}\left(\frac{\pi}{4}\right) + \dot{x}_{p2}\left(\frac{\pi}{4}\right)\\ &= -A_2 \mathrm{e}^{-\frac{\pi}{4}} \sin\left(\frac{3\pi}{4} + \phi_2\right) + 3A_2 \mathrm{e}^{-\frac{\pi}{4}} \cos\left(\frac{3\pi}{4} + \phi_2\right) + 0 = 0.53158\end{aligned}$$

To solve for the unknowns A_2 and ϕ_2 in this case, it is helpful to use a pair of trigonometric identities to simplify this system of equations:

$$\begin{aligned}\sin(\theta + \phi) &= \sin\theta \cos\phi + \cos\theta \sin\phi\\ \cos(\theta + \phi) &= \cos\theta \cos\phi - \sin\theta \sin\phi\end{aligned}$$

By using the identities, it is possible to write the initial condition equations in terms of the unknowns $A_2 \sin\phi_2$ and $A_2 \cos\phi_2$, and then format the system as a matrix equation.

$$A_2 e^{-\frac{\pi}{4}}\left[\sin\left(\frac{3\pi}{4}\right)\cos\phi_2 + \cos\left(\frac{3\pi}{4}\right)\sin\phi_2\right] = -0.13683$$
$$\Rightarrow -0.3224A_2\sin\phi_2 + 0.3224A_2\cos\phi_2 = -0.13683$$
$$A_2 e^{-\frac{\pi}{4}}\left\{-\left[\sin\left(\frac{3\pi}{4}\right)\cos\phi_2 + \cos\left(\frac{3\pi}{4}\right)\sin\phi_2\right] + 3\left[\cos\left(\frac{3\pi}{4}\right)\cos\phi_2 - \sin\left(\frac{3\pi}{4}\right)\sin\phi_2\right]\right\} = 0.53158$$
$$\Rightarrow -0.64479A_2\sin\phi_2 - 1.2896A_2\cos\phi_2 = 0.53158$$

$$\begin{bmatrix} -0.3224 & 0.3224 \\ -0.64479 & -1.2896 \end{bmatrix}\begin{bmatrix} A_2\sin\phi_2 \\ A_2\cos\phi_2 \end{bmatrix} = \begin{bmatrix} -0.13683 \\ 0.53158 \end{bmatrix}$$
$$\Rightarrow \quad A_2\sin\phi_2 = 0.0081413, \qquad A_2\cos\phi_2 = -0.41628$$
$$A_2 = \sqrt{(A_2\sin\phi_2)^2 + (A_2\cos\phi_2)} = 0.41636$$
$$\phi_2 = \tan^{-1}\left(\frac{A_2\sin\phi_2}{A_2\cos\phi_2}\right) = 3.122 \text{ rad}$$

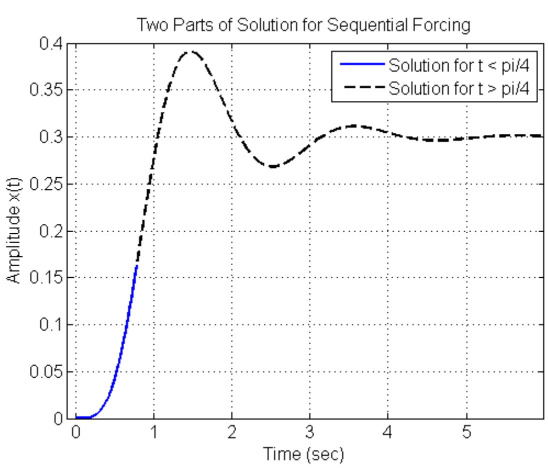
The second segments of the homogeneous solution and complete solution are therefore

$$x_{h2}(t) = 0.41636e^{-t}\sin(3t + 3.122)$$
$$x_2(t) = x_{h2}(t) + x_{p2}(t) = 0.41636e^{-t}\sin(3t + 3.122) + 0.3$$

One can combine the two segments to give a piecewise definition of the complete solution:

$$x(t) = \begin{cases} 0 & t < 0 \\ 0.25298e^{-t}\sin(3t + 5.0341) + 0.15 - 0.12\sin(4t) + 0.09\cos(4t) & 0 \le t < \frac{\pi}{4} \\ 0.41636e^{-t}\sin(3t + 3.122) + 0.3 & t \ge \frac{\pi}{4} \end{cases}$$

Figure 2 shows the complete solution for $0 \le t < 6$, with the two segments of the solution shown in different colors.




```
x1 = A1*exp(s*t1).*sin(w*t1+phi1)...
    + C0 + C1*sin(4*t) + C2*cos(4*t);
x2 = A2*exp(s*t2).*sin(w*t2+phi2) + C3;

plot(t,x1,'b',t,x2,'k--');
grid on; hold on;
set(gca,'FontSize',12);
legend('Solution for t < pi/4',...
     'Solution for t > pi/4');
xlabel('Time (sec)');
ylabel('Amplitude x(t)');
title('Two Parts of Solution...');
```

Figure 2: Solution for Sequential Forcing Example

Part 3: Comparison to Step Response

From Part 2, we already have the particular solution for a constant input of 3:

$$x_{p,step}(t) = 0.3$$

The homogeneous solution has the same form as in both previous sections, but with a new set of constants, to be determined:

$$\begin{aligned} x_{h,step}(t) &= A_{step}\mathrm{e}^{-t}\sin(3t+\phi_{step}) \\ \dot{x}_{h,step}(t) &= -A_{step}\mathrm{e}^{-t}\sin(3t+\phi_{step}) + 3A_{step}\mathrm{e}^{-t}\cos(3t+\phi_{step}) \end{aligned}$$

Applying initial conditions (at $t = 0$) gives

$$\begin{aligned} x_{step}(0) &= x_{h,step}(0) + x_{p,step}(0) \\ &= A_{step}\sin\phi_{step} + 0.3 = 0 \\ \dot{x}_{step}(0) &= \dot{x}_{h,step}(0) + \dot{x}_{p,step}(0) \\ &= -A_{step}\sin\phi_{step} + 3A_{step}\cos\phi_{step} + 0 = 0 \end{aligned}$$

The matrix-form equation and corresponding solution for A_{step} and ϕ_{step} are

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} A_{step}\sin\phi_{step} \\ A_{step}\cos\phi_{step} \end{bmatrix} &= \begin{bmatrix} -0.3 \\ 0 \end{bmatrix} \\ \Rightarrow \qquad A_{step}\sin\phi_{step} = -0.3, &\qquad A_{step}\cos\phi_{step} = -0.1 \\ A_{step} = \sqrt{(A_{step}\sin\phi_{step})^2 + (A_{step}\cos\phi_{step})} &= 0.31623 \\ \phi_{step} = \tan^{-1}\left(\frac{A_{step}\sin\phi_{step}}{A_{step}\cos\phi_{step}}\right) &= -1.8925 \text{ rad} \\ \phi_{step} = \phi_{step} + 2\pi &= 4.3906 \text{ rad} \end{aligned}$$

Therefore the homogeneous solution for the step response is

$$x_{h,step}(t) = 0.31623\mathrm{e}^{-t}\sin(3t + 4.3906)$$

Finally, the complete step response solution is

$$x_{step}(t) = 0.31623\mathrm{e}^{-t}\sin(3t + 4.3906) + 0.3$$

Figure 3 (next page) shows the sequential forcing solution and the step response on the same plot. As evident in the plot, the sequential forcing solution takes slightly longer to reach the steady-state level of 0.3, which could be expected since it takes longer for $f(t)$ to reach a constant level of 3. However, the sequential forcing solution has a lower overshoot, which is highly desirable in some situations.

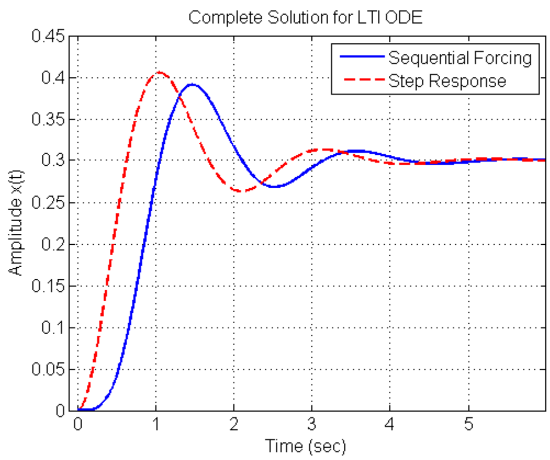


Figure 3: Comparison of Solutions

```
x = [x1 x2];
xstep = Astep*exp(s*t).*sin(w*t+phistep)...
     + C3;

plot(t,x,'b',t,xstep,'r--');
set(gca,'FontSize',12);
legend('Sequential Forcing',...
       'Step Response');
xlabel('Time (sec)');
ylabel('Amplitude x(t)');
title('Complete Solution for LTI ODE');
```