Find state-space models for each of the following systems:

Solution In each case, we use element equations, loop equations, and node equations to convert to the form

$$\underline{\dot{z}}(t) = A\underline{z}(t) + B\underline{u}(t) \quad , \quad \underline{y}(t) = C\underline{z}(t) + D\underline{u}(t)$$

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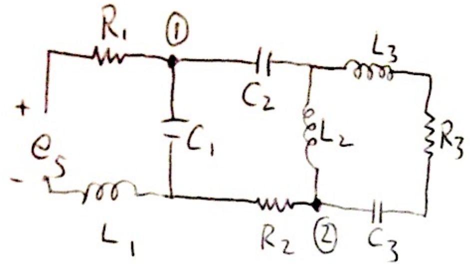


Output = $e_{1} - e_{2}$ $(= e_{1-2})$

Input voltage source e_s

We begin by defining the state, input, and output vectors:

$$\underline{z}(t) = \begin{bmatrix} e_{C1} \\ e_{C2} \\ e_{C3} \\ i_{L1} \\ i_{L2} \\ i_{L3} \end{bmatrix} \qquad \underline{u}(t) = (e_s) \qquad \underline{y}(t) = e_{1\to 2} = \begin{bmatrix} e_{C2} + e_{L2} \\ \to or \leftarrow \\ e_{C1} - e_{R2} \end{bmatrix}$$

Now proceed to find the state and output equations and fill in A, B, C, and D:

$$\dot{e}_{C1} = \frac{1}{C1} i_{C1} = \frac{1}{C_1}\left[i_{L1} - i_{L2} - i_{L3}\right] \qquad \text{node law below } C_1$$

$$\dot{e}_{C2} = \frac{1}{C2} i_{C2} = \frac{1}{C_2}\left[i_{L2} + i_{L3}\right] \qquad \text{node law (right from } C_2)$$

$$\dot{e}_{C3} = \frac{1}{C_3} i_{C3} = \frac{1}{C_3} i_{L3}$$

$$\dot{i}_{L1} = \frac{1}{L1} e_{L1} = \frac{1}{L_1}\left[e_s - e_{R1} - e_{C1}\right] = \frac{1}{L_1}\left[e_s - R_1 i_{L1} - e_{C1}\right] \qquad \text{loop law, } R_1 \text{ law, node law}$$

$$\dot{i}_{L2} = \frac{1}{L2} e_{L2} = \frac{1}{L_2}\left[e_{C1} - e_{C2} - e_{R2}\right] = \frac{1}{L_2}\left[e_{C1} - e_{C2} - R_2(i_{L2} + i_{L3})\right] \qquad \text{loop law, } R_2 \text{ law, node law}$$

$$\begin{aligned} \dot{i}_{L3} &= \frac{1}{L3} e_{L3} = \frac{1}{L_3}\left[e_{L2} - e_{R3} - e_{C3}\right] = \frac{1}{L_3}\left[e_{C1} - e_{C2} - R_2(i_{L2} + i_{L3}) - R_3 i_{L3} - e_{C3}\right] \\ &= \frac{1}{L_3}\left[e_{C1} - e_{C2} - e_{C3} - R_2 i_{L2} - (R_2 + R_3) i_{L3}\right] \qquad \text{loop law, } R_3 \text{ law, node law} \end{aligned}$$

Thus,

$$A = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{C1} & -\frac{1}{C1} & -\frac{1}{C1} \\ 0 & 0 & 0 & 0 & \frac{1}{C2} & \frac{1}{C2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{C3} \\ -\frac{1}{L1} & 0 & 0 & -\frac{R1}{L1} & 0 & 0 \\ \frac{1}{L2} & -\frac{1}{L2} & 0 & 0 & -\frac{R2}{L2} & -\frac{R2}{L2} \\ \frac{1}{L3} & -\frac{1}{L3} & -\frac{1}{L3} & 0 & -\frac{R2}{L3} & -\frac{(R2+R3)}{L3} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{L1} \\ 0 \\ 0 \end{bmatrix}$$

The output is

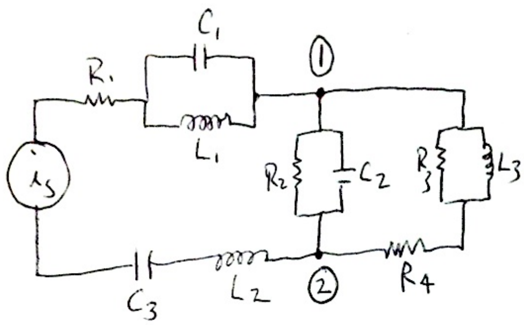
$$y = e_{C2} + e_{L2} = e_{C2} + \left[e_{C1} - e_{C2} - R_2(i_{L2} + i_{L3})\right] = e_{C1} - R_2 i_{L2} - R_2 i_{L3}$$

We can double-check by calculating the output as

$$y = e_{C1} - e_{R2} = e_{C1} - R_2 i_{R2} = e_{C1} - R_2(i_{L2} + i_{L3}) = e_{C1} - R_2 i_{L2} - R_2 i_{L3}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & -R_2 & -R_2 \end{bmatrix} \qquad D = [0]$$

(2)



Outputs: (i) $e_{1\text{-}2}$

(ii) i_{R4}

Input current source i_s

We begin by defining the usual state, input, and output vectors:

$$\underline{z}(t) = \begin{bmatrix} e_{C1} \\ e_{C2} \\ e_{C3} \\ i_{L1} \\ i_{L2} \\ i_{L3} \end{bmatrix} \qquad \underline{u}(t) = (i_s) \qquad \underline{y}(t) = \begin{pmatrix} e_{1\to 2} = e_{C2} \\ \\ i_{R4} \end{pmatrix}$$

NOTE We do not need a state i_{L2} in this case, because it is the same as i_s:

$$\underline{z}(t) = \begin{bmatrix} e_{C1} \\ e_{C2} \\ e_{C3} \\ i_{L1} \\ i_{L3} \end{bmatrix}$$

We now obtain the state and output matrices:

$$\begin{aligned}
\dot{e}_{C1} &= \frac{1}{C_1} i_{C1} = \frac{1}{C_1}\left[i_s - i_{L1}\right] \\
\dot{e}_{C2} &= \frac{1}{C_2} i_{C2} = \frac{1}{C_2}\left[i_s - i_{R2} - i_{R3} - i_{L3}\right] \\
&\to\ i_{R2} = \frac{e_{R2}}{R_2} = \frac{e_{C2}}{C_2} \\
&\to\ i_{R3} = \frac{e_{R3}}{R_3} = \frac{e_{C2} - e_{R4}}{R_3} = \frac{e_{C2} - R_4 i_{R4}}{R3} = \frac{e_{C2} - R_4(i_{R3} + i_{L3})}{R3} \\
&\text{Collect terms, solve to find } \ i_{R3} = \frac{e_{C2} - R_4 i_{L3}}{R_3 + R_4}
\end{aligned}$$

Now substitute for i_{R2} , i_{R3} :

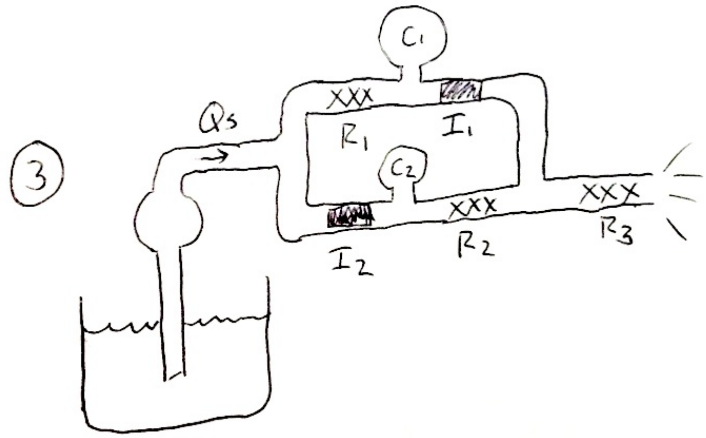
$$\begin{aligned}
\dot{e}_{C2} &= \frac{1}{C_2}\left[i_s - \frac{e_{C2}}{R_2} - \frac{e_{C2} - R_4 i_{L3}}{R_3 + R_4} - i_{L3})\right] \\
&= \frac{1}{C_2}\left[i_s - \left(\frac{R_2 + R_3 + R_4}{R_2(R_3 + R_4)}\right) e_{C2} - \left(\frac{R_3}{R_3 + R_4}\right) i_{L3}\right] \\
\dot{e}_{C3} &= \frac{1}{C_3} i_{C3} = \frac{1}{C3} i_s \\
\dot{i}_{L1} &= \frac{1}{L_1} e_{L1} = \frac{1}{L_1} e_{C1} \\
\dot{i}_{L3} &= \frac{1}{L_3} e_{L3} = \frac{1}{L_3} e_{R3} = \frac{1}{L_3} R_3 i_{R3} = \frac{R_3}{L_3}\left(\frac{e_{C2} - R_4 i_{L3}}{R_3 + R_4}\right) \\
&= \frac{1}{L_3}\left[\frac{R_3}{R_3 + R_4} e_{C2} - \frac{R_3 R_4}{R_3 + R_4} i_{L3}\right]
\end{aligned}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{C1} & 0 \\ 0 & -\left(\frac{R2+R3+R4}{C2R2(R3+R4)}\right) & 0 & 0 & -\frac{R3}{C2(R3+R4)} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{L1} & 0 & 0 & 0 & 0 \\ 0 & \frac{R3}{L3(R3+R4)} & 0 & 0 & -\frac{R3R4}{L3(R3+R4)} \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{C1} \\ \frac{1}{C2} \\ -\frac{1}{C3} \\ 0 \\ 0 \end{bmatrix}$$

The output equations are:

$$y(t) = \begin{bmatrix} e_{C2} \\ i_{R4} \end{bmatrix} = \begin{bmatrix} e_{C2} \\ i_{R3} + i_{L3} \end{bmatrix} = \begin{bmatrix} e_{C2} \\ \frac{e_{C2} - R_4 i_{L3}}{R_3 + R_4} + i_{L3} \end{bmatrix} = \begin{bmatrix} e_{C2} \\ \frac{1}{R3+R4}\left(e_{C2} + R_3 i_{L3}\right) \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{R3+R4} & 0 & 0 & \frac{R3}{R3+R4} \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Outputs:
(i) Q_{C_1} (ii) Q_{C_2}
(iii) Q_{R_3}

Input flow source Q_s

We begin by defining the usual state, input, and output vectors:

$$\underline{z}(t) = \begin{bmatrix} P_{C1} \\ P_{C2} \\ Q_{I1} \\ Q_{I2} \end{bmatrix} \qquad \underline{u}(t) = (Q_s) \qquad \underline{y}(t) = \begin{pmatrix} Q_{C1} \\ Q_{C2} \\ Q_{R3} \end{pmatrix}$$

We now obtain the state and output matrices:

$$\dot{P}_{C1} = \frac{1}{C_1} Q_{C1} = \frac{1}{C1}(Q_{R1} - Q_{I1}) = \frac{1}{C_1}\left[(Q_s - Q_{I2}) - Q_{I1}\right]$$

$$\dot{P}_{C2} = \frac{1}{C_2} Q_{C2} = \frac{1}{C_2}(Q_{I2} - Q_{R2}) = \frac{1}{C2}\left[Q_{I2} - \frac{(P_{C2} - P_3)}{R_2}\right]$$

where $P_3 \equiv$ the pressure on the high side of R_3

$$\rightarrow \; P_3 = R_3 Q_{R3} = R_3(Q_{I1} + Q_{R2}) = R_3\left[Q_{I1} + \frac{P_{C2} - P_3}{R_2}\right]$$

Collect terms, solve to find $P_3 = \dfrac{R_2 R_3 Q_{I1} + R_3 P_{C2}}{R_2 + R_3}$

Now substitute for P_3 :

$$\dot{P}_{C2} = \frac{1}{C_2}\left[Q_{I2} - \frac{P_{C2}}{R_2} + \frac{R_2 R_3 Q_{I1} + R_3 P_{C2}}{R_2(R_2 + R_3)}\right]$$

$$= \frac{1}{C_2}\left[Q_{I2} - \left(\frac{1}{R_2 + R_3}\right) P_{C2} + \left(\frac{R_3}{R_2 + R_3}\right) Q_{I1}\right]$$

$$\dot{Q}_{I1} = \frac{1}{I_1} P_{I1} = \frac{1}{I_1}(P_{C1} - P_3) = \frac{1}{I_1}\left[P_{C1} - \left(\frac{R_2 R_3 Q_{I1} + R_3 P_{C2}}{R_2 + R_3}\right)\right]$$

$$\dot{Q}_{I2} = \frac{1}{I_2} P_{I2} = \frac{1}{I_2}\left[(P_{C1} + R_1 Q_{R1}) - P_{C2}\right] = \frac{1}{I_2}\left[P_{C1} + R_1(Q_s - Q_{I2}) - P_{C2}\right]$$

Thus,

$$A = \begin{bmatrix} 0 & 0 & -\frac{1}{C1} & -\frac{1}{C1} \\ 0 & -\left(\frac{1}{C2(R2+R3)}\right) & \frac{R3}{C2(R2+R3)} & 0 \\ \frac{1}{I1} & -\frac{R3}{I1(R2+R3)} & -\frac{R2R3}{I1(R2+R3)} & 0 \\ \frac{1}{I2} & -\frac{1}{I2} & 0 & -\frac{R1}{I2} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{C1} \\ 0 \\ 0 \\ \frac{R1}{I2} \end{bmatrix}$$

The output equations are:

$$y(t) = \begin{pmatrix} Q_{C1} \\ Q_{C2} \\ Q_{R3} \end{pmatrix} = \begin{bmatrix} Q_s - Q_{I2} - Q_{I1} \\ \\ Q_{I2} - \left(\frac{1}{R_2+R_3}\right) P_{C2} + \left(\frac{R_3}{R_2+R_3}\right) Q_{I1} \\ \\ \frac{P_3}{R_3} = \frac{1}{(R_2+R_3)} P_{C2} + \frac{R_2}{(R_2+R_3)} Q_{I_1} \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 0 & 0 & -1 & -1 \\ \\ 0 & -\frac{1}{R2+R3} & \frac{R3}{R2+R3} & 1 \\ \\ 0 & \frac{1}{R2+R3} & \frac{R2}{R2+R3} & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 \\ \\ 0 \\ \\ 0 \end{bmatrix}$$