**Problem 1:** For the LTI ODE

$$5\ddot{x}(t) + 9\dot{x}(t) + 45x(t) = 0 \quad , \quad x(0) = 4 \quad , \quad \dot{x}(0) = 2$$

**Solution**: The Characteristic Equation and its roots are used to fill in the blanks below:

$$5\lambda^2 + 9\lambda + 45 = 0 = \lambda^2 + 1.8\lambda + 9 = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2$$

$$\Rightarrow \lambda_{1,2} = -0.9 \pm i2.86 \quad , \quad \omega_n = \sqrt{9} = 3 \quad , \quad \zeta = \frac{1.8}{2 * \omega_n} = 0.3$$

1a: *(20 points)* Find the following:

Natural frequency: $3\ rad/s$ Damping ratio: $0.3$

Period of oscillation: $2\pi/2.86 = 2.196\ s$ Settling time: $4/0.9 = 4.44\ s$

1b. *(15 points)* Sketch the homogeneous solution from $t = 0$ to the settling time; be sure to label both axes numerically, and to make your sketch consistent with your axis numbers

**Solution**: The general form of the homogeneous solution is $x_h(t) = Ae^{-0.9t}sin(2.86t + \phi)$

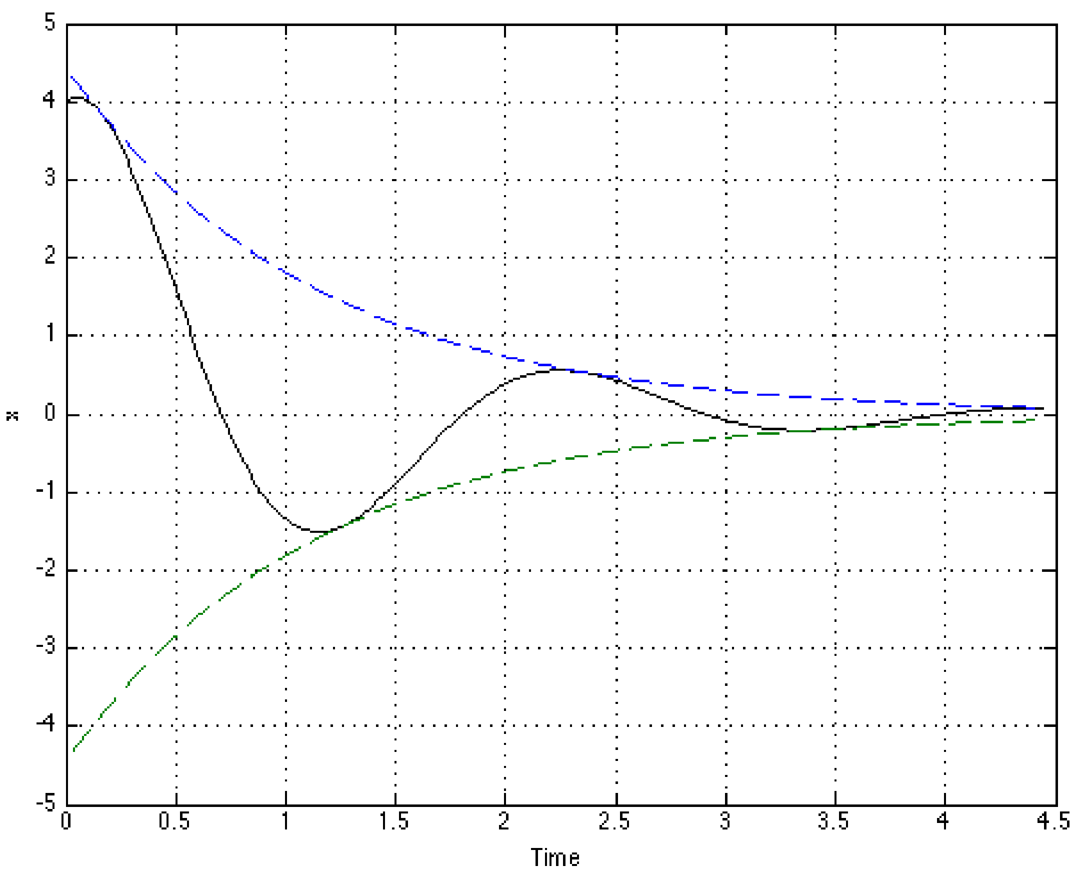
Apply initial conditions to find $A$ and $\phi$:

$$x(0) = 4 = Asin\phi \quad , \quad \dot{x}(0) = 2 = -0.9Asin\phi + 2.86Acos\phi \quad \Rightarrow Acos\phi = \frac{2 + 0.9(4)}{2.86} = 1.96$$

$$\text{Thus } \phi = tan^{-1}\frac{4}{1.96} = 1.1156\ rad \ \ (\approx 63^o) \quad , \quad A = \frac{4}{sin(1.1156\ rad)} = 4.45$$

Therefore, we plot

$$x_h(t) = 4.45e^{-0.9t}sin(2.86t + 1.1156) \quad 0 \le t \le 4.44s$$



**Problem 2:** For the LTI ODE

$$\dddot{x}(t) + 10\ddot{x}(t) + K\dot{x}(t) + cx(t) = 0 \quad , \quad x(0) = 4 \quad , \quad \dot{x}(0) = 2$$

Determine all conditions that must be satisfied by $K$ and $c$ to ensure that the system
2a. *(10 points)* is **stable**

**Solution**: Construct the Routh table and avoid sign changes in the first column

$$\begin{pmatrix} 1 & K \\ 10 & c \\ 10K - c & \\ c & \end{pmatrix} \quad \Rightarrow \quad \text{For stability, we require} \quad \underline{c > 0} \quad \textit{and} \quad \underline{10K > c}$$

2b. *(15 points)* has a settling time no greater than **2 seconds**
**Solution** Settling time $\leq 2$ seconds when real part of roots $\leq -2$ , so we axis-shift the polynomial to $-2$ , then use Routh on the shifted polynomial:

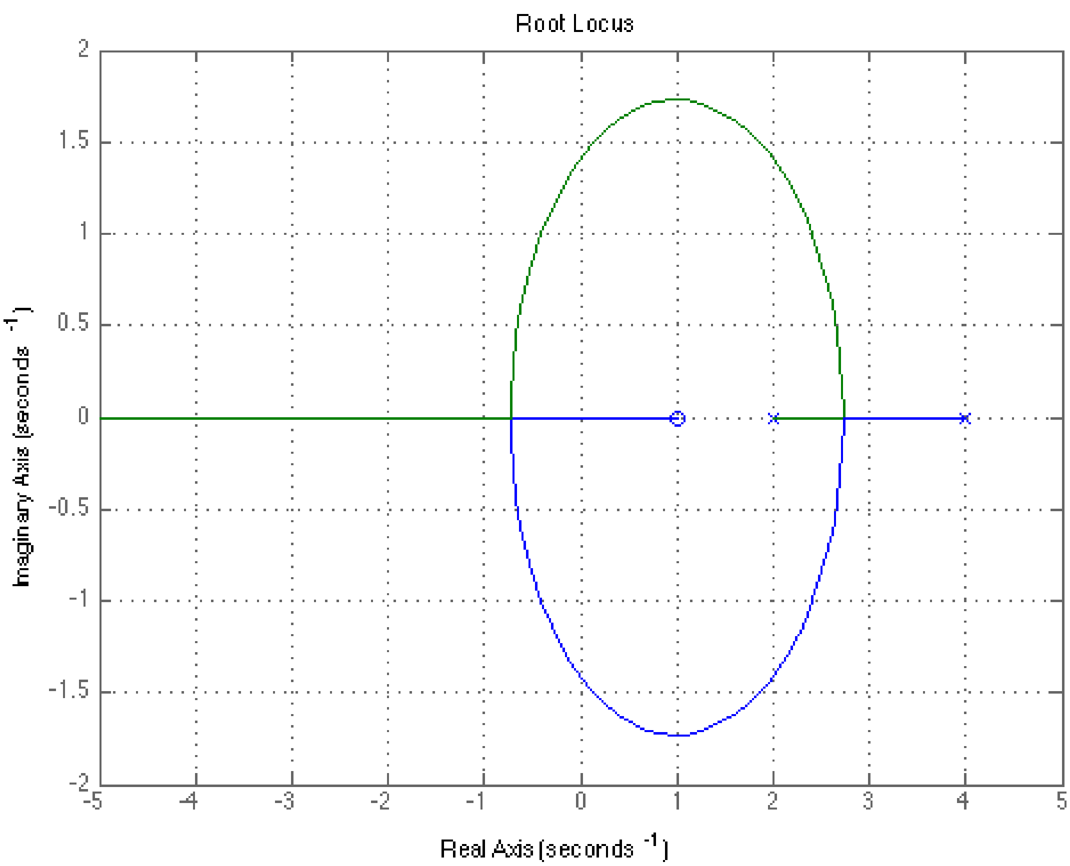
$$(\lambda - 2)^3 + 10(\lambda - 2)^2 + K(\lambda - 2) + c = 0 = \lambda^3 + 4\lambda^2 + (K - 28) * \lambda + (c + 32 - 2K)$$

$$\begin{pmatrix} 1 & K - 28 \\ 4 & c + 32 - 2K \\ 6K - c - 144 & \\ c + 32 - 2K & \end{pmatrix} \quad \Rightarrow \quad \text{For settling} \leq 2 \text{ s, we require} \quad \underline{c + 32 - 2K > 0} \quad \textit{and} \quad \underline{6K > c + 144}$$

**Problem 3:** For the LTI ODE

$$\ddot{x}(t) - 6\dot{x}(t) + 8x(t) + K\dot{x}(t) - Kx(t) = 0$$

3a (*20 points*) Draw the root locus plot for $0 \leq K$




The branches begin at the roots of $D$, end at roots of $N$ or $\infty$:

$$D = \lambda^2 - 6\lambda + 8 = 0 \quad \Rightarrow \quad X_{1,2} = 2, 4 \quad ; \quad N = \lambda - 1 \quad \Rightarrow \quad O_1 = 1, \text{ other branch } \to \infty$$

Since only one branch goes to $\infty$, the asymptote is the $-Re$ axis.
The BI/BO points are found from

$$D'N - N'D = 0 = (2\lambda - 6)(\lambda - 1) - (\lambda^2 - 6\lambda + 8) = \lambda^2 - 2\lambda - 2$$

$$\Rightarrow \quad (\lambda = -0.73, K = 7.46) \;\; BI \quad , \quad (\lambda = 2.73, K = 0.54) \;\; BO$$

Branches cross the imaginary axis when

$$-\omega^2 + i(K - 6)\omega + 8 - K = 0 \quad \Rightarrow \quad (\omega = 0, K = 8) \quad , \quad (\omega = \pm\sqrt{2}, K = 6)$$

3b (*5 points*) The value(s) of $K$ (if any) for which the system is stable is $6 < K < 8$

3c (*5 points*) The value(s) of $K$ (if any) for which all roots are real is $K \leq 0.54$ , $K \geq 7.46$

3d (*5 points*) The fastest settling time is $5.46s$

3e (*5 points*) The value(s) of $K$ that produce(s) the fastest settling time is $K = 7.46$