(30 pts) Write a state-space model for the system:

$$\ddot{x}_1(t) + 5\big[\dot{x}_1(t) - \dot{x}_2(t)\big] + 10\big[x_1(t) - x_2(t)\big] + 4\dot{x}_1(t) + 20x_1(t) = 8 - 4sin(2t)$$

$$\ddot{x}_2(t) - 5\big[\dot{x}_1(t) - \dot{x}_2(t)\big] - 10\big[x_1(t) - x_2(t)\big] = 6sin(2t) + 8$$

The two inputs are $u_1(t) = 1$, $u_2(t) = sin(2t)$ and the system has two outputs, defined as

$$y_1(t) = 12x_2(t) - 8\dot{x}_1(t)$$
$$y_2(t) = -2\dot{x}_2(t) + 8\big[x_2(t) - x_1(t) + 8\big]$$

A complete solution must correctly define the 3 vectors $\underline{z}(t), \underline{u}(t), \underline{y}(t)$ and the 4 matrices A, B, C, D

Solution: Define $\underline{z}(t)$ and $\underline{u}(t)$; write $\underline{y}(t)$; and then simply fill in A, B, C, and D. Two possibilities are shown:

1.

$$\underline{z}(t) \equiv \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} \quad , \quad \underline{u}(t) \equiv \begin{pmatrix} 1 \\ sin(2t) \end{pmatrix} \quad , \quad \underline{y}(t) = \begin{pmatrix} 12x_2 - 8\dot{x}_1 \\ -2\dot{x}_2 + 8x_2 - 8x_1 + 64 \end{pmatrix}$$

$$\Rightarrow \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -30 & -9 & 10 & 5 \\ 0 & 0 & 0 & 1 \\ 10 & -10 & 5 & -5 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 0 \\ 8 & -4 \\ 0 & 0 \\ 8 & 6 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 0 & -8 & 12 & 0 \\ -8 & 0 & 8 & -2 \end{pmatrix} \quad , \quad D = \begin{pmatrix} 0 & 0 \\ 64 & 0 \end{pmatrix}$$

2.

$$\underline{z}(t) \equiv \begin{pmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \quad , \quad \underline{u}(t) \equiv \begin{pmatrix} 1 \\ sin(2t) \end{pmatrix} \quad , \quad \underline{y}(t) = \begin{pmatrix} 12x_2 - 8\dot{x}_1 \\ -2\dot{x}_2 + 8x_2 - 8x_1 + 64 \end{pmatrix}$$

$$\Rightarrow \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -30 & 10 & -9 & 5 \\ 10 & 5 & -10 & -5 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 8 & -4 \\ 8 & 6 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 0 & 12 & -8 & 0 \\ -8 & 8 & 0 & -2 \end{pmatrix} \quad , \quad D = \begin{pmatrix} 0 & 0 \\ 64 & 0 \end{pmatrix}$$

(30 pts) Find the complete solution $x(t)$ and sketch it to twice the settling time. For full credit, you must clearly label numerical values on your axes, and sketch a reasonably accurate plot:

$$\ddot{x}(t) + 0.6\dot{x}(t) + 9.09x(t) = 18.18 \quad , \quad x(0) = 0 \ , \ \dot{x}(0) = 0$$

Solution:
The particular solution is a constant, $x_p = 2$. The general form of the homogeneous solution is:

$$\lambda^2 + 0.6\lambda + 9.09 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -0.3 \pm 3i \quad \Rightarrow \quad x_h(t) = Ae^{-0.3t}sin(3t + \phi)$$

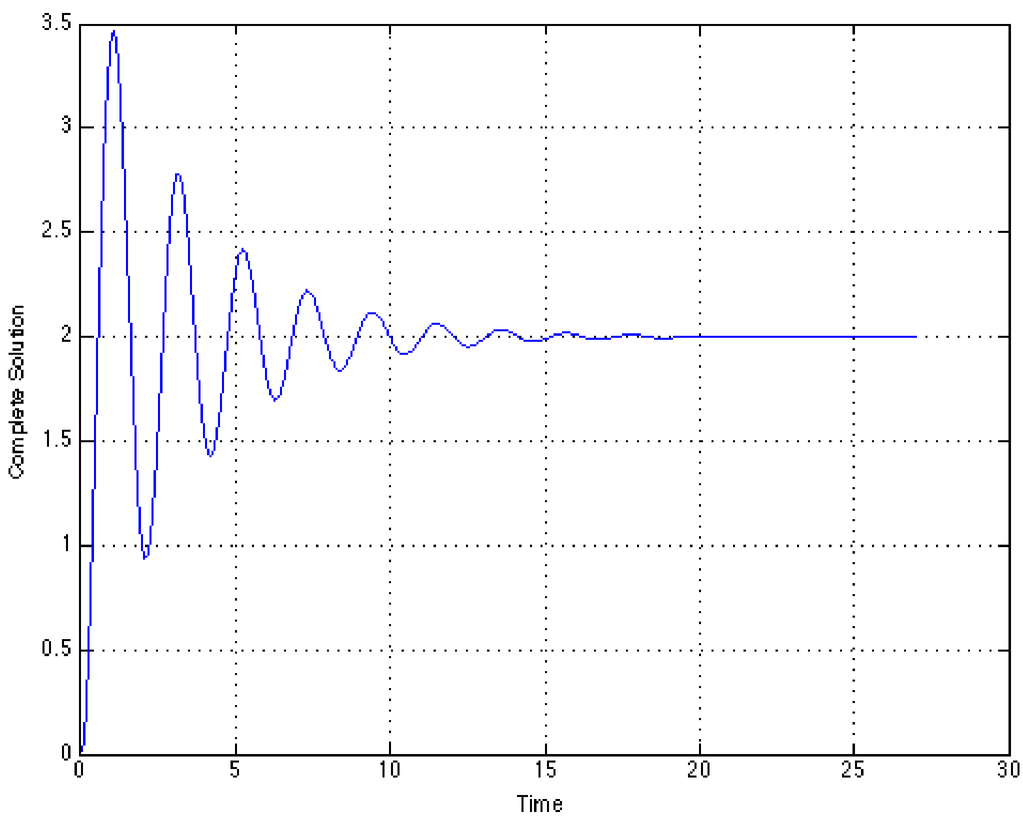
Now apply the initial conditions to determine A, ϕ :

$$x(0) = 0 = Asin\phi + 2 \qquad Asin\phi = -2$$
$$\dot{x}(0) = 0 = -0.3Asin\phi + 3Acos\phi \qquad Acos\phi = -0.2$$
$$\Rightarrow \phi = 1.4711 \ , \ A = -2.01$$

The complete solution is thus

$$\underline{x(t) = -2.01e^{-0.3t}sin(3t + 1.4711) + 2}$$

The settling time is 13.33 seconds. The solution begins at $x = 0$ with $\dot{x} = 0$, then settles at $x = 2$. The period of oscillation is $2.09s$, so the homogeneous part of the solution undergoes approximately 6 cycles before it settles, and is quite prominent The crossings of $x = 2$ occur every half-period, i.e., every $t = 1.045s$, beginning at $t = 0.5568s$. The following graph was generated by MATLAB, so your sketch may not match exactly, but it should be reasonably close.



(40 pts) For the system modeled by

$$\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = 5u(t)$$

with output defined as

$$y_1(t) = 4x(t) + \dot{x}(t) \quad , \quad y_2(t) = 3\ddot{x}(t)$$

a) Find the system's transfer function(s)

b) Find the system's pole(s) (if any) and zero(s) (if any)

c) Find $y_1(t \to \infty)$ **if** $u(t) = \begin{pmatrix} 0 & t < 0 \\ 1 & t \geq 0 \end{pmatrix}$

d) Find $\left[y_2(t_0^+) - y_2(t_0^-)\right]$ **if** $u(t) = \begin{pmatrix} 0 & t < 0 \\ 1 & t \geq 0 \end{pmatrix}$

e) Find the frequency response function corresponding to output y_2

f) Find steady-state $y_2(t)$ **if** $u(t) = 3sin(2t)$

Solution

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a) Take the Laplace Transforms and set the initial conditions to 0:

$$\mathcal{L}\left[\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = 5u(t)\right] \quad \Rightarrow \quad s^2X(s) + 3sX(s) + 2X(s) = 5U(s)$$

$$\mathcal{L}\left[y_1(t) = 4x(t) + \dot{x}(t)\right] \;\Rightarrow\; Y_1(s) = 4X(s) + sX(s) \quad , \quad \mathcal{L}\left[y_2(t) = 3\ddot{x}(t)\right] \;\Rightarrow\; Y_2(s) = -3s^2X(s)$$

Use algebra to find the TF:

$$\frac{X(s)}{U(s)} = \frac{5}{s^2+3s+2} \quad \Rightarrow \quad \frac{Y_1(s)}{U(s)} = \frac{4X(s)+sX(s)}{U(s)} = \boxed{\frac{20+5s}{s^2+3s+2}} \quad , \quad \frac{Y_2(s)}{U(s)} = \frac{3s^2X(s)}{U(s)} = \boxed{\frac{15s^2}{s^2+3s+2}}$$

b)

$$\text{For } \frac{Y_1}{U}, \; \boxed{\text{Zero} = -4 \;\; , \; \text{Poles} = -1, -2} \quad , \quad \text{For } \frac{Y_2}{U}, \; \boxed{\text{Two Zeroes} = 0, 0 \;\; , \; \text{Poles} = -1, -2}$$

c) To find $y_1(\infty)$, we use the final-value theorem:

$$y_1(\infty) = lim_{s \to 0}(\frac{Y_1}{U}) = lim_{s \to 0}\left[\frac{20+5s}{s^2+3s+2}\right] = \boxed{10}$$

d) To find $\left[y_2(t_0^+) - y_2(t_0^-)\right]$, we use the initial-value theorem:

$$\left[y_2(t_0^+) - y_2(t_0^-)\right] = lim_{s \to \infty}(\frac{Y_2}{U}) = lim_{s \to \infty}\left[\frac{15s^2}{s^2+3s+2}\right] = \boxed{15}$$

e) We evaluate the TF at $s = i\omega$:

$$\frac{15s^2}{s^2+3s+2} \quad \to \quad \frac{-15\omega^2}{2-\omega^2+i3\omega} * \frac{(2-\omega^2)-i3\omega}{(2-\omega^2)-i3\omega} = \boxed{\frac{(15\omega^4 - 30\omega^2) + i(45\omega^3)}{(\omega^4 + 5\omega^2 + 4)}}$$

f) Substitute $\omega = 2$, find $Re = 3$, $Im = 9$, thus $Mag = 9.487$, $\phi = 1.249$

$$\Rightarrow y_2(t) = 9.487 * 3sin(2t + 1.249) = \boxed{28.46sin(2t + 1.249)}$$