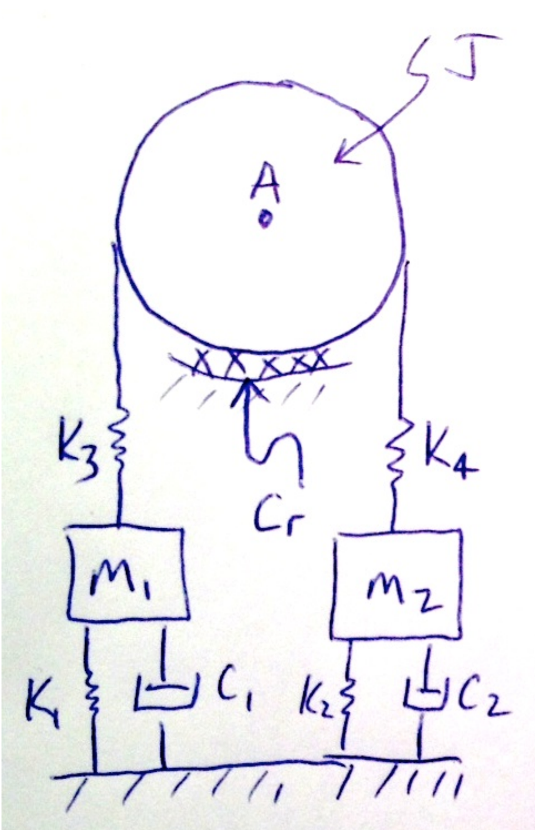
1.



Masses m_1 and m_2 are connected to the ground through the springs and dampers as shown. Each is also connected to a flexible attachment to the pulley with inertia J, center A, radius R, and rotational damping C_r. The attachments of the mass to the pulley are through the springs k_3 and k_4 as shown, which are connected over the pulley by an inextensible cord. The cord does NOT slide relative to the pulley. Gravity acts downward as usual.

You may assume that the geometry/construction is such that

1. There exists a configuration where all springs are in their undeformed positions simultaneously
2. All connections are made in such a way that every spring can exert both tension and compression forces

The pulley is subjected to an external torque, T (not shown). The whole system is subjected to gravity.

The outputs are (i) the velocity of m_2, (ii) the angular velocity of the pulley, and (iii) the tension in spring k_3

Solution

Define $x_1 = 0$ (positive up), $x_2 = 0$ (positive down), and $\theta = 0$ (positive counterclockwise) at the undeformed positions of the springs. The positive directions of these three variables correspond to the same motion. From the corresponding FBD's,

$$m_1\ddot{x}_1 = -m_1 g - k_1 x_1 - c_1\dot{x}_1 + k_3(R\theta - x_1)$$

$$m_2\ddot{x}_2 = m_2 g - k_2 x_2 - c_2\dot{x}_2 - k_4(x_2 - R\theta)$$

$$J\ddot{\theta} = T + R\big[k_4(x_2 - R\theta) - k_3(R\theta - x_1)\big] - c_r\dot{\theta}$$

Define the state, input, and output vectors as

$$\vec{z}(t) = \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ \theta \\ \dot{\theta} \end{pmatrix} \quad , \quad \vec{u}(t) = \begin{pmatrix} T \\ g \end{pmatrix} \quad , \quad \vec{y}(t) = \begin{pmatrix} \dot{x}_2 \\ \dot{\theta} \\ k_3(R\theta - x_1) \end{pmatrix}$$

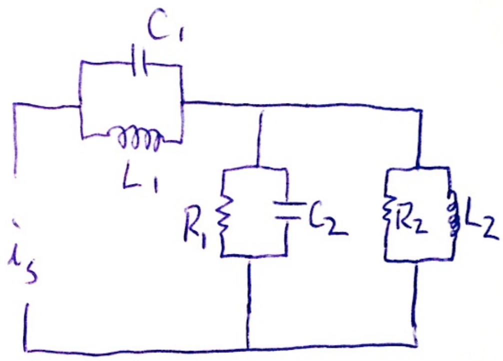
The state equation is

$$\dot{\vec{z}}(t) = A\vec{z} + B\vec{u} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -(k_1+k_3)/m_1 & -c_1/m_1 & 0 & 0 & k_3 R/m_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -(k_2+k_4)/m_2 & -c_2/m_2 & k_4 R/m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ k_3 R/J & 0 & k_4 R/J & 0 & -R^2(k_3+k_4)/J & -C_r/J \end{pmatrix} \vec{z} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1/J & 0 \end{pmatrix} \vec{u}$$

The output equation is

$$\vec{y}(t) = C\vec{y} + D\vec{u} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k_3 & 0 & 0 & 0 & k_3 R & 0 \end{pmatrix} \vec{z} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{u}$$

2.



The system's input is an ideal current source i_s; the output is the current through capacitor C_2.

NOTE: An ideal current source is one that maintains the current shown at all times (here, i_s); it has a voltage change associated with it, which may vary (in fact, it will vary, because the loop law must be satisfied at all times for every closed loop)

Solution

Define the state, input, and output vectors as

$$\vec{z}(t) = \begin{pmatrix} e_{C1} \\ e_{C2} \\ i_{L1} \\ i_{L2} \end{pmatrix} \quad , \quad \vec{u}(t) = \begin{pmatrix} i_s \end{pmatrix} \quad , \quad \vec{y}(t) = \begin{pmatrix} i_{C2} \end{pmatrix}$$

Write the element laws for the states and use loop laws, node laws, and other element laws:

$$\begin{aligned}
\dot{e}_{C1} &= \frac{1}{C_1} i_{C1} = \frac{1}{C_1}(i_s - i_{L1}) \\
\dot{e}_{C2} &= \frac{1}{C_2} i_{C2} = \frac{1}{C_2}(i_s - i_{R1} - i_{R2} - i_{L2}) = \frac{1}{C_2}(i_s - \frac{e_{R1}}{R_1} - \frac{e_{R2}}{R_2} - i_{L2}) = \frac{1}{C_2}(i_s - \frac{e_{C2}}{R_1} - \frac{e_{C2}}{R_2} - i_{L2}) \\
\dot{i}_{L1} &= \frac{1}{L1} e_{L1} = \frac{1}{L1} e_{C1} \\
\dot{i}_{L2} &= \frac{1}{L2} e_{L2} = \frac{1}{L2} e_{C2}
\end{aligned}$$

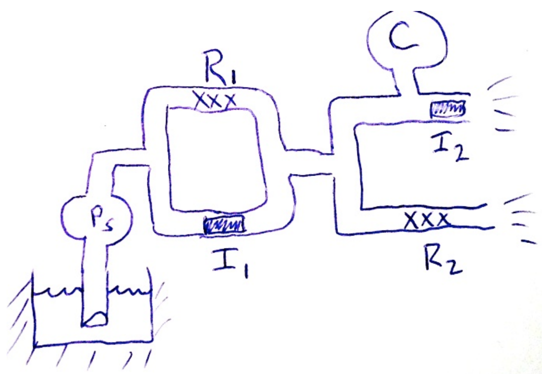
The state equation is

$$\dot{\vec{z}}(t) = A\vec{z} + B\vec{u} = \begin{pmatrix} 0 & 0 & -1/C_1 & 0 \\ 0 & (-1/C_2R_1 - 1/C_2R_2) & 0 & -1/C_2 \\ 1/L_1 & 0 & 0 & 0 \\ 0 & 1/L_2 & 0 & 0 \end{pmatrix} \vec{z} + \begin{pmatrix} 1/C_1 \\ 1/C_2 \\ 0 \\ 0 \end{pmatrix} \vec{u}$$

The output equation is

$$\vec{y}(t) = C\vec{y} + D\vec{u} = \begin{pmatrix} 0 & (-1/R_1 - 1/R_2) & 0 & -1 \end{pmatrix} \vec{z} + \begin{pmatrix} 1 \end{pmatrix} \vec{u}$$

3.



The input is the ideal hydraulic pump, P_s. An ideal pump adds exactly P_s of pressure (like a voltage source in an electric circuit). There is flow through the pump; this flow may vary with time, and may need to be determined as part of the solution.

The system's outputs are (i) the flow through resistor R_2, and (ii) the pressure in the capacitor, C

Solution

Define the state, input, and output vectors as

$$\vec{z}(t) = \begin{pmatrix} P_C \\ Q_{I1} \\ Q_{I2} \end{pmatrix} \quad , \quad \vec{u}(t) = \begin{pmatrix} P_s \end{pmatrix} \quad , \quad \vec{y}(t) = \begin{pmatrix} Q_{R2} \\ P_C \end{pmatrix}$$

Write the element laws for the states and use loop laws, node laws, and other element laws:

$$\begin{aligned}
\dot{P}_C &= \frac{1}{C}Q_C = \frac{1}{C}(Q_{I1} + Q_{R1} - Q_{R2} - Q_{I2} = \frac{1}{C}[Q_{I1} + \frac{P_s - P_C}{R_1} - \frac{P_C}{R_2} - Q_{I2}] \\
\dot{Q}_{I1} &= \frac{1}{I_1}P_{I1} = \frac{1}{I_1}(P_s - P_C) \\
\dot{Q}_{I2} &= \frac{1}{I_2}P_{I2} = \frac{1}{I_2}P_C
\end{aligned}$$

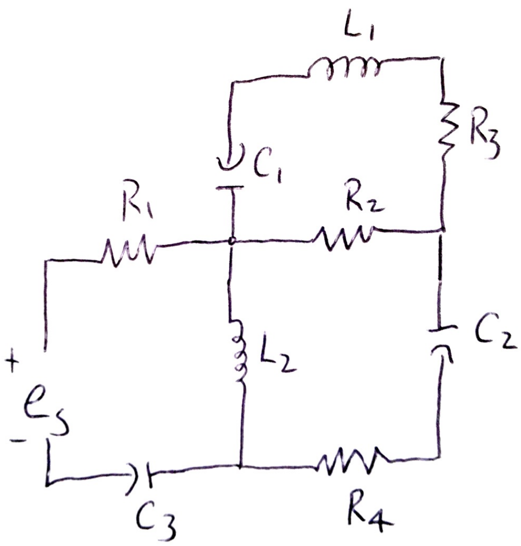
The state equation is

$$\dot{\vec{z}}(t) = A\vec{z} + B\vec{u} = \begin{pmatrix} (\frac{-1}{CR_1} - \frac{1}{CR_2}) & \frac{1}{C} & -\frac{1}{C} \\ -\frac{1}{I_1} & 0 & 0 \\ \frac{1}{I_2} & 0 & 0 \end{pmatrix} \vec{z} + \begin{pmatrix} \frac{1}{CR_1} \\ \frac{1}{I_1} \\ 0 \end{pmatrix} \vec{u}$$

The output equation is

$$\vec{y}(t) = C\vec{y} + D\vec{u} = \begin{pmatrix} -\frac{1}{R_2} & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \vec{z} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \vec{u}$$

4.



The input to the electrical system is the voltage source, e_s.

The system's outputs are (i) the current through inductor L_2, (ii) the voltage across resistor R_4, and the voltage across inductor L_1.

Solution

We begin by defining the state, input, and output vectors as

$$\vec{z}(t) = \begin{pmatrix} e_{C1} \\ e_{C2} \\ e_{C3} \\ i_{L1} \\ i_{L2} \end{pmatrix} \quad , \quad \vec{u} = (e_s) \quad , \quad \vec{y} = \begin{pmatrix} i_{L2} \\ e_{R4} \\ i_{L1} \end{pmatrix}$$

Now use a combination of element laws, node laws, and loop laws to find the state equations.

In more complicated problems, it's often useful to begin with elements that are common to more than one loop. For example, element L_2 is common to several loops. The element law is

$$\dot{i}_{L2} = \frac{1}{L_2} e_{L2}$$

so we will need to substitute for e_{L2}. Using the loop law for the lower-left loop:

$$e_{L2} = e_s - e_{C3} - e_{R1} = e_s - e_{C3} - R_1 * i_{R1} = e_s - e_{C3} - R_1 * (i_{L1} + i_{L2} + i_{R2}) \tag{1}$$

where i_{R2} is defined as positive from left-to-right. Now use the loop law for the lower-right loop, since L_2 is part of it as well (then use additional element laws and node laws as needed):

$$e_{L2} = e_{R2} + e_{C2} + e_{R4} = e_{C2} + R_2 * i_{R2} + R_4 * (i_{L1} + i_{R2}) \tag{2}$$

In the RHS's of both Eqs. (5) and (6), the only non-state, non-input variable is i_{R2}. We can set the RHS of Eqs. (5) and (6) equal to each other, then solve for i_{R2}:

$$i_{R2} = \frac{1}{(R_1 + R_2 + R_4)} * \{e_s - e_{C2} - e_{C3} - (R_1 + R_4) i_{L1} - R_1 i_{L2}\} \tag{3}$$

We may now proceed to obtain all of the state equations:

$$\dot{e}_{C1} = \frac{1}{C_1} i_{C1} = \frac{1}{C_1} i_{L1}$$

$$\dot{e}_{C2} = \frac{1}{C_2} i_{C2} = \frac{1}{C_2}(i_{L1} + i_{R2}) = \frac{1}{C_2}(i_{L1} + \frac{1}{(R_1 + R_2 + R_4)} * \{e_s - e_{C2} - e_{C3} - (R_1 + R_4)i_{L1} - R_1 i_{L2}\})$$

$$\begin{aligned}\dot{e}_{C3} =& \frac{1}{C_3} i_{C3} = \frac{1}{C_3}(i_{L2} + i_{C2}) \\ =& \frac{1}{C_3} * (i_{L2} + i_{L1} + \frac{1}{(R_1 + R_2 + R_4)} * \{e_s - e_{C2} - e_{C3} - (R_1 + R_4)i_{L1} - R_1 i_{L2}\})\end{aligned}$$

$$\begin{aligned}\dot{i}_{L1} =& \frac{1}{L_1} e_{L1} = \frac{1}{L1} * (e_{R2} - e_{C1} - e_{R3}) \\ =& \frac{1}{L1} * \Big(R_2 * \big[i_{L1} + \frac{1}{(R_1 + R_2 + R_4)} * \{e_s - e_{C2} - e_{C3} - (R_1 + R_4)i_{L1} - R_1 i_{L2}\}\big] - e_{C1} - R_3 i_{L1}\Big)\end{aligned}$$

$$\dot{i}_{L2} = \frac{1}{L_2} e_{L2} = \frac{1}{L2}\big(e_s - e_{C3} - R_1\big[i_{L1} + i_{L2} + \frac{1}{(R_1 + R_2 + R_4)} * \{e_s - e_{C2} - e_{C3} - (R_1 + R_4)i_{L1} - R_1 i_{L2}\}\big]\big)$$

The terms above may be collected into the A and B matrices, and then the output equation can be written easily.

$$A = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & -\frac{1}{C_2(R_1+R_2+R_4)} & -\frac{1}{C_2(R_1+R_2+R_4)} & \frac{1}{C_2} - \frac{R_1+R_4}{C_2(R_1+R_2+R_4)} & -\frac{R_1}{C_2(R_1+R_2+R_4)} \\ 0 & -\frac{1}{C_3(R_1+R_2+R_4)} & -\frac{1}{C_3(R_1+R_2+R_4)} & \frac{1}{C_3} - \frac{R_1+R_4}{C_3(R_1+R_2+R_4)} & \frac{1}{C_3} - \frac{R_1}{C_2(R_1+R_2+R_4)} \\ -\frac{1}{L_1(R_1+R_2+R_4)} & -\frac{1}{C_2(R_1+R_2+R_4)} & -\frac{1}{C_2(R_1+R_2+R_4)} & -\frac{R_1+R_4}{C_2(R_1+R_2+R_4)} & -\frac{R_1}{C_2(R_1+R_2+R_4)} \end{bmatrix}$$