This example illustrates the procedure that may be used with MATLAB to incorporate initial conditions. Suppose we have a fourth-order LTI ODE. We find the characteristic equation, then solve for its roots to find

$$\lambda_1 \ , \ \lambda_2 = \text{distinct real numbers, and} \ \ \lambda_3, \lambda_4 = \sigma \pm i\omega \ \ \text{complex conjugate pair}$$

For these roots,, the general form of the homogeneous solution may be written as

$$x_h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\sigma t} sin(\omega t + \phi)$$

where A_1, A_2, A_3, and ϕ are constants.

Suppose we are now asked to determine the values of the constants A_1, A_2, A_3, and ϕ using initial conditions.

Let the four initial conditions (whose values we are given) be defined as $x_0 = x(0)$, $\dot{x}_0 = dx(0)/dt$, $\ddot{x}_0 = d^2x(0)/dt^2$, and $\dddot{x}(0) = d^3x(0)/dt^3$. We set the solution and its derivatives at $t = 0$ equal to these four values:

$$\begin{aligned}
x_0 &= A_1 + A_2 + A_3 sin\phi \\
\dot{x}_0 &= \lambda_1 A_1 + \lambda_2 A_2 + \sigma A_3 sin\phi + \omega A_3 cos\phi \\
\ddot{x}_0 &= \lambda_1^2 A_1 + \lambda_2^2 A_2 + (\sigma^2 - \omega^2) A_3 sin\phi + 2\sigma\omega A_3 cos\phi \\
\dddot{x}_0 &= \lambda_1^3 A_1 + \lambda_2^3 A_2 + (\sigma^3 - 3\sigma\omega^2) A_3 sin\phi + (3\sigma^2\omega - \omega^3) A_3 cos\phi
\end{aligned}$$

These four algebraic equations are *linear* in the four unknowns defined as A_1, A_2, $A_3 sin\phi$, and $A_3 cos\phi$. Note that we solve for the products $A_3 sin\phi$ and $A_3 cos\phi$ as if they are two unknowns; after we obtain their values, we can use trigonometry to separate A_3 and ϕ. The four equations are not directly linear in A_3 and ϕ, so we leave them in the form shown for ease of solution.

Using linear algebra to set up the MATLAB solution, we define the vector of unknowns (to be found) as

$$\vec{z} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 sin\phi \\ A_3 cos\phi \end{pmatrix}$$

and the vector of known initial conditions as

$$\vec{y} = \begin{pmatrix} x_0 \\ \dot{x}_0 \\ \ddot{x}_0 \\ \dddot{x}_0 \end{pmatrix}$$

Using these definitions, our four equations may be written in matrix notation as

$$\vec{y} = \begin{pmatrix} x_0 \\ \dot{x}_0 \\ \ddot{x}_0 \\ \dddot{x}_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & \sigma & \omega \\ \lambda_1^2 & \lambda_2^2 & (\sigma^2 - \omega^2) & 2\sigma\omega \\ \lambda_1^3 & \lambda_2^3 & (\sigma^3 - 3\sigma\omega^2) & (3\sigma^2\omega - \omega^3) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 sin\phi \\ A_3 cos\phi \end{pmatrix} \equiv B\vec{z}$$

Solving for the unknowns,

$$\vec{z} = B^{-1}\vec{y}$$

This approach may be used for any system. Note that the general form of $x_h(t)$ depends on the actual roots of the characteristic equation, and the size of the system may be different, so the exact definitions of $\vec{y}$, $\vec{z}$, and B are different for every problem. But the steps are always the same:

1. Find the CE, solve for its roots, and write the general form of the homogeneous solution $x_h(t)$
2. Using the given initial conditions and the derivatives of $x_h(t)$, define $\vec{y}$ and B
3. Input $\vec{y}$ and B into MATLAB
4. The solution is given by the MATLAB command >> z = inv(B)*y

Once we know numerical values for $A_3 sin\phi$ and $A_3 cos\phi$, we can separate A_3 and ϕ using:

$$tan\phi = \frac{A_3 sin\phi}{A_3 cos\phi} \quad \Rightarrow \quad \text{solve for } \phi \ , \quad \text{then solve for } A_3$$

Alternative formulation: Recall that for a complex conjugate pair of roots, we could also write the general form of the homogeneous solution in this example as

$$x_h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + e^{\sigma t}(B_3 sin(\omega t) + B_4 cos(\omega t))$$

where A_1, A_2, B_3, and B_4 are constants. Again, let the four initial conditions be defined as $x_0 = x(0)$, $\dot{x}_0 = dx(0)/dt$, $\ddot{x}_0 = d^2x(0)/dt^2$, and $\dddot{x}(0) = d^3x(0)/dt^3$. We set the solution and its derivatives at $t = 0$ equal to these four values:

$$\begin{aligned}
x_0 &= A_1 + A_2 + B_4 \\
\dot{x}_0 &= \lambda_1 A_1 + \lambda_2 A_2 + \omega B_3 + \sigma B_4 \\
\ddot{x}_0 &= \lambda_1^2 A_1 + \lambda_2^2 A_2 + (2\sigma\omega)B_3 - (\sigma^2 + \omega^2)B_4 \\
\dddot{x}_0 &= \lambda_1^3 A_1 + \lambda_2^3 A_2 + (3\sigma^2\omega - \omega^3)B_3 + (\sigma^3 - 3\sigma\omega^2)B_4
\end{aligned}$$

These four algebraic equations are *linear* in the four unknowns defined as A_1, A_2, B_3, and B_4. Using these definitions, our four equations may be written in matrix notation as (note the order of B_3 and B_4, which is arbitrary):

$$\vec{y} = \begin{pmatrix} x_0 \\ \dot{x}_0 \\ \ddot{x}_0 \\ \dddot{x}_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & \sigma & \omega \\ \lambda_1^2 & \lambda_2^2 & -(\sigma^2+\omega^2) & 2\sigma\omega \\ \lambda_1^3 & \lambda_2^3 & (\sigma^3 - 3\sigma\omega^2) & (3\sigma^2\omega - \omega^3) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ B_4 \\ B_3 \end{pmatrix} \equiv B\vec{z}$$

Solving for the unknowns,

$$\vec{z} = \begin{pmatrix} A_1 \\ A_2 \\ B_4 \\ B_3 \end{pmatrix} = B^{-1}\vec{y}$$

Notice the placement of the unknowns in this definition of $\vec{z}$; the placement is arbitrary (put the unknowns in whatever sequence you wish), but once it is established, the corresponding entries in the B matrix must match the placement.