

Laplace transform:

$$X(s) \equiv \mathcal{L}(x(t)) \equiv \int_0^{\infty} x(t)e^{-st}\,ds$$

⇒ Use capital letter for Laplace domain, lower case letter for time domain

Inverse Laplace transform:

$$x(t) = \mathcal{L}^{-1}(X(s))$$

- In engineering practice, it's rare to actually calculate Laplace Transforms or inverse Laplace Transforms

  → Laplace Transforms of common functions are well-known

- Concept typically serves as an *intermediate* step to more widely-used results

  → Transfer Function

  → Frequency Response Function

- Laplace transform of a derivative:

$$\mathcal{L}(\dot{x}(t)) = sX(s) - x(0)$$

NOTES

$X(s)$ is a function of s $\iff$ BUT $x(0)$ is the **time-domain** initial condition, i.e., a number
Time domain differentiation $\approx$ multiplication by s in Laplace domain

Laplace Transform performs conversion from differential to algebraic

- Laplace transform of a second derivative (apply the above twice):

$$\mathcal{L}(\ddot{x}(t)) = sL(\dot{x}(t)) - \dot{x}(0) = s^2X(s) - sx(0) - \dot{x}(0)$$

- Laplace transform of an n^{th} derivative

$$\mathcal{L}(\frac{d^n}{dt^n}x(t)) = s^nX(s) - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \cdots - \frac{d^{n-1}}{dt^{n-1}}x(0)$$

- Laplace transform of an integral:

$$\mathcal{L}(\int_0^t x(\tau)d\tau) = \frac{X(s)}{s}$$

- For any *real* constant c: $\mathcal{L}(cx(t)) = cX(s)$
- Distributive: $\mathcal{L}(f(t) + g(t)) = F(s) + G(s)$
- $\mathcal{L}(1)$ = Laplace Transform of unit step input = $\frac{1}{s}$
- $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

$$\Rightarrow \mathcal{L}(ramp) = \mathcal{L}(at + b) = \frac{a}{s^2} + \frac{b}{s} = \frac{as + b}{s^2}$$

- $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$
- $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$
- Note sign: $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

Laplace Transform solution of LTI ODE:

$$a_n \frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + ... + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t)$$

Take LT of both sides:

$$a_n(s^n X(s) - i.c.'s) + a_{n-1}(s^{n-1} X(s) - i.c.'s) + ... + a_1(sX(s) - i.c.) + a_0 X(s) = F(s)$$

Solve for $X(s)$:

$$X(s) = \frac{F(s) + (lots\ of\ i.c.\ terms)}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \tag{1}$$

Then

$$x(t) = \mathcal{L}^{-1}(X(s))$$

Note that the RHS of $X(s)$ may be split into

$$X(s) = \frac{F(s) + (lots\ of\ i.c.\ terms)}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$= \frac{F(s)}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} + \frac{(lots\ of\ i.c.\ terms)}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \tag{2}$$

$$= \mathcal{L}(x_p) + \mathcal{L}(x_h)$$

Note also that

$$(lots\ of\ i.c.\ terms) = (n-1)^{th} - \text{order polynomial in } s$$

$$\begin{aligned} &= [a_n s^{n-1} + a_{n-1} s^{n-2} + \cdots + a_2 s + a_1] x(0) \\ &\quad + [a_n s^{n-2} + a_{n-1} s^{n-3} + \cdots + a_3 s + a_2] \dot{x}(0) \\ &\quad + [a_n s^{n-3} + a_{n-1} s^{n-4} + \cdots + a_4 s + a_3] \ddot{x}(0) \\ &\quad \cdots \\ &\quad + [a_n] \frac{d^{n-1} x(0)}{dt^{n-1}} \end{aligned}$$

$$\begin{aligned} &= s^{n-1}[a_n x(0)] \\ &\quad + s^{n-2}[a_n \dot{x}(0) + a_{n-1} x(0)] \\ &\quad + s^{n-3}[a_n \ddot{x}(0) + a_{n-1} \dot{x}(0) + a_{n-2} x(0)] \\ &\quad \cdots \\ &\quad + s[a_n \frac{d^{n-2} x(0)}{dt^{n-2}} + a_{n-1} \frac{d^{n-3} x(0)}{dt^{n-3}} + \cdots + a_2 x(0)] \\ &\quad + [a_n \frac{d^{n-1} x(0)}{dt^{n-1}} + a_{n-1} \frac{d^{n-2} x(0)}{dt^{n-2}} + \cdots + a_2 \dot{x}(0) + a_1 x(0)] \end{aligned} \tag{3}$$

Define the **Transfer Function:**

$$TF \equiv \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})} \quad \text{with all i.c.'s set equal to 0}$$

Examples

From LTI ODE on previous page, we found

$$X(s) = \frac{F(s) + (lots\ of\ i.c.\ terms)}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Input is $f(t)$, and in all cases, we set $i.c.'s = 0$

Define output for TF

1. If output $= x(t)$:
$$TF = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})} = \frac{X(s)}{F(s)} = \frac{1}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$
2. If output $= \dot{x}(t)$:
$$TF = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})} = \frac{sX(s)}{F(s)} = \frac{s}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$
3. If output $= \ddot{x}(t)$:
$$TF = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})} = \frac{s^2X(s)}{F(s)} = \frac{s^2}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$
4. If output $= k_1 x(t) + c_1 \dot{x}(t)$:
$$TF = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})} = \frac{k_1 X(s) + c_1 s X(s)}{F(s)} = \frac{k_1 + c_1 s}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

One system ⇔ Many TF's

The number of $TF's =$ **(number of inputs)**$\times$**(number of outputs)**

For example, system with 3 inputs, 4 outputs has 12 $TF's$

If there are multiple inputs, set all$= 0$ **except for one to find** $TF's$

All $TF's$ **in a system have same denominator; difference is numerator**

$$\textbf{Define } TF \equiv \frac{N(s)}{D(s)}$$

Definitions:

$$\textbf{Poles} \equiv \text{ roots of the TF denominator } D(s)$$
$$\textbf{Zeros} \equiv \text{ roots of the TF numerator } N(s)$$

- All TF's for system have same POLES
- Different TF's may have different ZEROS

Example

Find the transfer functions, and their poles and zeros, for the system modeled by

$$\ddot{x}(t) + 5\dot{x}(t) + 6 = 10t + 4$$

where the outputs are

$$y_1(t) = 7\dot{x}$$
$$y_2(t) = x + 5\dot{x}$$
$$y_3(t) = -3x$$

Let's define the inputs separately, as $u_1(t) = t$ and $u_2(t) = 1$

Since there are 3 outputs and 2 inputs, we will have 6 TF's

$\mathcal{L}\left[\ddot{x}(t) + 5\dot{x}(t) + 6 = 10t + 4\right] = s^2X(s) - sx(0) - \dot{x}(0) - 5X(s) - 5x(0) + 6X(s) = 10U_1(s) + 4U_2(s)$

For all $TF's$, set initial conditions to zero, collect terms to find:

$$(s^2 + 5s + 6)X(s) = 10U_1(s) + 4U_2(s)$$

The 6 $TF's$ and their poles and zeros are:

$$\frac{Y_1(s)}{U_1(s)} = \frac{7sX(s)}{U_1(s)} = \frac{70s}{s^2 + 5s + 6} \qquad \text{Poles} = -2, -3 \text{ ; Zeros} = 0$$

$$\frac{Y_1(s)}{U_2(s)} = \frac{7sX(s)}{U_2(s)} = \frac{28s}{s^2 + 5s + 6} \qquad \text{Poles} = -2, -3 \text{ ; Zeros} = 0$$

$$\frac{Y_2(s)}{U_1(s)} = \frac{(1 + 5s)X(s)}{U_1(s)} = \frac{(10 + 50s)}{s^2 + 5s + 6} \qquad \text{Poles} = -2, -3 \text{ ; Zeros} = -0.2$$

$$\frac{Y_2(s)}{U_2(s)} = \frac{(1 + 5s)X(s)}{U_2(s)} = \frac{(4 + 20s)}{s^2 + 5s + 6} \qquad \text{Poles} = -2, -3 \text{ ; Zeros} = -0.2$$

$$\frac{Y_3(s)}{U_1(s)} = \frac{-3X(s)}{U_1(s)} = \frac{-30}{s^2 + 5s + 6} \qquad \text{Poles} = -2, -3 \text{ ; No zeros}$$

$$\frac{Y_3(s)}{U_2(s)} = \frac{-3X(s)}{U_1(s)} = \frac{-12}{s^2 + 5s + 6} \qquad \text{Poles} = -2, -3 \text{ ; No zeros}$$

NOTE: Having no zeros is not the same as having a zero $= 0$

Poles of TF (Laplace domain) = roots of Characteristic Equation (time domain); same meaning!!

Explanation: Write TF using the *Partial Fraction Expansion*:

Let $p_i \equiv i^{th}$ pole of TF, and $z_i = i^{th}$ zero of TF. The transfer function may be factored as

$$TF = K\frac{(s-z_1)(s-z_2)\dots}{(s-p_1)(s-p_2)\dots} \equiv K\left[\frac{N_1}{s-p_1} + \frac{N_2}{s-p_2} + \dots\right]$$

where the right-hand side is called the *Partial Fraction Expansion*.

$\mathcal{L}^{-1}$ on the RHS gives the identical time-domain terms found in the general expression for the homogeneous solution of the original LTI ODE.

Knowledge of poles of any TF = knowledge of the roots of the characteristic equation

Example

A system is known to have the transfer function

$$\frac{Y(s)}{U(s)} = \frac{4s+4}{s^2+2s+10}$$

The roots of the system's characteristic equation are $\lambda_{1,2} = -1 \pm 3i$

The system is $2nd$-order

The homogeneous solution of the system states is in the form

$$x_h(t) = Ae^{-t}sin(3t+\phi)$$

A TF may have one or more poles = one or more zeros

$\Rightarrow$ **It is still a pole!!**

$\Rightarrow$ **Do not be confused by algebraic cancellation of pole/zero**

Example

A system is known to have the transfer functions

$$\frac{Y_1(s)}{U(s)} = \frac{4s+4}{(s+1)(s^2+2s+10)}$$

$$\frac{Y_2(s)}{U(s)} = \frac{s+5}{(s+1)(s^2+2s+10)}$$

Algebraically, we may simplify:

$$\frac{Y_1(s)}{U(s)} = \frac{4s+4}{(s+1)(s^2+2s+10)} = \frac{4}{s^2+2s+10} \rightarrow \text{Poles } -1 \pm 3i \text{ , No zeros}$$

But the roots of the system's characteristic equation are still $\lambda_{1,2,3} = -1 \ , \ -1 \pm 3i$

The system is still $3rd$-order

The homogeneous solution of the system states is still in the form

$$x_h(t) = A_1 e^{-t} + A_2 e^{-t} sin(3t + \phi)$$

$\Rightarrow$ **The behavior of output $y_1(t)$ will look like $2nd - order$ system,**

but the behavior of the states is still $3rd$-order!

State-Space: Transfer Function Matrix

Using a state-space approach, we may calculate all TF's simultaneously follows:

$$\text{L.T. of state equation:} \qquad \mathcal{L}\left[\dot{\vec{z}}(t) = A\vec{z}(t) + B\vec{u}(t)\right] \Rightarrow s\vec{Z}(s) - \vec{x}(0) = A\vec{Z}(s) + B\vec{U}(s)$$

$$\text{T.F. has i.c.'s set} = 0 : \quad s\vec{Z}(s) = A\vec{Z}(s) + B\vec{U}(s)$$

$$\text{Collect terms on } \vec{Z}(s) : \qquad sZ(s) - A\vec{Z}(s) = (sI - A)\vec{Z}(s) = BU(s) \;\;, \text{ where } I_{n\times n} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\Rightarrow \vec{Z}(s) = (sI - A)^{-1}B\vec{U}(s)$$

We want TF=$\frac{\mathcal{L}(output)}{\mathcal{L}(input)}$:

$$\mathcal{L}\left[\vec{y}(t) = C\vec{z}(t) + D\vec{u}(t)\right] \Rightarrow \vec{Y}(s) = C\vec{Z}(s) + D\vec{U}(s)$$

Substituting,

$$\vec{Y}(s) = \left[C(sI - A)^{-1}B + D\right]\vec{U}(s)$$

The Transfer Function Matrix is defined as

$$TFM_{p\times m} = \left[C(sI - A)^{-1}B + D\right]$$

$$\Rightarrow \text{The element in row } i \text{, column } j \text{ is the transfer function } \frac{\mathcal{L}(Y_i)}{\mathcal{L}(U_j)}$$

The MATLAB function **ss2tf** (the letters stand for 'state space to transfer function') finds the system transfer functions for a state space model:

```
>> [NUM, DEN] = ss2tf (A,B,C,D,iu)
```

This function is a little strange compared with related state space functions in MATLAB:

– it doesn't recognize a state-space 'object' created by the **ss** command; instead, it requires the user to input the state space matrices A, B, C, and D explicitly

– it will only find transfer functions corresponding to one input at a time (the input indicated by the integer, iu) ; therefore, multiple calls are required for systems with more than one input

The output of **ss2tf** consists of all of the transfer functions corresponding to input u_i. Since all TF's have the same denominator, DEN is a vector whose elements are the coefficients of the denominator of all TF's. But since each TF has its own numerator, NUM is a matrix, with one row per output; and the elements of each row are the coefficients of the numerator of the TF for that output

Example

Use **ss2tf** to find the TF's for the system modeled by

$$\ddot{x}(t) + 5\dot{x}(t) + 6x(t) = f_1(t) + 3f_2(t)$$

where the inputs are f_1 and f_2, and the outputs are $y_1(t) = 7\dot{x}$, $y_2(t) = x + 5\dot{x}$, $y_3(t) = -3x$

Converting this system to state space, we have

$$\dot{\vec{z}} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \vec{z}(t) + \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix} \vec{u}(t) \qquad \vec{y}(t) = \begin{pmatrix} 0 & 7 \\ 1 & 5 \\ -3 & 0 \end{pmatrix} \vec{z}(t) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{u}(t)$$

We can find the transfer functions from

```
>> A=[0 1;-6 -5];B=[0 0;1 3];C=[0 7;1 5;-3 0];D=[0 0;0 0;0 0];
>> [Num1,Den1]=ss2tf(A,B,C,D,1)
   Num1  =

       0   7.0000   - 0.0000
       0   5.0000   1.0000
       0      0     - 3.0000

   Den1  =

       1.0000   5.0000   6.0000


>> [Num2,Den2]=ss2tf(A,B,C,D,2)
   Num2  =

       0   21.0000   - 0.0000
       0   15.0000   3.0000
       0      0      - 9.0000

   Den2  =

       1.0000   5.0000   6.0000
```

The system transfer functions are thus:

$$\frac{Y_1}{U_1} = \frac{7s}{s^2 + 5s + 6} \quad , \quad \frac{Y_2}{U_1} = \frac{5s + 1}{s^2 + 5s + 6} \quad , \quad \frac{Y_3}{U_1} = \frac{-3}{s^2 + 5s + 6}$$

$$\frac{Y_1}{U_2} = \frac{21s}{s^2 + 5s + 6} \quad , \quad \frac{Y_2}{U_2} = \frac{15s + 3}{s^2 + 5s + 6} \quad , \quad \frac{Y_3}{U_2} = \frac{-9}{s^2 + 5s + 6}$$

The Final Value Theorem:

$$\lim_{s\to 0}\left(TF \equiv \frac{Y(s)}{U(s)}\right) = y(t\to\infty) \text{ for } u(t) = \begin{pmatrix} 0 & t<0 \\ 1 & t\geq 0 \end{pmatrix}$$

Example

Find the outputs as $t\to\infty$ for the system modeled by

$$\ddot{x}(t) + 5\dot{x}(t) + 6x(t) = f(t)$$

where the outputs are

$$\begin{aligned} y_1(t) &= 7\dot{x} \\ y_2(t) &= x + 5\dot{x} \\ y_3(t) &= -3x \end{aligned}$$

Define the input as $u_(t) = f(t)$

Since there are 3 outputs and 1 input, we have 3 TF's:

$$\mathcal{L}\left[\ddot{x}(t) + 5\dot{x}(t) + 6x(t) = f(t)\right] = s^2X(s) - sx(0) - \dot{x}(0) - 5X(s) - 5x(0) + 6X(s) = U(s)$$

For all $TF's$, set initial conditions to zero, collect terms to find:

$$(s^2+5s+6)X(s) = U(s)$$

The 3 $TF's$ are evaluated as $s\to 0$ to find $y_i(t\to\infty)$:

$$\frac{Y_1(s)}{U(s)} = \frac{7sX(s)}{U(s)} = \frac{7s}{s^2+5s+6} \qquad y_1(t\to\infty) = \lim_{s\to 0}\frac{7s}{s^2+5s+6} = 0$$

$$\frac{Y_2(s)}{U(s)} = \frac{(1+5s)X(s)}{U(s)} = \frac{1+5s}{s^2+5s+6} \qquad y_2(t\to\infty) = \lim_{s\to 0}\frac{1+5s}{s^2+5s+6} = \frac{1}{6}$$

$$\frac{Y_3(s)}{U(s)} = \frac{-3X(s)}{U(s)} = \frac{-3}{s^2+5s+6} \qquad y_3(t\to\infty) = \lim_{s\to 0}\frac{-3}{s^2+5s+6} = -\frac{1}{2}$$

Check by calculating the solution in time domain:
The characteristic equation is $\lambda^2 + 5\lambda + 6 = 0$, so the roots are $\lambda_{1,2} = -2, -3$. Therefore,

$$x_h(t) = A_1e^{-2t} + A_2e^{-3t}$$

Since $f(t)$=constant, the particular solution is $x_p = \textit{constant} = c$, so $\dot{x}_p = \ddot{x}_p = 0$. Substitute into the LTI ODE to find:

$$0 + 5*0 + 6c = f(t) = 1 \to c = \frac{1}{6} = x_p(t)$$

Combining the homogeneous and particular solutions,

$$x(t) = A_1e^{-2t} + A_2e^{-3t} + \frac{1}{6}$$

Thus, we see that $x(t\to\infty) = \frac{1}{6}$; substituting into the output equations, we confirm the result.

Initial Value Theorem:

$$\lim_{s\to\infty}\left(TF \equiv \frac{Y(s)}{U(s)}\right) = y(0^+) - y(0^-) \text{ for } u(t) = \begin{pmatrix} 0 & t<0 \\ 1 & t\geq 0 \end{pmatrix}$$

Example

Find the instantaneous change in outputs when a step input is applied to the system modeled by

$$\ddot{x}(t) + 5\dot{x}(t) + 6 = f(t)$$

where the outputs are

$$\begin{aligned} y_1(t) &= 7\dot{x} \\ y_2(t) &= x + 5\dot{x} \\ y_3(t) &= -3\ddot{x} \end{aligned}$$

The first 2 $TF's$ were found in the previous example. The third TF is found using the identical procedure. Then, all 3 TF's are evaluated as $s \to \infty$ to find $y_i(0^+) - y_i(0^-)$:

$$\frac{Y_1(s)}{U(s)} = \frac{7sX(s)}{U(s)} = \frac{7s}{s^2+5s+6} \qquad y_1(0^+) - y_1(0^-) = \lim_{s\to\infty}\frac{7s}{s^2+5s+6} = 0$$

$$\frac{Y_2(s)}{U(s)} = \frac{(1+5s)X(s)}{U(s)} = \frac{1+5s}{s^2+5s+6} \qquad y_2(0^+) - y_2(0^-) = \lim_{s\to\infty}\frac{1+5s}{s^2+5s+6} = 0$$

$$\frac{Y_3(s)}{U(s)} = \frac{-3s^2X(s)}{U(s)} = \frac{-3s^2}{s^2+5s+6} \qquad y_3(0^+) - y_3(0^-) = \lim_{s\to\infty}\frac{-3s^2}{s^2+5s+6} = -3$$

Check by calculating the solution in time domain:
For $t < 0$, we have $u = 0$ and thus $x_p = 0$. Since $x_h(t) = A_1e^{-2t} + A_2e^{-3t}$, applying initial conditions, we find that for $t < 0$,

$$\begin{pmatrix} x(0^-) & = 0 = A_1 + A_2 \\ \dot{x}(0^-) & = 0 = -2A_1 - 3A_2 \end{pmatrix} \Rightarrow A_1 = 0, A_2 = 0 \Rightarrow \ddot{x}(0^-) = 0$$

The solution for input $u = 1$ $(t \geq 0)$ was found in the previous example

$$x(t) = A_1e^{-2t} + A_2e^{-3t} + \frac{1}{6}$$

Applying initial conditions of $x(0) = 0$, $\dot{x}(0) = 0$, we find that for $t \geq 0$:

$$\begin{pmatrix} x(0^+) & = 0 = A_1 + A_2 + \frac{1}{6} \\ \dot{x}(0^+) & = 0 = -2A_1 - 3A_2 \end{pmatrix} \Rightarrow A_1 = 0.5, A_2 = -\frac{1}{3} \Rightarrow \ddot{x}(0^+) = -4A_1 - 9A_2 = 1$$

Combining the solutions from $t = 0^-$ and $t = 0^+$, we see that

$$\begin{aligned} y_1(0^+) - y_1(0^-) &= 7\dot{x}(0^+) - 7\dot{x}(0^-) = 0 \\ y_2(0^+) - y_2(0^-) &= x(0^+) + 5\dot{x}(0^+) - \big(x(0^-) + 5\dot{x}(0^-)\big) = 0 \\ y_3(0^+) - y_3(0^-) &= -3\ddot{x}(0^+) - (-3\ddot{x}(0^-)) = -3 \end{aligned}$$