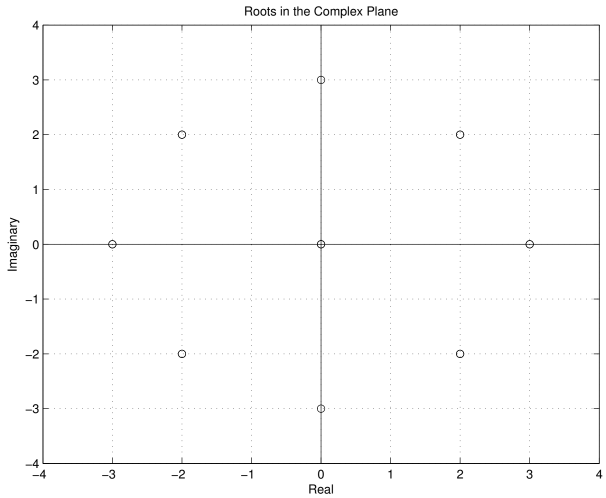
The Complex Plane



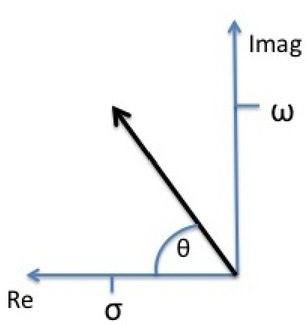
All roots are either *real* or *complex conjugate pairs*. The figure to the left shows a complex plane with a total of 6 different types of roots. Make sure that you could label each root in the complex plane as one of the following:

1. A real root at zero
2. A negative real root
3. A positive real root
4. A complex conjugate pair of roots with (+) real part
5. A complex conjugate pair of roots with (-) real part
6. A complex conjugate pair of roots with (0) real part

The complex plane is very convenient for illustrating roots, so it's important to correlate the behavior of the system with regions of this plane

- Since all complex roots must form a complex conjugate pair, the plot of roots in the complex plane must be symmetric about the real axis
- The *left-half plane* (LHP for short) is the region associated with stability, since all roots of a stable system must have negative real parts. Alternatively, the right-half plane (RHP) is associated with unstable systems. Thus, we may say that a *necessary condition* for a stable system is that all roots appear in the LHP.
- *Settling time* (speed of response) is proportional to the inverse of the real parts of stable roots; therefore, the farther to the left that roots appear in the complex plane, the faster the settling time.
  - Since the *system* settling time is based on the slowest root, we may obtain it directly from the plot using the root that is *farthest to the right* in a stable system
  - Thus, we may say that a design goal - to obtain the fastest possible system - is to select the system whose roots are such that the root that is farthest to the right, is as far to the left as possible
- *Damped natural frequency* is equal to the imaginary part of complex conjugate pairs of roots
  - All damped natural frequencies in a system are easily read from the same complex plane
  - If all roots appear on the real axis, the homogeneous solution has no oscillation
- The *damping ratio* for any pair of complex conjugate roots is equal to the cosine of the angle θ
  Consider roots $\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} \equiv -\sigma \pm i\omega$, with $\lambda = -\sigma + i\omega$ shown in the figure. From geometry,

$$cos\theta = \sigma/\sqrt{\sigma^2+\omega^2} = \zeta\omega_n/\sqrt{(\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2} = \zeta\omega_n/\sqrt{\omega_n^2} = \zeta = \text{damping ratio!}$$



- A steeper angle (larger θ) means more cycles of oscillation before settling time
- Most often, we don't want oscillations, so a common design goal is to keep θ as small as possible
- For $\zeta > 1$, $cos\theta$ cannot provide actual ζ value
- Undamped system has $\theta = 90^o$
- The distance from the origin to the root is ω_n