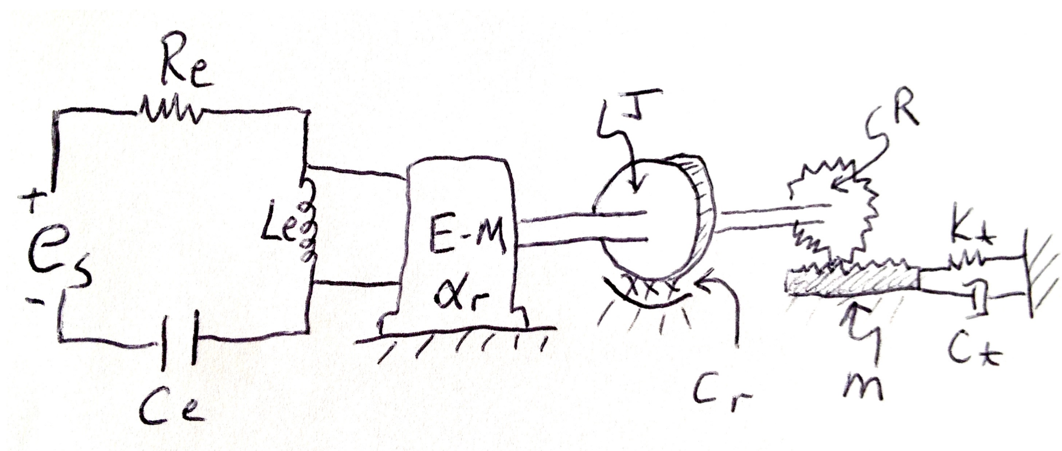


The system shows an electrically-powered rack-and-pinion system. The input is the voltage source. The electric system drives the output shaft of the EM transducer, which ends at the pinion gear. The shaft has inertia J and rotational damping C_r; the gear has radius R. The rack has mass m and is connected to the wall through the spring and damper as shown.

The outputs are (1) the voltage e_{Le}, and (2) the force exerted on the wall by the spring and damper.

1. Find a state-space model of the system. Do not use more than the minimum necessary number of states.
2. Assume that the system is at rest (all states equal to zero) with $e_s = 0$, when suddenly at $t = 0$, $e_s = V =$ constant (for example, a switch is thrown). What are the initial conditions of your state vector?

Solution

1. Using the *usual* subsystem states, we have

$$\vec{z}(t) = \begin{pmatrix} e_{Ce} \\ i_{Le} \\ \theta_J \\ \dot{\theta}_J \\ x \\ \dot{x} \end{pmatrix} \quad , \quad \vec{u}(t) = \begin{pmatrix} e_s \end{pmatrix} \quad , \quad \vec{y}(t) = \begin{pmatrix} e_{Le} \\ k_t x + c_t \dot{x} \end{pmatrix}$$

However, we don't need them all. The rack and pinion give $x = R\theta_J$ and $\dot{x} = R\dot{\theta}_J$, so we can eliminate either x or θ_J, and either $\dot{x}$ or $\dot{\theta}_J$. Let's keep x and $\dot{x}$ since the output uses them directly. Our state vector is now:

$$\vec{z}(t) = \begin{pmatrix} e_{Ce} \\ i_{Le} \\ x \\ \dot{x} \end{pmatrix}$$

The electrical system state equations are:

$$\dot{e}_{Ce} = \frac{1}{C_e} i_{Ce} = \frac{1}{C_e}\left(\frac{e_{Re}}{R_e}\right) = \frac{1}{C_e}\left(\frac{e_s - e_{Ce} - e_{EM}}{R_e}\right) = \frac{1}{C_e}\left(\frac{e_s - e_{Ce} - \dot{\theta}/\alpha_r}{R_e}\right) = \frac{1}{C_e}\left(\frac{e_s - e_{Ce} - \dot{x}/(\alpha_r R)}{R_e}\right)$$

$$\dot{i}_{Le} = \frac{1}{L_e} e_{Le} = \frac{1}{L_e}(e_{EM}) = \frac{1}{L_e}\frac{\dot{\theta}}{\alpha_r} = \frac{1}{L_e}\left(\frac{\dot{x}}{\alpha_r R}\right)$$

Note that if $i_{Le}(0) = x(0)$, then we can integrate to find $i_{Le} = \frac{1}{L_e}\left(\frac{x}{\alpha_r R}\right)$

Let's assume that we have $i_{Le}(0) = x(0) = 0$, so we again keep x, and eliminate i_{Le}. Thus,

$$\vec{z}(t) = \begin{pmatrix} e_{Ce} \\ x \\ \dot{x} \end{pmatrix}$$

Moving to the mechanical side, let f denote the contact force between rack and pinion. The pinion equation is

$$J\ddot{\theta}_J = T_{EM} - c_r\dot{\theta}_J - Rf$$

where

$$\begin{aligned} T_{EM} &= \frac{1}{\alpha_r} i_{EM} = \frac{1}{\alpha_r}(i_{Ce} - i_{Le}) = \frac{1}{\alpha_r}\left[\frac{e_s - e_{Ce} - \dot{x}/(\alpha_r R)}{R_e} - i_{Le}\right] \\ &= \frac{1}{\alpha_r}\left[\frac{e_s - e_{Ce} - \dot{x}/(\alpha_r R)}{R_e} - \frac{1}{L_e}(\frac{x}{\alpha_r R})\right] \end{aligned}$$

Substituting back into the pinion equation and solving for f,

$$f = \frac{1}{R}(T_{EM} - c_r\dot{\theta}_J - J\ddot{\theta}_J) = \frac{1}{R}(\frac{1}{\alpha_r}\left[\frac{e_s - e_{Ce} - \dot{x}/(\alpha_r R)}{R_e} - \frac{1}{L_e}(\frac{x}{\alpha_r R})\right] - \frac{c_r}{R}\dot{x} - \frac{J}{R}\ddot{x})$$

The rack equation can now be obtained:

$$\begin{aligned} m\ddot{x} =& f - k_t x - c_t\dot{x} \\ =& \frac{1}{R}(\frac{1}{\alpha_r}\left[\frac{e_s - e_{Ce} - \dot{x}/(\alpha_r R)}{R_e} - \frac{1}{L_e}(\frac{x}{\alpha_r R})\right] - \frac{c_r}{R}\dot{x} - \frac{J}{R}\ddot{x}) - k_t x - c_t\dot{x} \end{aligned}$$

Solve for

$$\ddot{x} = (\frac{1}{m + J/R^2})\Big(\frac{1}{R}\left[\frac{1}{\alpha_r}(\frac{e_s - e_{Ce} - \dot{x}/(\alpha_r R)}{R_e} - \frac{1}{L_e}(\frac{x}{\alpha_r R})) - \frac{c_r}{R}\dot{x}\right] - k_t x - c_t\dot{x}\Big)$$

The state equation is

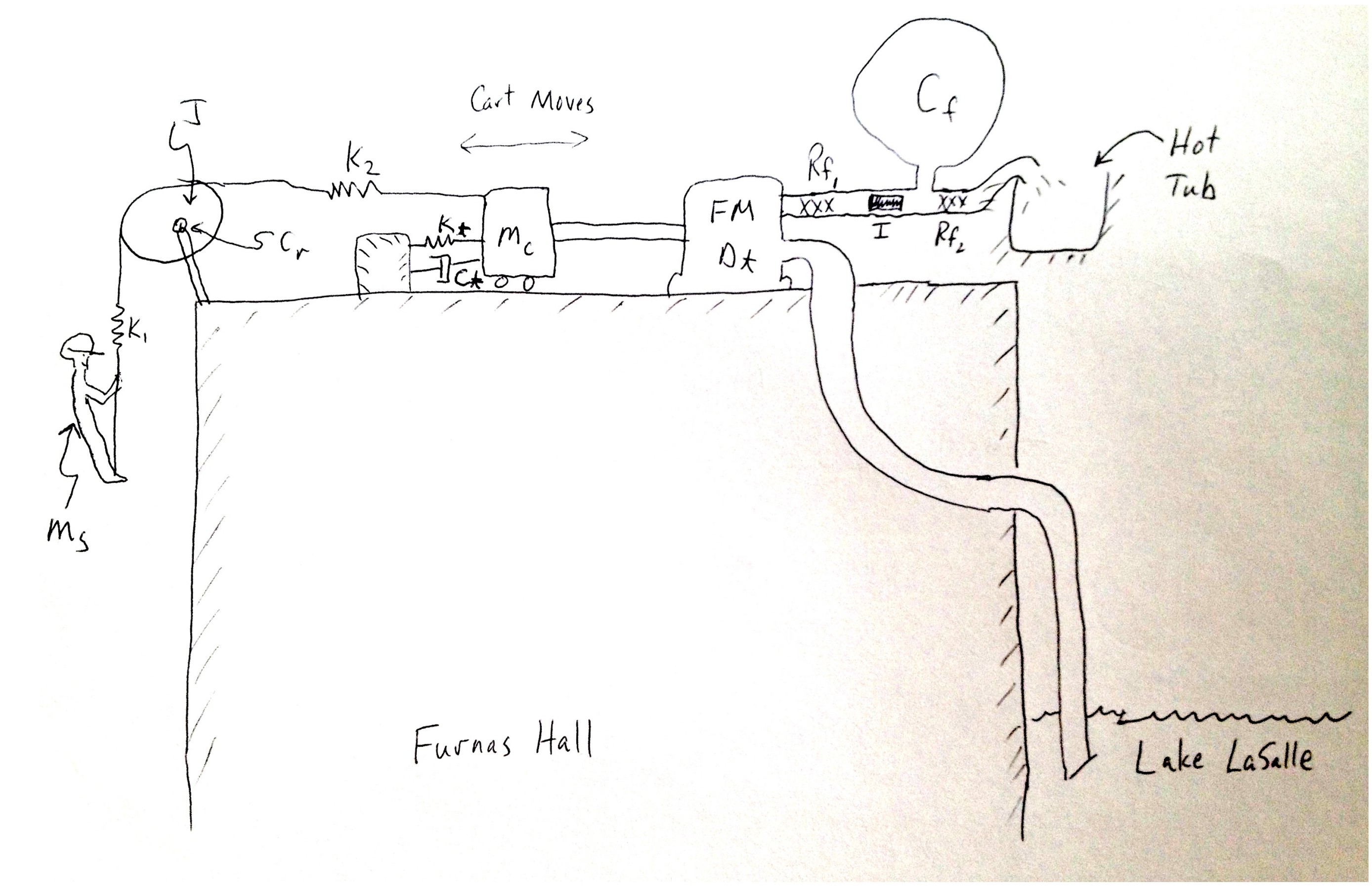
$$\begin{aligned} \dot{\vec{z}}(t) =& A\vec{z} + B\vec{u} \\ =& \begin{pmatrix} -\frac{1}{C_e R_e} & 0 & -\frac{1}{\alpha_r R C_e R_e}) \\ 0 & 0 & 1 \\ -(\frac{1}{m+J/R^2})(\frac{1}{R\alpha_r R_e}) & -(\frac{1}{m+J/R^2})(\frac{1}{L_e\alpha_r^2 R^2} + k_t) & -(\frac{1}{m+J/R^2})(\frac{1}{R^2\alpha_r^2 R_e} + \frac{c_r}{R^2} + c_t) \end{pmatrix}\vec{z} + \begin{pmatrix} \frac{1}{C_e R_e} \\ 0 \\ (\frac{1}{m+J/R^2})(\frac{1}{R\alpha_r R_e}) \end{pmatrix}\vec{u} \end{aligned}$$

The output equation is

$$\vec{y}(t) = C\vec{y} + D\vec{u} = \begin{pmatrix} 0 & 0 & \frac{1}{\alpha_r R} \\ 0 & k_t & c_t \end{pmatrix}\vec{z} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\vec{u}$$

2. Now consider the initial conditions. In the instant that the switch is thrown, we know from element laws that neither e_C nor i_L can change value instantaneously, so both remain 0. Thus, $z_1(0) = e_{Ce}(0) = 0$ and $z_2(0) = x(0) = L_e\alpha_r R i_{Le}(0) = 0$. Similarly, on the mechanical side, due to the inertia, we know that neither x nor $\dot{x}$ can change value instantaneously, so $z_3(0) = \dot{x}(0) = 0$. Therefore, the IC's are

$$\vec{z}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$



The valedictorian of MAE 340 decides to build a rooftop hot tub on Furnas Hall.

To fill the hot tub, free water will be pumped out of Lake Lasalle; the input for the pump will come from gravity. The student lowers himself over the side of the building on a cable with some stretch, modeled by k_1. The student's mass is m_s. The cable goes over a large drum (cylinder) with inertia J and radius R, mounted on an axle with rotational damping C_r. The cable then attaches to spring k_2 which is attached to the cart m_c. The cart can move translationally. It is attached to a wall through k_t and c_t, and it is attached to the shaft of the translational M-F pump with parameter D_t. The pump draws water out of Lake Lasalle and pushes it into the fluid system. For future use, there is a tank with capacitance C_f. The water flowing into the hot tub from the end of the pipe as shown.

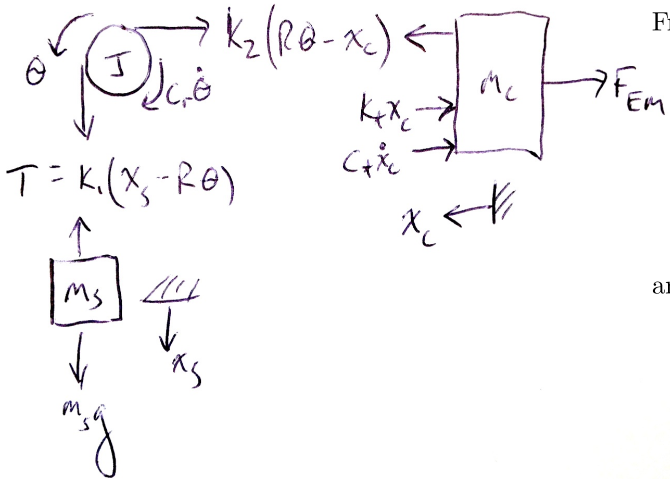
You may neglect the effect of gravity on the water.

The output of the system is the flow rate into the hot tub.

1. Find a state-space model of the system. Do not use more than the minimum necessary number of states.
2. Assume that the student is standing on a ledge with all springs in the undeformed position, and all states equal to zero, when suddenly at $t = 0$, he steps off of the ledge and the cord attached to him instantaneously has tension $= m_s g$. What are the initial conditions of your state vector?

Solution

1.



From the FBD's, we can write:

$$m_s\ddot{x}_s = m_s g - k_1(x_s - R\theta)$$
$$J\ddot{\theta} = R\big[k_1(x_s - R\theta) - k_2(R\theta - x_c)\big] - c_r\dot{\theta}$$
$$m_c\ddot{x}_c = k_2(R\theta - x_c) - k_t x_c - c_t\dot{x}_c - F_{MF}$$

and from the fluid side,

$$\dot{P}_{Cf} = \frac{1}{C_f}(Q_{Cf}) = \frac{1}{C_f}(Q_{If} - \frac{P_{Cf}}{R_{f2}})$$
$$\dot{Q}_{If} = \frac{1}{I_f}P_{If} = \frac{1}{I_f}(P_{MF} - R_{f1}Q_{If} - P_{Cf})$$

Our *usual* state vector, constructed from the *usual* subsystem states, is

$$\vec{z}(t) = \begin{pmatrix} x_s \\ \dot{x}_s \\ \theta \\ \dot{\theta} \\ x_c \\ \dot{x}_c \\ P_{Cf} \\ Q_{If} \end{pmatrix}$$

However, from the transducer equations, we have

$$\dot{v} = \frac{1}{D_t}Q_{MF} \quad \Rightarrow \; \dot{x}_c = \frac{1}{D_t}Q_{If}$$

Thus, we may eliminate one of these two states, say, Q_{If}. To eliminate F_{MF}, we use the other transducer equation:

$$P_{MF} = \frac{1}{D_t}F_{MF} = I_f\dot{Q}_{If} + R_{f1}Q_{If} + P_{Cf} \quad \Rightarrow F_{MF} = D_t\big[I_f D_t\ddot{x}_c + R_{f1}D_t\dot{x}_c + P_{Cf}\big]$$

Substituting into the $\ddot{x}_c$ equation,

$$m_c\ddot{x}_c = k_2(R\theta - x_c) - k_t x_c - c_t\dot{x}_c - D_t\big[I_f D_t\ddot{x}_c + R_{f1}D_t\dot{x}_c + P_{Cf}\big]$$

Collect terms to find

$$\ddot{x}_c = \frac{1}{(m_c + I_f D_t^2)}\big[k_2 R\theta - (k_2 + k_t)x_c - (c_t + R_{f1}D_t^2)\dot{x}_c - D_t P_{Cf}\big]$$

The state equation is (defining $\vec{u} \equiv (g)$)

$$\begin{aligned}\dot{\vec{z}}(t) =& A\vec{z} + B\vec{u} \\ =& \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -k_1/m_s & 0 & k_1R/m_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ Rk_1/J & 0 & -R^2(k_1+k_2)/J & -c_r/J & Rk_2/J & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{Rk_2}{(m_c+I_fD_t^2)} & 0 & -\frac{(k_2+k_t)}{(m_c+I_fD_t^2)} & -\frac{c_t+R_{f1}D_t^2}{(m_c+I_fD_t^2)} & -\frac{D_tP_{Cf}}{(m_c+I_fD_t^2)} \end{pmatrix}\vec{z} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\vec{u}\end{aligned}$$

The output equation is

$$\vec{y}(t) = (Q_{Rf2}) = (\frac{P_{Cf}}{R_{f2}}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1/R_{f2} \end{pmatrix} \vec{z} + \begin{pmatrix} 0 \end{pmatrix} \vec{u}$$

2. The effect of the step input is an instantaneous change in $\ddot{x}_c$; therefore $\vec{z}(0) = 0$.