Concept: Convert all LTI ODE models into a standardized form that is:

- Analytically efficient - many generic results for standard state-space form
- Computationally efficient - many built-in MATLAB functions for standard state-space form
- Model construction efficient - state-space model form and solution is independent of underlying physical system

Standard linear state-space form consists of two equations:

$$\text{State Equation:} \qquad \dot{\vec{z}}(t) = A\vec{z} + B\vec{u} \tag{1}$$

$$\text{Output Equation:} \qquad \vec{y} = C\vec{z} + D\vec{u} \tag{2}$$

The seven (7) components of the standard state-space model:

1. **State Vector** $\vec{z}(t)$: an $n \times 1$ vector of time-dependent variables that represent the fundamental dynamics of the system, i.e., the homogeneous LTI ODE.
2. **Input Vector** $\vec{u}(t)$: an $m \times 1$ vector that represents any external inputs to the LTI ODE
3. **Output Vector** $\vec{y}(t)$: an $p \times 1$ vector that represents the system outputs (more discussion to follow)
4. **State Matrix** A: an $n \times n$ matrix that relates $\dot{\vec{z}}(t)$ to $\vec{z}(t)$, i.e., the homogeneous system equation
5. **Input Matrix** B: an $n \times m$ matrix that relates the inputs $\vec{u}(t)$ to $\dot{\vec{z}}(t)$
6. **Output Matrix** C: an $p \times n$ matrix that relates the states $\vec{z}(t)$ to $\vec{y}(t)$
7. **Direct Transmission Matrix** D: an $p \times m$ matrix that relates the inputs $\vec{u}(t)$ to $\vec{y}(t)$

Example:
Suppose our LTI ODE is given by

$$\dddot{x}(t) + 5\ddot{x}(t) + 12\dot{x}(t) + 3x(t) = 4sin(3t) + 8$$

We can define the state vector and the input vector as

$$\vec{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} \equiv \begin{pmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix} \qquad , \qquad \vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \equiv \begin{pmatrix} sin(3t) \\ 1 \end{pmatrix}$$

Using these definitions, we may write:

$$\dot{\vec{z}}(t) = \begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \dot{z}_3(t) \end{pmatrix} \equiv \begin{pmatrix} z_2(t) \\ z_3(t) \\ -3z_1(t) - 12z_2(t) - 5z_3(t) + 4u_1(t) + 8u_2(t) \end{pmatrix}$$

We can now construct the state equation in the standard form of Eq. (1):

$$\dot{\vec{z}}(t) = A\vec{z} + B\vec{u} \qquad \Rightarrow \quad \dot{\vec{z}}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -12 & -5 \end{pmatrix} \vec{z}(t) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 4 & 8 \end{pmatrix} \vec{u}(t)$$

$\Rightarrow$ The *state equation* is always a set of 1st-order LTI ODE's

Outputs: In most real-world systems, we are not really interested in every state; we are only interested in a relatively small number of *outputs*. For example,

1. Modeling the full dynamics of a car might require many states, but we may be only interested in the force in the shock absorber
2. Modeling the full dynamics of the HVAC system in a large building may require many states, but we may be only interested in room temperature and humidity
3. Modeling the full dynamics of an electrical system may require many states, but we may be interested only in the voltage at some specific location within the system

In selecting outputs, we note the following:

1. We often choose outputs that are measurable $\Rightarrow$ hopefully by cheap, reliable, accurate instruments!
2. Usually $p < n$; often $p << n$; sometimes $p = 1$ $\Rightarrow$ the number of states required to accurately model the dynamic behavior could be very large, but we sometimes only care about a single output
3. The choice of outputs may change many times during design and/or operation of a system

Example:
Consider again the system with LTI ODE given by

$$\dddot{x}(t) + 5\ddot{x}(t) + 12\dot{x}(t) + 3x(t) = 4sin(3t) + 8$$

We defined the state vector and the input vector as

$$\vec{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} \equiv \begin{pmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix} \quad , \quad \vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \equiv \begin{pmatrix} sin(3t) \\ 1 \end{pmatrix}$$

leading to the state equation in the standard form of Eq. (1):

$$\dot{\vec{z}}(t) = A\vec{z} + B\vec{u} \quad \Rightarrow \quad \dot{\vec{z}}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -12 & -5 \end{pmatrix} \vec{z}(t) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 4 & 8 \end{pmatrix} \vec{u}(t)$$

Now suppose the original model is of a car suspension system, and that we are interested in the force in the spring and the force in the shock absorber. We define these as

$$\begin{aligned} \text{Force in spring } &= kx(t) \\ \text{Force in shock } &= c\dot{x}(t) \end{aligned}$$

To put these two outputs into the form of Eq. (2), we write

$$\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} kx(t) \\ c\dot{x}(t) \end{pmatrix} = \begin{pmatrix} kz_1(t) \\ cz_2(t) \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & c & 0 \end{pmatrix} \vec{z}(t) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{u}(t) = C\vec{z}(t) + D\vec{u}(t)$$

Notes:

1. **State Vector** represents the fundamental dynamics of the system, i.e., the homogeneous LTI ODE. For a single n^{th}-order LTI ODE, we may always define the state vector as

$$\vec{z}(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \\ \dots \\ \frac{d^{n-1}x(t)}{dt^{n-1}} \end{pmatrix}$$

   This definition always works, but is not the only possibility. **For now, we will always use it.**

   Shortly, we will expand our capabilities to systems of higher-order LTI ODE's, but in each case, the basic pattern shown above will work. Later in the course, we will see how to define states directly in terms of physical variables for certain types of systems.

   The use of the word *state* is exactly analogous to its use in the *State of the Union* speech that the US President gives every year. The *state* is a description of what shape (or condition) things are in; and this is exactly what a state vector is for a dynamic system, a description of the current condition of the system.

2. **Input Vector** $\vec{u}(t)$: an $m \times 1$ vector that represents any external inputs to the LTI ODE.

   There is always mathematical ambiguity in the definition of $\vec{u}(t)$; we only need the product $B\vec{u}(t)$ to be precise. Thus, we may always adjust constants in B and $\vec{u}(t)$, as long as $B\vec{u}(t)$ is correct. However, physically, there is often a precise definition of $u(t)$

3. **Output Vector** $\vec{y}(t)$: an $p \times 1$ vector that represents the system outputs. As noted, we typically have fewer outputs than states, and, we typically select outputs that are relatively cheap and easy to measure. The reason for this is that we want to control the world, so we choose outputs that we can easily monitor in order to see that our world domination is working.

4. **State Matrix** A: an $n \times n$ matrix that relates $\dot{\vec{z}}(t)$ to $\vec{z}(t)$, i.e., the homogeneous system equation.

   Since the state matrix represents the homogeneous solution, we might expect that we can find the roots of the characteristic equation directly from it; and indeed, it can be shown that the *eigenvalues* of A are equal to the roots of the characteristic equation!

5. **Input Matrix** B: an $n \times m$ matrix that relates the inputs $\vec{u}(t)$ to $\dot{\vec{z}}(t)$

   As discussed above, the product $B\vec{u}(t)$ must be precise, but any combination of B and $\vec{u}(t)$ that satisfies this is mathematically correct. If we have a physical definition of $\vec{u}(t)$, we should use it and define B accordingly.

6. **Output Matrix** C: an $p \times n$ matrix that relates the states $\vec{z}(t)$ to $\vec{y}(t)$

   This matrix essentially selects from the state vector to form the output vector. In most cases, the output vector is smaller than the state vector. The choice of outputs is often based on measurements, so the output matrix is used to create the proper physical measurement out of the states.

7. **Direct Transmission Matrix** D: an $p \times m$ matrix that relates the inputs $\vec{u}(t)$ to $\vec{y}(t)$

   This matrix is called *direct transmission* because it represents situations where the external inputs have some direct effect on the outputs, without going through the actual system (but note, the same inputs may also be affecting the outputs through the system). Typically, we visualize the situation as the input acting on the system, which causes some system response, and our output is a measure of that response. But sometimes inputs directly affect outputs, not through the system, and the direct transmission matrix D accounts for that possibility.

   D is often zero because direct transmission is not so common, but note that the dimensions must be $p \times m$

For an LTI ODE, we may always define states using the pattern:

$$\vec{z}(t) \equiv \begin{pmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \\ \dots \\ \frac{d^{n-1}}{dt^{n-1}}(x(t)) \end{pmatrix}$$

Using this pattern, write the state-space model for the following system:

$$\dddot{x}(t) + 4\ddot{x}(t) - 3\dot{x}(t) - 12x(t) = 5sin(8t) + 12 + 3e^{-t}$$

where the output is $7\ddot{x} + 3\dot{x} + 2x$

Solution: Define $\vec{z}(t)$, $\vec{u}(t)$, substitute, and collect terms:

$$\vec{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{pmatrix} \equiv \begin{pmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix} \quad , \quad \vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} \equiv \begin{pmatrix} sin(8t) \\ 1 \\ e^{-t} \end{pmatrix}$$

Using these definitions, we may write the state equation:

$$\dot{\vec{z}}(t) = \begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \dot{z}_3(t) \end{pmatrix} \equiv \begin{pmatrix} z_2(t) \\ z_3(t) \\ 12z_1(t) + 3z_2(t) - 4z_3(t) + 5u_1(t) + 12u_2(t) + 3u_3(t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & 3 & -4 \end{pmatrix} \vec{z}(t) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 5 & 12 & 3 \end{pmatrix} \vec{u}(t) = A\vec{z} + B\vec{u}$$

and the output equation:

$$\vec{y}(t) = \begin{pmatrix} 7\ddot{x} + 3\dot{x} + 2x \end{pmatrix} = \begin{pmatrix} 2z_1 + 3z_2 + 7z_3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 7 \end{pmatrix} \vec{z}(t) + \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \vec{u}(t)$$

$$= C\vec{z}(t) + D\vec{u}(t)$$

Consider the generic LTI ODE:

$$a_n \frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + ... + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t)$$

We may define n states using the pattern described above:

$$\vec{z}(t) \equiv \begin{pmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \\ \dots \\ \frac{d^{n-1}}{dt^{n-1}}(x(t)) \end{pmatrix}$$

Defining $\vec{u}(t) \equiv f(t)$, we can write the standard form of the state-space equation as

$$\dot{\vec{z}}(t) = \begin{pmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dddot{x}(t) \\ \dots \\ \frac{d^n}{dt^n}x(t) \end{pmatrix} \equiv \begin{pmatrix} z_2(t) \\ z_3(t) \\ z_4(t) \\ \dots \\ -\frac{a_0}{a_n}z_1 - \frac{a_1}{a_n}z_2 - \dots - \frac{a_{n-1}}{a_n}z_n + \frac{1}{a_n}u_1(t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ & & & \dots & & \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & -\frac{a_3}{a_n} & \dots & -\frac{a_{n-1}}{a_n} \end{pmatrix} \vec{z}(t) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ \frac{1}{a_n} \end{pmatrix} \vec{u}(t)$$

$$= A\vec{z}(t) + B\vec{u}(t)$$

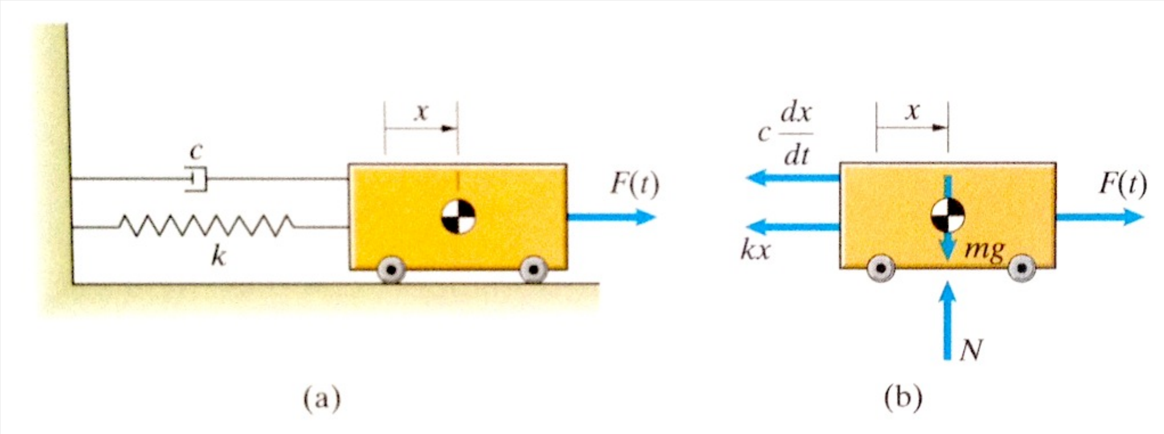
Now suppose there are 2 outputs, defined as $4x(t) - 5\dot{x}(t) + 7\ddot{x}(t)$, and $\frac{d^n}{dt^n}x(t)$. We construct the standard-form output equation as follows:

$$\vec{y}(t) = \begin{pmatrix} 4x(t) - 5\dot{x}(t) + 7\ddot{x}(t) \\ \frac{d^n}{dt^n}x(t) \end{pmatrix} = \begin{pmatrix} 4z_1 - 5z_2 + 7z_3 \\ -\frac{a_0}{a_n}z_1 - \frac{a_1}{a_n}z_2 - \dots - \frac{a_{n-1}}{a_n}z_n + \frac{1}{a_n}u_1(t) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -5 & 7 & 0 & \dots & 0 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & -\frac{a_3}{a_n} & \dots & -\frac{a_{n-1}}{a_n} \end{pmatrix} \vec{z}(t) + \begin{pmatrix} 0 \\ \frac{1}{a_n} \end{pmatrix} \vec{u}(t)$$

$$= C\vec{z}(t) + D\vec{u}(t)$$

Example: Vibrating cart



Write a state-space model for the system shown, with outputs equal to (1) the force in the spring, defined as positive in compression, and (2) the acceleration of the cart.

Solution
Writing $f = ma$ in the horizontal direction, with forces shown in the free-body diagram on the right, yields the LTI ODE

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

The states are defined as

$$z_1 \equiv x(t)$$
$$z_2 \equiv \dot{x}(t)$$

The inputs are defined as

$$u_1 = F(t)$$

Using these definitions, the state equations are

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = -\frac{k}{m}z_1 - \frac{c}{m}z_2 + \frac{1}{m}F(t)$$

Written in standard form $\dot{\vec{z}} = A\vec{z} + B\vec{u}$,

$$A_{2\times 2} = \begin{pmatrix} 0 & 1 \\ -k/m & -c/m \end{pmatrix}$$

$$B_{2\times 1} = \begin{pmatrix} 0 \\ 1/m \end{pmatrix}$$

For the spring to be in compression, we must have $x < 0$. Thus, writing spring force positive in compression, the output vector is

$$\vec{y}(t) = \begin{pmatrix} -kx(t) \\ \ddot{x}(t) \end{pmatrix} = \begin{pmatrix} -kz_1 \\ -\frac{k}{m}z_1 - \frac{c}{m}z_2 + \frac{1}{m}F(t) \end{pmatrix} = \begin{pmatrix} -k & 0 \\ -k/m & -c/m \end{pmatrix}\vec{z}(t) + \begin{pmatrix} 0 \\ 1/m \end{pmatrix}\vec{u}(t)$$

Thus,

$$C_{2\times 2} = \begin{pmatrix} -k & 0 \\ -k/m & -c/m \end{pmatrix} \quad , \quad D_{2\times 1} = \begin{pmatrix} 0 \\ 1/m \end{pmatrix}$$

In-class Exercise:
Write a standard-form state-space model for the system with 3^{rd}-order LTI ODE model

$$5\dddot{x} + 3\ddot{x} + 12\dot{x} + x = 2t - 3e^{-2t}$$

and outputs defined as

$$y_1 = 2\ddot{x} + 3t \quad , \quad y_2 = x(t) + \dot{x}(t) \qquad y_3 = 10e^{-2t}$$

Exercise solution:

$$5\dddot{x} + 3\ddot{x} + 12\dot{x} + x = 2t - 3e^{-2t}$$

The states are defined as

$$\begin{aligned} z_1 &\equiv x(t) \\ z_2 &\equiv \dot{x}(t) \\ z_3 &\equiv \ddot{x}(t) \end{aligned}$$

The inputs are defined as

$$u_1 = t$$

$$u_2 = e^{-2t}$$

Using these definitions, the state equations are

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= -0.2z_1 - 2.4z_2 - 0.6z_3 + 0.4u_1 - 0.6u_2 \end{aligned}$$

Written in standard form $\dot{\vec{z}} = A\vec{z} + B\vec{u}$, the matrices are

$$A_{3\times3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.2 & -2.4 & -0.6 \end{pmatrix}$$

$$B_{3\times2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.4 & -0.6 \end{pmatrix}$$

The output vector is

$$\vec{y}(t) = \begin{pmatrix} 2\ddot{x} + 3t \\ x(t) + \dot{x}(t) \\ 10e^{-2t} \end{pmatrix} = \begin{pmatrix} 2z_3 + 3u_1 \\ z_1 + z_2 \\ 10u_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{z}(t) + \begin{pmatrix} 3 & 0 \\ 0 & 0 \\ 0 & 10 \end{pmatrix} \vec{u}(t)$$

Thus,

$$C_{3\times3} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$D_{3\times2} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \\ 0 & 10 \end{pmatrix}$$