Although roots of algebraic equations (such as characteristic equations) are typically found using built-in functions on calculators or in programs such as MATLAB, it is useful to review how such roots may be found by hand.

We define the problem as follows: Find the *roots* of a known function $f(x)$, i.e., find the values of x such that

$$f(x) = 0$$

Euler's Method

There are actually a number of slightly different algorithmic versions of this basic idea, all of which use the first derivative of the function to estimate a root. The first derivative of a function indicates the *rate of change* of the function with respect to x. Using this information along with an initial guess x_0, we can estimate a root as follows:

1. Make an initial guess of a root, x_0
2. Calculate $f(x_0)$
3. Calculate $f' \equiv (df/dx)_{x_0}$
4. Set $x = x_0 - f/f'$, and repeat steps 1-4 using this new value as x_0

   Notes:

   (a) The underlying logic is simple: f' indicates the rate of change of f with respect to x, so based on the first derivative, we predict that $f(x_0 + \Delta x) = f(x) + \Delta x * f'(x_0)$. To find roots beginning from x_0, we solve for Δx such that $f(x_o + \Delta x) = 0$; this yields $\Delta x = x - x_0 = -f/f'$ as shown in step 4

   (b) If the first derivative is constant throughout the entire range of x (i.e., if $f(x)$ is linear in x), then a single cycle through steps (1-4) will find the exact root.

   (c) If the first derivative is not constant through the range of x (i.e., if $f(x)$ is nonlinear in x), then the predicted correction in step 4 is not exact, and we may need to repeat steps (1-4) some number of times, using each updated value of x_0 from one cycle as the beginning value in the next cycle, in order to converge to a root.

   (d) Inherent in this simple algorithm is the assumption that the sign of f' correctly indicates the direction of the root from the current value of x_0. In fact, for many nonlinear functions, this may be false, which has led to a large number of modifications of the basic algorithm over the years. When f is a polynomial, f' does indeed reliably indicate the direction of a root, but the calculated correction in step (4) may be so large that the root gets leapfrogged. Therefore, a sensible modification of Euler's method for polynomial root-finding is to arbitrarily scale the correction calculated in step (4) by some positive number less than 1, say, 0.1, resulting in a step in the right direction that is less likely to leapfrog the nearest root.

Example Find the root of the function

$$y(x) = 5x + 5$$

using Euler's method.

Suppose we use $x_0 = 0$ as our initial guess. Applying the steps outlined above,

1. $x_0 = 0$
2. $f(x_0) = 5 * 0 + 5 = 5$
3. $f'(x_0) = 5$
4. $x = x_0 - f/f' = 0 - 5/5 = -1$

Substituting, we see that $f(-1) = 5 * (-1) + 5 = 0$, so we found the root. Note that this function was linear in x; thus, Euler's method found the root in one cycle, and it was exact. The initial guess can be set to any value, and Euler's method will converge on the root in one cycle.

Example Use Euler's method to find the roots of the function

$$f(x) = x^3 + 9x^2 + 26x + 24 = 0$$

Again, let's begin with a guess of $x_0 = 0$ and cycle through steps (1-4) outlined above:

1. $x_0 = 0$
2. $f(x_0) = 24$
3. $f'(x_0) = 26$
4. $x = x_0 - f/f' = -24/26 = -0.9231$

Substituting this value yields $f(-0.9231) = 6.8821$, so we need to repeat steps (1)-(4) to get closer to a root. Repeating the cycle using the end of each one as the beginning of the next one, we obtain the following sequence of $x's$: -1.4994 , -1.8258 , -1.9676 , -1.9985 , -2.0000

Substituting $x = -2$, we see that $f(-2) = -8 + 36 - 52 + 24 = 0$, so $x = -2$ is indeed a root. Note that the convergence may be monitored either by checking the value of $f(x)$ to be sure that it is becoming close to or equal to 0, and/or, checking the value of x_0 to see if it is becoming constant (indicating that we've converged on a root).

So we found a root at $x = -2$, but since $f(x)$ is third-order in x, there are three roots to find. How do we find the other two? We could try repeating Euler's method using a new x_0, which *may* (or may not) converge to a different root. There's no way to guarantee that we will find other roots this way. In order to be sure to find the other roots, we instead use *synthetic division* (i.e., factorization) to divide $f(x)$ by the factor $x = -2$, which removes the root $x = -2$ and reduces the order of the remaining function by one:

$$\frac{f(x)}{x+2} = \frac{x^3 + 9x^2 + 26x + 24}{x+2} = x^2 + 7x + 12 \equiv g(x)$$

We can of course use the quadratic formula to obtain the roots of $g(x)$:

$$x_{1,2} = \frac{-7 \pm \sqrt{7^2 - 4 * 12}}{2} = -3 \ , \ -4$$

So the three roots of $f(x)$ are $x_{1,2,3} = -2$, -3 , -4

We could also apply Euler's method to $g(x)$ to find one of the remaining two roots, then factor that root out of $g(x)$ to find the last root.