Transducers represent physical devices that couple different types of subsystems together.

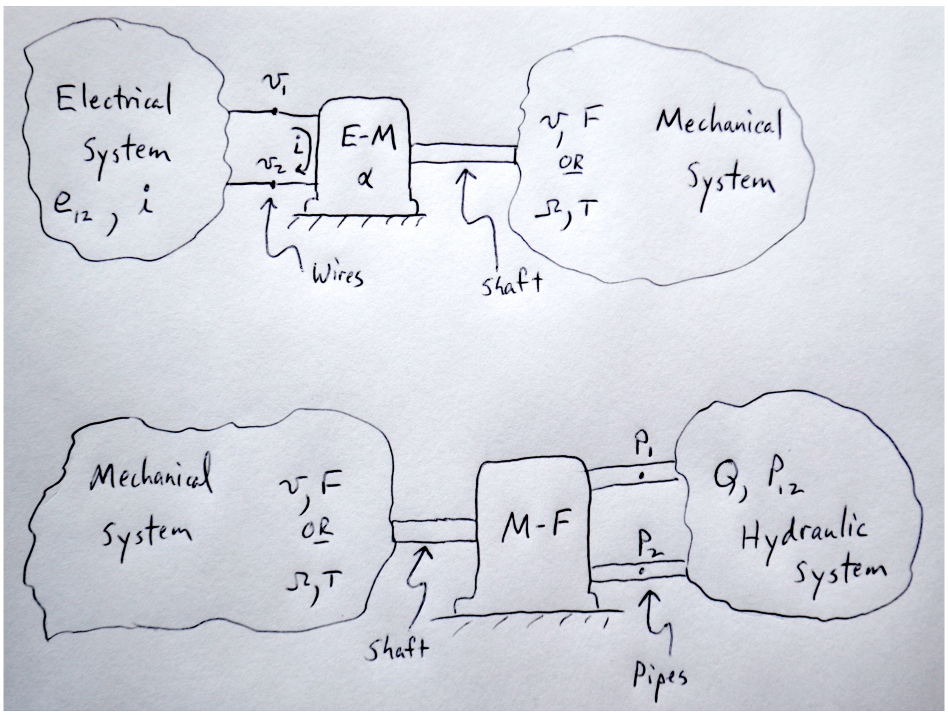
Transducers are called **ideal** if they do not consume any energy or power.

Since ideal transducers do not consume any energy or power, **they do not require additional states.**

Transducers:

1. are used like other elements to complete state-space models
2. always link two *physically-different* subsystems - for example, mechanical to electrical
3. utilize a constant physical parameter, like other elements
4. have *two* element equations, not one
5. each element equation relates a variable from one subsystem to a variable from the other subsystem

We define and use the following transducers:



Electrical-Mechanical (E-M or M-E; parameter α):

Translation	Rotation
$v = \alpha_t e_{12}$	$\Omega = \alpha_r e_{12}$
$F = \dfrac{1}{\alpha_t} i$	$T = \dfrac{1}{\alpha_r} i$

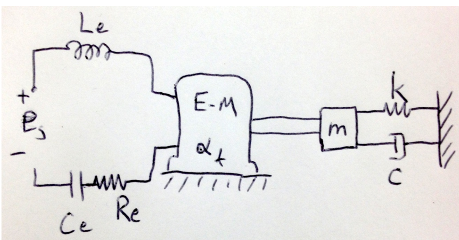
Mechanical-Hydraulic (F-M or M-F; parameter D):

Translation	Rotation
$v = \dfrac{1}{D_t} Q$	$\Omega = \dfrac{1}{D_r} Q$
$F = D_t P_{12}$	$T = D_r P_{12}$

The standard approach to obtaining state-space models for mixed systems is:

1. Identify the *usual* states and inputs within each subsystem
2. Define the full system state and input vectors as the collection of all subsystem states and inputs
3. Begin to obtain state equations in the *usual* way within each subsystem (element laws, first principles)
4. Sooner or later, transducer variables will appear in the subsystem state equations
5. Use transducer equations to substitute states and/or inputs from neighboring subsystems to finish

Example:



Find a standard-form state-space model for the system shown, consisting of a voltage source which powers the electrical subsystem, which then drives the translational mechanical system through the E-M motor.

The input is the voltage source e_s

The output is the velocity of the mass

The states for this system come from the electrical subsystem (e_{C_e}, i_{L_e}) and from the mechanical subsystem $(x, \dot{x})$. The output of the system is the velocity of the mass, $\dot{x}$:

$$\vec{z}(t) = \begin{pmatrix} e_{C_e} \\ i_{L_e} \\ x \\ \dot{x} \end{pmatrix} \quad , \quad u(t) = (e_S) \quad , \quad y(t) = (\dot{x})$$

On the electrical side, we have

$$\dot{e}_{C_e} = \frac{1}{C_e} i_{C_e} = \frac{1}{C_e} i_{L_e}$$
$$\dot{i}_{L_e} = \frac{1}{L_e} e_{L_e} = \frac{1}{L_e}(e_s - e_{C_e} - e_{R_e} - e_{EM})$$

On the RHS of the inductor state equation, we have e_{Re} and e_{EM}, which are neither states nor inputs, so we need to substitute for them. We now utilize the resistor and transducer element laws:

$$\dot{i}_{L_e} = \frac{1}{L_e}(e_s - e_{C_e} - R_e i_{L_e} - \frac{1}{\alpha_t}\dot{x})$$

The RHS variables are now all states and inputs, so we're done. On the mechanical side, use $F = ma$ on the mass:

$$\ddot{x} = \frac{1}{m}(F_{EM} - kx - c\dot{x})$$

The transducer force F_{EM} is neither a state nor an input, so use the transducer equation to substitute for F_{EM}:

$$\ddot{x} = \frac{1}{m}(\frac{1}{\alpha_t} i_{L_e} - kx - c\dot{x})$$

The RHS variables are now all states. Collecting terms, the system state equations are:

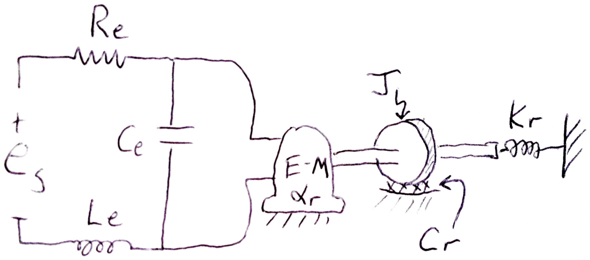
$$\dot{\vec{z}}(t) = \begin{pmatrix} \dot{e}_{C_e} \\ \dot{i}_{L_e} \\ \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{C_e} & 0 & 0 \\ -\frac{1}{L_e} & -\frac{R_e}{L_e} & 0 & -\frac{1}{L_e \alpha_t} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{m\alpha_t} & -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \vec{z}(t) + \begin{pmatrix} 0 \\ \frac{1}{L_e} \\ 0 \\ 0 \end{pmatrix} \vec{u}(t)$$

$$\vec{y}(t) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \vec{z}(t) + (0)\vec{u}(t)$$

Reduction of system states

Sometimes, the transducer equation relates two variables that are both states within their subsystems. In these cases, we only need to include one of these states in the system state vector.

Example



Find a standard-form state-space model for the system shown, consisting of a voltage source which powers the electrical subsystem, which then drives the rotational mechanical system through the E-M motor.

The input is the voltage source e_s

The output is the angular velocity of the disk (shaft)

The *usual* electrical and mechanical subsystem states are

$$\vec{z}(t) = \begin{pmatrix} e_{Ce} \\ i_{Le} \\ \theta_J \\ \dot{\theta}_J \end{pmatrix}$$

so it appears to be a 4th-order system. However, from the transducer equations, we know that

$$\dot{\theta}_J = \alpha_r e_{EM}$$

From a loop law, we can see that

$$e_{EM} = e_{Ce}$$

Combining these two, we have

$$\dot{\theta}_J = \alpha_r e_{Ce}$$

Thus, knowledge of either $\dot{\theta}_J$ or e_{Ce} is sufficient to determine the other one, so we only include one; this system is actually **3rd**-order, not 4th-order. For example, suppose we choose

$$\vec{z}(t) = \begin{pmatrix} e_{Ce} \\ i_{Le} \\ \theta_J \end{pmatrix} \quad \Rightarrow \quad \dot{\vec{z}} = \begin{pmatrix} \frac{1}{Ce} i_{Ce} \\ \frac{1}{Le} e_{Le} \\ \dot{\theta}_J \end{pmatrix} = \begin{pmatrix} \frac{1}{Ce}(i_{Le} - \alpha_r T_{EM}) \\ \frac{1}{Le}(e_s - R_e i_{Le} - e_{Ce}) \\ \alpha_r e_{Ce} \end{pmatrix}$$

The second and third equations are already in the correct form, but we need to eliminate T_{EM} on the RHS of the first state equation, so we write the moment equation for the rotational subsystem:

$$J\ddot{\theta}_J = T_{EM} - K_r\theta_J - C_r\dot{\theta}_J$$

We now use the relationship between $\dot{\theta}_J$ and e_{Ce} to eliminate $\dot{\theta}_J$:

$$\Rightarrow \quad T_{EM} = J\alpha_r\dot{e}_{Ce} + K_r\theta_J + C_r\alpha_r e_{Ce}$$

Now substitute into the first state equation and collect terms:

$$\dot{e}_{Ce} = \frac{1}{Ce}\big(i_{Le} - \alpha_r(J\alpha_r\dot{e}_{Ce} + K_r\theta_J + C_r\alpha_r e_{Ce})\big) \quad \Rightarrow \dot{e}_{Ce} = \frac{1}{Ce + \alpha_r^2 J}\left[i_{Le} + \alpha_r K_r\theta_J + \alpha_r^2 C_r e_C\right]$$

The RHS variables are all states. Combining this with the other two state equations,

$$\dot{\vec{z}}(t) = A\vec{z} + B\vec{u} = \begin{pmatrix} \frac{\alpha_r^2 C_r}{C_e+\alpha_r^2 J} & \frac{1}{C_e+\alpha_r^2 J} & \frac{\alpha_r K_r}{C_e+\alpha_r^2 J} \\ -\frac{1}{Le} & -\frac{Re}{Le} & 0 \\ \alpha_r & 0 & 0 \end{pmatrix}\vec{z} + \begin{pmatrix} 0 \\ \frac{1}{Le} \\ 0 \end{pmatrix}(e_s) \quad , \quad \vec{y}(t) = (1/\alpha_r \;\; 0 \;\; 0)\vec{z} + (0)\vec{u}$$

We could instead choose to keep $\dot{\theta}_J$ rather than e_{Ce}:

$$\vec{z}(t) = \begin{pmatrix} i_{Le} \\ \theta_J \\ \dot{\theta}_J \end{pmatrix} \quad \Rightarrow \quad \dot{\vec{z}} = \begin{pmatrix} \frac{1}{Le} e_{Le} \\ \dot{\theta}_J \\ \ddot{\theta}_J \end{pmatrix} = \begin{pmatrix} \frac{1}{Le}(e_s - R_e i_{Le} - e_{Ce}) \\ \dot{\theta}_J \\ \frac{1}{J}(T_{EM} - K_r\theta_J - C_r\dot{\theta}_J) \end{pmatrix} = \begin{pmatrix} \frac{1}{Le}(e_s - R_e i_{Le} - \dot{\theta}_J/\alpha_r) \\ \dot{\theta}_J \\ \frac{1}{J}(T_{EM} - K_r\theta_J - C_r\dot{\theta}_J) \end{pmatrix}$$

We need to eliminate T_{EM} on the RHS of the third state equation, so we use the transducer equation:

$$T_{EM} = \frac{1}{\alpha_r} i_{EM} = \frac{1}{\alpha_r}(i_{Le} - i_{Ce})$$

We now use the relationship between $\dot{\theta}_J$ and e_{Ce} to eliminate i_{Ce}:

$$i_{Ce} = C_e \dot{e}_{Ce} = \frac{C_e}{\alpha_r}\ddot{\theta}_J$$

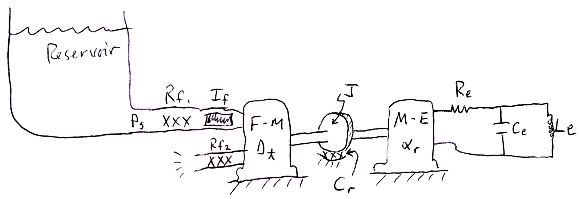
Now substitute into the third state equation and collect terms:

$$\ddot{\theta}_J = \frac{1}{J}\Big[\frac{1}{\alpha_r}(i_{Le} - \frac{C_e}{\alpha_r}\ddot{\theta}_J) - K_r\theta_J - C_r\dot{\theta}_J\Big] \quad \Rightarrow \ddot{\theta}_J = \frac{1}{J + C_e/\alpha_r^2}(\frac{1}{\alpha_r} i_{Le} - K_r\theta_J - C_r\dot{\theta}_J)$$

The RHS variables are all states. Combining this with the other two state equations,

$$\dot{\vec{z}}(t) = A\vec{z} + B\vec{u} = \begin{pmatrix} -\frac{Re}{Le} & 0 & -\frac{1}{L_e\alpha_r} \\ 0 & 0 & 1 \\ \frac{1}{\alpha_r(J + C_e/\alpha_r^2)} & -\frac{K_r}{J + C_e/\alpha_r^2} & -\frac{C_r}{J + C_e/\alpha_r^2} \end{pmatrix}\vec{z} + \begin{pmatrix} \frac{1}{Le} \\ 0 \\ 0 \end{pmatrix}(e_s) \quad , \quad \vec{y}(t) = (0 \;\; 0 \;\; 1)\vec{z} + (0)\vec{u}$$

Example



The hydroelectric generation system has a reservoir that is tapped at a specific depth to create a constant pressure source P_s. The pressure drives a turbine generator modeled by the fluid system→ FM transducer→mechanical shaft→ ME transducer→ electrical system. Find a standard-form state-space model for the system.

The output is the voltage in R_e

Beginning with the *usual* subsystem states, the system state vector would be defined by

$$\vec{z}(t) = \begin{pmatrix} Q_{If} \\ \theta_J \\ \dot{\theta}_J \\ e_{Ce} \\ i_{Le} \end{pmatrix}$$

However, we don't need to include θ_J because there is no angular position term necessary (no rotational spring).

Moreover, note that $Q_{If} = Q_{FM} = D_r\dot{\theta}_J$, so we don't need both Q_{If} and $\dot{\theta}_J$. Let's keep, say, Q_{If}.

Using element laws and first principles,

$$\begin{aligned} \dot{Q}_{If} &= \frac{1}{C_f}(P_{If}) = \frac{1}{C_f}\left[(P_s - P_{Rf1}) - (P_{Rf2} + P_{FM})\right] \\ &= \frac{1}{Cf}\left[P_s - R_{f1}Q_{If} - R_{f2}Q_{If} - \frac{1}{D_r}T_{FM}\right] \qquad (1) \\ \dot{e}_{Ce} &= \frac{1}{C_e}(i_{Ce}) = \frac{1}{C_e}(i_{EM} - i_{Le}) \qquad (2) \\ \dot{i}_{Le} &= \frac{1}{L_e}(e_{Le}) = \frac{1}{L_e}(e_{Ce}) \qquad (3) \end{aligned}$$

On the RHS of Eqs. (1)-(2), we need to eliminate T_{FM} and i_{EM}. For example,

$$i_{EM} = i_{Re} = \frac{e_{Re}}{R_e} = \frac{e_{EM} - e_{Ce}}{R_e} = \frac{\dot{\theta}_J/\alpha_r - e_{Ce}}{R_e} = \frac{Q_{If}/D_r\alpha_r - e_{Ce}}{R_e} \qquad (4)$$

Now substitute Eq. (4) → Eq. (2), after which all variables on the RHS of Eq. (2) are states. Also,

$$J\ddot{\theta}_J = T_{FM} - C_r\dot{\theta}_J - T_{EM} = T_{FM} - C_r\dot{\theta}_J - \frac{i_{EM}}{\alpha_r} \quad \Rightarrow T_{FM} = \frac{J}{D_r}\dot{Q}_{If} + \frac{C_r}{D_r}Q_{If} + \frac{1}{\alpha_r}i_{EM} \qquad (5)$$

Substitute Eq. (4) → Eq.(5), then Eq. (5) → Eq. (1), then collect terms and put into correct form.