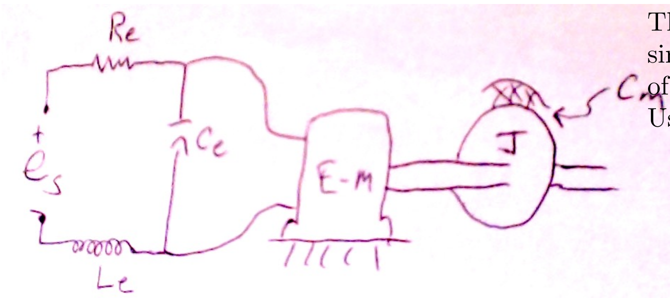
1. Simple electric motor:



The battery (voltage source) input e_s is used to power a simple electric motor as shown in the figure. The output of the system is the rotational speed of the motor shaft. Use the symbol α_{em} for the motor constant.

1. Write a state-space model for this system, using the minimum required number of states

Solution: Choosing the usual states in each subsystem, I define

$$\underline{z} = \begin{pmatrix} e_{Ce} \\ i_{Le} \\ \dot{\theta} \end{pmatrix} \quad , \quad \underline{u} = (e_s) \quad , \quad \underline{y} = (\dot{\theta})$$

However, from the loop law for the motor-capacitor loop, and the transducer equations, we know that

$$e_{em} = e_{Ce} = \dot{\theta}/\alpha_{em}$$

We can eliminate either e_{Ce} or $\dot{\theta}$. I'll keep $\dot{\theta}$ since it's the output. Now I proceed to determine the state-space equations:

$$\underline{z} = \begin{pmatrix} i_{Le} \\ \dot{\theta} \end{pmatrix} \quad , \quad \underline{u} = (e_s) \quad , \quad \underline{y} = (\dot{\theta})$$

$$\dot{z}_1 = \dot{i}_{Le} = \frac{1}{L_e}(e_{Le}) = \frac{1}{L_e}(e_s - e_{Re} - e_{Ce}) \qquad \text{loop law}$$

$$= \frac{1}{L_e}(e_s - R_e i_{Le} - \frac{\dot{\theta}}{\alpha_{em}}) \qquad \text{resistor element, transducer element}$$

$$\dot{z}_2 = \ddot{\theta} = \frac{1}{J}(T_{em} - C_m\dot{\theta}) \qquad \text{Newton}$$

$$= \frac{1}{J}(\frac{i_{em}}{\alpha_{em}} - C_m\dot{\theta}) \qquad \text{transducer}$$

$$\text{find } i_{em} = i_{Le} - i_{Ce} = i_{Le} - C_e\dot{e}_{Ce} = i_{Le} - C_e\frac{\ddot{\theta}}{\alpha_{em}}$$

$$\rightarrow \ \ddot{\theta} = \frac{1}{J}(\frac{i_{Le}}{\alpha_{em}} - \frac{C_e\ddot{\theta}}{\alpha_{em}^2} - C_m\dot{\theta})$$

collect terms:

$$(1 + \frac{C_e}{J\alpha_{em}^2})\ddot{\theta} = \frac{i_{Le}}{J\alpha_{em}} - \frac{C_m\dot{\theta}}{J}$$

$$\Rightarrow \ddot{\theta} = \frac{\alpha_{em}}{J\alpha_{em}^2 + C_e}(i_{Le} - C_m\alpha_{em}\dot{\theta})$$

$$A = \begin{pmatrix} -\frac{R_e}{L_e} & -\frac{1}{L_e\alpha_{em}} \\ \frac{\alpha_{em}}{J\alpha_{em}^2 + C_e} & -\frac{C_m\alpha_{em}^2}{J\alpha_{em}^2 + C_e} \end{pmatrix} \quad , \quad B = \begin{pmatrix} \frac{1}{L_e} \\ 0 \end{pmatrix} \quad , \quad C = (0\ 1) \quad , \quad D = (0)$$

2. Using symbolic variables, find the Characteristic Equation for this system

Solution: The roots of the characteristic equation are the eigenvalues of A, so for this 2×2 system it can be easily found by hand:

$$CE = det(A - \lambda I) = \lambda^2 + [\frac{R_e}{L_e} + \frac{C_m\alpha^2}{J\alpha^2 + C_e}]\lambda + [\frac{R_e C_m \alpha^2}{L_e(J\alpha^2 + C_e)} + \frac{1}{L_e(J\alpha^2 + C_e)}] = 0$$

This can also be calculated using MATLAB, e.g.:

```
>> syms R L Ce Cm J a lam
>> A=[-R/L -1/(L*a) ; a/(J*a^2+Ce) -a^2*Cm/(J*a^2+Ce)]
>> det(A-lam*eye(2))
```

The MATLAB result gives the same CE (of course!).

3. Using symbolic variables, find the transfer function

Solution: The calculation is again straightforward, and can be done by hand, or by MATLAB:

$$TF = C(sI - A)^{-1}B + D = \frac{\frac{\alpha}{L_e(J\alpha^2+C_e)}}{s^2 + \left(\frac{R_e}{L_e} + \frac{\alpha^2 C_m}{J\alpha^2+C_e}\right)s + \left(\frac{1+\alpha^2 R_e C_m}{L_e(J\alpha^2+C_e)}\right)}$$

using, for example, MATLAB:

```
>> B=[1/L 0] ; C=[0 1] ; D=[0]
>> TF=C*inv(lam*eye(2)-A)*B+D
```

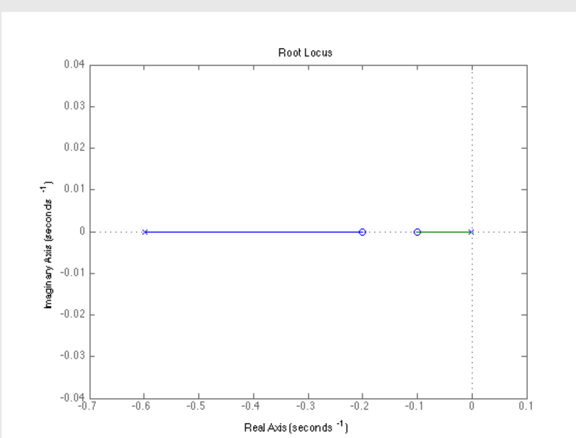
4. Assign the following values, and then draw the root locus plot for α_{em} (note: you may use **rlocus**): $R_e = 10$, $C_e = 50$, $L_e = 50$, $J = 100$, $C_m = 10$

Solution: For convenience in separating the α terms from the others in the CE, I multiply the form in (2) above by the term $L_e(J\alpha^2 + C_e)$:

$$CE = L_e(J\alpha^2 + C_e)\lambda^2 + \left[R_e(J\alpha^2 + C_e) + L_e C_m \alpha^2\right]\lambda + \left[R_e C_m \alpha^2 + 1\right] = 0$$

Substituting, and rearranging terms into the standard root locus form $D(\lambda) + KN(\lambda) = 0$ with $K = \alpha^2$,

$$CE = (2500\lambda^2 + 500\lambda + 1) + \alpha^2(5000\lambda^2 + 1500\lambda + 100) = 0$$



5. Select the 'best' value for α_{em} from your root locus plot; explain your choice!!

Solution: According to the root locus plot, the roots will always be real and the system is always stable. The fastest settling time occurs for $\alpha \to \infty$ so I would pick the largest α possible.

6. Using your selected value of α_{em}, find the steady-state rotational speed of the shaft as a function of a constant value for e_s . Your answer should appear as

$$\text{constant } * e_s$$

Solution: The steady-state response for input $e_s = 1$ is given by $lim_{s\to 0}(TF)$:

$$lim_{s\to 0}(TF) = \frac{\alpha}{1 + 100\alpha^2}$$

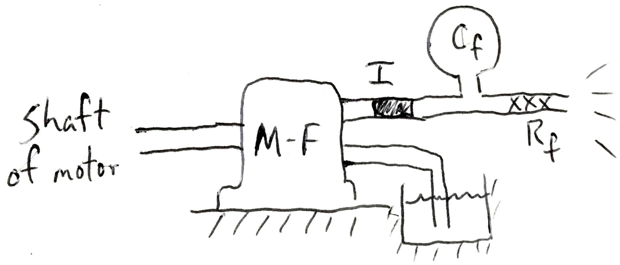
Note that this goes to 0 for large α.

7. Select e_s such that the rotational speed of the shaft is $2500rpm$

Solution: From the results of (6), we select

$$e_s = \frac{2500\alpha}{1 + 100\alpha^2}$$

2.



Now assume that the fluid pump is attached to the end of the shaft of the previous problem. The output of the system is the flow out of the end of the pipe. For the pump, use transducer constant α_{mf}

1. Write a state-space model for this system, using the minimum required number of states

Solution: I begin by adding the usual fluid states to the results of Problem 1:

$$\underline{z} = \begin{pmatrix} i_{Le} \\ \dot{\theta} \\ P_{Cf} \\ Q_I \end{pmatrix} \quad , \quad \underline{u} = (e_s) \quad , \quad \underline{y} = (Q_{Rf})$$

However, from the transducer equations for the pump, we know that

$$\dot{\theta} = \frac{1}{\alpha_{mf}} Q_I$$

We can eliminate either Q_I or $\dot{\theta}$. I'll keep Q_I since it's closely related to the output. Now I proceed to determine the state-space equations:

$$\underline{z} = \begin{pmatrix} i_{Le} \\ P_{Cf} \\ Q_I \end{pmatrix} \quad , \quad \underline{u} = (e_s) \quad , \quad \underline{y} = (Q_{Rf})$$

$$\dot{z}_1 = \dot{i}_{Le} = \frac{1}{L_e}(e_{Le}) = \frac{1}{L_e}(e_s - e_{Re} - e_{Ce}) \qquad \text{loop law}$$

$$= \frac{1}{L_e}\big(e_s - R_e i_{Le} - \frac{\dot{\theta}}{\alpha_{em}}\big) \qquad \text{resistor element, transducer element}$$

$$= \frac{1}{Le}\big(e_s - R_e i_{Le} - \frac{Q_I}{\alpha_{em}\alpha_{mf}} \qquad \dot{\theta} = \frac{1}{\alpha_{mf}} Q_I)$$

$$\dot{z}_2 = \dot{P}_{Cf} = \frac{1}{C_f} Q_{Cf} = \frac{1}{C_f}(Q_I - Q_{Rf}) = \frac{1}{C_f}(Q_I - \frac{P_{Cf}}{R_f}) \qquad \text{node law, element law} R_f$$

$$\dot{z}_3 = \dot{Q}_I = \frac{1}{I} P_I = \frac{1}{I} P_{mf} = \frac{1}{I}\frac{T_{mf}}{\alpha_{mf}} = \frac{1}{I}\big(T_{em} - C_m\dot{\theta} - J\ddot{\theta}\big) \qquad \text{transducer eq, Newton)}$$

$$= \frac{1}{I}\big(\frac{i_{em}}{\alpha_{em}} - C_m\frac{Q_I}{\alpha_{mf}} - J\frac{\dot{Q}_I}{\alpha_{mf}}\big)$$

$$\text{find } i_{em} = i_{Le} - i_{Ce} = i_{Le} - C_e\dot{e}_{Ce} = i_{Le} - C_e\frac{\ddot{\theta}}{\alpha_{em}} = i_{Le} - C_e\frac{\dot{Q}_I}{\alpha_{em}\alpha_{mf}}$$

Substituting,

$$\dot{z}_3 = \dot{Q}_I = \frac{1}{I}\Big(\frac{i_{Le} - C_e\frac{\dot{Q}_I}{\alpha_{em}\alpha_{mf}}}{\alpha_{em}} - C_m\frac{Q_I}{\alpha_{mf}} - J\frac{\dot{Q}_I}{\alpha_{mf}}\Big)$$

Collecting terms,

$$\big(I + \frac{C_e}{\alpha_{em}^2\alpha_{mf}} + \frac{J}{\alpha_{mf}}\big)\dot{Q}_I = \frac{i_{Le}}{\alpha_{em}} - \frac{C_m}{\alpha_{mf}} Q_I$$

We may now write the state-space model, where I define $K = (I + \frac{C_e}{\alpha_{em}^2 \alpha_{mf}} + \frac{J}{\alpha_{mf}})$ for convenience:

$$A = \begin{pmatrix} -\frac{R_e}{L_e} & 0 & -\frac{1}{L_e \alpha_{em} \alpha_{mf}} \\ 0 & -\frac{1}{C_f R_f} & \frac{1}{C_f} \\ \frac{1}{K \alpha_{em}} & 0 & -\frac{C_m}{K \alpha_{mf}} \end{pmatrix} \quad , \quad B = \begin{pmatrix} \frac{1}{L_e} \\ 0 \\ 0 \end{pmatrix} \quad , \quad C = (0 \quad \frac{1}{R_f} \quad 0) \quad , \quad D = (0)$$

2. Using symbolic variables, find the Characteristic Equation for this system
3. Using symbolic variables, find the transfer function
4. Assign the values in the previous problem, along with the following values, and then draw the root locus plot for α_{mf} (note: you may use **rlocus**):
   $R_f = 5$, $C_f = 100$, $I = 50$, $\alpha_{em} = 10$
5. Select the 'best' value for α_{mf} from your root locus plot; explain your choice!!
6. Using your selected value of α_{mf}, find the steady-state flow out of the pipe as a function of a constant value for e_s . Your answer should appear as

$$\text{constant} \; * e_s$$