LTI ODE General Form:

$$a_n \frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + ... + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t) \tag{1}$$

The solution is sum of homogeneous and particular:

$$x(t) = x_h(t) + x_p(t)$$

ICBS that

$$x_h(t) = Ae^{\lambda t} \tag{2}$$

where

$$\begin{aligned} A, \lambda &= \text{constants, whose values we will determine soon} \\ e &= 2.718281828.... \\ \lambda &= \text{ values found from } a_i\text{'s (}\textbf{inherent}\text{ to the model/system)} \\ A &= \text{ value found from initial conditions (not inherent to system/model)} \end{aligned}$$

Determination of λ: the Characteristic Equation

First, note that

$$\frac{d(x_h(t))}{dt} = \frac{d}{dt}(Ae^{\lambda t}) = \lambda Ae^{\lambda t}$$

$$\frac{d^2(x_h(t))}{dt^2} = \frac{d^2}{dt^2}(\lambda Ae^{\lambda t}) = \lambda^2 Ae^{\lambda t}$$

$$...$$

$$\frac{d^n(x_h(t))}{dt^n} = \frac{d^n}{dt^n} = \lambda^n Ae^{\lambda t}$$

Substitute into Eq. (1) and collect terms:

$$(a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + ... + a_1\lambda + a_0)Ae^{\lambda t} = 0$$

$$\Rightarrow \qquad Ae^{\lambda t} = 0 \qquad \textbf{OR} \qquad a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + ... + a_1\lambda + a_0 = 0$$

We cannot set $Ae^{\lambda t} = 0$, since this would mean every $x_h(t)$ is always zero, which is not true.

Therefore, we must have

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + ... + a_1\lambda + a_0 = 0$$

This is called the **Characteristic Equation** (CE)

In-class activity: Find the characteristic equation of the LTI ODE given by

$$3\frac{d^4x(t)}{dt^4} + 6\frac{d^3x(t)}{dt^3} + 4\frac{dx(t)}{dt} + 5x(t) = 10sin(2t)$$

In-class activity: Find the characteristic equation of the LTI ODE given by

$$3\frac{d^4x(t)}{dt^4} + 6\frac{d^3x(t)}{dt^3} + 4\frac{dx(t)}{dt} + 5x(t) = 10sin(2t)$$

Begin by writing the form of the homogeneous solution:

$$x_h(t) = Ae^{\lambda t}$$

Take derivatives for substitution into the system model:

$$\frac{d(x_h(t))}{dt} = \frac{d}{dt}(Ae^{\lambda t}) = \lambda Ae^{\lambda t} \quad , \quad \frac{d^3(x_h(t))}{dt^3} = \lambda^3 Ae^{\lambda t} \quad , \quad \frac{d^4(x_h(t))}{dt^4} = \lambda^4 Ae^{\lambda t}$$

Substitute into the system model, with the right-hand side set to 0:

$$3\lambda^4 Ae^{\lambda t} + 6\lambda^3 Ae^{\lambda t} + 4\lambda Ae^{\lambda t} + 5Ae^{\lambda t} = 0$$

Collect terms:

$$(3\lambda^4 + 6\lambda^3 + 4\lambda + 5)Ae^{\lambda t} = 0$$

We cannot assume that $Ae^{\lambda t} = 0$, so we must have

$$3\lambda^4 + 6\lambda^3 + 4\lambda + 5 = 0$$

$\Rightarrow$ This is the Characteristic Equation

Roots of the Characteristic Equation

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + ... + a_1\lambda + a_0 = 0$$

is an n^{th}-order polynomial in λ, and thus it has n roots

$$\lambda_1, \lambda_2, ...\lambda_n$$

The CE can be divided by a_n and factored using its roots as

$$(\lambda - \lambda_1) * (\lambda - \lambda_2) * (\lambda - \lambda_3) * \cdots * (\lambda - \lambda_n) = 0$$

NOTES:

1. **If all a_i's are real, then each of the n roots must be real, or, occur in complex conjugate pairs**. We represent a complex conjugate pair of roots as:

$$\begin{aligned}\lambda_k, \lambda_{k+1} &= \sigma \pm i\omega \\ \sigma &= \text{the real part of the complex conjugate pair} \\ \omega &= \text{the imaginary part of the complex conjugate pair} \\ i &= \sqrt{-1} \\ k &= \text{an arbitrary subscript to denote root k}\end{aligned}$$

   → In physical models (all MAE 340 models), all $a_i's$ are real

   → If n is odd, there must be at least one real root

2. **If $n > 1$, we write the general $x_h(t)$ as the sum of terms that go with each root:**

$$x_h(t) = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t} + ... + A_ne^{\lambda_n t}$$

3. **There may be *repeated* roots.**
   For example, the polynomial

$$\lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0$$

   has roots $\lambda_1 = -2$, $\lambda_2 = -2$, and $\lambda_3 = -2$. **There are 3 roots of the 3rd-order polynomial**

4. **Roots are usually found using built-in functions, but Euler's method and synthetic division may also be used**
   → See separate attachment

In-class activity: Perform synthetic division to obtain the $2nd$-order polynomial that results from

$$\frac{\lambda^3 + 6\lambda^2 + 12\lambda + 8}{(\lambda + 2)}$$

The Characteristic Equation tells us the character of the system's inherent (homogeneous) behavior

Behavior of $x_h(t)$ by type of root

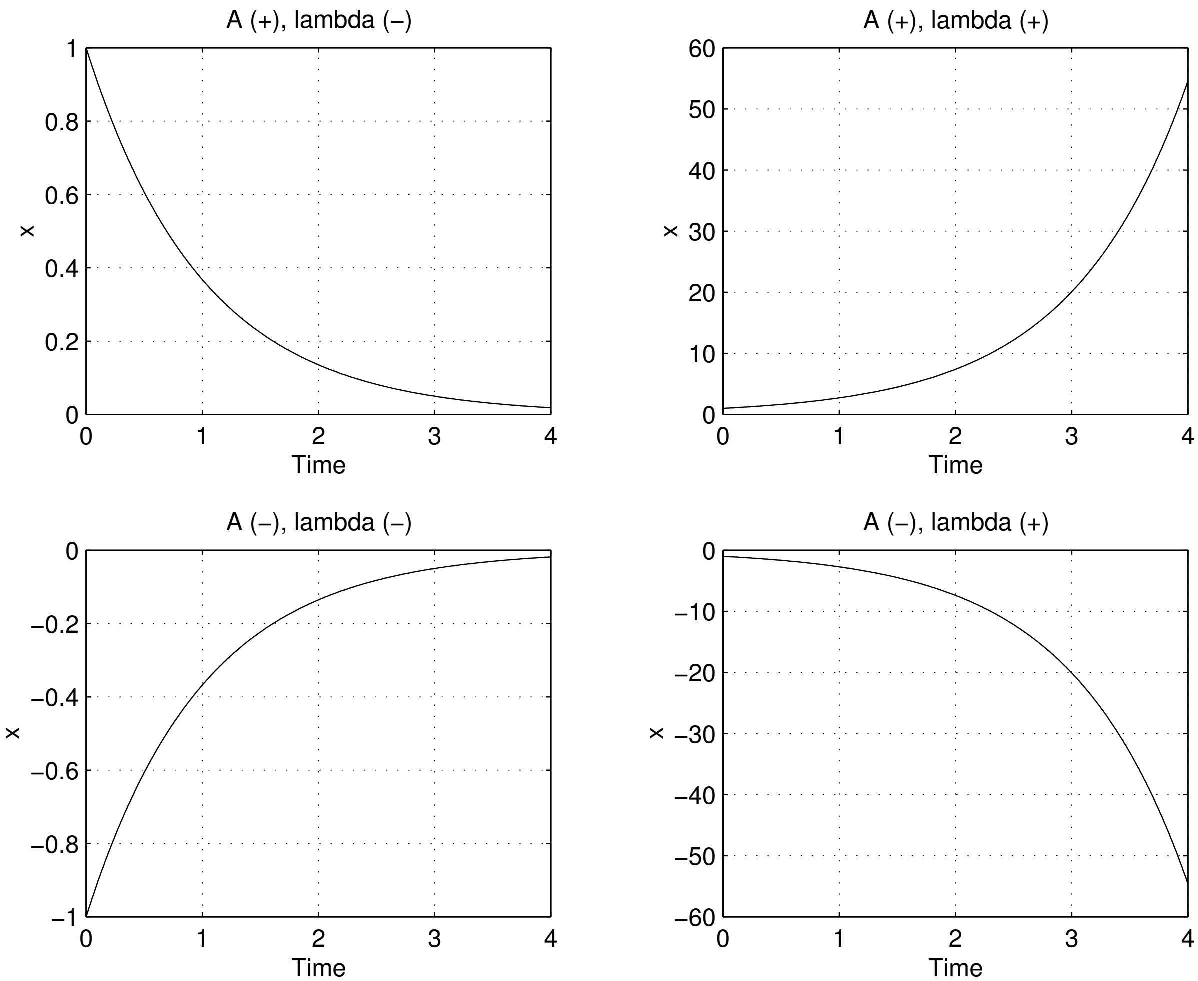
1. $\lambda =$ a *distinct* (i.e., not repeated) real number:

$$x_h(t) = Ae^{\lambda t}$$

A is found from initial conditions (may vary with every case)

and

λ may be zero, positive, or negative (but is inherent to system, calculated from system's a_i's)



The sign of λ is crucial

Note that if $\lambda = 0$, then $x_h(t) = Ae^{0*t} = A = constant$

2. Complex conjugate pair: $\lambda_k, \lambda_{k+1} = \sigma \pm i\omega$, where $i = \sqrt{-1}$ and σ, ω are real numbers

$$\begin{aligned} \Rightarrow \quad x_h(t) &= A_k e^{\lambda_k t} + A_{k+1} e^{\lambda_{k+1} t} \\ &= A_k e^{(\sigma + i\omega)t} + A_{k+1} e^{(\sigma - i\omega)t} \\ &= e^{\sigma t}(C_1 sin\omega t + C_2 cos\omega t) \qquad \text{(using } \textit{Euler Identity}\text{)} \qquad (3) \\ &= A e^{\sigma t} sin(\omega t + \phi) \qquad (4) \end{aligned}$$

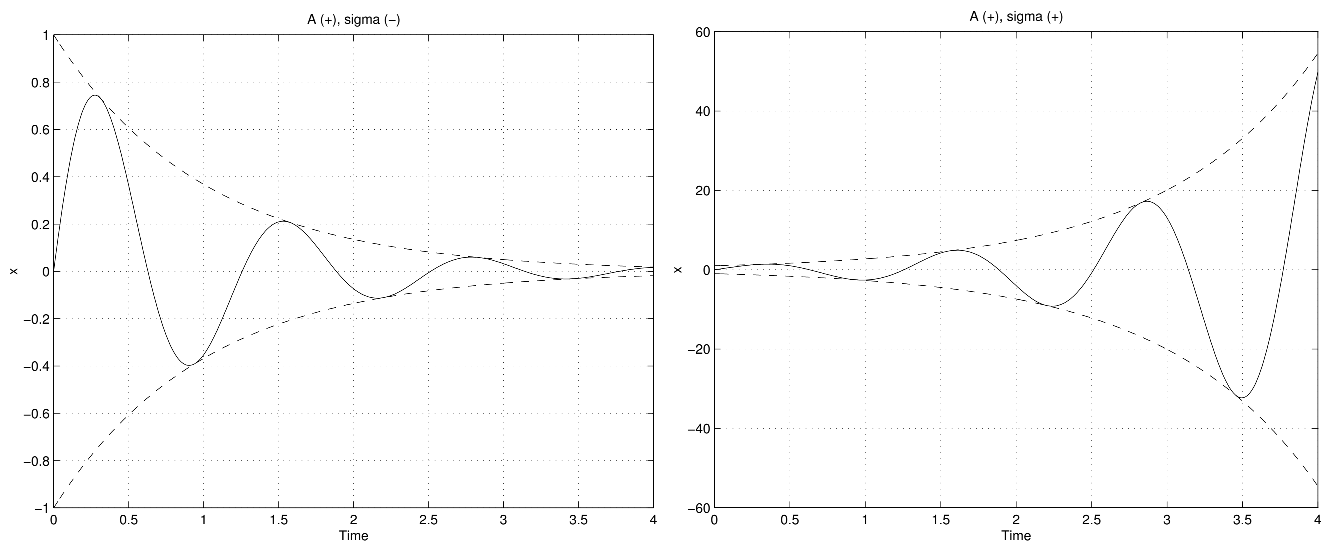
In Eq. (4),

$A \equiv$ **amplitude** or **magnitude**
$\omega \equiv$ **damped natural frequency**, often just called the **frequency**
$\phi \equiv$ **phase angle** or just **phase**

Eqs. (3) and (4) are mathematically equivalent, with (using trig identity $sin(\alpha + \beta) = sin\alpha \; cos\beta + sin\beta \; cos\alpha$)

$$C_1 = Acos\phi \qquad , \qquad C_2 = Asin\phi$$

It's physically easier to visualize a single periodic function, so most engineers use the form of Eq. (4) to describe the system. On the other hand, Eq. (3) is easier to use for evaluating initial conditions (if required).



<u>**The sign of σ is crucial!**</u>

Note that if $\sigma = 0$, then $x_h(t) = Ae^{0*t} sin(\omega t + \phi) = Asin(\omega t + \phi) =$ *constant* sinusoid

3. Real *nonzero* root λ, repeated so that it appears m times:

$$x_h(t) = (A_1 + A_2 t + ... + A_m t^{m-1})e^{\lambda t}$$

   (a) At $t = 0$, $x_h = A_1$

   (b) If $\lambda < 0$, then as $t \to \infty$, eventually $x_h(t) \to 0$ (looks like distinct real root case)

   (c) If $\lambda > 0$, then as $t \to \infty$, eventually $x_h(t) \to \infty$ (looks like distinct real root case)

   (d) $(A_2 t + A_3 t^2 + ... + A_m t^{m-1})$ may cause growth/decay initially, opposite of distinct real root case

   Behavior eventually looks similar to that of a distinct root, but may be weird until t gets large

4. Repeated complex conjugate pair: Assume that the pair is repeated such that it occurs m times (i.e., a total of $2m$ roots):

$$\begin{aligned} x_h(t) = e^{\sigma t} * ((&C_1 + C_2 t + C_3 t^2 + \cdots + C_m t^{m-1}) sin(\omega t) \\ &+ (D_1 + D_2 t + D_3 t^2 + \cdots + D_m t^{m-1}) cos(\omega t)) \end{aligned}$$

$$= e^{\sigma t}(A_1 sin(\omega t + \phi_1) + A_2 t sin(\omega t + \phi_2) + \cdots + A_m t^{m-1} sin(\omega t + \phi_m))$$

   (a) Behavior eventually settles to single sinusoid

   (b) If $\sigma < 0$, then as $t \to \infty$, eventually $x_h(t) \to 0$ as decaying sinusoid (looks like distinct case)

   (c) If $\sigma > 0$, then as $t \to \infty$, eventually $x_h(t) \to \infty$ as exponential sinusoid (looks like distinct case)

   (d) $(A_2 t + A_3 t^2 + ... + A_m t^{m-1})$ may cause growth/decay initially, opposite of distinct case

   Behavior eventually looks similar to that of a distinct complex conjugate pair, but amplitude may be weird until t gets large

5. Repeated real root $\lambda = 0$, total of m roots:

$$x_h(t) = (A_1 + A_2 t + A_3 t^2 + \cdots + A_m t^{m-1})$$

   For single root $\lambda = 0$, $x_h = constant$. Therefore, unlike cases (3) and (4) above, this behavior is usually completely different for all t compared with single root $\lambda = 0$

Homogeneous solution of large-order system:

- The actual behavior of $x_h(t)$ is a *linear combination* of the specific behavior associated with each of the roots
- For an LTI ODE with no repeated roots, the complete homogeneous behavior is ALWAYS a linear combination of *purely exponential* and/or *exponentially-changing sinusoidal* components!
- Suppose we have an 8th-order LTI ODE whose roots are given by

$$\begin{aligned}
\lambda_1, \lambda_2 &= \text{two distinct real roots} \\
\lambda_3, \lambda_4 &= \text{a repeated root, } \neq \lambda_1 \text{ or } \lambda_2 \\
\lambda_5, \lambda_6 &= \sigma_1 \pm i\omega_1, \text{ a complex conjugate pair} \\
\lambda_7, \lambda_8 &= \sigma_2 \pm i\omega_2, \text{ a complex conjugate pair} \neq \sigma_1 \pm \omega_1
\end{aligned}$$

In this example, combining the terms corresponding to each root, we find that

$$x_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + (C_3 + C_4 t) e^{\lambda_3 t} + C_5 e^{\sigma_1 t} sin(\omega_1 t + \phi_1) + C_6 e^{\sigma_2 t} sin(\omega_2 t + \phi_2)$$

where $C_1, C_2, ..., C_6$ and ϕ_1, ϕ_2 are constants.

$\rightarrow$ **The Characteristic Equation** is so-named because its roots give the character of the behavior $\leftarrow$

In-class activity: Sketch the behavior of the system modeled by

$$\frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} + 3x(t) = 0$$

where

$$x(0) = 4 \quad , \quad \dot{x}(0) = 0$$

In-class activity: Sketch the behavior of the system modeled by

$$\frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} + 3x(t) = 0$$

where

$$x(0) = 4 \quad , \quad \dot{x}(0) = 0$$

The homogeneous solution is in the form

$$x_h(t) = Ae^{\lambda t}$$

Take derivatives, substitute into the system model:

$$\lambda^2 Ae^{\lambda t} + 4\lambda Ae^{\lambda t} + 3Ae^{\lambda t} = 0$$

Collect terms:

$$(\lambda^2 + 4\lambda + 3)Ae^{\lambda t} = 0$$

We can't assume that $Ae^{\lambda t} = 0$, so

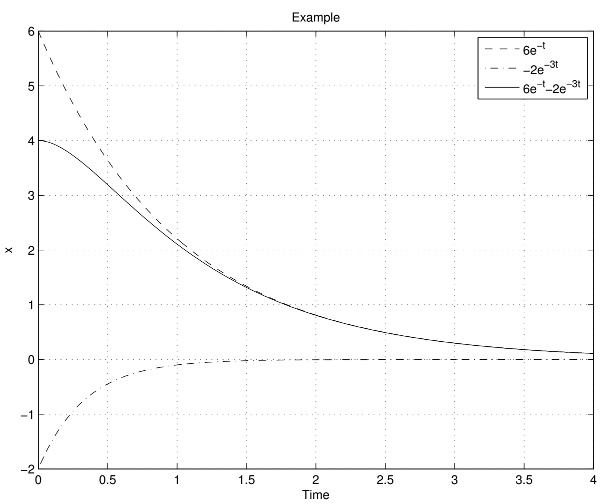
$$(\lambda^2 + 4\lambda + 3) = 0 \quad \rightarrow \quad \lambda_{1,2} = -1, -3$$

Therefore, the homogeneous solution is

$$x_h(t) = A_1 e^{-t} + A_2 e^{-3t}$$

Apply initial conditions to obtain A_1 and A_2:

$$\begin{aligned} 4 &= A_1 + A_2 \\ 0 &= -A1 - 3A_2 \\ &\Rightarrow A_1 = 6, \quad A_2 = -2 \quad \Rightarrow \quad x_h(t) = 6e^{-t} - 2e^{-3t} \end{aligned}$$



This figure was generated by the following MATLAB commands:

```
>> t=[0:.01:4];
>> x1=6*exp(-t)
>> x2=-2*exp(-3*t);
>> plot(t, x1, 'b–' , t, x2, 'g-.', t, x1+x2, 'k')
>> legend('6e^{-t}','-2e^{-3t}','6e^{-t}-2e^{-3t}')
>> grid; xlabel('Time');ylabel('x');title('Example')
```

In-class activity: Sketch the behavior of the system modeled by

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + 9x(t) = 0$$

where

$$x(0) = 4 \quad , \quad \dot{x}(0) = 0$$

In-class activity: Sketch the behavior of the system modeled by

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + 9x(t) = 0$$

where

$$x(0) = 4 \quad , \quad \dot{x}(0) = 0$$

Set $x_h(t) = Ae^{\lambda t}$; take derivatives ; substitute ; collect terms ; and obtain the Characteristic Equation:

$$\lambda^2 + \lambda + 9 = 0 \quad \rightarrow \quad \lambda_{1,2} = -0.5 \pm i2.958$$

$$x_h(t) = Ae^{-0.5t} sin(2.958t + \phi)$$

Apply initial conditions:

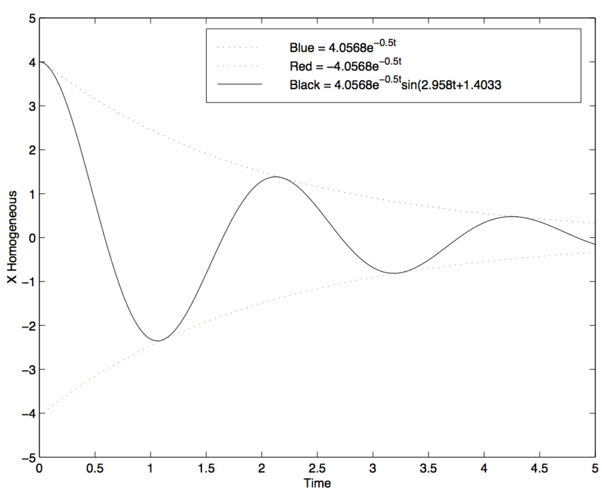
$$x(0) = 4 = Asin\phi$$
$$\dot{x}(0) = 0 = -0.5Asin\phi + 2.958Acos\phi$$

$$\Rightarrow Acos\phi = \frac{0.5sin\phi}{2.958} = 0.6761$$

$$tan\phi = \frac{Asin\phi}{Acos\phi} = 5.9161 \quad \Rightarrow \phi = 1.4033(\text{rad})$$

$$\Rightarrow A = \frac{4}{sin(1.4033} = 4.0568$$

$$\Rightarrow \quad x_h(t) = 4.0568e^{-0.5t} sin(2.958t + 1.4033)$$



This figure was generated by the following MATLAB commands:

```
>> t=[0:.01:5];
>> x=4.0568*exp(-0.5*t).*sin(2.958*t+1.4033)
>> xt=-4.0568*exp(-0.5*t);
>> plot(t, xt, 'b :' , t, -xt, 'r :', t, x, 'k')
>> legend('Blue = 4.0568e ˆ {-0.5t}', ...)
>> xlabel('Time')
>> ylabel('X Homogeneous')
```

Stability, Time constant, and Settling time

Unstable systems: $x_h(t) \to \infty$ as $t \to \infty$

$$\text{For real roots } \lambda, \text{behavior is unstable if } \lambda > 0$$

$$\text{For complex conjugate roots } \sigma \pm i\omega, \text{behavior is unstable if } \sigma > 0$$

$\Rightarrow$ System is unstable if **any** root has real part > 0

Stable systems: $x_h(t) \to 0$ as $t \to \infty$

$$\text{For real roots } \lambda, \text{behavior is stable if } \lambda < 0$$

$$\text{For complex conjugate roots } \sigma \pm i\omega, \text{behavior is stable if } \sigma < 0$$

$\Rightarrow$ System is stable if **all** roots have real part < 0

Marginal Stability (Instability): $x_h(t) \to$ nonzero, but $\neq \infty$, as $t \to \infty$

$$\text{For real roots } \lambda, \text{behavior is marginally stable (unstable) if } \lambda = 0$$

$$\text{For complex conjugate roots } \sigma \pm i\omega, \text{behavior is marginally stable (unstable) if } \sigma = 0$$

Time Constant and Settling Time: Speed of response, defined **ONLY** for stable systems

$$\begin{aligned} \text{Note that } e^{-1} &\approx 0.3679 \\ e^{-2} &\approx 0.1353 \\ e^{-4} &\approx 0.0183 \end{aligned}$$

Define **time constant** τ as

$$\tau = -\frac{1}{\lambda} \text{ for a real root } \lambda < 0$$

$$\tau = -\frac{1}{\sigma} \text{ for a complex conjugate pair with } \sigma < 0$$

$$\tau \text{ is the time required for the exponent in } x_h \text{ to equal -1}$$

Define **settling time** as 4τ ; this corresponds to a reduction of $98^+\%$ of the initial amplitude

A key consideration in many engineering applications is *speed of response*, measured by settling time

More negative real roots mean shorter settling time, i.e., less time required for $x_h \to 0$; faster system

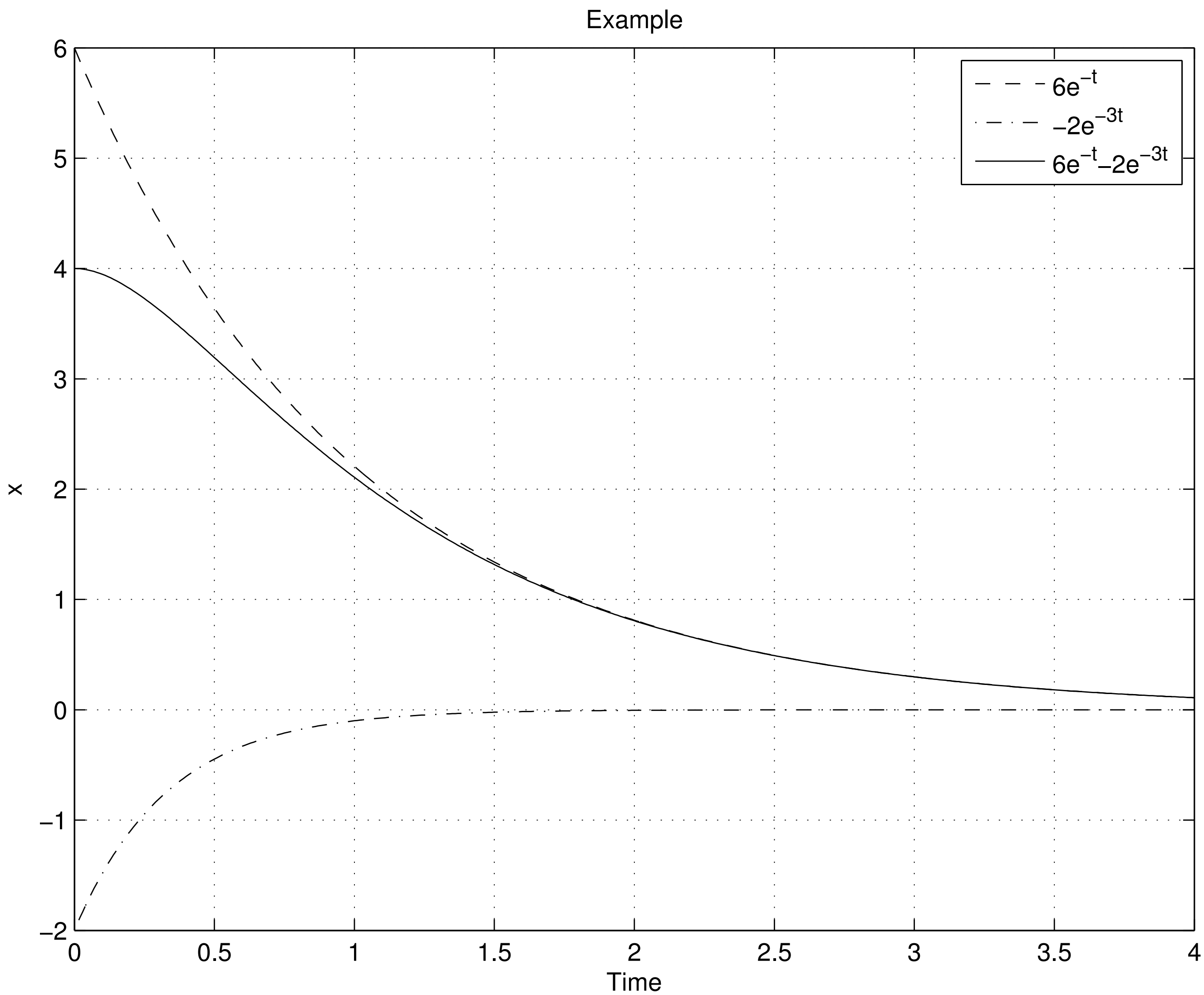
Large-order system speed of response is defined by slowest root (least negative root)

In-class activity: Find the time constant of the system modeled by

$$\frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} + 3x(t) = 0$$

where

$$x(0) = 4 \quad , \quad \dot{x}(0) = 0$$



Characteristic Equation: $\lambda^2 + 4\lambda + 3 = 0 \quad \rightarrow \quad \lambda_{1,2} = -1, -3$

$$x_h(t) = A_1 e^{-t} + A_2 e^{-3t}$$

Thus,

$$\tau_1 = 1 \text{ sec} \quad , \quad \tau_2 = 0.333 \text{ sec} \quad \rightarrow \quad \text{system time constant} = 1 \text{ sec}$$

Note:

Settling time for τ_1 is 4 secs

Settling time for τ_2 is 1.333 secs

Solution settles onto λ_1 component at $t \approx 1.333$

$\Rightarrow$ shows that system settles with slowest root

The actual time at which the part of $x_h(t)$ based on the *faster* root becomes negligible, and the total solution settles onto the *slowest* root, depends on the coefficient of each term in $x_h(t)$, which in turn depends on the IC's; but we define this time as the settling time of the faster root regardless of the IC's

In-class activity: Find the time constant of the system modeled by

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + 9x(t) = 0$$

where

$$x(0) = 4 \quad , \quad \dot{x}(0) = 0$$

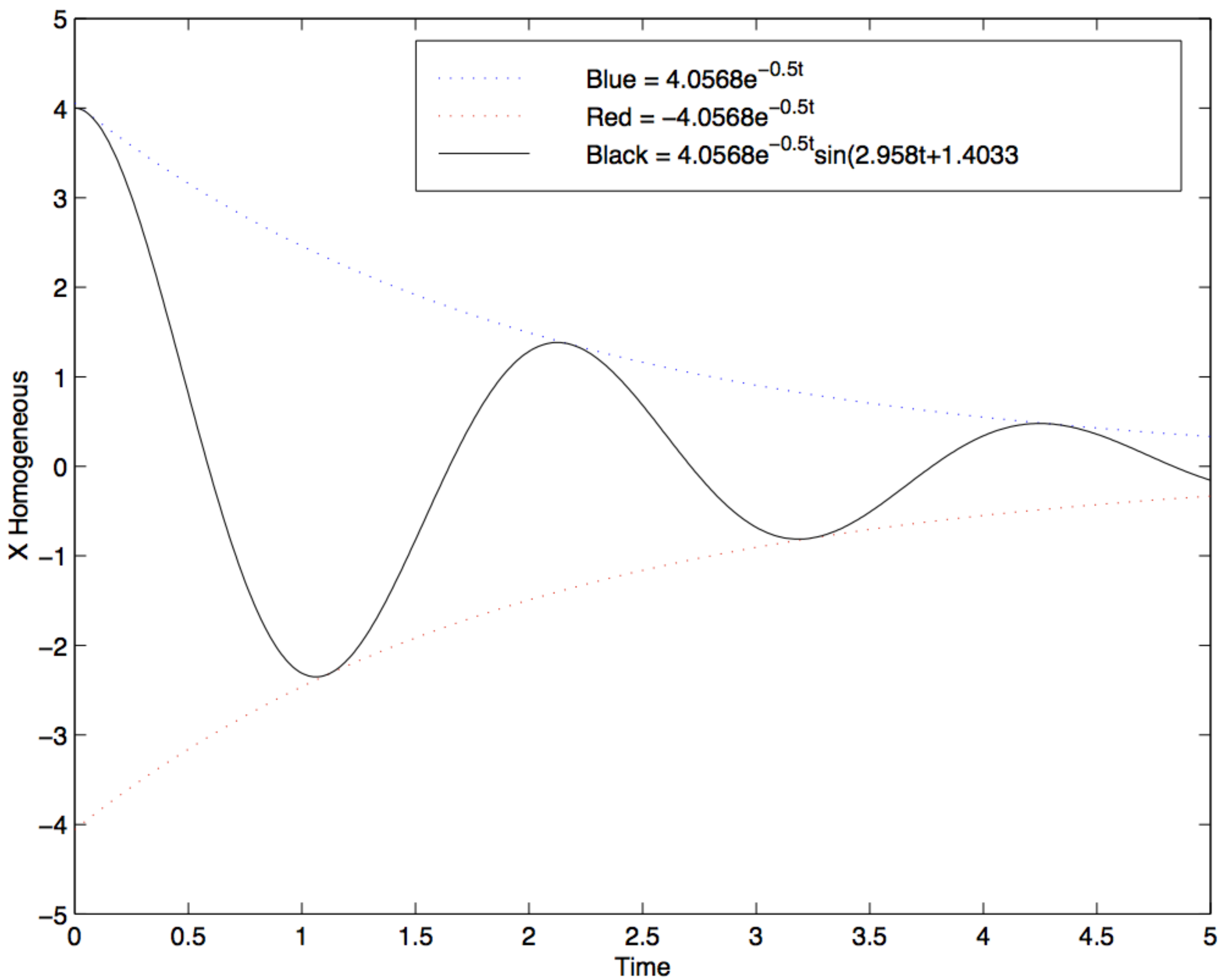
Characteristic Equation: $\lambda^2 + \lambda + 9 = 0 \quad \rightarrow \quad \lambda_{1,2} = -0.5 \pm i2.958$

$$x_h(t) = Ae^{-0.5t} sin(2.958t + \phi)$$

Thus, the time constant is $\tau = 2$ sec and the settling time is 8 sec; this is based *entirely* on the exponential envelope (the *real* part of the complex conjugate pair)

Except at the cyclical peaks of the sinusoid, the actual $x_h(t)$ is smaller in magnitude than the exponential envelope; therefore, the settling time calculation for complex conjugate terms is usually conservative. But the calculated phase angle is a function of initial conditions, which can vary (!), so we use the worst-case scenario - the exponential envelope.

The plot below does not go to the full settling time (8 sec).



Natural Frequency, Damping Ratio, and Damped Natural Frequency

Roots = **real** or **complex conjugate pairs** $\rightarrow$ Can always factor into **2nd-order** subpolynomials!
(plus one 1st-order if n=odd)

Also, many physical laws are **2nd-order**

Thus, there is a special focus on 2^{nd}-order polynomials, written as:

$$a_2\lambda^2 + a_1\lambda + a_0 = 0 = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \tag{5}$$

where

$$\omega_n \equiv \text{Natural Frequency} = \sqrt{\frac{a_0}{a_2}} \tag{6}$$

$$\zeta \equiv \text{Damping Ratio} = \frac{a_1}{2\omega_n a_2} \tag{7}$$

Roots:

$$\lambda_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{8}$$

Three cases in ζ:

1. $\zeta > 1$
   (a) Called **Overdamped**
   (b) Two real, distinct roots
   (c) Behavior is exponential decay (no oscillation)
   (d) Time constant is $\tau = \frac{1}{\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}$; system settles slower as ζ increases
   (e) As $\zeta \rightarrow \infty$, $\tau \rightarrow \infty$
2. $\zeta < 1$
   (a) Called **Underdamped**
   (b) Complex conjugate pair of roots $-\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} \equiv -\zeta\omega_n \pm i\omega_d$

   where
   $$\omega_d \equiv \omega_n\sqrt{1-\zeta^2} \equiv \textit{Damped Natural Frequency} \tag{9}$$
   (c) ω_d is the imaginary part of a complex conjugate pair of roots, so the corresponding behavior is an exponentially-decaying sinusoid of frequency ω_d (**NOT** ω_n)
   (d) $\omega_d < \omega_n$ for $0 < \zeta < 1$ $\rightarrow$ frequency is slower as ζ increases
   (e) Time constant is $\tau = \frac{1}{\zeta\omega_n}$ $\rightarrow$ system settles faster as ζ increases
3. $\zeta = 1$
   (a) Called **Critical damping**
   (b) Two real, repeated roots
   (c) Behavior is exponential decay, with some modification for low t due to repeated root
   (d) Time constant is $\tau = \frac{1}{\zeta\omega_n}$
   (e) For a given ω_n, fastest possible settling time
4. $\zeta = 0$
   (a) Called **Undamped**
   (b) Roots are $\pm i\omega_n$; complex conjugate pair with real part = 0
   (c) Sinusoidal behavior, constant amplitude, frequency ω_n , higher frequency than any $\zeta \neq 0$
   (d) Marginally (un)stable

In-class activity: Find the natural frequency and damping ratio of the system modeled by

$$\frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} + 3x(t) = 0$$

where

$$x(0) = 4 \quad , \quad \dot{x}(0) = 0$$

In-class activity: Find the natural frequency and damping ratio of the system modeled by

$$\frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} + 3x(t) = 0$$

where

$$x(0) = 4 \quad , \quad \dot{x}(0) = 0$$

Characteristic Equation:

$$\lambda^2 + 4\lambda + 3 = 0 = \lambda^2 + 2\zeta\omega_n + \omega_n^2$$

Solving,

$$\omega_n = \sqrt{3} = 1.732 \ \frac{rad}{sec} \quad , \quad \zeta = \frac{4}{2\omega_n} = 1.1547$$

This system is *overdamped* because $\zeta > 1$; therefore, the behavior is pure exponential decay (no sinusoid)

Note that we have previously found the roots of this system to be $\lambda_{1,2} = -1, -3$ and thus the behavior is pure exponential decay (no sinusoid); i.e., the behavior is the same no matter which approach we use to find it!

Of course, we find the roots to be the same. From Eq. (8), we may write the roots as

$$\lambda_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Substituting the values for ω_n and ζ,

$$\lambda_{1,2} = -(1.1547)(1.732) \pm (1.732)\sqrt{1.1547^2 - 1} = -1, -3$$

In-class activity: Find the natural frequency, damped natural frequency, and damping ratio of the system modeled by

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + 9x(t) = 0$$

where

$$x(0) = 4 \quad , \quad \dot{x}(0) = 0$$

In-class activity: Find the natural frequency, damped natural frequency, and damping ratio of the system modeled by

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + 9x(t) = 0$$

where

$$x(0) = 4 \quad , \quad \dot{x}(0) = 0$$

Characteristic Equation:

$$\lambda^2 + \lambda + 9 = 0 = \lambda^2 + 2\zeta\omega_n + \omega_n^2$$

Solving,

$$\omega_n = 3\ \frac{rad}{sec}\ ,\ \zeta = \frac{1}{2\omega_n} = 0.1667$$

This system is *underdamped* because $\zeta < 1$

Previously found $\lambda_{1,2} = -0.5 \pm i2.958 \quad \rightarrow$ underdamped; exponentially decaying sinusoid

The actual frequency of the solution is given by the imaginary part of the roots, 2.958

This can also be calculated directly as the *damped natural frequency*:

$$\omega_d \equiv \omega_n\sqrt{1-\zeta^2} = 3\sqrt{1-0.1667^2} = 2.958$$

For any $0 < \zeta < 1$,

$$\omega_d < \omega_n$$

Summary

1. The Characteristic Equation
   (a) The *character* of the behavior of the homogeneous solution is found from the roots of the characteristic equation
   (b) If the coefficients in the CE are real, all roots are either real or occur in complex conjugate pairs
   (c) A real root corresponds to purely exponential behavior; a complex conjugate pair corresponds to exponentially-changing sinusoidal behavior
   (d) The complete behavior is a linear combination of the individual behaviors of each root
   (e) The presence of a real root = zero, or complex conjugate pair with real part = zero, produces a constant, nonzero behavior
   (f) The presence of multiple roots with the same value (repeated roots) leads to unusual behavior for at least the beginning of the response (nonzero root); it may change the behavior throughout the entire response (zero root)
2. Stability
   (a) The stability is defined by roots of the characteristic equation; it is *inherent* to the system and not based on initial conditions or external forcing
   (b) The stability of each distinct real root depends on its *sign*; negative is stable
   (c) The stability of each complex conjugate pair of roots is defined by the *sign* of the real part
   (d) A system is stable **ONLY if all roots are stable**, which means that all roots must have negative real parts. If any single root out of the hundreds or thousands of roots in a large system has a positive real part, the system is unstable.
   (e) For engineering purposes, unstable or marginally stable systems are almost never satisfactory; thus, the first rule of design is to ensure stability
3. Time Constant and Settling Time:
   (a) These are *inherent* measures of the system; not based on initial conditions or external forcing
   (b) Can be calculated for every distinct real root and every distinct complex conjugate pair of roots
   (c) However, for a large-order system, **normally only use one value, defined by the slowest root** (the root with the longest time constant or settling time)
   (d) Settling time is most often defined as 4τ, corresponding to a 98% reduction in amplitude from the initial condition; but in some industries higher or lower multiples of τ may be used. The concept of settling time is clear, but the exact definition is somewhat arbitrary; we will use 4τ by default
   (e) Settling time is a measure of how quickly a nonzero initial value returns to zero in the homogeneous solution. However, we will see that it also applies to how quickly the system goes to any value due to forcing. Thus it is defined by the homogeneous solution, but is crucial in the forced response as well.
4. Natural Frequencies, Damping Ratios, and Damped Natural Frequencies:
   (a) These are *inherent* measures of the system; not based on initial conditions or external forcing
   (b) A set of these correspond to every distinct complex conjugate pair of roots
   (c) **Normally use all of them**, unlike time constants, because each one has potential importance in the forced response (particular solution)
   (d) Frequencies and damping ratios are defined by the homogeneous solution, but are crucial in the forced response as well.