The *unit step response* is the response of an LTI ODE with *zero initial conditions* to the *unit step input*

$$f(t) = \begin{pmatrix} 0 & t < 0 \\ 1 & t \geq 0 \end{pmatrix} \equiv \text{ Step Input}$$

Consider the standard 2^{nd}-order LTI ODE:

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = f(t)$$

subject to the unit step input defined in Eq. (1).

The complete solution for $x(t)$ is always equal to the sum of the homogeneous and particular solutions. The homogeneous solution for the underdamped case $\zeta < 1$ is

$$x_h(t) = Ae^{-\zeta\omega_n t} sin(\omega_d t + \phi)$$

where we determine A and ϕ from initial conditions. The homogeneous solution for the overdamped case $\zeta > 1$ is

$$x_h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \qquad , \quad \lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

where again, we determine A_1 and A_2 from initial conditions.

The particular solution for $t \geq 1$ is in the form $x_p = C$, since $f(t)$ is constant. Clearly, for $t < 0$, we have $x_p(t) = 0$. Since the initial conditions are zero, we also have $x_h(t) = 0$. Thus, for $t \leq 0$, the complete solution is $x(t) = 0$.

For $t \geq 0$, the we have $f(t) = 1$. Substituting into the LTI ODE, we find

$$\ddot{x}_p(t) + 2\zeta\omega_n\dot{x}_p(t) + \omega_n^2 x_p(t) = \omega_n^2 C = 1 \qquad \Rightarrow x_p = C = 1/\omega_n^2$$

The complete solution is thus

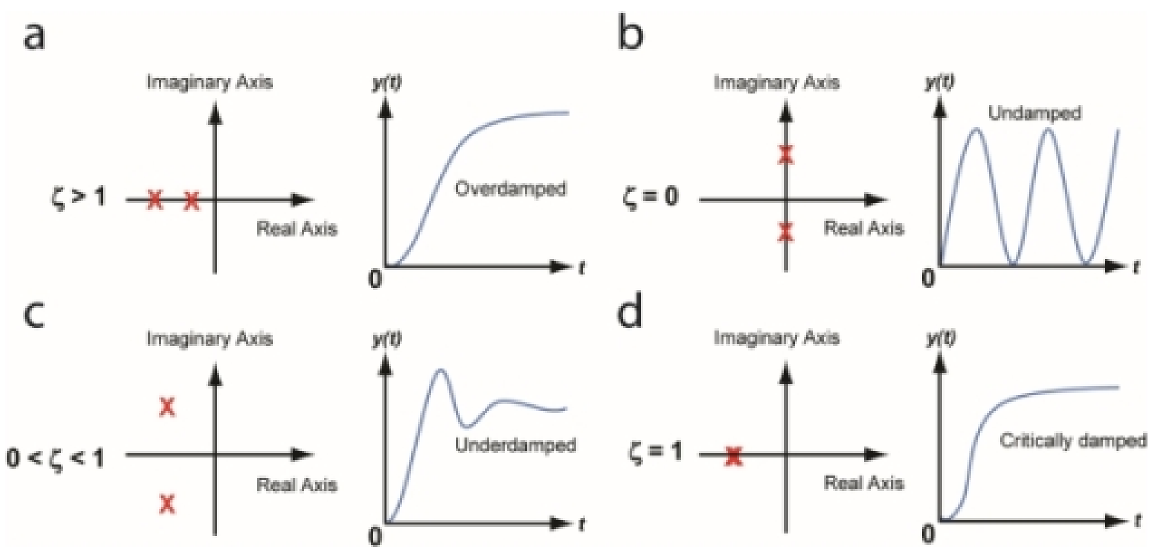
$$x(t) = x_h(t) + x_p(t) = Ae^{-\zeta\omega_n t} sin(\omega_d t + \phi) + 1/\omega_n^2 \quad \text{underdamped}$$

$$x(t) = x_h(t) + x_p(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + 1/\omega_n^2 \quad \text{overdamped}$$

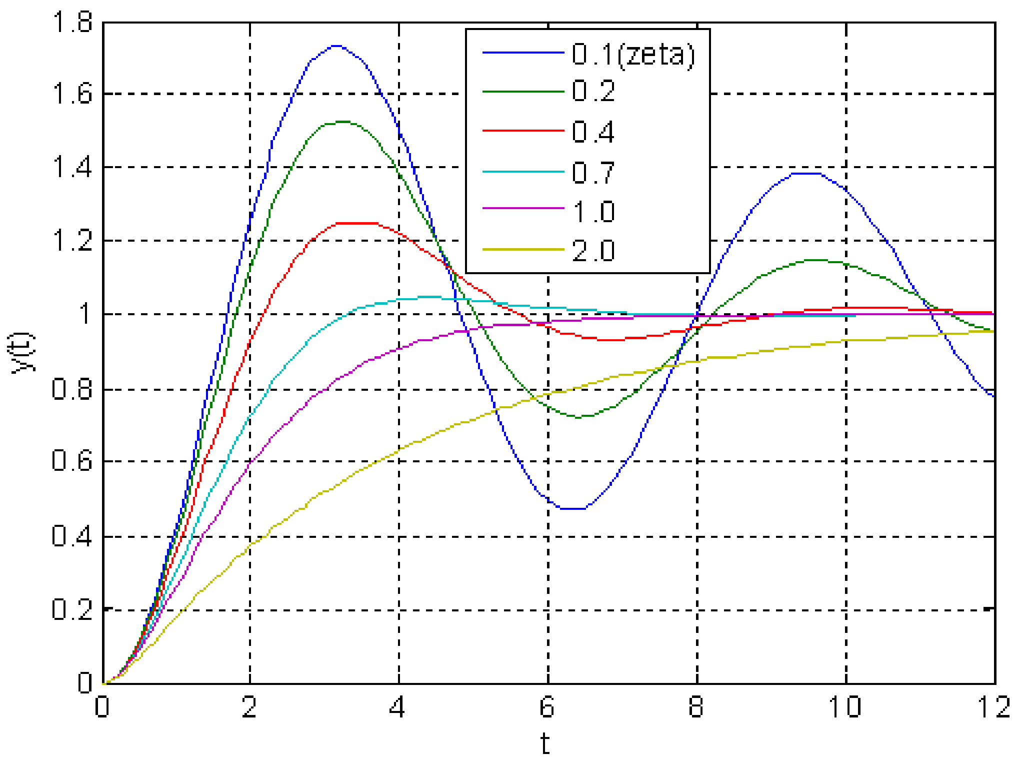
$$x(t) = x_h(t) + x_p(t) = (A_1 + A_2 t)e^{-\zeta\omega_n t} + 1/\omega_n^2 \quad \text{critically damped}$$

where in each case, the initial conditions are used to find the constants in the homogeneous portion.

The following graph illustrates the general shape of the complete solution:



The following graph shows the solution for the case $\omega_n = 1$, for various values of ζ



The homogeneous solution is triggered by the application of $f = 1$ at $t = 0$. For a stable system, the homogeneous solution disappears as $t \to$ settling time. Thus, the complete solution settles onto the particular solution.

We define the following new descriptors of the unit step response:

1. Transient response: For a stable system with forcing, this is the homogeneous response; the portion of the complete solution that fades away after some initial time
2. Steady-state response: For a stable system, this is the particular solution; the portion of the complete solution that does not fade away
3. Overshoot (or *maximum* overshoot): For an underdamped system, the percentage by which the maximum step response exceeds the steady-state response
4. Peak time: The time required for a step response to reach its maximum value (underdamped systems)
5. Rise time: (different precise definitions in different contexts) - the time required for the complete solution to first reach some prescribed percentage of the final value, e.g., 90%
6. Settling time: In the plot above, this is defined by the time after which the complete solution remains within $\pm 2\%$ of the steady-state solution

Note the following trends in the plot above:

- The particular solution is the same for all cases; all cases have the same steady-state response
- For $\zeta \geq 1$, there is no overshoot or peak time
- As $\zeta < 1$ decreases, the maximum overshoot increases; the peak time and rise time decrease; the settling time increases; the period of oscillation decreases
- The fastest settling time occurs for $\zeta = 1$

Example:

Find, plot, and analyze the step response of the system

$$\ddot{x} + 0.6\dot{x} + 9x = f(t)$$

Solution: The form of the particular solution is

$$x_p(t) = constant = C$$

Substituting into the ODE,

$$x_p = \begin{pmatrix} 0 & t < 0 \\ 1/9 & t \geq 0 \end{pmatrix}$$

To find the homogeneous solution, we need the roots of the characteristic equation:

$$\lambda^2 + 0.6\lambda + 9 = 0 \qquad \Rightarrow \quad \lambda_{1,2} = -0.3 \pm i2.985$$

Thus, the general form of the complete solution is

$$x_h(t) = Ae^{-0.3t} sin(2.985t + \phi) + 1/9$$

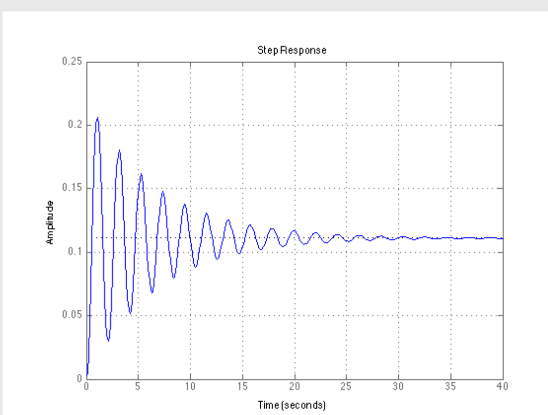
We use the IC's to find the constants in the homogeneous part of the solution:

$$x(0) = 0 = Asin\phi + 1/9 \qquad Asin\phi = -1/9 = -0.1111$$
$$\dot{x}(0) = 0 = -0.3Asin\phi + 2.985Acos\phi + 1/9 \qquad Acos\phi = -0.0484$$
$$\phi = tan^{-1}(\frac{Asin\phi}{Acos\phi}) = 1.16\ rad$$
$$A = \frac{sin\phi}{-0.1111} = -0.1212$$

The complete solution is

$$\underline{x(t) = -0.1212e^{-0.3t} sin(2.985t + 1.16) + 0.1111}$$

and is shown in the following plot



Maximum overshoot ≈ 100%
Peak time ≈ 1 second
Rise time ≈ 0.8 second
Settling time ≈ 28 seconds