Basis pursuit II

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Summary

Review Exact Reconstruction

- For $A \in \mathbb{K}^{m \times N}$ and $s \in \mathbb{N}$ we have that

$$\left.\begin{array}{c} \forall x \in \mathbb{C}^N \; s\text{-sparse} : \\ x \text{ is the unique minimizer} \\ \text{of } \left\{ \|z\|_1 \,\middle|\, Az = Ax \right\} \end{array}\right\} \iff A \text{ satisfies the NSP of order } s.$$

- Also,

$$\left.\begin{array}{c} \forall x \in \mathbb{C}^N \; s\text{-sparse} : \\ x \text{ is the unique minimizer} \\ \text{of } \left\{ \|z\|_1 \,\middle|\, Az = Ax \right\} \end{array}\right\} \implies \left\{\begin{array}{c} \forall x \in \mathbb{C}^N \; s\text{-sparse} : \\ \arg\min \left\{ \|z\|_1 \,\middle|\, Az = Ax \right\} \\ = \arg\min \left\{ \|z\|_0 \,\middle|\, Az = Ax \right\} \end{array}\right.$$

Review Stability

- The basis pursuit is stable under a sparsity defect in the vector x, if the measurement matrix satisfies the stable null space property (SNSP).
- For a matrix $A \in \mathbb{C}^{m \times N}$, we have

$$\left. \begin{array}{c} \forall x, z \in \mathbb{C}^N : \\ \|z - x\|_1 \leq \frac{1+\rho}{1-\rho} \left(\|z\|_1 - \|x\|_1 + 2\|x_{\overline{S}}\|_1 \right) \end{array} \right\} \iff \begin{array}{c} A \text{ satisfies} \\ \mathrm{SNSP}(\rho, S) \end{array} .$$

- In particular, if A satisfies the SNSP(ρ, S), any minimizer $x^\#$ of $\left\{ \|z\|_1 \,\middle|\, Ax = Az \right\}$ satisfies

$$\|x - x^\#\|_1 \leq \frac{2(1+\rho)}{1-\rho} \sigma_s(x)_1 \,.$$

Robustness

In this chapter we also want to handle noise in the measurement in addition to a sparsity deficit of the data.
What additional assumptions do we need to obtain similar results, if we consider the problem

$$\min \left\{ \|z\|_1 \,\middle|\, z \in \mathbb{C}^N,\ \ \|Az - (Ax + e)\| \le \eta \right\} \tag{$P_{1,\eta}$}$$

for $\|e\| \le \eta$? It depends on the norm, in which we measure the error, i.e. the distance of a minimizer to x.

At first, we consider the situation where we measure the error in the ℓ^1-norm.

Definition 1.1: Robust null space property

A matrix $A \in \mathbb{C}^{m\times N}$ is said to satisfy the **robust null space property** with respect to $\|\cdot\|$ with the constants $\rho \in (0,1)$ and $\tau > 0$ relative to a set $S \subseteq [N]$ iff

$$\forall v \in \mathbb{C}^N : \quad \|v_S\|_1 \leq \rho\|v_{\overline{S}}\|_1 + \tau\|Av\|\,. \qquad (\mathrm{RNSP}(\|\cdot\|, \rho, \tau, S))$$

A satisfies the robust null space property of **order** s with respect to $\|\cdot\|$ with the constants $\rho \in (0,1)$ and $\tau > 0$ $\mathrm{RNSP}(\|\cdot\|, \rho, \tau, s)$ iff A satisfies $\mathrm{RNSP}(\|\cdot\|, \rho, \tau, S)$ for all sets $S \subseteq [N]$ with $|S| \leq s$.

The main result is the following theorem.

Theorem 1.2: (Theorem 4.20 in the book)

A matrix $A \in \mathbb{C}^{m \times N}$ satisfies RNSP$(\|\cdot\|, \rho, \tau, S)$ if and only if

$$\forall x, z \in \mathbb{C}^N :$$
$$\|z - x\|_1 \leq \frac{1+\rho}{1-\rho}(\|z\|_1 - \|x\|_1 + 2\|x_{\overline{S}}\|_1) + \frac{2\tau}{1-\rho}\|A(x-z)\|\,.$$

This is a generalization of the previously discussed theorem for the SNSP.

Before proving this theorem, note the following corollary.

Corollary 1.3: (Theorem 4.19 in the book)

Assume that $A \in \mathbb{C}^{m \times N}$ satisfies $\mathrm{RNSP}(\|\cdot\|, \rho, \tau, s)$ with $0 < \rho < 1$ and $\tau > 0$, $e \in \mathbb{C}^m$ with $\|e\| \leq \eta$ and let $x \in \mathbb{C}^N$. Then, if $\mathcal{L}_{x,\eta}$ is the set of all minimizers of the problem

$$\min \left\{ \|z\|_1 \,\middle|\, z \in \mathbb{C}^N, \;\; \|Az - (Ax + e)\| \leq \eta \right\} \,,$$

then

$$\sup_{x^\# \in \mathcal{L}_x} \|x - x^\#\|_1 \leq \frac{2(1+\rho)}{1-\rho} \sigma_s(x)_1 + \frac{4\tau}{1-\rho} \eta \,,$$

i.e. the solution set $\mathcal{L}_x$ is contained in a ball of radius $\frac{2(1+\rho)}{1-\rho}\sigma_s(x)_1 + \frac{4\tau}{1-\rho}\eta$ around x in the ℓ^1-norm, which is small for small η and small sparsity defect $\sigma_s(x)_1$.

To establish an ℓ^q-bound for the error, we need a slightly stronger version of the RNSP:

Definition 1.4: ℓ^q-robust null space property

Let $q \geq 1$. A matrix $A \in \mathbb{C}^{m \times N}$ satisfies the **ℓ^q-robust null space property** of **order** $s \in \mathbb{N}$ (with respect to $\|\cdot\|$) with the constants $\rho \in (0,1)$ and $\tau \geq 0$ iff

$$\forall S \subseteq N,\ |S| \leq s\ \forall v \in \mathbb{C}^N :\quad \|v_S\|_q \leq \frac{\rho}{s^{1-\frac{1}{q}}} \|v_{\overline{S}}\|_1 + \tau \|Av\|\,.$$

- For $v \in \mathbb{C}^N$ and $1 \leq p \leq q$, we have

$$\|v_S\|_p \leq s^{\frac{1}{p}-\frac{1}{q}} \|v_S\|_q \,.$$

- Hence,

$$\ell^q\text{-RNSP } (\|\cdot\|, \rho, \tau, s) \implies \ell^p\text{-RNSP}\big(s^{\frac{1}{p}-\frac{1}{q}}\|\cdot\|, \rho, \tau, s\big) \,.$$

- In particular, the previously known ℓ^1-RNSP is implied by the ℓ^q-RNSP for $1 \leq q < \infty$.

Robustness of quadratically constrained basis pursuit

Corollary 1.5: (Theorem 4.22 in the book)

Assume $A \in \mathbb{C}^{m\times N}$ satisfies ℓ^2-RNSP$(\|\cdot\|_2, \rho, \tau, s)$ and let $x \in \mathbb{C}^N$, $e \in \mathbb{C}^m$ with $\|e\| \leq \eta$. Then, if $\mathcal{L}_{x,\eta}$ is the set of all minimizers of the problem

$$\min\left\{\|z\|_1 \,\middle|\, z \in \mathbb{C}^N,\;\; \|Az - (Ax+e)\|_2 \leq \eta\right\},$$

then

$$\sup_{x^{\#}\in\mathcal{L}_{x,\eta}} \|x - x^{\#}\|_p \leq \frac{C}{s^{1-\frac{1}{p}}}\sigma_s(x)_1 + Ds^{\frac{1}{p}-\frac{1}{2}}\eta$$

for $p \in [1,2]$ and for some constants $C, D > 0$ that only depend on ρ, τ.

Remarks

- The powers of s are interpolated between the ℓ^1- and the ℓ^2-case.
- A theorem in section 11 shoes that the same error estimate involving $\sigma_s(x)_2$ instead of $\sigma_s(x)_1$ is impossible for $N \gg m$.
- However, if $\|x\|$ belongs to a ℓ^q-unit ball with $q < 1$, which are good models for compressible vectors, then we have seen that

$$\sigma_s(x)_p \leq s^{\frac{1}{p} - \frac{1}{q}}.$$

- Hence, assuming $\eta = 0$, our error bound yields

$$\|x - x^{\#}\|_p \leq \frac{C}{s^{1-\frac{1}{p}}} \sigma_s(x)_1 \leq C s^{\frac{1}{p} - \frac{1}{q}},$$

  i.e. it decays in s like $\sigma_s(x)_p$.

Theorem 1.6: (Theorem 4.25 in the book)

If a matrix $A \in \mathbb{C}^{m\times N}$ satisfies ℓ^q-RNSP$(\|\cdot\|, \rho, \tau, s)$ for some $1 \leq q < \infty$, $\rho \in (0,1)$, $\tau > 0$, then

$$\forall x, z \in \mathbb{C}^N,\ \forall p \in [1,q] :$$
$$\|z - x\|_p \leq \frac{C}{s^{1-\frac{1}{p}}}(\|z\|_1 - \|x\|_1 + 2\sigma_s(x)_1) + Ds^{\frac{1}{p}-\frac{1}{q}}\|A(x-z)\|,$$

where $C = \frac{(1+\rho)^2}{1-\rho}$ and $D := (3+\rho)\tau/(1-\rho)$.

Recovery of individual vectors

- So far, we found conditions on A for the unique reconstruction of **all** vectors $x \in \mathbb{C}^N$ with sparsity s (or with support $S \subseteq [N]$).
- Now, we want to find conditions on A **and** x such that the unique reconstruction of the **one** vector $x \in \mathbb{C}^N$ with sparsity s (or with support $S \subseteq [N]$) is possible.

In the complex case, we have the following theorem:

Theorem 2.1: (Theorem 4.26 in the book)

Let $A \in \mathbb{C}^{m \times N}$ and $x \in \mathbb{C}^N$ with support $S \subseteq [N]$ be given. Then, the following conditions are equivalent:

(a) $\forall v \in \ker(A) \setminus \{0\} : \quad |\langle v, \operatorname{sgn}(x) \rangle| < \|v_{\overline{S}}\|_1$.

(b) A_S is injective and

$$\exists h \in \mathbb{C}^m : \begin{cases} (A^* h)_j = \operatorname{sgn}(x_j), & j \in S \quad \text{i.e. } A_S^* h = \operatorname{sgn}(x_S) \\ (A^* h)_j < 1, & j \in \overline{S} \end{cases} .$$

If one (and hence both) of these conditions hold, then x is the unique minimizer of the problem

$$\min \left\{ \|z\|_1 \,\middle|\, Az = Ax \right\} .$$

Remarks

- A satisfies the NSP relative $S \iff \forall x \in \mathbb{C}^N$ with $\operatorname{supp}(x) \subseteq S$: (a) holds.
- If (a) holds for $x \in \mathbb{C}^N$ with $\operatorname{supp}(x) \subseteq S$, (a) also holds for all

$$\{x' \in \mathbb{C}^N \,|\, \operatorname{supp}(x') \subseteq S \text{ and } \operatorname{sgn}(x)_{\operatorname{supp}(x')} = \operatorname{sgn}(x')_{\operatorname{supp}(x')}\}\,.$$

- There are stable and robust versions of this theorem, which yield slightly weaker error bounds than of the previous theorems.

- If A_S is injective, then $A_S^* A_S$ is invertible.
- Furthermore, the Moore-Penrose pseudo-inverse of A_S is then a left-inverse of A_S and given by

$$A_S^\dagger = (A_S^* A_S)^{-1} A_S^* \,.$$

- Therefore the condition $A_S^* h = \operatorname{sgn}(x_S)$ is satisfied by $h := (A_S^\dagger)^* \operatorname{sgn}(x_S)$.
- Hence, if this choice of h also satisfies the rest of (b), then the Theorem holds:

Corollary 2.2: (Corollary 4.28 in the book)

Let $A = (a_1, \ldots, a_N) \in \mathbb{C}^{m \times N}$ and $x \in \mathbb{C}^N$ with support $S \subseteq [N]$ be given. If A_S is injective and if

$$\forall l \in \overline{S}: \ \left| \left\langle A_S^\dagger a_l, \operatorname{sgn}(x_S) \right\rangle \right| = \left| \left\langle a_l, (A_S^\dagger)^* \operatorname{sgn}(x_S) \right\rangle \right| \\ = \Big(A^* (A_S^\dagger)^* \operatorname{sgn}(x_S) \Big)_l < 1\,,$$

then x is the unique minimizer of the problem

$$\min \left\{ \|z\|_1 \,\middle|\, Az = Ax \right\} \,.$$

The converse direction

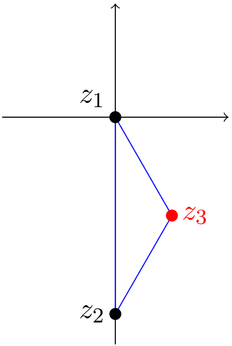
The converse of the Theorem 4.26 is not true in general, e.g. take

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} \mathrm{e}^{-\pi \mathrm{i}/3} \\ \mathrm{e}^{\pi \mathrm{i}/3} \\ 0 \end{pmatrix}.$$

The solution space of $Az = Ax$ is then

$$\left\{ \begin{pmatrix} 0 \\ \sqrt{3}\mathrm{i} \\ -\mathrm{e}^{-\mathrm{i}\pi/3} \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \,\middle|\, z \in \mathbb{C} \right\},$$

and hence, the minimization problem consists in finding the point $z \in \mathbb{C}$ such that the sum of the distances to the points $z_1 = 0$, $z_2 = -\sqrt{3}\mathrm{i}$, $z_3 = \mathrm{e}^{-\mathrm{i}\pi/3}$ is minimal.



In the picture one can see that the sum of the distances is minimal exactly for $z = z_3$ (cf. first Fermat point, the angle at z_3 is $120°$). Hence, $x = (0, \sqrt{3}\mathrm{i}, \mathrm{e}^{-\mathrm{i}\pi/3})^T + z_3(1,1,1)^T$ is the unique minimizer of the problem.

However, the condition (a) does not hold: For $v = (z, z, z) \in \ker(A) \setminus \{0\}$, we have that

$$|\left\langle \mathrm{sgn}(x), v_{\{1,2\}} \right\rangle| = |\mathrm{e}^{\pi\mathrm{i}/3} z + \mathrm{e}^{-\pi\mathrm{i}/3} z| = |z| = \|v_{\{3\}}\|_1 .$$

In the real case, however, the converse does actually hold:

Theorem 2.3: (Theorem 4.30 in the book)

Let $A \in \mathbb{R}^{m \times N}$ and $x \in \mathbb{R}^N$ with support $S \subseteq [N]$ be given. Then, the following conditions are equivalent:

(i) $\forall v \in \ker(A) \setminus \{0\} : \quad |\langle v, \operatorname{sgn}(x)\rangle| < \|v_{\overline{S}}\|_1$.

(ii) A_S is injective and

$$\exists h \in \mathbb{R}^m : \; \begin{cases} (A^*h)_j = \operatorname{sgn}(x_j), & j \in S \\ (A^*h)_j < 1, & j \in \overline{S} \end{cases} .$$

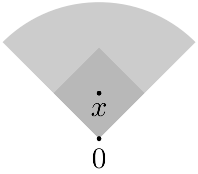
(iii) x is the unique minimizer of

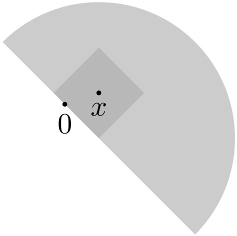
$$\min\left\{ \|z\|_1 \,\middle|\, z \in \mathbb{R}^N, \; Az = Ax \right\} .$$

Definition 2.4

Let $x \in \mathbb{R}^N$, then the **tangent cone** to the ℓ^1-ball at x is defined as

$$T(x) = \operatorname{cone}\left\{z - x \,\middle|\, z \in \mathbb{R}^N,\ \|z\|_1 \leq \|x\|_1\right\} = \operatorname{cone}\left(\overline{B_{\|x\|_1}^{\|\cdot\|_1}(x)}\right).$$





Theorem 2.5: (Theorem 4.35 in the book)

Let $A \in \mathbb{R}^{m \times N}$ and $x \in \mathbb{R}^N$. Then

$$\left.\begin{array}{c} x \text{ is the unique minimizer} \\ \text{of } \left\{ \|z\|_1 \,\middle|\, Az = Ax \right\} \end{array}\right\} \iff \ker(A) \cap T(x) = \{0\}\,.$$

Theorem 2.6: (Theorem 4.36 in the book)

Let $A \in \mathbb{R}^{m\times N}$, $x \in \mathbb{R}^N$ and $e \in \mathbb{R}^m$ such that $\|e\|_2 \leq \eta$. Assume

$$\exists \tau > 0 : \inf_{v \in T(x) \cap \partial B_1^{\|\cdot\|_2}} \|Av\|_2 \geq \tau\,.$$

Then, if $x^\#$ is a minimizer of the problem

$$\min\left\{\|z\|_1 \,\middle|\, z \in \mathbb{R}^N,\ \ \|Az - (Ax + e)\|_2 \leq \eta\right\}\,,$$

we have that

$$\|x - x^\#\|_2 \leq \frac{2\eta}{\tau}\,.$$