

## Basis Pursuit Part II

### Review of the null space property and stability

The basis pursuit the convex relaxation of the  $\ell^0$ -minimization problem, which is NP-hard in general. The null space property (NSP) of order  $s$  is a necessary and sufficient condition for the exact reconstruction of every  $s$ -sparse vector via the basis pursuit. Here, “exact reconstruction” means that every vector  $x \in \mathbb{C}^N$  with at most  $s$  non-zero entries is the unique minimizer of the following optimization problem:

$$\min \{ \|z\|_1 \mid Az = Ax \} .$$

Furthermore, one can see that if this is the case, then  $x$  is also the unique minimizer of the corresponding  $\ell^0$ -minimization problem.

The basis pursuit is stable under a sparsity defect in the vector  $x$ , if the measurement matrix satisfies the stable null space property (SNSP).

### 4.3 Robustness

The goal of this section is to take the noise in the measurements into account, additinally to the sparsity defect in the vector  $x$ . More precisely, we want to investigate the following optimization problem, for a given noise level  $\eta$ :

$$\min \left\{ \|z\|_1 \mid z \in \mathbb{C}^N, \|Az - (Ax + e)\| \leq \eta \right\} . \quad (P_{1,\eta})$$

For that we need to strengthen the assumptions on the measurement matrix  $A$  to satisfy the robust null space property (RNSP), depending on the  $\ell^q$ -norm, in which the error bound is desired. We start with  $q = 1$ :

#### Definition: Robust null space property

A matrix  $A \in \mathbb{C}^{m \times N}$  is said to satisfy the *robust null space property* with respect to  $\|\cdot\|$  with the constants  $\rho \in (0, 1)$  and  $\tau > 0$  relative to a set  $S \subseteq [N]$  iff

$$\forall v \in \mathbb{C}^N : \|v_S\|_1 \leq \rho \|v_{\bar{S}}\|_1 + \tau \|Av\| . \quad (\text{RNSP}(\|\cdot\|, \rho, \tau, S))$$

$A$  satisfies the robust null space property of *order*  $s$  with respect to  $\|\cdot\|$  with the constants  $\rho \in (0, 1)$  and  $\tau > 0$  RNSP( $\|\cdot\|, \rho, \tau, s$ ) iff  $A$  satisfies RNSP( $\|\cdot\|, \rho, \tau, S$ ) for all sets  $S \subseteq [N]$  with  $|S| \leq s$ .

We note that the RNSP directly implies the SNSP. The main result is the following theorem.

#### Theorem: (Theorem 4.20 in the book)

A matrix  $A \in \mathbb{C}^{m \times N}$  satisfies RNSP( $\|\cdot\|, \rho, \tau, S$ ) if and only if

$$\begin{aligned} & \forall x, z \in \mathbb{C}^N : \\ & \|z - x\|_1 \leq \frac{1 + \rho}{1 - \rho} (\|z\|_1 - \|x\|_1 + 2\|x_{\bar{S}}\|_1) + \frac{2\tau}{1 - \rho} \|A(x - z)\| . \end{aligned}$$

This is a generalization of the previously discussed theorem for the SNSP (cf. Theorem 4.14 in the book). The following corollary, an  $\ell^1$ -bound for the error, is a key result:

#### Corollary: (Theorem 4.19 in the book)

Assume that  $A \in \mathbb{C}^{m \times N}$  satisfies RNSP( $\|\cdot\|, \rho, \tau, s$ ) with  $0 < \rho < 1$  and  $\tau > 0$ ,  $e \in \mathbb{C}^m$  with  $\|e\| \leq \eta$  and let  $x \in \mathbb{C}^N$ . Then, if  $\mathcal{L}_{x,\eta}$  is the set of all minimizers of the problem

$$\min \left\{ \|z\|_1 \mid z \in \mathbb{C}^N, \|Az - (Ax + e)\| \leq \eta \right\} ,$$

then

$$\sup_{x^\# \in \mathcal{L}_x} \|x - x^\#\|_1 \leq \frac{2(1 + \rho)}{1 - \rho} \sigma_s(x)_1 + \frac{4\tau}{1 - \rho} \eta ,$$

i.e. the solution set  $\mathcal{L}_x$  is contained in a ball of radius  $\frac{2(1+\rho)}{1-\rho}\sigma_s(x)_1 + \frac{4\tau}{1-\rho}\eta$  around  $x$  in the  $\ell^1$ -norm, which is small for small  $\eta$  and small sparsity defect  $\sigma_s(x)_1$ .

Now, let us move on to establish an error bound in  $\ell^q$  for  $q \in [1, \infty)$ . We need:

**Definition:  $\ell^q$ -robust null space property**

Let  $q \geq 1$ . A matrix  $A \in \mathbb{C}^{m \times N}$  satisfies the  $\ell^q$ -robust null space property of order  $s \in \mathbb{N}$  (with respect to  $\|\cdot\|$ ) with the constants  $\rho \in (0, 1)$  and  $\tau \geq 0$  iff

$$\forall S \subseteq [N], |S| \leq s \forall v \in \mathbb{C}^N : \|v_S\|_q \leq \frac{\rho}{s^{1-\frac{1}{q}}} \|v_{S^c}\|_1 + \tau \|Av\|.$$

**Remark**

- For  $v \in \mathbb{C}^N$ ,  $S \subseteq [N]$  with  $|S| \leq s$  and  $1 \leq p \leq q$ , we have

$$\|v_S\|_p \leq s^{\frac{1}{p}-\frac{1}{q}} \|v_S\|_q.$$

- Hence,

$$\ell^q\text{-RNSP}(\|\cdot\|, \rho, \tau, s) \implies \ell^p\text{-RNSP}(s^{\frac{1}{p}-\frac{1}{q}}\|\cdot\|, \rho, \tau, s)$$

for  $1 \leq p \leq q$ .

- In particular, the previously known  $\ell^1$ -RNSP is implied by the  $\ell^q$ -RNSP for  $1 \leq q < \infty$ .

We have the following result:

**Theorem: (Theorem 4.25 in the book)**

If a matrix  $A \in \mathbb{C}^{m \times N}$  satisfies  $\ell^q$ -RNSP( $\|\cdot\|, \rho, \tau, s$ ) for some  $1 \leq q < \infty$ ,  $\rho \in (0, 1)$ ,  $\tau > 0$ , then

$$\begin{aligned} & \forall x, z \in \mathbb{C}^N, \forall p \in [1, q] : \\ & \|z - x\|_p \leq \frac{C}{s^{1-\frac{1}{p}}} (\|z\|_1 - \|x\|_1 + 2\sigma_s(x)_1) + D s^{\frac{1}{p}-\frac{1}{q}} \|A(x - z)\|, \end{aligned}$$

where  $C = \frac{(1+\rho)^2}{1-\rho}$  and  $D := (3 + \rho)\tau/(1 - \rho)$ .

The main result is the following corollary of this theorem:

**Corollary 0.1: Robustness of the quadratically constrained basis pursuit**

Assume  $A \in \mathbb{C}^{m \times N}$  satisfies  $\ell^2$ -RNSP( $\|\cdot\|_2, \rho, \tau, s$ ) and let  $x \in \mathbb{C}^N$ ,  $e \in \mathbb{C}^m$  with  $\|e\| \leq \eta$ . Then, if  $\mathcal{L}_{x,\eta}$  is the set of all minimizers of the problem

$$\min \left\{ \|z\|_1 \mid z \in \mathbb{C}^N, \|Az - (Ax + e)\|_2 \leq \eta \right\},$$

then

$$\sup_{x^\# \in \mathcal{L}_{x,\eta}} \|x - x^\#\|_p \leq \frac{C}{s^{1-\frac{1}{p}}} \sigma_s(x)_1 + D s^{\frac{1}{p}-\frac{1}{2}} \eta$$

for  $p \in [1, 2]$  and for some constants  $C, D > 0$  that only depend on  $\rho, \tau$ .

## 4.4 Recovery of individual vectors