CMPT-335 Discrete Structures

ORDERED SETS. RELATIONS

Ordered Pair

- When listing the elements of a set, the order of the elements does not matter
- In the ordered pair of elements a and b, denoted (a, b), the order in which the entries are written, matters
- Hence $(1,2) \neq (2,1)$ and $(a,b) = (c,d)$ if and only if $a=c$ and $b=d$

Cartesian Product

- The Cartesian Product of sets A and B is the set consisting of all the ordered pairs (a, b), where $a \in A, b \in B$:

$$A \times B = \{(a,b) \mid (a \in A) \wedge (b \in B)\}$$

- The most popular and simple example of the Cartesian product is the Euclidean plane, which is the set of all ordered pairs of real numbers $R \times R$

- It is important that in general $A \times B \neq B \times A$

Cartesian Product

- The Cartesian Product of the sets $A_1, A_2, \dots, A_n$ is the set consisting of ordered n-tuples $\left(a_1, a_2, \dots, a_n\right)$ where $a_i \in A_i; i = 1, \dots, n$:

$$A_1 \times A_2 \times \dots \times A_n =$$

$$= \left\{\left(a_1, a_2, \dots, a_n\right) \mid a_i \in A_i; i = 1, \dots, n\right\}$$

Relation

- A **relation** $\boldsymbol{R}$ from a set A to a set B is any subset of the Cartesian product $A \text{ x } B$.
- Thus, if $P{=}A \text{ x } B$, $(x, y) \in R; R \subseteq P$, then we say that x is related to y by R and use the following notation: $x\ R\ y$.

Relation

- For example, let A={Smith, Johnson} be a set of students and B={Calculus, Discrete Structures, History, Programming} be a set of classes. Let student Smith takes Calculus and History, while student Johnson takes Calculus, Discrete Structures and Programming. Let us determine a relation R as follows: "student *x* takes class *y*"
- Thus, R={(Smith, Calc), (Smith, Hist), (Johnson, Calc), (Johnson, Discrete Structures), (Johnson, Programming)}
- Hence, for example Smith R Calc is true, while Johnson R Hist is false.

Relation on a set

- A relation from set *S* to itself, that is a subset of the Cartesian product *S* x *S*, is called a relation on *S*.
- Let *S*={A, B, C, D, E, F, G} be a set of classes from some program curriculum. Let us define relation as follows: *x* is related to *y* if class *x* is a prerequisite for class *y*.
- If A and B are prerequisites for C and C, D are prerequisites for E, F, G, then the resulting relation on S is {(A,C), (B,C), (C,E), (C,F), (C,G), (D,E), (D,F), (D,G) } is a just defined relation on S.

Properties of a Relation on a set

- A relation R on a set S may have any of the following three special properties:

➢ If $\forall x \in S, \quad x \, R \, x$ is true, then R is called reflexive.

➢ If $y \, R \, x$ is true whenever $x \, R \, y$ is true, then R is called symmetric.

➢ If $x \, R \, z$ is true whenever $x \, R \, y$ and $y \, R \, z$ are both true, then R is called transitive.

Strict Order Relation

- Let $S=\{1,2,3,4\}$. Define a relation R on S by letting $x\ R\ y$ mean $x < y$.
- Then, for example, $1\ R\ 4$, $2\ R\ 4$, $1\ R\ 3$, $1\ R\ 2$, $3\ R\ 4$ are true, but $4\ R\ 1$, $4\ R\ 2$, $3\ R\ 1$, $2\ R\ 1$, $4\ R\ 3$ are false.
- The relation “<“ and the relation “>“ on the set S (and the set Z) are the relations of a strict order.
- Thus a strict order relation is a relation, which is not reflexive, not symmetric, but it is **transitive**.

Properties of a Relation on a set

- The relation “<“ on the set of integer numbers or any its subset containing not less than 3 elements, is not reflexive, it is also not symmetric, but it is transitive.
  In fact, if $x < y$ and $y < z$ then always $x < z$.
- The relation “=“ on any numbering set is reflexive ($x=x$), symmetric ($x=y$ whenever $y=x$), and transitive (if $x=y$ and $y=z$ then $x=z$).

Equivalence Relation

- A relation, which is reflexive, symmetric, and transitive is called an equivalence relation.
- Example 1: on the set of students of a particular university, define one student to be related to another student if they both take the Discrete Structures class.
- Example 2: on the set of students of a particular university, define one student to be related to another whenever their last names begin with the same letter.

Equivalence Relation

- Example 3: On the set of polygons define $x\ R\ y$ to mean that x has the same area as y.
- Example 4: on the set of students of a particular university, define one student to be related to another one if they both got “A” in the Calculus 1 class.

Equivalence Relation

- Example 5: On the set $S = \{2, 3, 4, 5, \ldots\}$ of integers greater than 1 define $x\ R\ y$ to mean that x has the same number of distinct prime devisors as y.
- Thus, 6 R 15 (6=2x3 and 15=3x5) and 12 R 55 (12=2x2x3 and 55=5x11).
- Evidently, 6 R 12, 6 R 55, 15 R 12, and 15 R 55 are also true. It is clear that this relation is reflexive, symmetric and transitive.

Equivalence Relation

- Example 5 (continuation). On the set $S = \{2, 3, 4, 5, \ldots\}$ of integers greater than 1 define $x\ R\ y$ to mean that x has the same number of distinct prime devisors as y.
- Thus, 30 R 60 (30=2x3x5 and 60=2x2x3x5) and 90 R 150 (90=2x3x3x5 and 150=2x3x5x5). Evidently, 30 R 150, 60 R 90, 30 R 90, and 60 R 150 are also true.
- However, $6 \not R\ 30;\ 55 \not R\ 150;\ 12 \not R\ 90$!!!

Equivalence Class

- If $x \in S$ and R is an equivalence relation on a set S, the set of elements of S $\{y \in S \mid y\ R\ x\}$ that are related to *x* is called the equivalence class containing *x* and is denoted [*x*]. Thus

$$[x] = \{y \in S \mid y\ R\ x\}$$

Equivalence Class

- Returning to the previous example, we may conclude that we have considered two equivalence classes [6]={6, 12, 15, 55,…} and [30]={30, 60, 90, 150,…}
- Evidently, the different equivalence classes are always disjoint. If we suppose that two or more different equivalence classes have a non-empty intersection, we immediately have to conclude that these classes cannot be different and they must contain the same elements.

Equivalence Relation

- Example. On the set $S = \{2, 3, 4, 5, \ldots\}$ of integers greater than 1 define $x\, R\, y$ to mean that *x* has the same largest prime devisor as *y*. Then R is an equivalence relation on S.
- The equivalence class of R containing 2 consists of all elements in S that are related to 2 – that is all positive integers whose largest prime devisor is 2: $[2] = \{2^k : k = 1, 2, 3, \ldots\}$

Theorem on Equivalence Classes

- Let R be an equivalence relation on a set S. Then:

➢ If $x \in S,\ y \in S$, then x is related to y by R only if $[x]=[y]$.

➢ Two equivalence classes of R are either equal or disjoint.

Equivalence Relation and Partition

- It follows from the second part of Theorem that the equivalence classes of an equivalence relation R on set S divide S into disjoint subsets. This family of subsets is called a partition of S and has the following properties:

➢ No subset is empty.

➢ Each element of S belongs to some subset.

➢ Two distinct subsets are disjoint.

Equivalence Relation and Partition

- Every equivalence relation on S gives rise to a partition of S by taking the family of subsets in the partition to be the equivalence classes of the equivalence relation.
- If P is a partition of S, we can define a relation R on S by letting $x\ R\ y$ mean that x and y belong to the same member of P.

Equivalence Relation and Partition

- Let *S*={1,2,3,4,5,6}. Let *A*={1,3,4}, *B*={2,6}, and *C*={5}. Let some equivalence relation is defined on these sets. Evidently,

$$A \bigcup B \bigcup C = S; A \bigcap B = A \bigcap C = B \bigcap C = \varnothing$$

- Then $P = \{A, B, C\}$ is a partition of *S*={1,2,3,4,5,6}.
- Then we can establish a relation "$x\ R\ y$ means that x and y belong to the same member of *P*":

R={(1,1),(1,3),(1,4),(3,1),(3,3),(3,4),(4,1),(4,3), (4,4), (2,2), (2,6), (6,2), (6,6), (5,5)}

Equivalence Relation and Partition

- Theorem.

➢ An equivalence relation R on S gives rise to a partition P of S, in which the members of P are the equivalence classes of R.

➢ A partition P of S induces an equivalence relation R in which any two elements x and y are related by R whenever they belong to the same member of P. Moreover, the equivalence classes of this relation are members of P.

Antisymmetric Relation

- A relation R on a set S is called antisymmetric if, whenever $x\ R\ y$ and $y\ R\ x$ are both true, then $x=y$.
- Examples. Relations "≤" and "≥" on the set Z of integer numbers. If $x \leq y$ and $y \leq x$ then always $x=y$. If $x \geq y$ and $y \geq x$ then always $x=y$.

Partial Ordering Relations

- A relation R on a set S is called a partial ordering relation, or simply a partial order, if the following 3 properties hold for this relation:

1) R is reflexive, that is, $x\ R\ x$ is true $\forall x \in S$
2) R is antisymmetric, that is $(x\ R\ y) \wedge (y\ R\ x) \rightarrow x = y$
3) R is transitive, that is $(x\ R\ y) \wedge (y\ R\ z) \rightarrow (x\ R\ z)$

Partial Ordering Relations. Examples

- Relations "≤" and "≥" are partial orders on sets Z of integer numbers and R of real numbers.
- Let $S=\{A,B,C,\ldots\}$ be a set whose elements are other sets. For $A,B\in S$ define $A\ \boldsymbol{R}\ B$ if $A\subseteq B$

➢ R is reflexive ($A\subseteq A$),

antisymmetric $A\subseteq B, B\subseteq A\rightarrow A=B$

and transitive $A\subseteq B, B\subseteq C\rightarrow A\subseteq C$

Thus R is a partial order.

Partial Ordering Relations. Examples

- Let us consider a set of n-dimensional binary vectors $E_2{}^n$={(0,…,0), (0,…,01),…,(1,…,1)}. We say that vector x precedes to vector y $x \prec y$ if for all n components of these two vectors the following property holds $x_i \le y_i, i = 1,\dots,n$. For example, if n=3: $(0,0,0) \prec (0,0,1) \prec (1,0,1) \prec (1,1,1)$, but $(0,1,1) \not\prec (1,0,0); (0,1,0) \not\prec (1,0,1)$.
- The relation $"\prec"$ is a partial order on the set of n-dimensional binary vectors.

Partial Ordering Relations. Examples

- Let $S=\{A, B, C, D, E, F, G\}$ be a set of classes from some program curriculum. Let us define relation as follows: *x* is related to *y* if class *x* is a prerequisite for class *y*.
- This relation is a partial order.

Tolerance Relations

- A relation, which is reflexive and symmetric, but not necessary is transitive is called a tolerance relation.
- A tolerance relation establishes closeness of some objects with each other, but this closeness cannot be expanded to other objects.
- Example 1: on the set of students attending a particular university, define one student to be related to another one if they both have a grandmother in common.