COSC-335 Discrete Structures

FUNCTIONS

Function. Definition

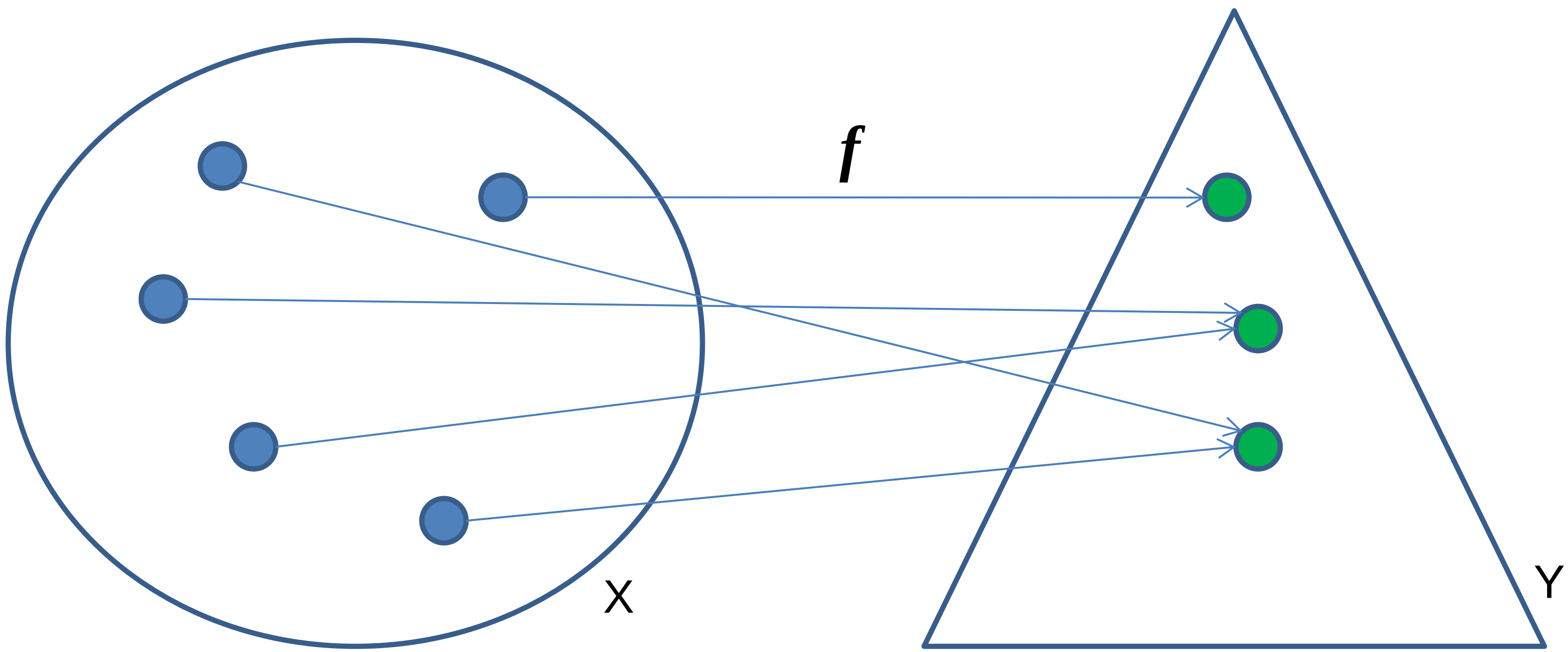
- Let *X* and *Y* be sets. A **function** *f* from *X* to *Y* is a relation from *X* to *Y* with the property that, **for each** element $x \in X$, there is **exactly one** element $y \in Y$ such that $\boldsymbol{x\ f\ y}$:

$$f : X \to Y;\ y = f(x)$$

- Since any relation from *X* to *Y* is a subset of X x Y, a function is a subset *S* of X x Y such that for each $x \in X$ there is a unique $y \in Y$ with $(x, y) \in S$

Functions

- Functions are also called mappings or transformations. We say that f maps X to Y



Functions

- If f is a function from X to Y
$$f : X \to Y;\ y = f(x),$$
the sets X and Y are called the domain and codomain of the function, respectively.
- The unique element $y \in Y$ is called the image of $x \in X$ and $x \in X$ is a preimage of $y \in Y$ under f.
- The range of f is the set of all images of elements of X in Y.

Range

- If f is a function from *X* to *Y*

$$f : X \to Y;\ y = f(x),$$

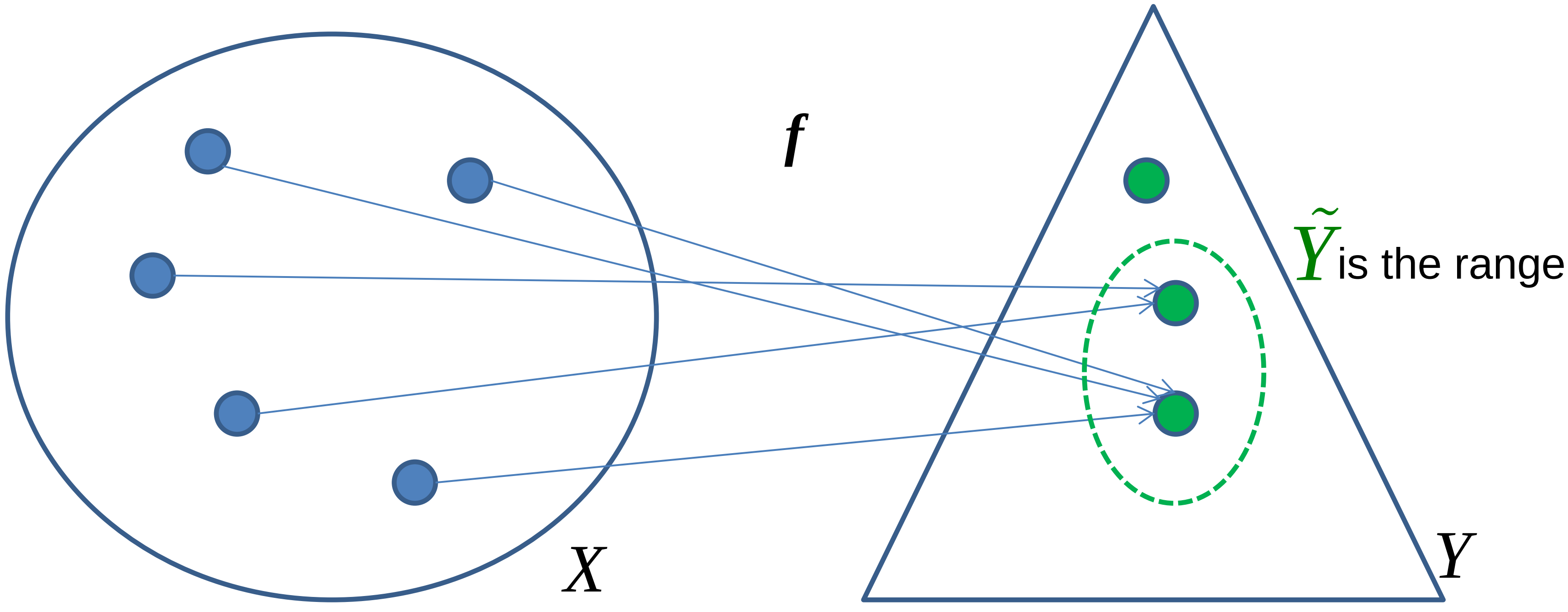
and $\hat{Y} \subseteq Y$ is a subset of the codomain such that

$$\forall y \in \hat{Y}\ \exists x \in X : y = f(x)$$

(thus $\hat{Y}$ contains all the elements from *Y* that are paired with elements of the domain *X*), then $\hat{Y}$ is called the range of the function.

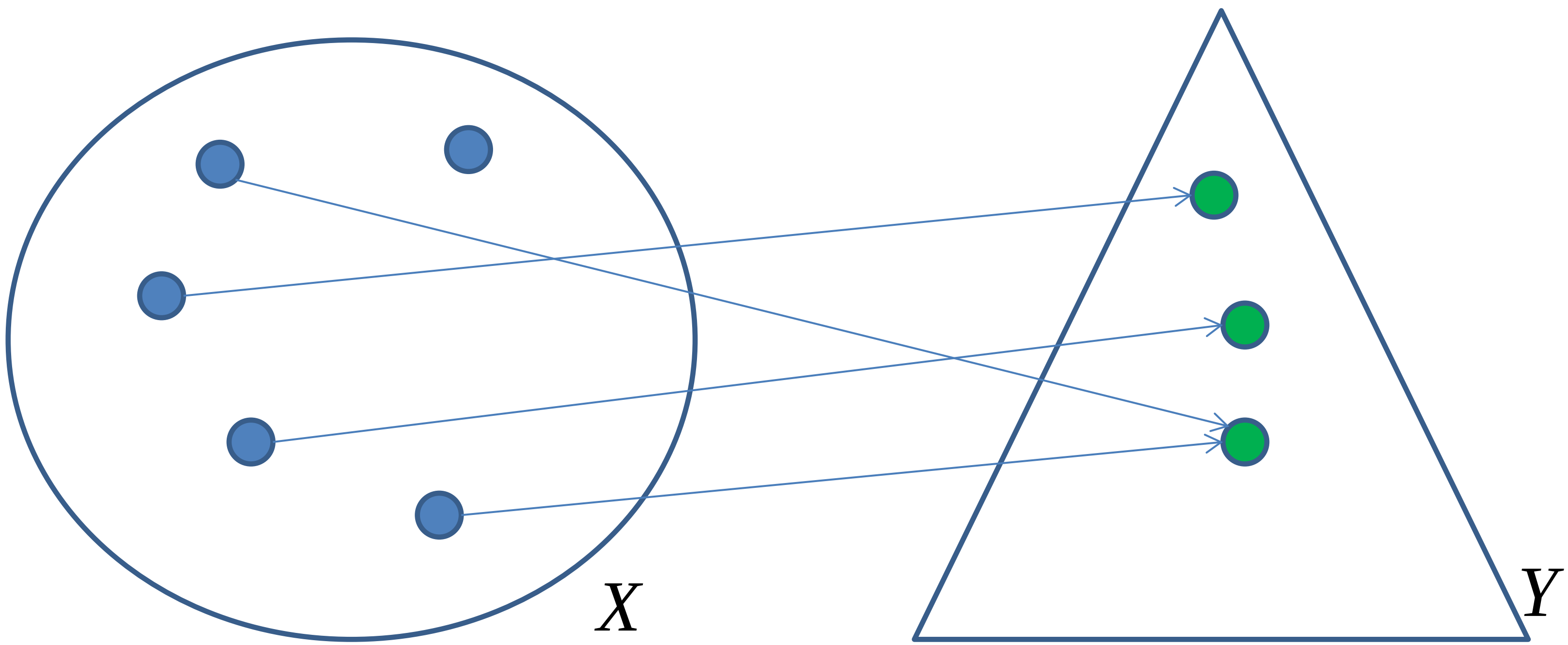
Function

- $f : X \rightarrow Y$, X is the domain, Y is the codomain, $\tilde{Y}$ is the range



Not a Function

- This relation is not a function because there is an element in X, which is not a preimage of any element in Y



Functions

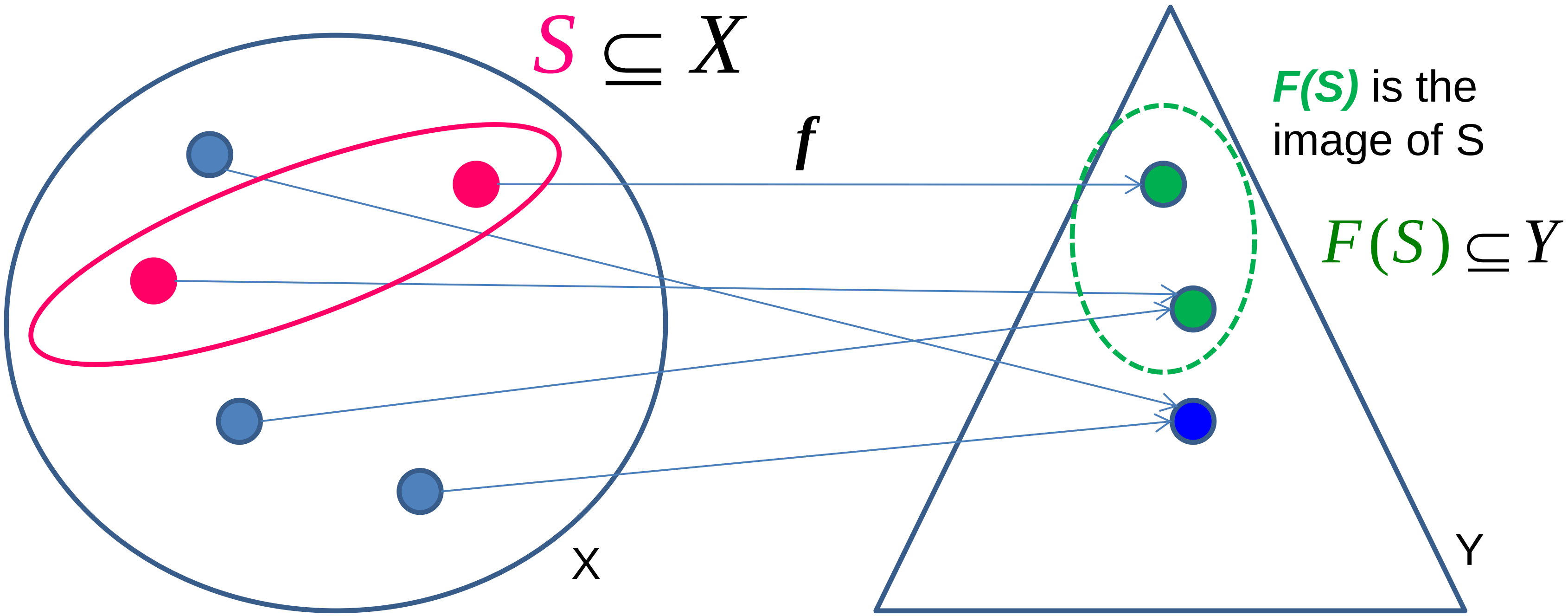
- To define a function, it is necessary to specify its domain, its codomain and the mapping of elements of the domain to elements in the codomain
- Two functions are equal when they have the same domain, have the same codomain, and map elements of their common domain to the elements of their common codomain in the same way
- If we change either the domain or the codomain of a function, then we obtain a different function

Image

- Let *f* be a function from the set *X* to the set *Y* and $S \subseteq X$. The image of *S* under the function *f* is a subset of *Y* that consists of the images of the elements of *S*

$$f\left(S\right) = \left\{t \mid x \in S, t = f(x)\right\}$$

Image



One-to-one Function

- If f is a function from X to Y

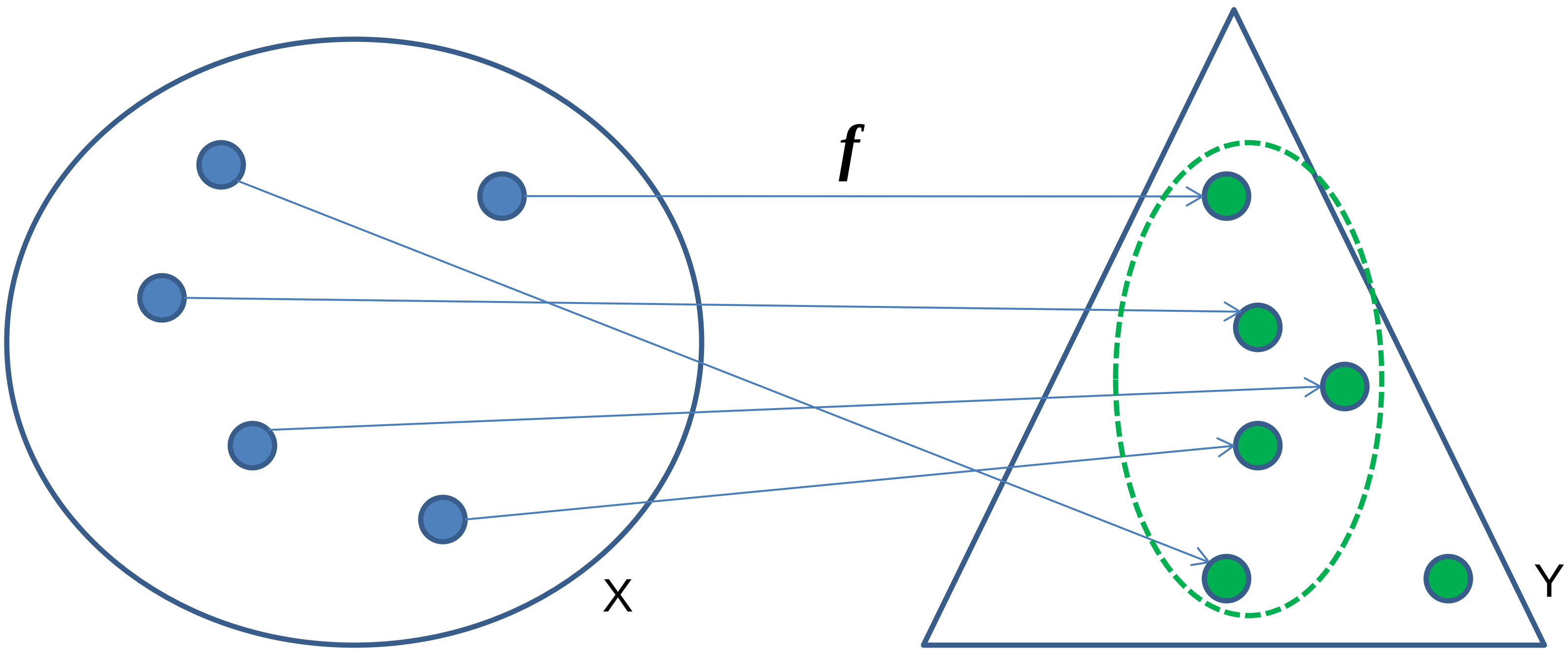
$$f : X \to Y;\ y = f(x),$$

  and no two distinct elements of the domain are assigned the same element in the codomain, then the function is called **one-to-one** (or **injective**).

- To show that a function f is one-to-one, it is necessary to show that

$$\forall x_1 \in X \ \ \forall x_2 \in X:\ f\left(x_1\right) = f\left(x_2\right) \to x_1 = x_2 = x$$

One-to-one Function

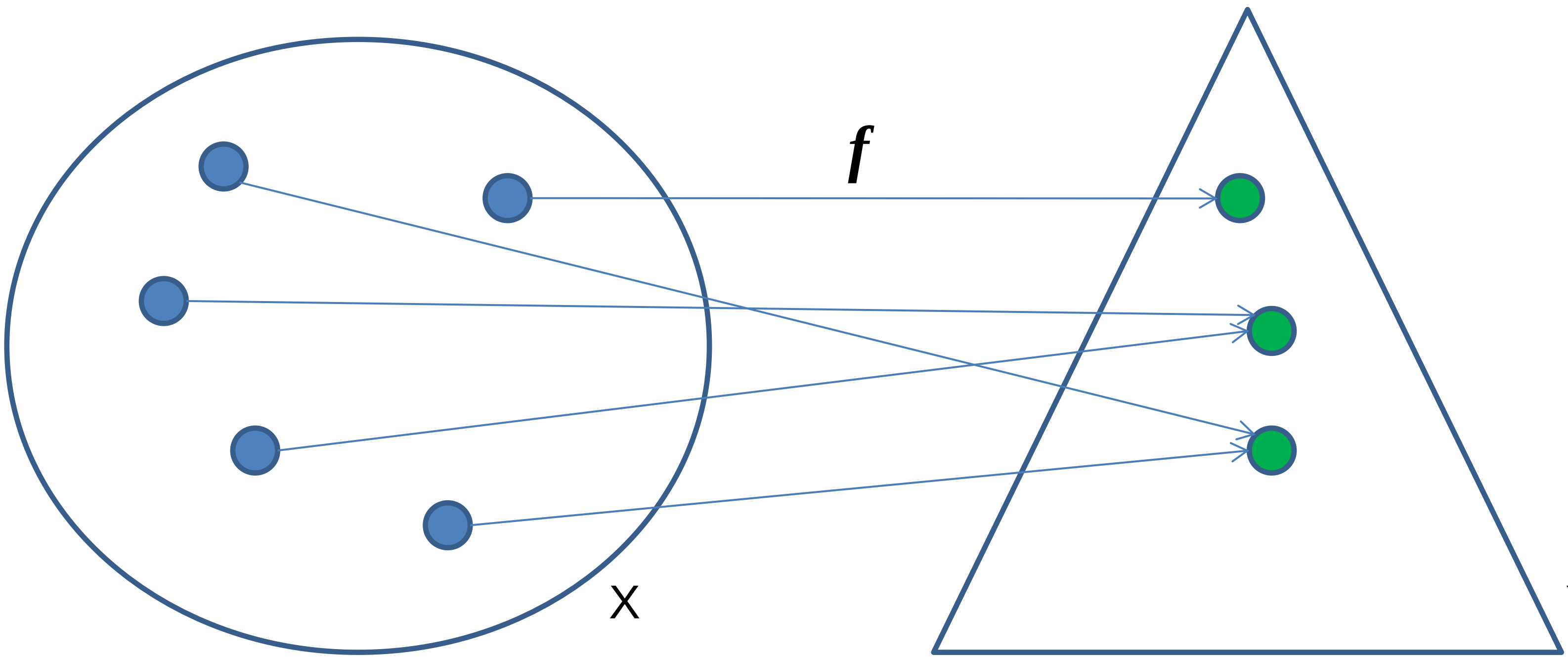


Onto Function

- If the range and codomain of a function are equal, the function is called **onto** (or **surjective**)
- To show that a function *f* is onto, it is necessary to show that

$$\forall y \in Y \ \exists x \in X : y = f(x)$$

Onto Function



One-to-one Correspondence

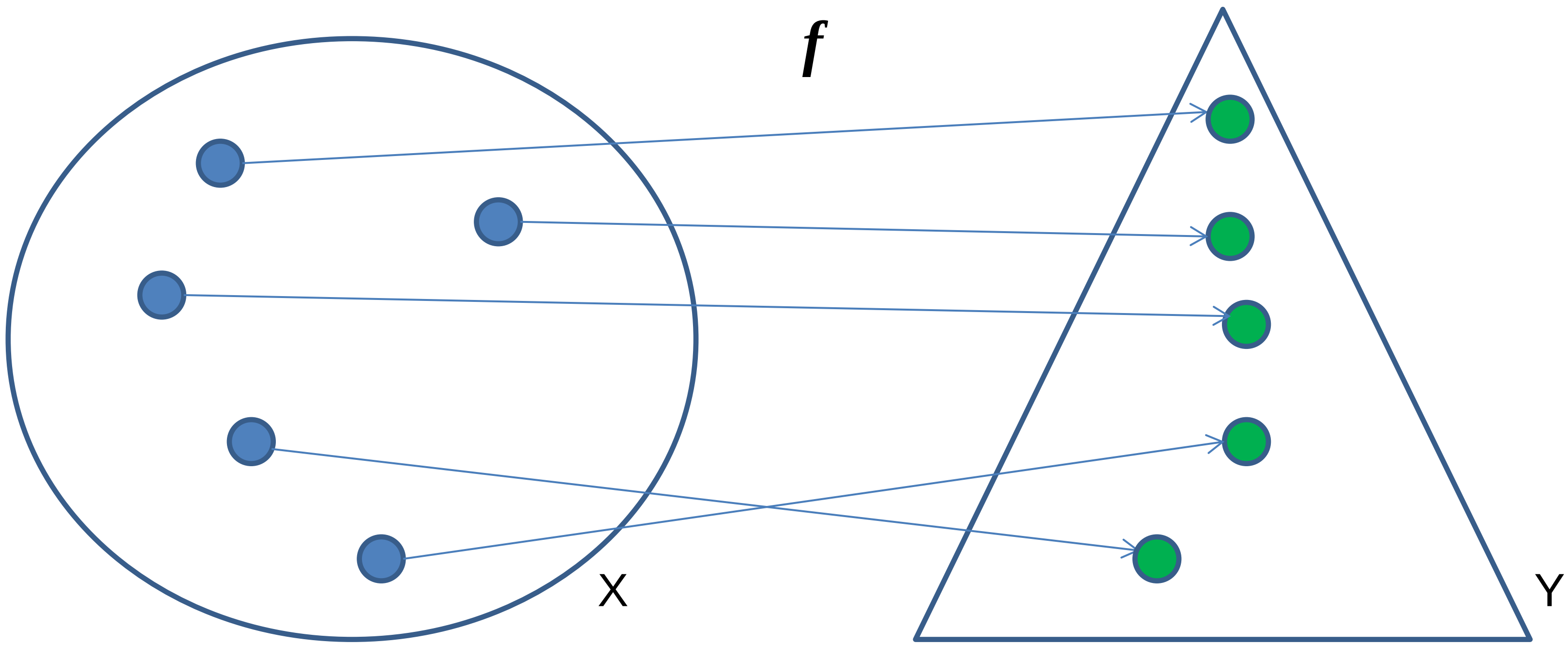
- A function, which is both one-to-one and onto is called a **one-to-one correspondence** (or **bijection**)
- To show that a function f is a one-to-one correspondence, it is necessary to show that

$$\forall y \in Y \ \exists! x \in X : y = f(x)$$

thus, for each $y \in Y$ there is exactly one $x \in X$ such that $y = f(x)$

$\exists!$ means "there exists exactly one"

One-to-one Correspondence



Useful properties

- Any one-to-one correspondence is onto and one-to-one
- Onto and one-to-one are not necessary one-to-one correspondences
- If onto is a one-to-one correspondence, it is also one-to-one
- If one-to-one is a one-to-one correspondence, it is also onto

Identity Function

- For any set *X*, the function

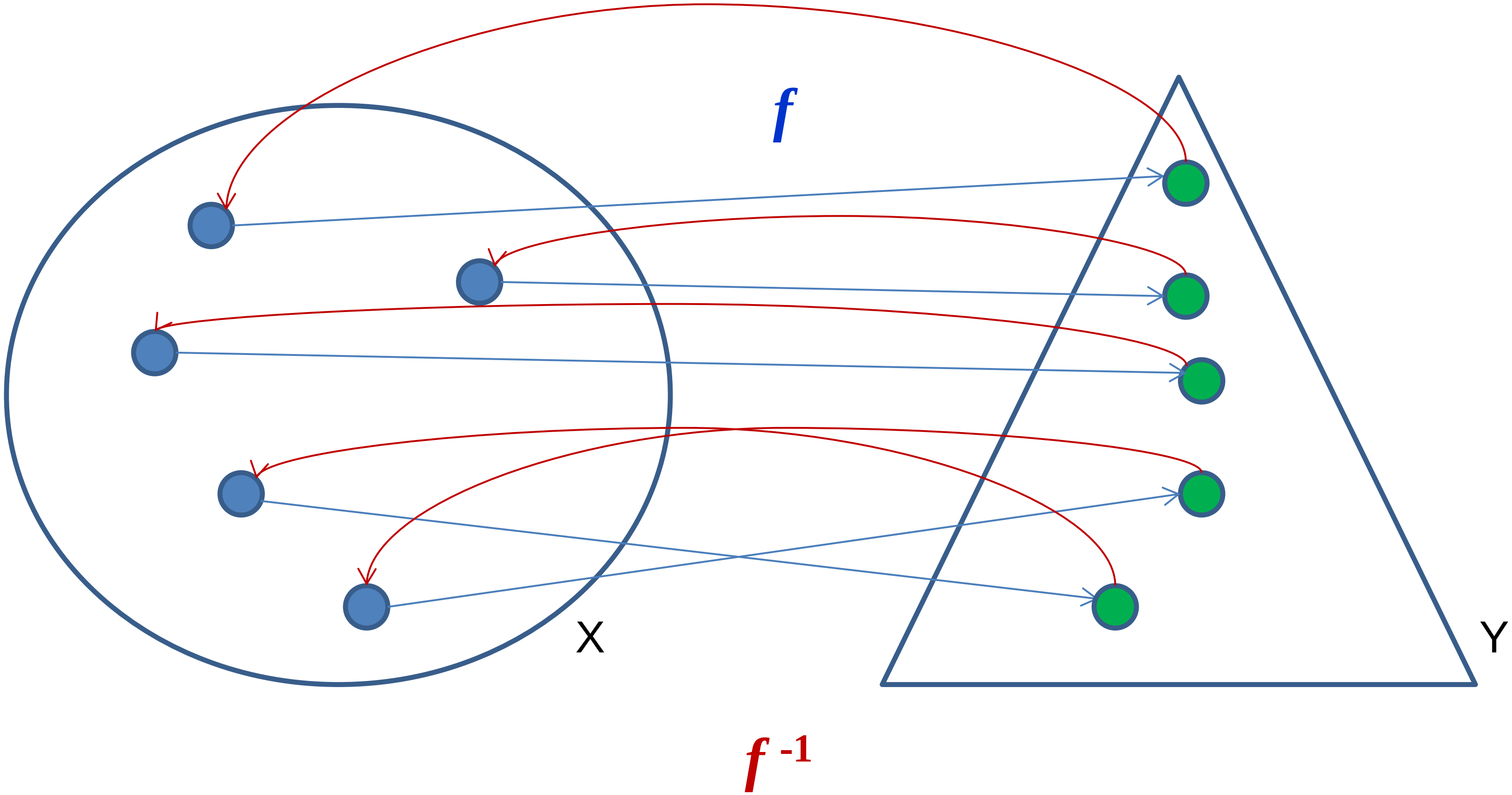
$$I_X : X \to X; \forall x \in X \ \ I_X(x) = x$$

is a one-to-one correspondence. This function is called the identity function on *X*.

Inverse Function

- Let $f : X \to Y$ be a one-to-one correspondence, then for each $y \in Y$ there is exactly one $x \in X$ such that $y = f(x)$.
- Hence we may define a function with domain Y and codomain X by associating to each $y \in Y$ the unique $x \in X$ such that $y = f(x)$. This function is denoted by f^{-1} and is called the inverse of function *f*.

Inverse Function



Inverse Function

- Theorem. Let $f : X \to Y$ is one-to-one correspondence. Then:
- $f^{-1} : Y \to X$ is one-to-one correspondence
- The inverse function of f^{-1} is f.
- $\forall x \in X, f^{-1}\left(f(x) \right) = x;\ \forall y \in Y, f\left(f^{-1}(y) \right) = y$, that is

$$f^{-1} \circ f = I_X;\ f \circ f^{-1} = I_Y$$

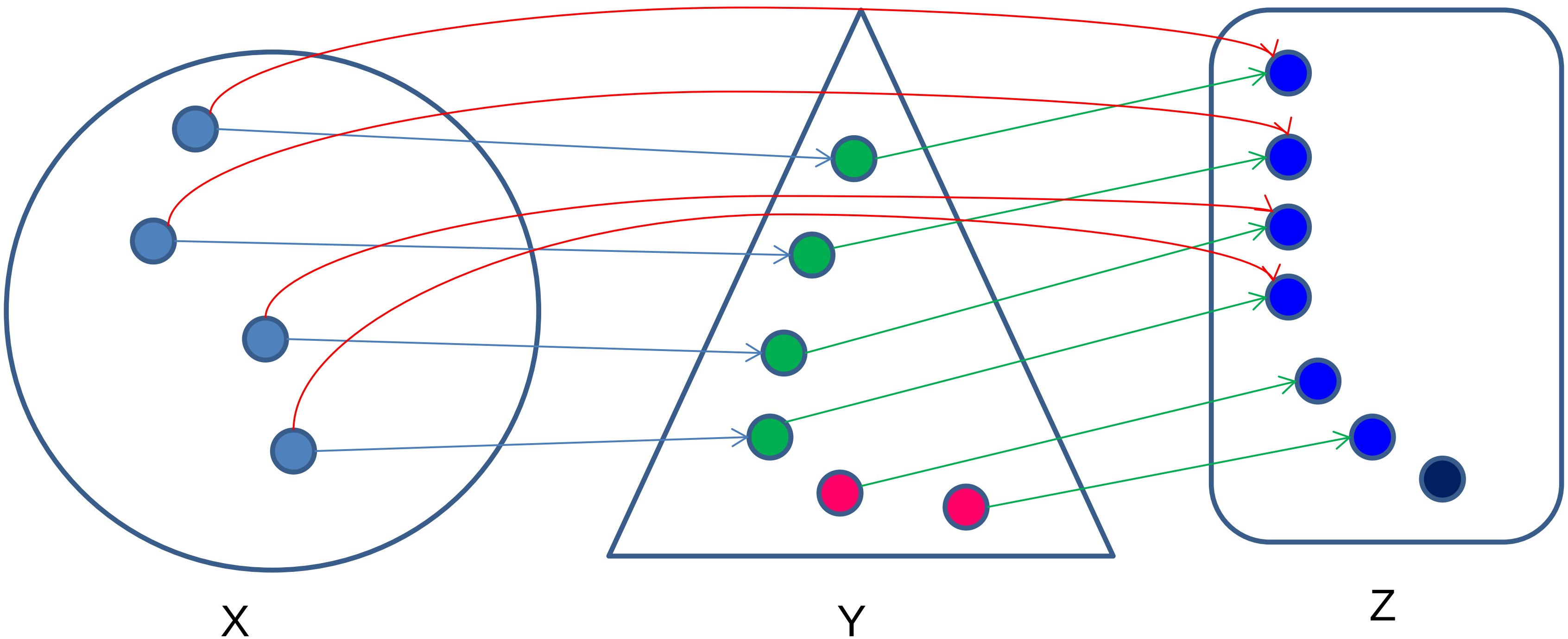
Composition of Functions

- Let g be a function from X to Y and f be a function from Y to Z. Then it is possible to combine these two functions in a function $f \circ g$ from X to Z.
- The function $f \circ g$ is called the composition of f and g and is defined by taking the image of x under $f \circ g$ to be $f(g(x))$:

$$\left(f \circ g\right)(x) = f\left(g(x)\right) \;\; \forall x \in X$$

Composition of Functions

- Composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f



Exponential Function with base 2

- The equation $f(x) = 2^x, x \in R$ defines a function with the set of real numbers as its domain and the set of positive real numbers as its codomain. This function is called the exponential function with base 2.

Exponential Function with base *n*

- In general, the equation $f(x) = n^x, x \in R$ defines a function with the set of real numbers as its domain and the set of positive real numbers as its codomain. This function is called the exponential function with base *n*.

Logarithmic Function with base *n*

- Evidently, the exponential function with base n is a one-to-one correspondence because each element of the codomain is associated with exactly one element of the domain.
- This means that the exponential function with base n has and inverse g called the logarithmic function with base *n*: $g(x) = \log_n x$

Logarithmic Function with base *n*

- Particularly, the logarithmic function with base 2 $g(x) = \log_2 x$ is the inverse of the exponential function with base 2.
- The definition of an inverse function implies that $y = \log_n x$ if and only if $x = n^y$:

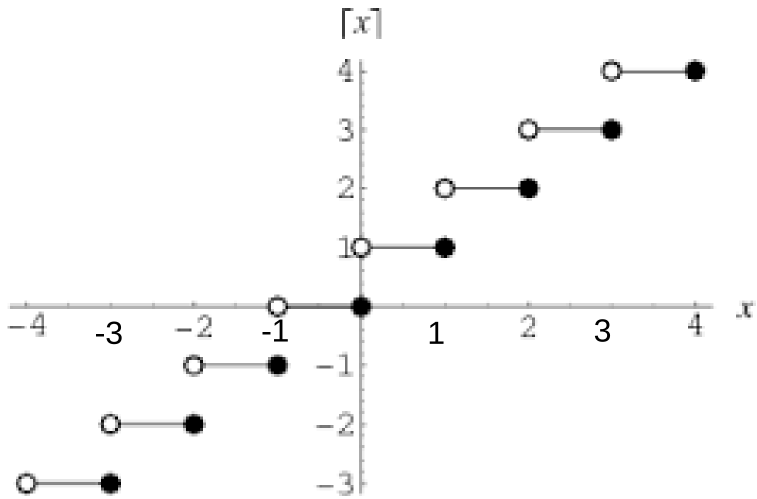
$$y = \log_n x \leftrightarrow x = n^y$$

Some other useful functions

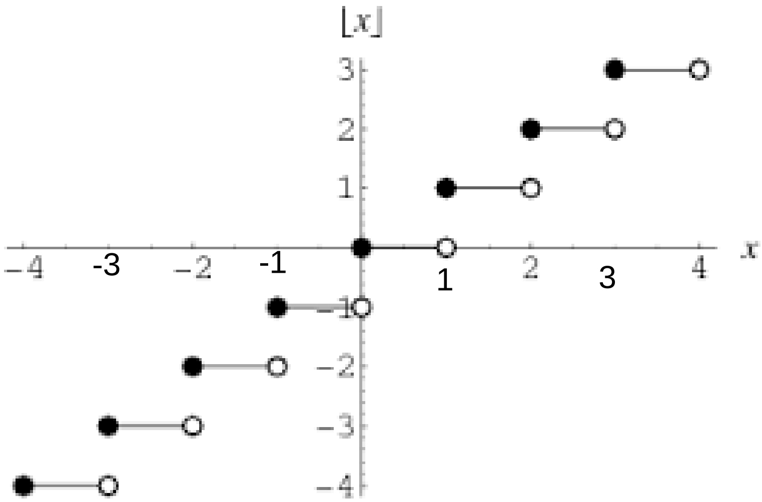
- The floor function assigns to the real number *x* the largest integer that is $\leq x$. The floor function at *x* is denoted $\lfloor x \rfloor$
- The ceiling function assigns to the real number *x* the smallest integer that is $\geq x$. The ceiling function at *x* is denoted $\lceil x \rceil$
- Examples:

$$\lfloor 5.9 \rfloor = 5; \lceil 5.9 \rceil = 6; \lfloor 5.1 \rfloor = 5; \lceil 5.1 \rceil = 6$$

Floor and Ceiling Functions



$$f(x) = \lceil x \rceil$$



$$f(x) = \lfloor x \rfloor$$