CMPT-439
Numerical Computation

Fall 2020

Solving Nonlinear Equations
Secant Method
Newton’s Method

Nonlinear Equation Solvers

- **Bracketing**
  - Bisection
  - False Position (Regula-Falsi)
- **Graphical**
- **Open Methods**
  - Newton Raphson
  - Secant

All Methods are iterative

Nonlinear Equation Solvers

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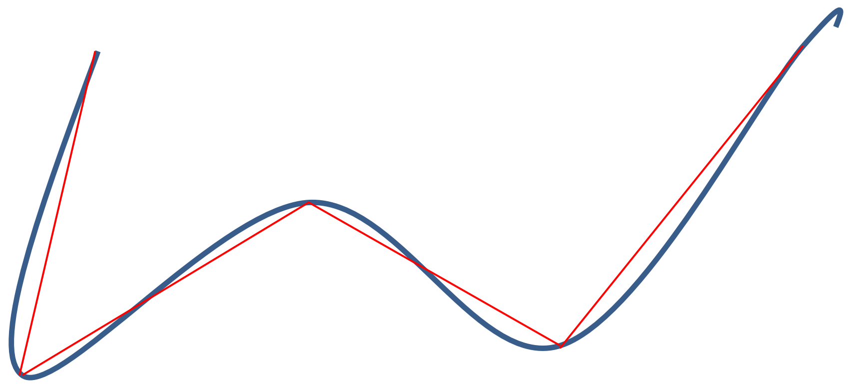
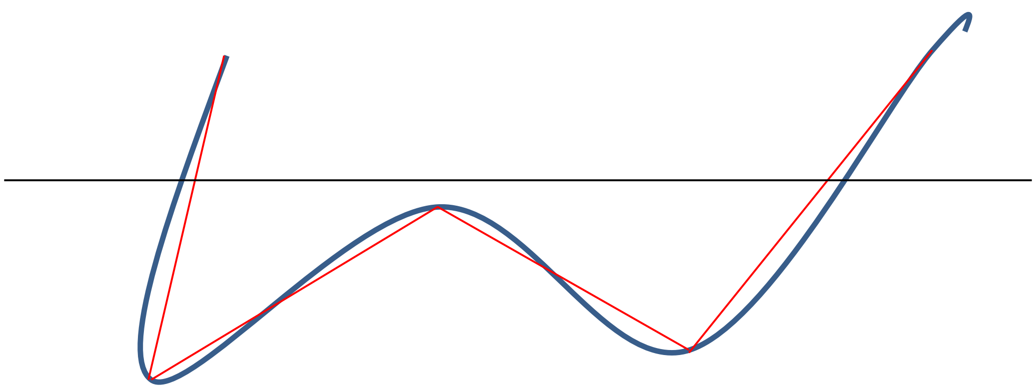
The Secant Method

- Basic idea: most nonlinear functions can be approximated by a set of straight lines over small intervals:



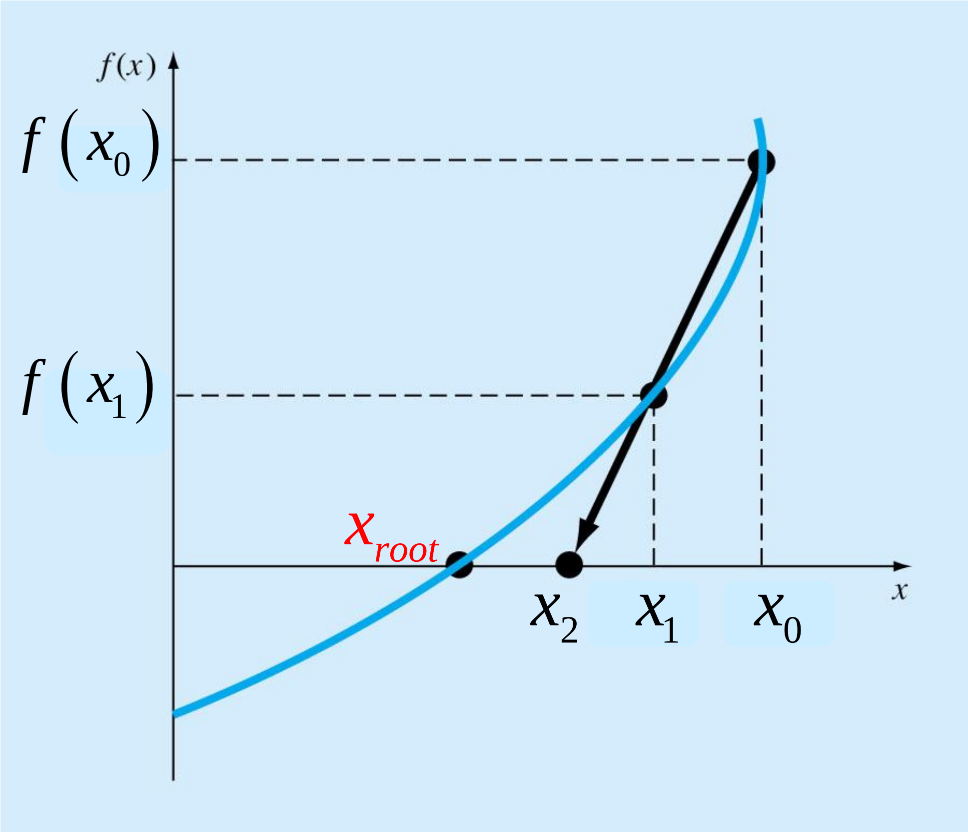
The Secant Method

- The secant method reduces finding a root of a nonlinear equation to finding a point where some linear equation determined by the approximating linear function has a root as close as possible to the root we are looking for



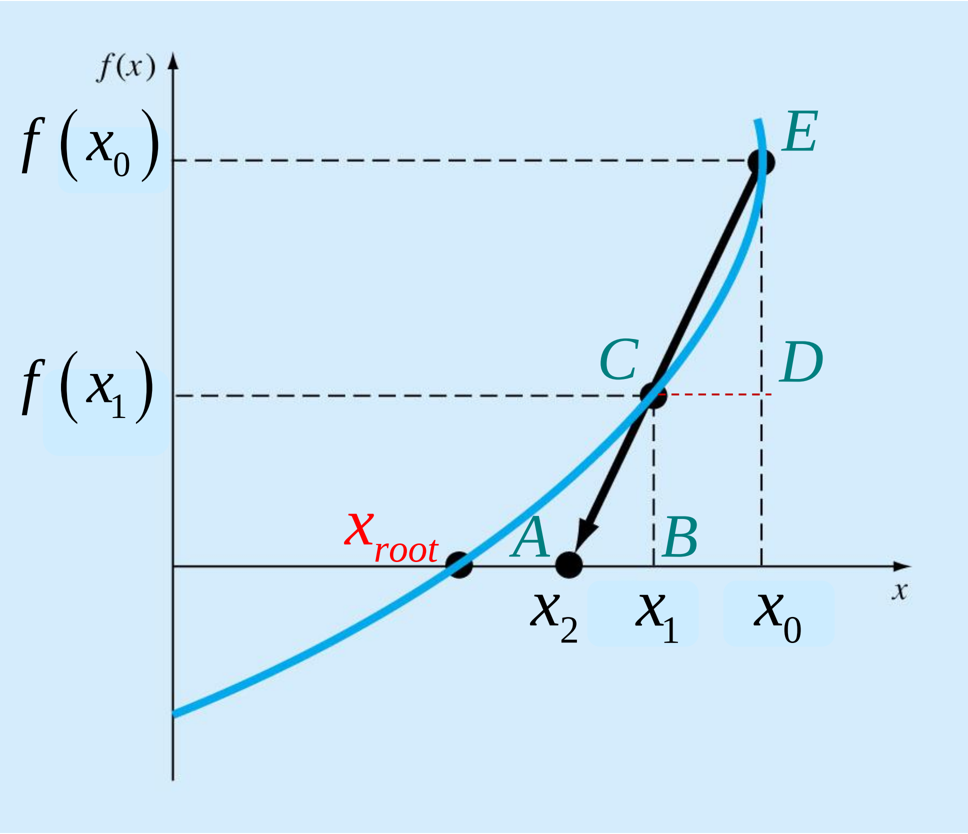
The Secant Method

- Requires two initial estimates of x : x_o, x_1. However, because $f(x)$ is not required to change signs between estimates, it is not classified as a "bracketing" method
- Convergence is not guaranteed for an arbitrary chosen x_o.



The Secant Method

$$\triangleleft ABC \sim \triangleleft CDE \rightarrow \frac{|AB|}{|BC|} = \frac{|CD|}{|DE|}$$



$$|AB| = x_1 - x_2 \quad |BC| = f(x_1)$$

$$|CD| = x_0 - x_1 \quad |DE| = f(x_0) - f(x_1)$$

The Secant Method

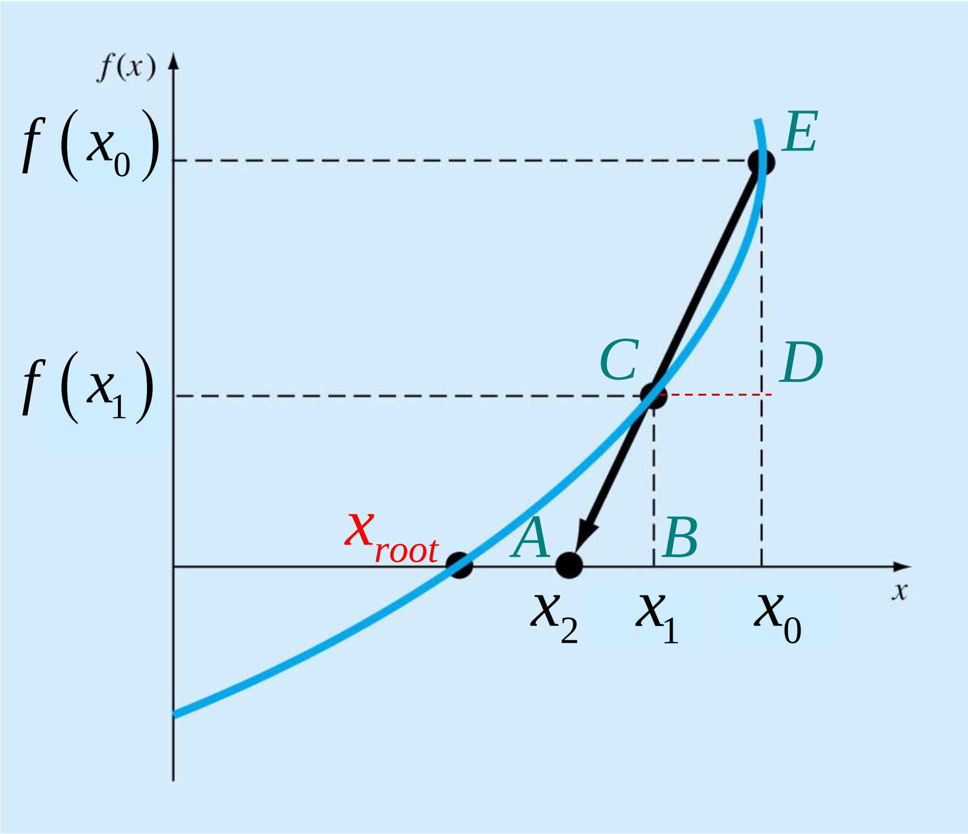
$$\triangleleft\, ABC \sim \triangleleft\, CDE \rightarrow \frac{|AB|}{|BC|} = \frac{|CD|}{|DE|}$$

$$\frac{(x_1 - x_2)}{f(x_1)} = \frac{(x_0 - x_1)}{f(x_0) - f(x_1)}$$

$$x_2 = x_1 - f(x_1)\frac{(x_0 - x_1)}{f(x_0) - f(x_1)}$$

… … …

$$x_{n+1} = x_n - f(x_n)\frac{(x_{n-1} - x_n)}{f(x_{n-1}) - f(x_n)}$$



$$|AB| = x_1 - x_2 \quad |BC| = f(x_1)$$

$$|CD| = x_0 - x_1 \quad |DE| = f(x_0) - f(x_1)$$

The Secant Method: Preliminaries

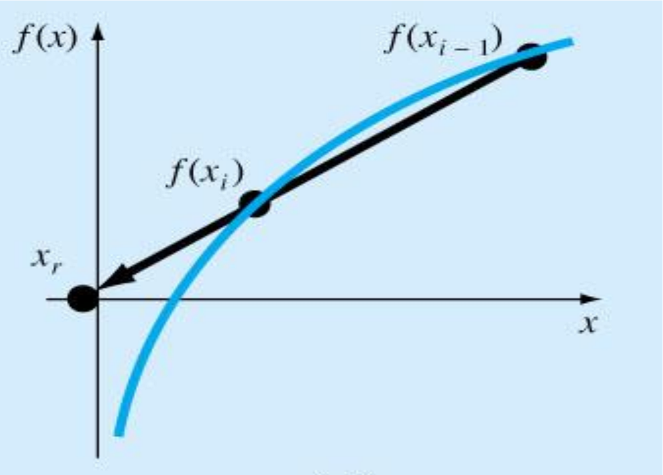
- The secant method should be used to find a root of $f(x)=0$
- A root should be estimated first. It can be done by plotting a graph of $f(x)$ and choosing an initial interval close to a point where $f(x)=0$

The Secant Method: Algorithm

- Determine δ - the tolerance value to ensure the appropriate true or approximate accuracy of a solution
- Determine x_0 and x_1 close enough to the projected root
- If $\left|f\left(x_0\right)\right| < \left|f\left(x_1\right)\right|$ then swap x_0 and x_1
- Step 1. Set $x_2 = x_1 - f\left(x_1\right)\dfrac{\left(x_0 - x_1\right)}{f\left(x_0\right) - f\left(x_1\right)};\ x_0 = x_1;\ x_1 = x_2$
- Step 2 (true error). If $\left|f\left(x_2\right)\right| \le \delta$, then stop and x_2 is a root, (estimation) else go to Step 1
- Alternative Step 2. If $\left|x_0 - x_1\right| \le \delta$, then stop and x_2 is a root, (absolute approximate error*) else go to Step 1
- *A relative approximate error should also be used here

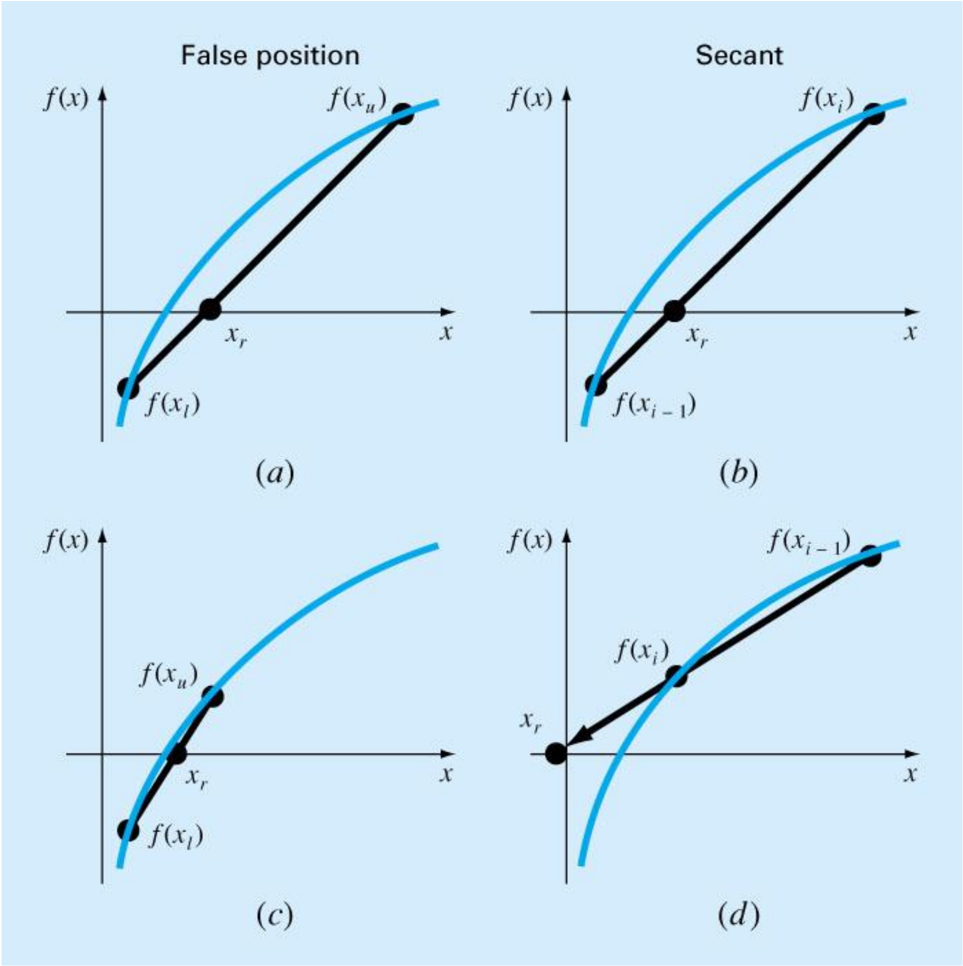
The Secant Method: Analysis

- The secant method converges faster than the bisection method
- However, if the function is far from linear near the root, the successive iterations can fly off to points far from the root:



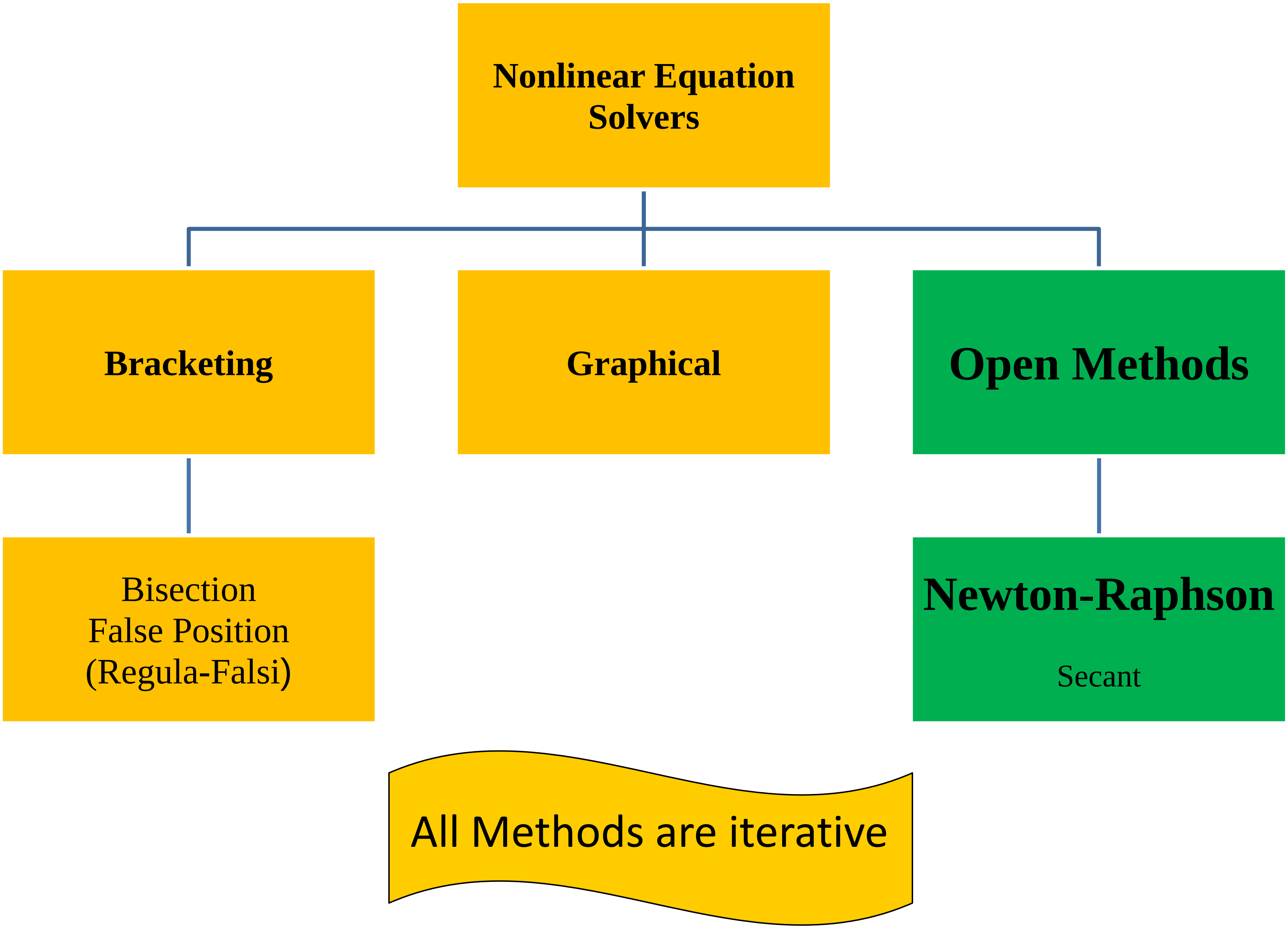
(d)

Secant and False Position: Comparison



Which method is better?

- There is no best method. All of them have advantages and disadvantages.
- Bisection → the number of iterations can be exactly predicted from a desired accuracy, but it yields to the secant and false positions methods in speed of convergence
- Secant → the fastest method in general, but may fly off a root
- False position → faster than bisection, but does not work (as well as the bisection) when a root can't be bracketed


Nonlinear Equation Solvers
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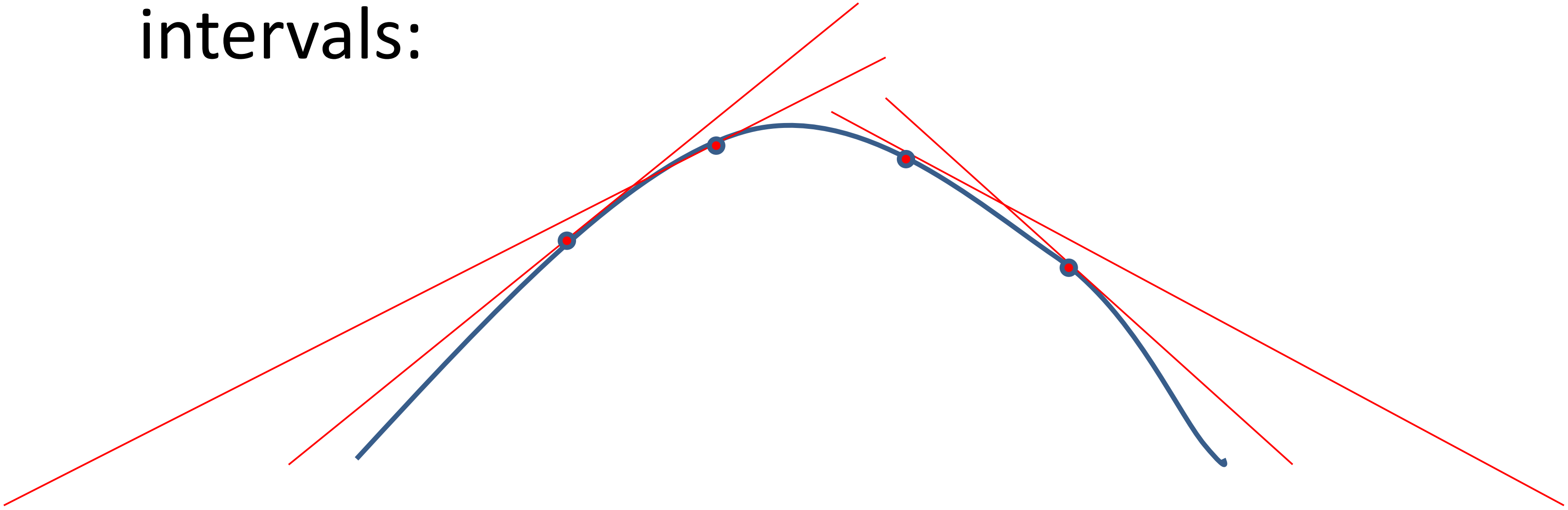
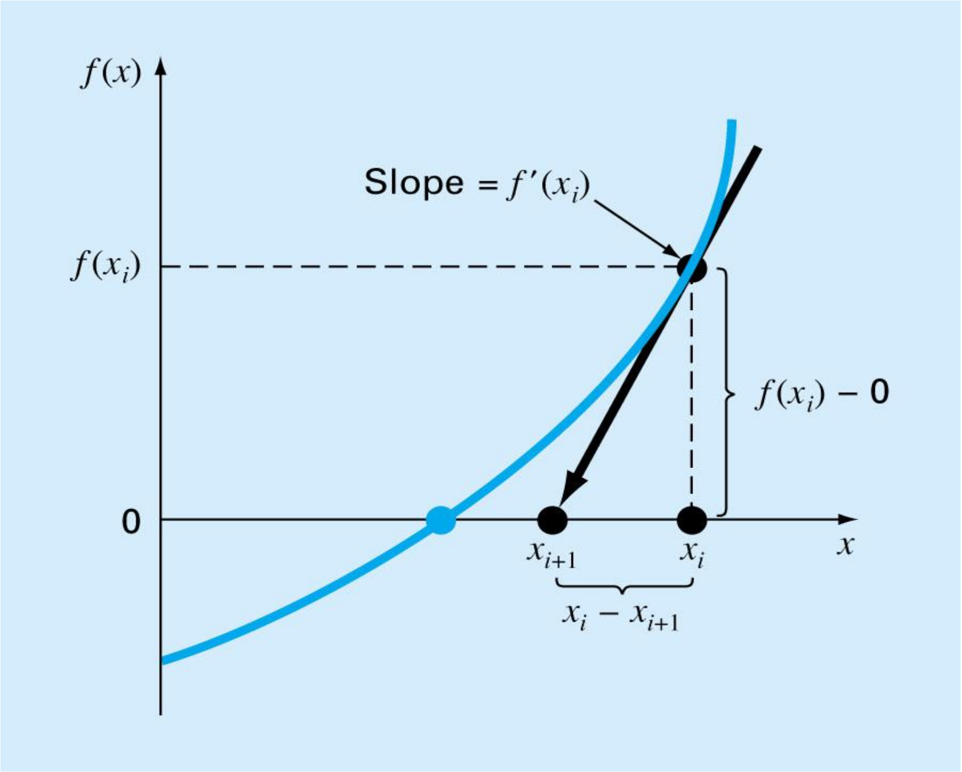
The Newton's Method

- Basic idea: most nonlinear functions can be approximated by a set of tangents over small intervals:



The Newton's Method

- A convenient method for functions whose derivatives can be evaluated analytically. It may not be convenient for functions whose derivatives cannot be evaluated analytically.

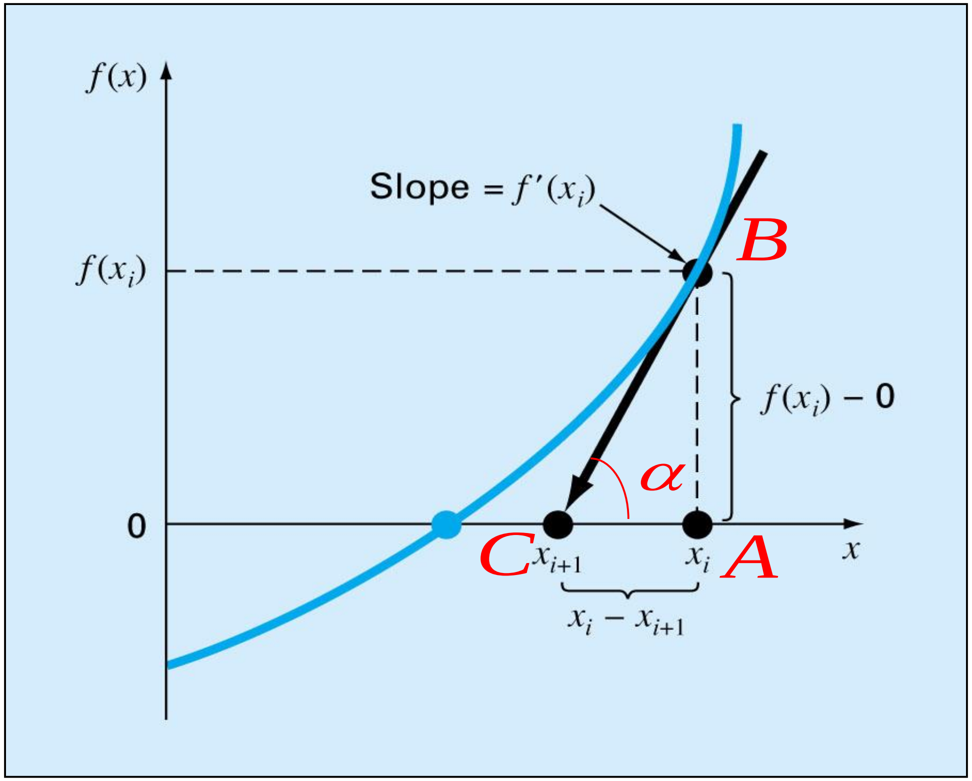


The Newton's Method

$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \tan(\alpha) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



The Newton-Raphson Formula

- Based on Taylor series expansion:

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + f''(x_i)\frac{\Delta x^2}{2!} + O\Delta x^3$$

The root is the value of x_{i+1} when $f(x_{i+1}) = 0$

Rearranging,

$$0 = f\left(x_i\right) + f'\left(x_i\right)\left(x_{i+1} - x_i\right)$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Newton vs. Secant

Newton-Raphson:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Secant:

$$x_{i+1} = x_i - f\left(x_i\right)\frac{\left(x_{i-1} - x_i\right)}{f\left(x_{i-1}\right) - f\left(x_i\right)} = x_i - \frac{f\left(x_i\right)}{\dfrac{f\left(x_{i-1}\right) - f\left(x_i\right)}{\left(x_{i-1} - x_i\right)}}$$

$$\lim_{\left(x_{i-1} - x_i\right)\to 0} \frac{f\left(x_{i-1}\right) - f\left(x_i\right)}{\left(x_{i-1} - x_i\right)} = f'\left(x_i\right)$$

- The secant method transforms to the Newton method when a secant transforms into a tangent

The Newton Method: Preliminaries

- The Newton's method should be used to find a root of $f(x)=0$
- A root should be estimated first. This can be done by plotting a graph of $f(x)$ and estimating x_0 close to the projected x where $f(x)=0$
- The derivative $f'(x)$ of $f(x)$ must be evaluated

Evaluation of the derivative in MATLAB

- In MATLAB, the derivative of any function (if it exists) can be evaluated symbolically as follows:
- Define x as a symbolic variable using syms x, then define $f(x)$ as a symbolic function, e.g. f=sin(x)+3*x^2
- Then apply the command diff (f), which returns a symbolic expression containing the derivative of f
- For example, fprime=diff(f) for the f determined above returns cos(x)+6*x
- Then you may use the eval command to find the value of fprime, e.g. x=1; eval(fprime) returns fprime(1)

The Newton Method: Algorithm

- Determine δ - the tolerance value to ensure the appropriate accuracy of a solution
- Determine x_0 close enough to the projected root and evaluate the derivative $f'(x)$
- If $f(x_0) = 0$ then stop; If $f'(x_0) = 0$ then choose another x_0
- Step 1. Set $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
- Step 2 (true error). If $|f(x_1)| \le \delta$, then stop and x_1 is a root,
  (estimation) else set $x_0 = x_1$ and go to Step 1
- Alternative Step 2. If $|x_0 - x_1| \le \delta$, then stop and x_1 is a root,
  (absolute approximate error*) else set $x_0 = x_1$ and go to Step 1
- *A relative approximate error should also be used here

The Newton Method: Advantages

- Converges faster than other methods (quadratic convergence – the error of each step approaches a constant *K* times the square of the error of the previous step), if it converges at all
- Requires only one guess
- Allows for finding complex roots of an equation

The Newton Method: Drawbacks

1. Divergence at inflection points
   Selection of the initial guess or an iteration value of the root that is close to the inflection (concavity change) point of the function $f(x)$ may start diverging away from the root in the Newton-Raphson method.

For example, to find the root of the equation $f(x) = (x-1)^3 + 0.512 = 0$.

The Newton-Raphson method reduces to $x_{i+1} = x_i - \dfrac{(x_i^3 - 1)^3 + 0.512}{3(x_i - 1)^2}$.

Table 1 shows the iterated values of the root of the equation.

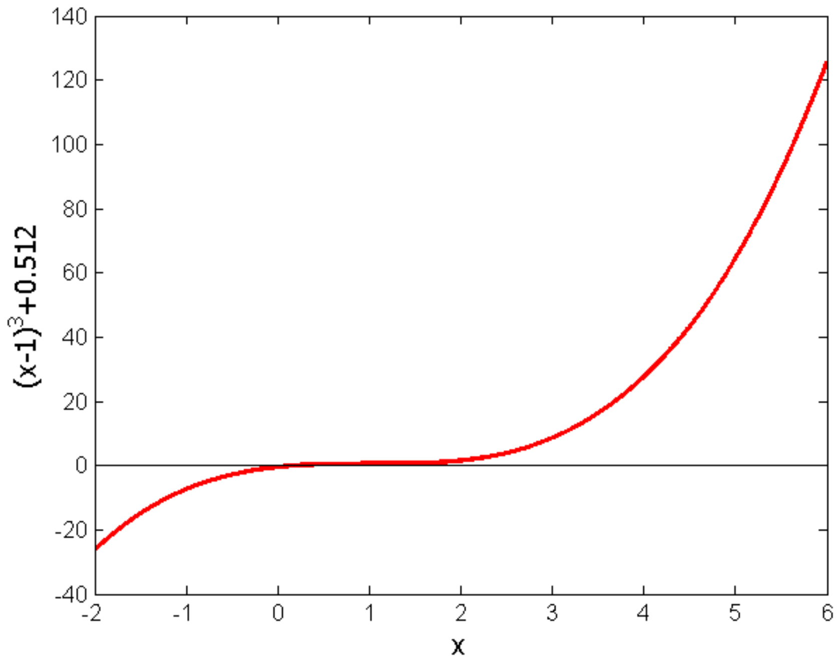
The root starts to diverge at Iteration 6 because the previous estimate of 0.92589 is close to the inflection point of $x = 1$.

Eventually after 12 more iterations the root converges to the exact value of $x = 0.2$.

Drawbacks – Inflection Points

Table 1 Divergence near inflection point.

Iteration Number	x_i
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	−30.119
7	−19.746
18	0.2000



Divergence at inflection point for

$$f(x) = (x-1)^3 + 0.512 = 0$$

Drawbacks – Division by Zero

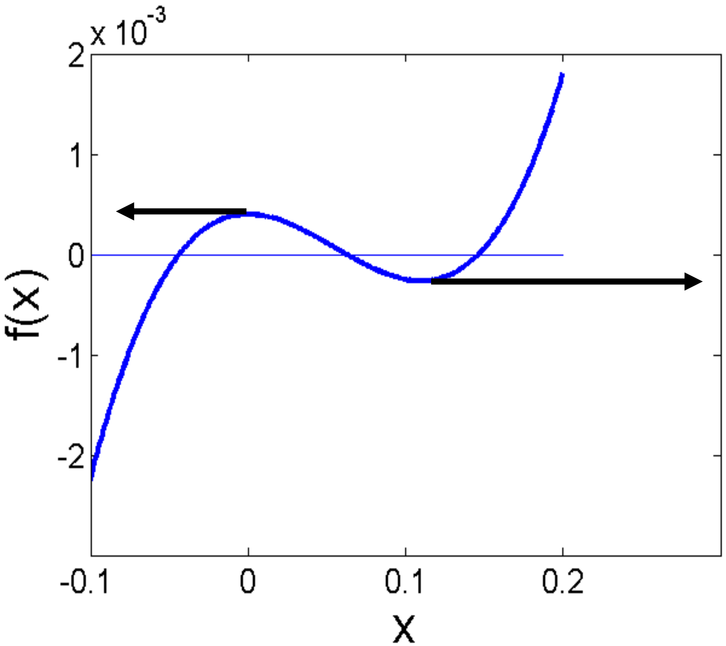
2. <u>Division by zero</u>
   For the equation

$$f(x) = x^3 - 0.03x^2 + 2.4\times10^{-6} = 0$$

   the Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4\times10^{-6}}{3x_i^2 - 0.06x_i}$$

   For $x_0 = 0$ or $x_0 = 0.02$, the denominator will equal zero.



Pitfall of division by zero or near a zero number

Drawbacks – Oscillations near local maximum and minimum

3. <u>Oscillations near local maximum and minimum</u>

Results obtained from the Newton-Raphson method may oscillate about the local maximum or minimum without converging on a root but converging on the local maximum or minimum.

Eventually, it may lead to division by a number close to zero and may diverge.

For example, the equation $f(x) = x^2 + 2 = 0$ has no real roots.

Drawbacks – Root Jumping

4. <u>Root Jumping</u>
In some cases where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

For example

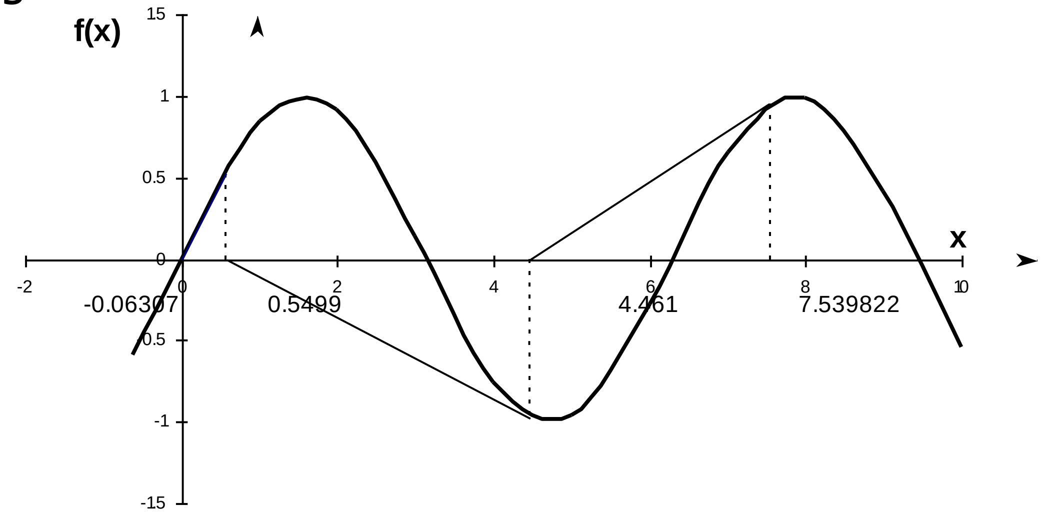
$$f(x) = \sin x = 0$$

Choose

$$x_0 = 2.4\pi = 7.539822$$

It will converge to $x = 0$

instead of $x = 2\pi = 6.2831853$



Root jumping from intended location of root for $f(x) = \sin x = 0$.

Drawbacks – may not converge on some occasions:

when $f'(x_i)=0$
or when it may get stuck in the endless loop when x_i equals on of $x_0, x_1, \ldots, x_{i-1}$