CMPT-439
Numerical Computation

Fall 2020

Solving Systems of Linear Equations
Introduction, Cramer's Rule

Linear Algebraic Equations

- An equation of the form $ax + by + c = 0$ or $ax + by = -c$ is called a linear equation in unknowns x and y
- $ax+by+cz = d$ is a linear equation in three unknowns x, y, and z
- Thus, a linear equation in n unknowns is

  $a_1x_1+a_2x_2+ \ldots +a_nx_n = b \qquad a_1,\ldots,a_n,b \in R$ or $a_1,\ldots,a_n,b \in C$
- A solution of such an equation consists of numbers

  $c_1, c_2, c_3, \ldots , c_n$
- If we have to deal with more than one linear equation, a set (system) of linear equations must be solved simultaneously

Systems of Linear Equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots &= b_3 \\
&\vdots
\end{aligned}
$$

- In a matrix form:

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}
\longrightarrow Ax = b
$$

Systems of Linear Equations

- Solve $Ax=b$, where A is an $n \times n$ matrix and b is an $n \times 1$ column vector
- Can also talk about non-square systems where A is $m \times n$, b is $m \times 1$, and x is $n \times 1$
  - *Overdetermined* if $m>n$:
    "more equations than unknowns"
  - *Underdetermined* if $n>m$:
    "more unknowns than equations"

Square Matrices

- A square matrix of order n contains the same number of rows and columns

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

- Square matrices are particularly important when a system of equations to be solved

A Symmetric Matrix

- A square matrix is called a symmetric matrix when the pairs of elements with the symmetric indexes across the diagonal are equal $a_{ij} = a_{ji}; i, j = 1, \ldots, n$

$$\begin{bmatrix} a_{11} & x & y \\ x & a_{22} & z \\ y & z & a_{33} \end{bmatrix}$$

The Transpose of a Matrix

- The transpose of a matrix is a matrix obtained by writing the rows as columns or (which is the same) by writing the columns as rows

$$A = \left[a_{ij} \right]; A^T = \left[a_{ji} \right]$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}; A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$$

Diagonal Matrix

- If all elements in a square matrix except those on the main diagonal are zero, the matrix is called a diagonal matrix

$$A = \begin{bmatrix} a_{11} & 0 & \dots & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & 0 & a_{n-1n-1} & 0 \\ 0 & \dots & \dots & 0 & a_{nn} \end{bmatrix}$$

Triangular Matrices

- A square matrix is called triangular if all the elements above or below diagonal are zeros
- Lower-triangular

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & 0\dots & 0 \\ \dots & \dots & \ddots & 0 \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

- Upper-triangular

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ 0 & 0 & \ddots & \dots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

Identity Matrix

- If the nonzero elements of a diagonal matrix of order n all are equal to 1 , the matrix is called the identity matrix of order n

$$I_n = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}$$

- For any square matrix A of order n $\quad I_n A = A I_n = A$

Transposition Matrix

- If two rows of an identity matrix are interchanged, it is called a transposition matrix

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xRightarrow[2nd \text{ and } 4th \text{ rows are interchanged}]{} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = P_{24} = P_{42}$$

Transposition Matrix

- If a transposition matrix is multiplied with a square matrix *A* of the same size, the product will be the *A* matrix, but with the same two rows interchanged as in the transposition matrix

$$A = \begin{bmatrix} 9 & 6 & 2 & 13 \\ 4 & 2 & 8 & 11 \\ 0 & 7 & 1 & 9 \\ 3 & 2 & 6 & 8 \end{bmatrix}; P_{24}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 9 & 6 & 2 & 13 \\ 4 & 2 & 8 & 11 \\ 0 & 7 & 1 & 9 \\ 3 & 2 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 2 & 13 \\ 3 & 2 & 6 & 8 \\ 0 & 7 & 1 & 9 \\ 4 & 2 & 8 & 11 \end{bmatrix}$$

Transposition Matrix

- If a square matrix A is multiplied with the transposition matrix P of the same size, the product will be the A matrix, but with the columns of A interchanged according to the rows of P interchanged

$$A = \begin{bmatrix} 9 & 6 & 2 & 13 \\ 4 & 2 & 8 & 11 \\ 0 & 7 & 1 & 9 \\ 3 & 2 & 6 & 8 \end{bmatrix}; \; AP_{24} = \begin{bmatrix} 9 & 6 & 2 & 13 \\ 4 & 2 & 8 & 11 \\ 0 & 7 & 1 & 9 \\ 3 & 2 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 13 & 2 & 6 \\ 4 & 11 & 8 & 2 \\ 0 & 9 & 1 & 7 \\ 3 & 8 & 6 & 2 \end{bmatrix}$$

Permutation Matrix

- A permutation matrix is obtained by multiplying several transposition matrices

$$P_{24}P_{12}=\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{bmatrix}=\tilde{P}$$

$$A=\begin{bmatrix}9&6&2&13\\4&2&8&11\\0&7&1&9\\3&2&6&8\end{bmatrix};\tilde{P}A=P_{24}\left(P_{12}A\right)=\begin{bmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}9&6&2&13\\4&2&8&11\\0&7&1&9\\3&2&6&8\end{bmatrix}=\begin{bmatrix}4&2&8&11\\3&2&6&8\\0&7&1&9\\9&6&2&13\end{bmatrix}$$

Permutation

- A permutation is a function that reorders an ordered set of integers. This function can be determined by a table containing two rows: the 1st row contains the integers in the initial order, the 2nd row contains the corresponding reordered set
$$\begin{pmatrix} 1 & 2 & \dots & n \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix}; \alpha_i \in \{1, \dots, n\}, i = 1, \dots, n$$
- Symbols r and s in a permutation create an inversion if $r > s$ and r precedes s :
$$\begin{pmatrix} 1 & 2 & \dots & & n \\ \alpha_1 & r & \dots s \dots & & \alpha_n \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$
2,1 and 4,3 are inversions
- A permutation is even if it contains the even number of inversions , and it is odd if it contains the odd number of inversions

Permutation and Permutation Matrix

- A permutation corresponding to a permutation matrix determines a new order of rows:

$$P_{24}P_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \tilde{P}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Determinant

- A determinant of order n corresponding to a given square matrix of order n is a value, which equals to the algebraic sum of $\boldsymbol{n}$**!** additive terms such that every term is a product of exactly the n elements from the matrix taken by 1 from every row and every column
- The additive term appears in the sum with the "+" sign if the permutation created from the numbers of rows (the 1st row of the permutation) and columns (the second row of the permutation) is even, and with the "-" sign if this permutation is odd
- det(A) is a common notation for the determinant of A

Determinant

- 2x2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix};$$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

0 invers. 1 invers.

- 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix};$$

$$\det(A) = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{12}a_{31}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

0 invers. 1 invers. 2 invers. 1 invers. 2 invers. 3 invers.

Singular Matrix

- A matrix is called singular if its determinant equals 0
- A singular matrix **is not invertible**!
- A matrix is always singular if some row (column) is a linear combination of some other rows (columns)

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ ka_{11} & ka_{12} & ka_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ k_1a_{11}+k_2a_{31} & k_1a_{12}+k_2a_{32} & k_1a_{13}+k_2a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- **A system of n linear equations in n unknowns has a single non-zero solution only if its matrix is not singular**

Singular Systems of Linear Equations

- Singular systems can be underdetermined:

$$2x_1 + 3x_2 = 5$$
$$4x_1 + 6x_2 = 10$$

(the second equation is coefficient-wise proportional to the first one)

- or inconsistent:

$$2x_1 + 3x_2 = 5$$
$$4x_1 + 6x_2 = 11$$

(the system matrix $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ is singular)

Minor

- Let us have an nxm matrix A. Let us choose in this matrix arbitrarily k rows and k columns $k \leq p; p = \min(m, n)$.
- The elements located in the intersection points of these k rows and k columns form a k x k matrix.
- The determinant of this k x k matrix is called a **minor** of order k of the matrix A.

Minor

- Any nxm matrix has more than 1 minor of order $k<n$ for any k.
- A **rank** of an nxm matrix is the highest order of a non-zero minor of this matrix

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\,a_{33} - a_{32}\,a_{23};$$

$$D_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\,a_{33} - a_{31}\,a_{23}; D_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\,a_{32} - a_{31}\,a_{22}$$

Systems of Linear Equations

- Solve $Ax=b$, where A is an $n \times n$ matrix and b is an $n \times 1$ column vector
- Usually not a good idea to compute $x=A^{-1}b$
  - Inefficient (because finding an inverse matrix is a special and computationally very costly procedure)
  - Prone to round-off errors
- So some numerical methods are needed…

Systems of Linear Equations

- Solve $Ax=b$, where A is an $n \times m$ matrix and b is an $m\times1$ column vector
- A is called a matrix of the system of linear equations
- $\begin{bmatrix} & b_1 \\ A & \dots \\ & b_m \end{bmatrix}$ is an augmented $n\times(m+1)$ matrix of the system

- Rouché-Kronecker-Capelli Theorem: A system of linear equations has a solution if and only if the rank of its coefficient matrix is equal to the rank of its augmented matrix

Methods for Solving Systems of Equations

- For small number of equations and unknowns $(n \leq 3)$ linear equations can be solved by simple techniques such as "method of elimination by hand"
- On the other hand, Linear Algebra provides the tools to solve such systems of linear equations
- Nowadays, easy access to computers makes the solution of large sets of linear algebraic equations possible and practical

Methods for Solving Systems of Equations

Solving Systems of Equations

- There are many ways to solve a system of linear equations:
  - Graphical method
  - Cramer's rule
  - Method of elimination
  - Directed elimination
  - Other Numerical methods

For small n

For large n

Example: Graphical Method

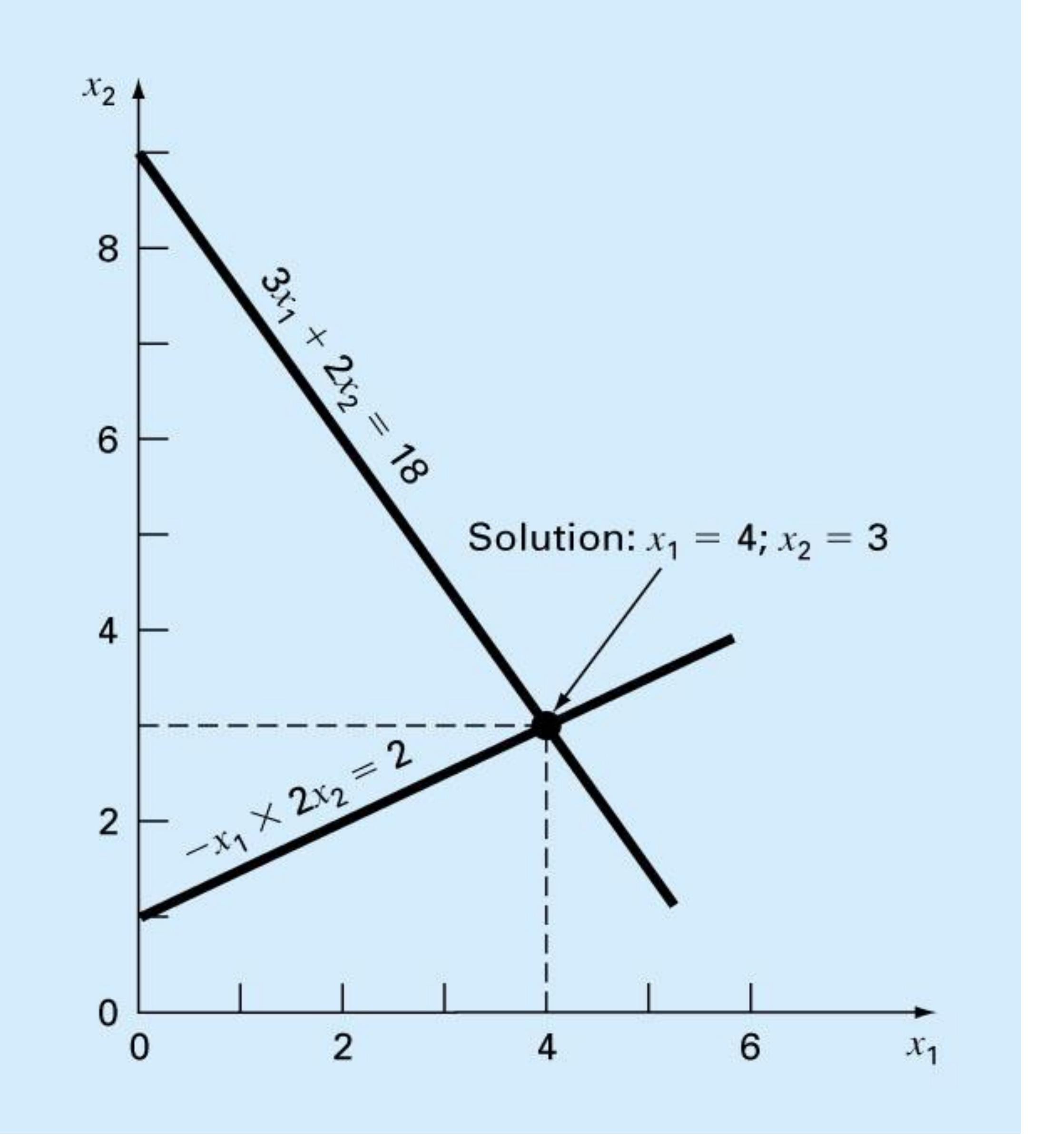
- For two equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

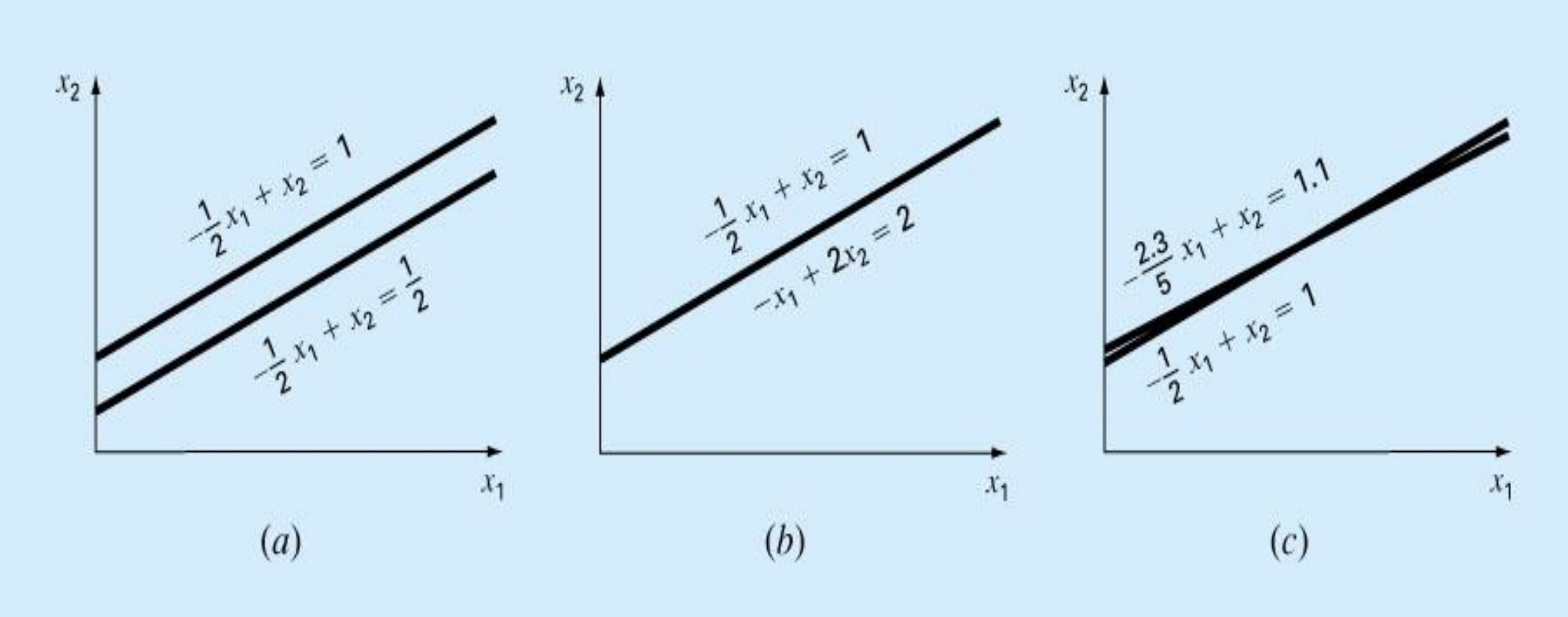
- Solve both equations for $x_{2:}$

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad \Rightarrow \quad x_2 = (\text{slope}_1)x_1 + \text{intercept}_1$$
$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}} \quad \Rightarrow \quad x_2 = (\text{slope}_2)x_1 + \text{intercept}_2$$

- Plot x_2 vs. x_1 for both equations.
- The intersection of the lines present the solution



Graphical Method



No solution

Infinite solutions

Ill-conditioned
(Slopes are too close)

Trivial Elimination Method

- For two equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Let us solve both equations for x_2:

Trivial Elimination Method

From the 1st eq. $$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$ From the 2nd eq.

Then $$\Rightarrow \left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} - \frac{b_2}{a_{22}} = 0$$

And now $$\Rightarrow x_1 = -\frac{\left(\frac{b_1}{a_{12}} - \frac{b_2}{a_{22}}\right)}{\left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)} = \frac{\left(\frac{b_2}{a_{22}} - \frac{b_1}{a_{12}}\right)}{\left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)}$$

x_2 can now easily be found from either of the two equations

Cramer's Rule

- Cramer's rule expresses the solution of a system $Ax=b$ (where A is an $n \times n$ matrix) of linear equations in terms of ratios of determinants
- The unknown x_i is equal to the ratio of the determinant created from the one of the matrix A by replacement of its ith column with a column of "free" coefficients to the determinant of the matrix A.

Cramer's Rule

- For example: $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}; D = |A| = \det(A)$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

Cramer's Rule Example

Find x_2 in the following system of equations:

$$0.3x_1 + 0.52x_2 + x_3 = -0.01$$

$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$

$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$$

Find the determinant *D*

$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix} = 0.3 \begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} - 0.52 \begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} + 1 \begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.4 \end{vmatrix} = -0.0022$$

Find determinant D_2 by replacing *D*'s second column with *b*

$$D_2 = \begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix} = 0.3 \begin{vmatrix} 0.67 & 1.9 \\ -0.44 & 0.5 \end{vmatrix} - 0.01 \begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} + 1 \begin{vmatrix} 0.5 & 0.67 \\ 0.1 & -0.44 \end{vmatrix} = -0.0649$$

Divide

$$x_2 = \frac{D_2}{D} = \frac{0.0649}{-0.0022} = -29.5$$

Cramer's Rule

- Advantage of the Cramer's rule is its formal algebraic simplicity
- Disadvantage of the Cramer's rule is a computational complexity of determinants computation for large values of n