CMPT-335 Discrete Structures

SETS AND SET OPERATIONS

Sets

- A **set**, in mathematics, is any collection of objects of any nature specified according to a well-defined rule.
- Each object in a set is called an element (a member, a point). If x is an element of the set X, (x belongs to X) this is expressed by

$$x \in X$$

- $x \notin X$ means that x does not belong to X

Sets

- Sets can be finite (the set of students in the class), infinite (the set of real numbers) or empty (null - a set of no elements, for example a node in a data structure with no data item inside – an empty node).
- A set can be specified by either listing all its elements in braces (a small finite set) or stating the requirements for the elements belonging to the set.
- $X=\{a, b, c, d\}$
- $X=\{x \mid x$ is a student taking the “Discrete Structures” class$\}$

Sets

- Sets are very important in Computer Science and Engineering
- All data structures used in computer programming are based on the concept of sets
- Sets are very important in databases where any entire database and its any record is in fact nothing else than sets

Sets: Standard Notations

- Z is the set of integer numbers
- Q is the set of rational numbers
- R is the set of real numbers
- C is the set of complex numbers
- N is the set of natural numbers
- $\varnothing$ is an empty set (contains no elements)
- $X = \{\varnothing\}$ is a set whose single element is an empty set (<u>Do not mix it with the empty set!!!</u>)

Sets

- What about a set of the roots of the equation

$$2x^2+1=0\,?$$

- The set of the real roots is empty: $\varnothing$
- The set of the complex roots is $\left\{i/\sqrt{2}, -i/\sqrt{2}\right\}$, where i is an imaginary unit

Subsets

- When every element of a set A is at the same time an element of a set B then A is a subset of B (A is contained in B):

$$A \subseteq B \Rightarrow \forall x \left(x \in A \rightarrow x \in B \right)$$

$$B \supseteq A \Rightarrow \forall x \left(x \in A \rightarrow x \in B \right)$$

- For example,

$$A = \left\{1, 2, 3, 7\right\}; B = \{1, 2, 3, 7, 9\}; A \subseteq B$$

$$A = \{1, 2, 5\}; B = \{1, 2, 5\}; A \subseteq B, A \supseteq B$$

$$A = \{1, 2, 5\}; B = \{1, 2, 7, 9\}; A \not\subseteq B$$

Subsets

- When every element of a set A is at the same time an element of a set B, but B definitely contains at least one element not belonging to A, then A is a proper subset of B

$$A \subset B \Rightarrow \forall x\left(x \in A \rightarrow x \in B\right) \wedge \exists x\left(x \in B \wedge x \notin A\right)$$

$$B \supset A \Rightarrow \forall x\left(x \in A \rightarrow x \in B\right) \wedge \exists x\left(x \in B \wedge x \notin A\right)$$

- For example,

$$Z \subset Q, Z \subset R, Q \subset R, R \subset C$$

Subsets

- The sets A and B are said to be equal if they consist of exactly the same elements.
- That is, $\left(A \subseteq B\right) \wedge \left(B \subseteq A\right) \leftrightarrow A = B$
- For instance, let the set A consists of the roots of equation

$$x(x+1)(x^2-4)(x-3)=0$$

$$B=\left\{-2,-1,0,2,3\right\}$$

$$C=\left\{x \mid x \in Z, |x|<4\right\}$$

- What about the relationships among A, B, C ?

Subsets

- Solution

$$A \subset C$$

$$B \subset C$$

$$\left.\begin{array}{l} A \subseteq B \\ B \subseteq A \end{array}\right\} \rightarrow A = B$$

Important Theorem on Subsets

- Theorem. Let S be an arbitrary set. Then

$$\forall S$$

$$\emptyset \subseteq S$$

$$S \subseteq S$$

➢ The empty set is always a subset of any set and any set is always a subset of itself.

Cardinality

- The cardinality of a finite set is the number of elements in the set.
- $|A|$ is the cardinality of A.
- A set with the same number of elements as any subset of the set of natural numbers is called a countable set.

The Power Set

- Given a set S, the power set $P(S)$ of S is the set of all subsets of the set S.
- Theorem. If the cardinality of a finite set is n, its power set always has the cardinality 2^n and therefore it contains 2^n elements, thus it has exactly 2^n subsets.
- For example, if $S=\{0,1,2\}$, then

$$P(S)=\left\{\varnothing,\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\right\}$$

Continuum

- For an infinite continuous linearly ordered set, which has a property that there is always an element between other two, we say that its cardinality is “continuum” (for example, interval [0,1], any other interval, or any line) or simply call this set continuum.
- Linear order means that there is a precedence relation between every two elements of a set (for example the relation “<“ or “>”)

Universal Set

- A large set, which includes some useful in dealing with the specific problem smaller sets, is called the universal set (universe). It is usually denoted by U.
- For instance, in the previous example, the set of real numbers R can be naturally considered as a universal set.

Operations on Sets: Union

- Let U be a universal set of any arbitrary elements and it contains all possible elements under consideration. The universal set may contain a number of subsets A, B, C, D,..., which individually are well-defined.
- The **union** (sum) of two sets *A* and *B* is the set of all those elements that belong to *A* or *B* or both: $A \cup B = \{x : (x \in A) \vee (x \in B)\}$

Operations on Sets: Union

$$A = \{a, b, c, d\}; B = \{e, f\}; A \bigcup B = \{a, b, c, d, e, f\}$$

$$A = \{a, b, c, d\}; B = \{c, d, e, f\}; A \bigcup B = \{a, b, c, d, e, f\}$$

$$A = \{a, b, c, d\}; B = \{c, d\}; A \bigcup B = \{a, b, c, d\} = A$$

Important property:

$$B \subseteq A \rightarrow A \bigcup B = A$$

Operations on Sets: Intersection

- The **intersection** (product) of two sets *A* and *B* is the set of all those elements that belong to both *A* and *B* (that are common for these sets):

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

- When $A \cap B = \varnothing$ the sets *A* and *B* are said to be mutually exclusive or disjoint.

Operations on Sets: Intersection

$$A = \{a, b, c, d\}; B = \{e, f\}; A \bigcap B = \varnothing$$

$$A = \{a, b, c, d\}; B = \{c, d, f\}; A \bigcap B = \{c, d\}$$

$$A = \{a, b, c, d\}; B = \{c, d\}; A \bigcap B = \{c, d\} = B$$

Important property:

$$B \subseteq A \rightarrow A \bigcap B = B$$

Operations on Sets: Difference

- The **difference** of two sets *A* and *B* is the set of all those elements that belong to the set *A* but do not belong to the set *B*:

$$A / B \text{ or } A - B$$

$$A - B = \{x : (x \in A) \ \wedge \ (x \notin B)\}$$

Operations on Sets: Complement

- The **complement** (negation) of any set *A* is the set *A'* ($\overline{A}$) containing all elements of the universe that are not elements of *A*.

Venn Diagrams

- A Venn diagram is a useful mean for representing relationships among sets (see pp. 118-120 in the text book) .
- In a Venn diagram, the universal set is represented by a rectangular region, and subsets of the universal set are represented by circular discs drawn within the rectangular region.

Algebra of Sets

- Let *A*, *B*, and *C* be subsets of a universal set *U*. Then the following laws hold.
- Commutative Laws: $A \cup B = B \cup A; A \cap B = B \cap A$
- Associative Laws:
$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
- Distributive Laws:
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Algebra of Sets

- Complementary:

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \varnothing$$
$$A \cup U = U$$
$$A \cap U = A$$
$$A \cup \varnothing = A$$
$$A \cap \varnothing = \varnothing$$

- Difference Laws:

$$(A \cap B) \cup (A - B) = A$$
$$(A \cap B) \cap (A - B) = \varnothing$$
$$A - B = A \cap \overline{B}$$

Algebra of Sets

- De Morgan's Laws (Dualisation):

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$$

$$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$$

- Involution Law: $\overline{\left(\bar{A}\right)} = A$
- Idempotent Law: For any set A:

$$A \cup A = A$$

$$A \cap A = A$$