Session 1

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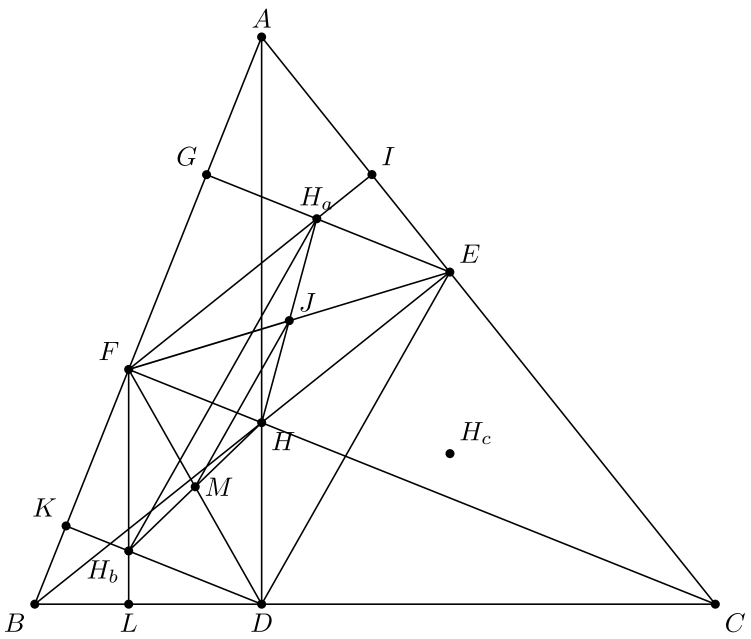
1. Let D, E, F be the foot of perpendicular from A, B, C of $\triangle ABC$ respectively. Suppose that H_a, H_b, H_c are the orthocenters of $\triangle AEF$, $\triangle BFD$, $\triangle CDE$. Prove that $H_aH_bH_c \cong DEF$.

Proof:



$$\left.\begin{array}{l} \left.\begin{array}{l} BE \perp AC \\ FJ \perp AE \end{array}\right\} \implies FH_a \parallel HE \\ \left.\begin{array}{l} CF \perp AB \\ EG \perp AF \end{array}\right\} \implies FH \parallel EH_a \end{array}\right\} \implies FHEH_a \text{ is parallelogram}$$

$$FHEH_a \text{ is parallelogram} \implies \begin{cases} FJ = JE \\ HJ = JH_a \end{cases}$$

By the same way

$$DHFH_b \text{ is parallelogram} \implies \begin{cases} H_bM = MH \\ FM = MD \end{cases}$$

Consider $\triangle DEF$. We have

$$\left.\begin{array}{l} FJ = JE \\ FM = FD \end{array}\right\} \implies MJ \parallel DE \tag{1}$$

Consider $\triangle HH_aH_b$. We have

$$\left.\begin{array}{l} HJ = JH_a \\ HM = MH_b \end{array}\right\} \implies MJ \parallel H_aH_b \tag{2}$$

Thus by (1) and (2) we have

$$\left.\begin{array}{l} MJ \parallel DE \\ MJ \parallel H_aH_b \end{array}\right\} \implies DE \parallel H_aH_b \tag{3}$$

Also

$$\left.\begin{array}{l} EG \perp AB \\ DK \perp AB \end{array}\right\} \implies EH_a \parallel DH_b \tag{4}$$

Finally by (3) and (4)

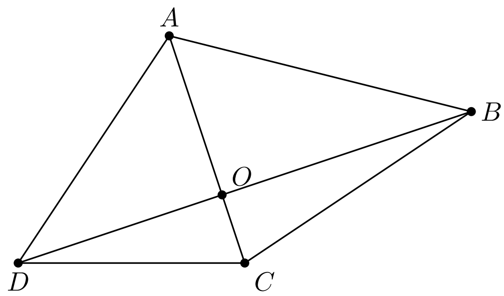
$$\left.\begin{array}{l} DE \parallel H_aH_b \\ EH_a \parallel DH_b \end{array}\right\} \implies H_aEDH_b \text{ is parallelogram } \implies H_aH_b = DE$$

By the same way we can conclude $H_bH_c = EF$ and $H_cH_a = FD$. Thus

$$\left.\begin{array}{l} H_aH_b = DE \\ H_bH_c = EF \\ H_cH_a = FD \end{array}\right\} \implies H_aH_bH_c \cong DEF$$

2. Let H be the orthocenter of $\triangle ABC$ and E, F be the foot of perpendicular BH and CH respectively. Suppose that M, N is the midpoint of BC and AH. Prove that $MN \perp EF$.

Lemma: $AB^2 + CD^2 = AD^2 + CB^2 \iff AC \perp BD$



Proof: Suppose $AC \perp BD$, thus by the Pythagoras theorem we have

$$AB^2 + CD^2 = \left(AO^2 + BO^2\right) + \left(CO^2 + DO^2\right)$$

$$= \left(AO^2 + DO^2\right) + \left(CO^2 + BO^2\right) = AD^2 + CB^2$$

Now Suppose $AB^2 + CD^2 = AD^2 + CB^2$, thus by the law of cosines we have

$$\left.\begin{array}{l} AB^2 = AO^2 + BO^2 - 2AO.BO\cos(\angle AOB) \\ CD^2 = CO^2 + DO^2 - 2CO.DO\cos(\angle COD) \\ AD^2 = AO^2 + DO^2 - 2AO.DO\cos(\angle AOD) \\ CB^2 = CO^2 + DO^2 - 2CO.DO\cos(\angle COB) \end{array}\right\}$$

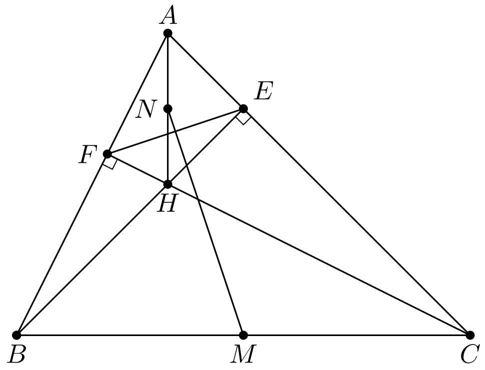
$$\xRightarrow{AB^2+CD^2=AD^2+CB^2} AO.BO\cos(\angle AOB) + CO.DO\cos(\angle COD)$$

$$= AO.DO\cos(\angle AOD) + CO.DO\cos(\angle COB)$$

$$\xRightarrow[\angle AOD=180^\circ-\angle AOB]{\angle AOB=\angle COD} \big(AO.BO + CO.DO + AO.DO + CO.DO\big)\cos(\angle AOB\big) = 0$$

$$\implies \angle AOB = 90^\circ \implies AB \perp CD$$

Proof of the problem:

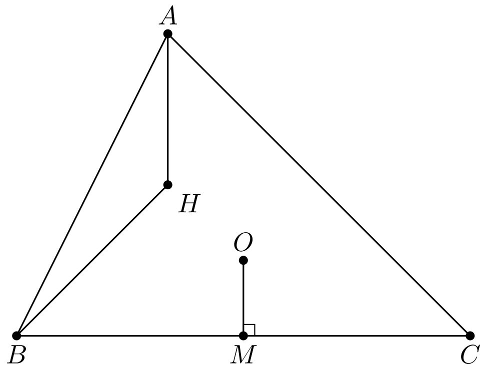


Consider $\triangle EBC$, EM is the median thus $EM = BM$. Conseder $\triangle FBC$, FM is the median thus $FM = BM$. So $FM = EM$. By the same way $EN = FN$ and if we apply this two equality in the lemma we have

$$\left.\begin{array}{l} FM = EM \\ EN = FN \end{array}\right\} \implies EN^2 + FM^2 = FN^2 + EM^2 \implies MN \perp FE$$

3. Let $ABCD$ be a cyclic quadrilateral. Suppose H_1 and H_2 are the orthocenters of $\triangle ACD$ and $\triangle BCD$. Prove that $H_1H_2 = AB$.

Lemma: Suppose H is the orthocenter, O is the circucenter of $\triangle ABC$. Let M be the midpoint of BC. Then we have $AH = 2OM$.
Proof 1:



Consider $\triangle COM$. Obviously $\angle MOC = \angle A$ thus we have

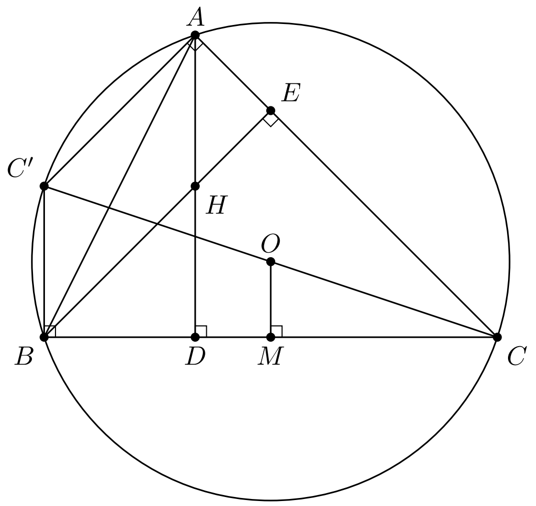
$$OM = \cot(\angle MOC)\frac{BC}{2} \implies OM = \frac{BC}{2}\cot(\angle A) = \frac{\cos(\angle A)}{2} \times \frac{BC}{\sin(\angle A)}$$

By the law of sines on $\triangle AHB$ we have

$$\frac{AH}{AB} = \frac{\sin(90^\circ - \angle A)}{\sin(180^\circ - \angle C)} = \frac{\cos(\angle A)}{\sin(\angle C)} \implies AH = \cos(\angle A) \times \frac{AB}{\sin(\angle C)}$$

Again By the law of on $\triangle ABC$ sines we have $\frac{AB}{\sin(\angle C)} = \frac{BC}{\sin(\angle A)}$. So $AH = 2OM$.

Proof 2:

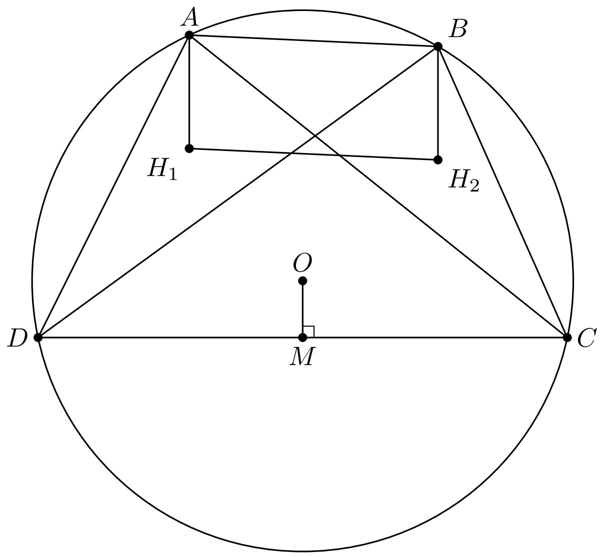


Suppose CO intersects the circumcircle for the second time at C'.

$$\left.\begin{array}{l}\left.\begin{array}{r}AH \perp BC\\ C'B \perp BC\end{array}\right\} \implies AH \parallel C'B\\ \left.\begin{array}{r}BH \perp AC\\ C'A \perp AC\end{array}\right\} \implies BH \parallel C'A\end{array}\right\} \implies BC'AH \text{ is parallelogram} \tag{5}$$

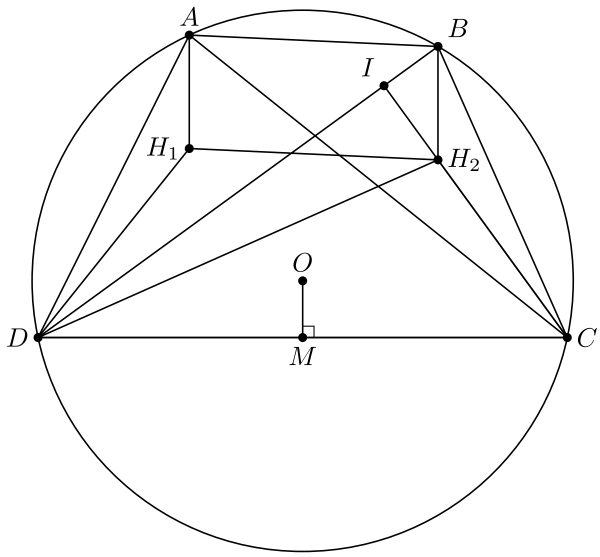
By the intercept theorem in $\triangle C'BC$ we have $C'B = 2OM$. Also by (5) we have $AH = BC'$ thus $AH = 2OM$.

Proof 1 of the Problem:



By the lemma we have $BH_1 = BH_2 = 2OM$ also since $BH_1 \perp DC$ and $BH_2 \perp DC$, we conclude that ABH_2H_1 is a parallelogram. So $H_1H_2 = AB$.

Proof 2 of the Problem:



$$\left.\begin{array}{l}\angle DH_2C = 180^\circ - \angle DBC \\ \angle DH_1C = 180^\circ - \angle DAC \\ \angle DAC = \angle DBC\end{array}\right\} \angle DH_1C = \angle DH_2C \implies DH_1H_2C \text{ is cyclic}$$

$$\left.\begin{array}{r}\angle IH_2B = \angle CDB \\ \angle CDB = \angle CAB\end{array}\right\} \implies \angle IH_2B = \angle CAB \tag{6}$$

Since DH_1H_2C is cyclic, $\angle H_1DC = \angle H_1H_2I$.

$$\left.\begin{array}{r}\angle H_1H_2I = \angle H_1DC \\ \angle H_1DC = \angle H_1AC\end{array}\right\} \implies \angle H_1H_2I = \angle H_1AC \tag{7}$$

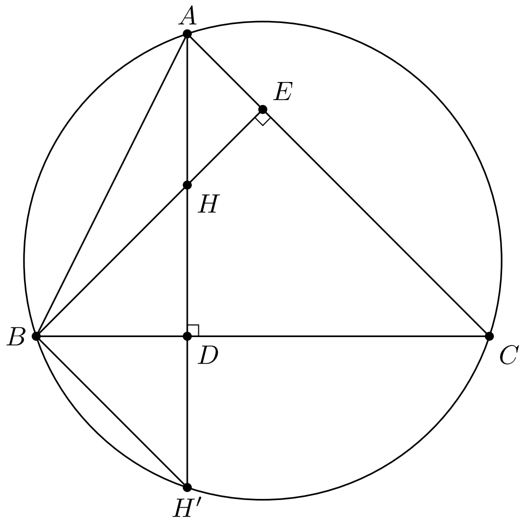
By (6) and (7) we have

$$\angle H_1H_2B = \angle H_1H_2I + \angle IH_2 = \angle H_1AC + \angle CAB = \angle H_1AB$$

Consider H_1H_2BA, $AH_1 \parallel BH_2$ and $\angle H_1H_2B = \angle H_1AB$. Thus H_1H_2B is parallelogram which means $H_1H_2 = AB$.

4. Suppose H is the orthocenter of $\triangle ABC$. Suppose that R_{ABH} is the circumradius of $\triangle ABH$. R_{BCH}, R_{CAH} and R_{ABC} are similarly defined. Prove that $R_{ABH} = R_{BCH} = R_{CAH} = R_{ABC}$.

Lemma 1: The symmetric points of orthocenter H of a triangle with respect to any side, resides on the triangle circumcircle.
Proof:

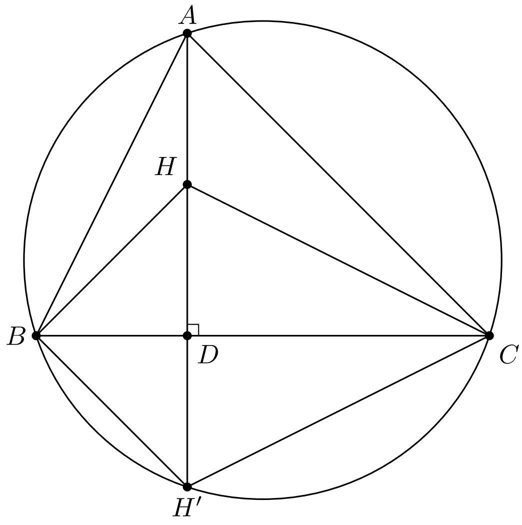


Suppose AD intersects the circumcircle at H' for the second time. We will show that $HD = H'D$. We have

$$\angle DBH' = \frac{\overset{\frown}{H'C}}{2} = \angle DAC = 90^\circ - \angle C$$

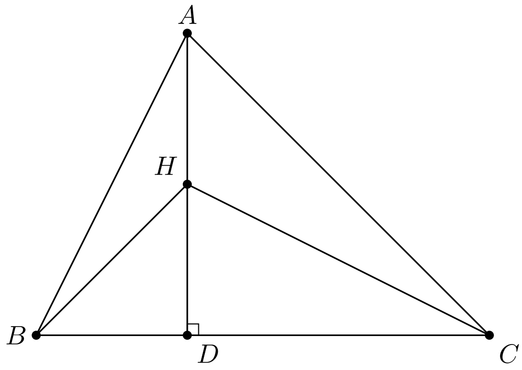
Also since $ABDE$ is cyclic $\angle HBD = \angle DAE = 90^\circ - \angle C$. Thus $\angle HBD = \angle H'BD$ which means BD is angle bisector. Since BD is both angle bisector and altitude, $\triangle HBH'$ is isocelesm, and $HD = H'D$.

Proof 1 of the problem:



By Lemma 1 we just proved we can easily see $\triangle BHC \cong \triangle BH'C$ thus $R_{BHC} = R_{BH'C} = R_{ABC}$. By the same way for other triangles, we have $R_{ABH} = R_{BCH} = R_{CAH} = R_{ABC}$.

Proof 2 of the problem:



By the law of sines in $\triangle ABC$ we have

$$2R_{ABC} = \frac{BC}{\sin(\angle A)}$$

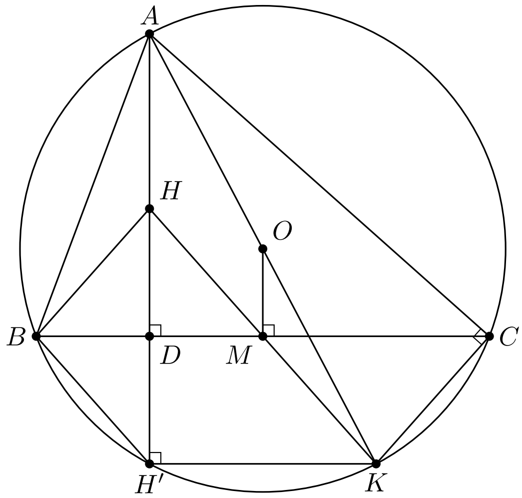
And by the law of sines in $\triangle BHC$ we have

$$2R_{BHC} = \frac{BC}{\sin(\angle BHC)} = \frac{BC}{\sin(180^\circ - \angle A)} = \frac{BC}{\sin(\angle A)}$$

Thus $R_{ABC} = R_{BHC}$. By the same way for other triangles, we have $R_{ABH} = R_{BCH} = R_{CAH} = R_{ABC}$.

5. Let AM be the median and H be the orthocenter of $\triangle ABC$. Let P be the foot of perpendicular from H to AM. Prove that $MP.MA = MB^2$.

Lemma: The symmetric point of orthocenter H of a triangle with respect to the midpoint of any side resides on the triangle's circumcircle.
Proof:



Choose K on $\overset{\frown}{BC}$ such that $\angle CAK = \angle BAD$.

$$\left.\begin{array}{l}\angle KAC = \angle H'AB = 90^\circ - \angle B \\ \angle AKC = \frac{\overset{\frown}{AC}}{2} = \angle B\end{array}\right\} \implies \angle ACK = 90^\circ$$

Thus AK is a diameter and O the circumcenter lie on it. Since $\angle H'AB = \angle KAC$, we have $\overset{\frown}{KC} = \overset{\frown}{H'B}$ thus $BC \parallel KH'$. Now suppose HK intersects BC at M, we will prove M is the midpoint of BC. Consider $\triangle HH'K$

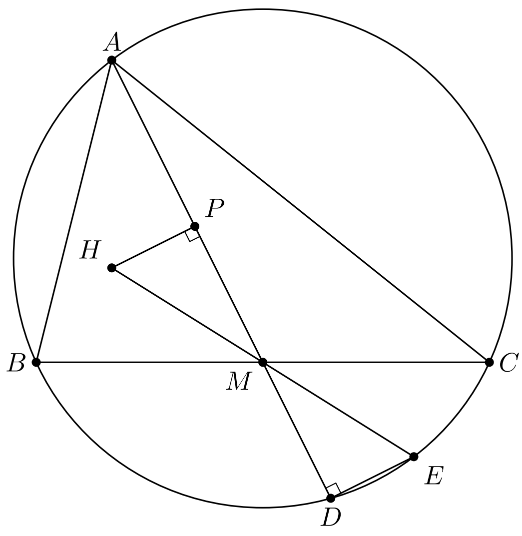
$$\left.\begin{array}{l}HD = \frac{1}{2}HH' \\ DM \parallel KC\end{array}\right\} \implies KM = \frac{1}{2}KH$$

Now consider $\triangle AKH$

$$\left.\begin{array}{l}\left.\begin{array}{l}HD = \frac{1}{2}HH' \\ KN = \frac{1}{2}KH\end{array}\right\} \implies OM \parallel AH \\ \qquad\qquad\qquad\quad AH \perp BC\end{array}\right\} \implies OM \perp BC$$

Thus M is the midpoint.

Proof of the problem:



Suppose HM intersects the circumcircle at E. By the lemma we have $HM = ME$. Again by the lemma AE is the diameter of the circle, so $\angle EDA = 90°$. Thus

$$\left.\begin{aligned} &HM = ME \\ &\angle EDM = \angle HPM \\ &\angle HMP = \angle EMD \end{aligned}\right\} \implies \triangle HMP \cong \triangle EDM \implies MD = MP$$

Now consider the power of M with respect to the circle.

$$MB.MC = MD.MA \xRightarrow[MB=MC]{MD=MP} MB^2 = MP.MA$$