Session 4

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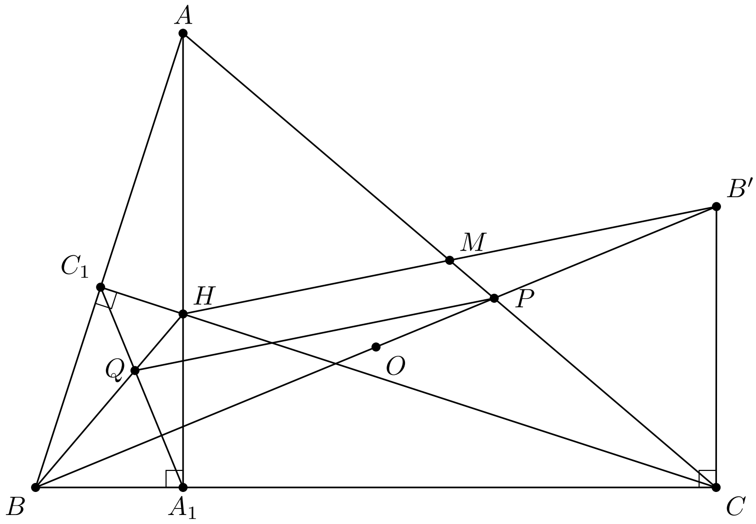
1. Given a non-isoscales triangle $\triangle ABC$ with orthocenter H and altitudes AA_1 and CC_1. O is the cicumcenter and M is the midpoint of AC. BO intersects AC at P and A_1C_1 intersects BH at Q. Prove that $MH \parallel PQ$.

Proof:



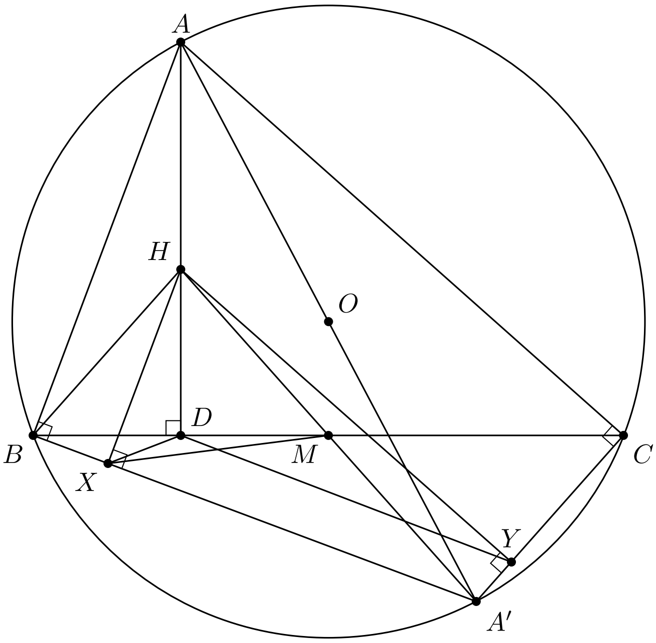
Suppose the extension of BP intersects the extension of HM at B'. Clearly B' lies on the circumcircle, thus $\angle B'CB = 90°$.

$$\left.\begin{array}{l}\angle BB'C = \angle BHC_1 = \angle A \\ \angle BCB' = \angle BC_1H = 90°\end{array}\right\} \implies \triangle B'CH \sim \triangle BHC_1 \implies \frac{BH}{BB'} = \frac{BC_1}{BC} \tag{1}$$

$$\left.\begin{array}{l}\angle CBP = \angle BQC_1 = 90° - \angle A \\ \angle BCP = \angle BHC = \angle C\end{array}\right\} \implies \triangle BQC_1 \sim \triangle BPC \implies \frac{BQ}{BP} = \frac{BC_1}{BC} \tag{2}$$

$$(1),(2) \implies \frac{BH}{BB'} = \frac{BQ}{BP} \implies \frac{BP}{BB'} = \frac{BQ}{BH} \implies MH \parallel PQ$$

2. Let AD be the altitude, H the orthocenter and O the circumcenter of $\triangle ABC$. Suppose M is the midpoint of BC and AO intersects the circumcircle for the second time at A'. Denote X, Y as the foot of perpendicular from H to BA' and CA' respectively. Prove that X, Y, D and M lie on a circle.



$$\left.\begin{array}{r} AB \parallel XH \\ YH \parallel AC \end{array}\right\} \implies \angle XHY = \angle A$$

$$\angle HYK = \angle HXK = 90^\circ \implies \begin{cases} XHYK \text{ is cyclic} \\ M \text{ is the center} \end{cases}$$

M is the circumcenter of HXY thus $\angle XMY = 2\angle A$.

$$\angle HDB = \angle HXB \implies HDXB \text{ is cyclic} \implies BDX = \angle BHX \tag{3}$$

$$\left.\begin{array}{l} \angle ABH = 90^\circ - \angle A \\ \angle ABA' = 90^\circ \end{array}\right\} \implies \angle HBX = \angle A \stackrel{(3)}{\implies} \angle BDX = 90^\circ - \angle A$$

By the same way $\angle CDY = 90^\circ - \angle A$ thus

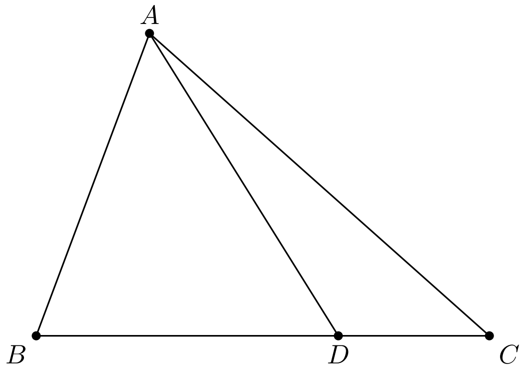
$$\angle XDY = 180^\circ - \angle BDX - \angle CDY = 2\angle A$$

$$\implies \angle XMY = \angle XDY \implies XDMY \text{ is cyclic}$$

3. Suppose $\triangle ABC$ is a right triangle ($\angle A = 90°$), also the angle bisectors BE and CF intersect each other at I. Let K be the foot of the perpendicular from I to BC. Prove that IK passes through the midpoint of EF.

Lemma 1:

$$\frac{BD}{CD} = \frac{AB\sin(\angle BAD)}{AC\sin(\angle CAD)}$$



By the law of sines in ABD we have

$$\frac{BD}{\sin(\angle BAD)} = \frac{AB}{\sin(\angle BDA)} \implies BD = \frac{AB\sin(\angle BAD)}{\sin(\angle BDA)}$$

By the same way in ACD we have

$$CD = \frac{AC\sin(\angle CAD)}{\sin(\angle CDA)}$$

thus

$$\frac{BD}{CD} = \frac{\frac{AB\sin(\angle BAD)}{\sin(\angle BDA)}}{\frac{AC\sin(\angle CAD)}{\sin(\angle CDA)}} \xrightarrow{\sin(CDA)=\sin(BDA)} \frac{BD}{CD} = \frac{AB\sin(\angle BAD)}{AC\sin(\angle CAD)}$$

Corollary 1: If AD is the median, then we have

$$BD = CD \iff AB\sin(\angle BAD) = AC\sin(\angle CAD)$$

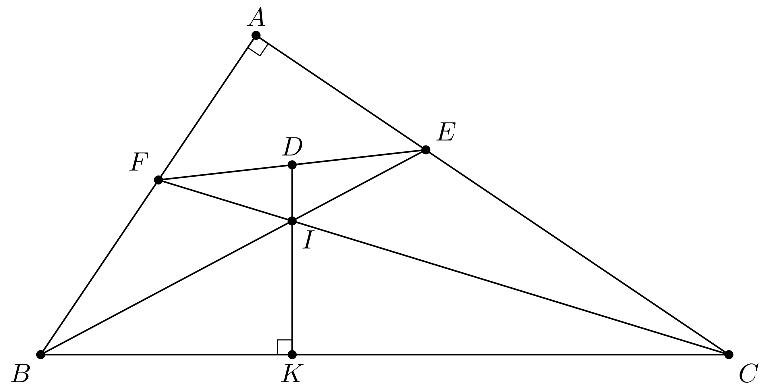
Corollary 2: If AD is the angle bisector, then we have

$$\angle BAD = \angle CAD \iff \frac{BD}{CD} = \frac{AB}{AC}$$

Corollary 3: if $\triangle ABC$ is an isosceles triangle, then we have

$$AB = AC \implies \frac{BD}{CD} = \frac{\sin(\angle BAD)}{\sin(\angle CAD)}$$

Proof:



By *Lemma 1 Corollary 1* in order to show ID is the median, we will show

$$IF\sin(\angle DIF) = IE\sin(\angle DIE) \iff IF\sin(\angle CIK) = IE\sin(\angle BIK)$$

$$\iff IF\sin\left(90^\circ - \frac{\angle C}{2}\right) = IE\sin\left(90^\circ - \frac{\angle B}{2}\right) \iff IE\cos\left(\frac{\angle B}{2}\right) = IF\cos(\left(\frac{\angle C}{2}\right) = \tag{4}$$

Clearly AI is the angle bisector and by the law of sines in $\triangle AFI$ we have

$$\frac{IF}{AI} = \frac{\sin(\angle AFI)}{\sin(45^\circ)} \implies IF = \frac{AI\sin(\angle AFI)}{\sin(45^\circ)} = \frac{AI\sin\left(\angle B + \frac{\angle C}{2}\right)}{\sin(45^\circ)} = \frac{AI\sin\left(90^\circ - \frac{\angle C}{2}\right)}{\sin(45^\circ)}$$

$$\implies IF = \frac{AI\cos\left(\frac{\angle C}{2}\right)}{\sin(45^\circ)} \tag{5}$$

By the same way in $\triangle AEI$ we have

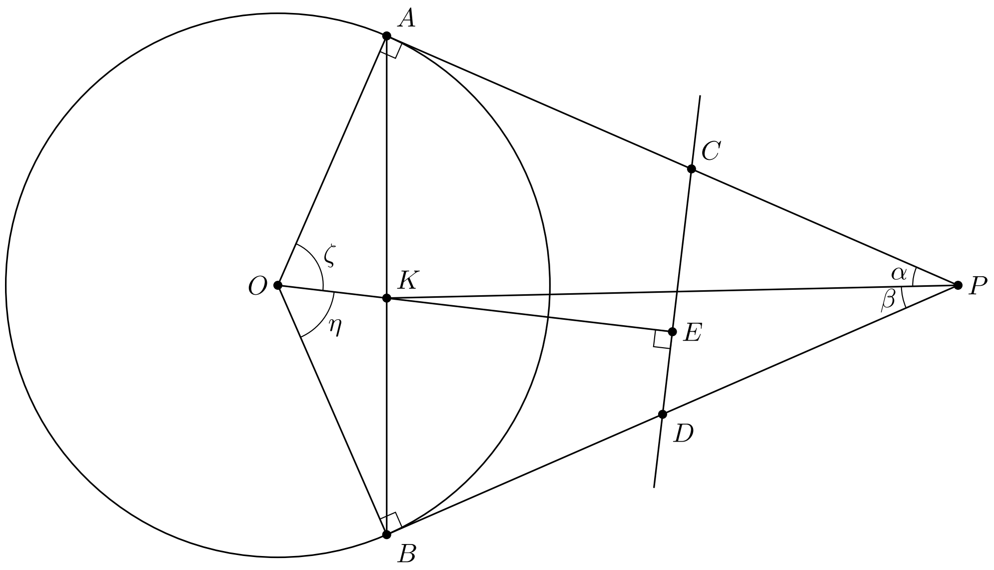
$$IE = \frac{AI\cos\left(\frac{\angle B}{2}\right)}{\sin(45^\circ)} \tag{6}$$

$$(4), (5), (6) \implies \begin{cases} IE\cos\left(\frac{\angle B}{2}\right) = \frac{AI\cos\left(\frac{\angle C}{2}\right)\cos\left(\frac{\angle B}{2}\right)}{\sin(45^\circ)} \\ IF\cos(\left(\frac{\angle C}{2}\right) = \frac{AI\cos\left(\frac{\angle B}{2}\right)\cos\left(\frac{\angle C}{2}\right)}{\sin(45^\circ)} \end{cases}$$

Thus the statement is true.

4. Let O be the center of circle ω and P is a point outside of ω. Suppose PA and PB are tangent to ω and an arbitrary line l intersects PA and PB at C and D respectively. Suppose the perpendicular from O to CD intersects AB at K. Prove that PK passes through the middle point of CD.

Proof:



In order to prove the statement we will show

$$PC\sin(\alpha) = PD\sin(\beta)$$

By *Lemma 1 Corollary 3* in $\triangle APB$ we have

$$\frac{AK}{BK} = \frac{\sin(\alpha)}{\sin(\beta)}$$

Thus we need to show

$$\frac{PC}{PD} = \frac{AK}{BK}$$

Again By *Lemma 1 Corollary 3* in $\triangle OAB$ we have

$$\frac{AK}{BK} = \frac{\sin(\zeta)}{\sin(\eta)}$$

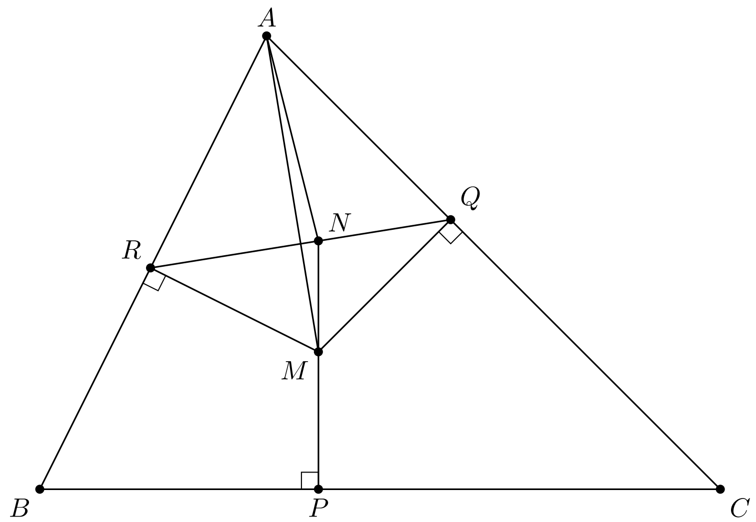
Also since $OEDB$ is cyclic, $\angle PDC = \eta$, by the same way $\angle PCD = \zeta$ thus by the law of sines in PCD we have

$$\frac{PC}{PD} = \frac{\sin(\angle PDC)}{\sin(\angle PCD)} = \frac{\sin(\zeta)}{\sin(\eta)}$$

Thus the statement is true.

5. Let M be an arbitrary point on the angle bisector of $\angle A$. Choose P, Q, and R on BC, CA and AB such that $\angle MPC = \angle MQA = \angle MRB = 90°$. the extension of PM intersects RQ at N. Prove that AN passes through the midpoint of BC.

Proof:



$$\left.\begin{array}{l}\angle MRA = \angle MQA = 90° \\ \angle RAM = \angle QAM \\ AM = AM\end{array}\right\} \implies \triangle ARM \cong \triangle AQM \implies \begin{cases} AR = AQ \\ MR = MQ \end{cases}$$

In $\triangle ARQ$ we have

$$\frac{RN}{QN} = \frac{\sin(\angle RAN)}{\sin(\angle QAN)} \tag{7}$$

In $\triangle MRQ$ we have

$$\frac{RN}{QN} = \frac{\sin(\angle RMN)}{\sin(\angle QMN)} \tag{8}$$

Since $RMPB$ and $MQCR$ is cyclic we have

$$\angle RMN = \angle B, \angle QMN = \angle C \tag{9}$$

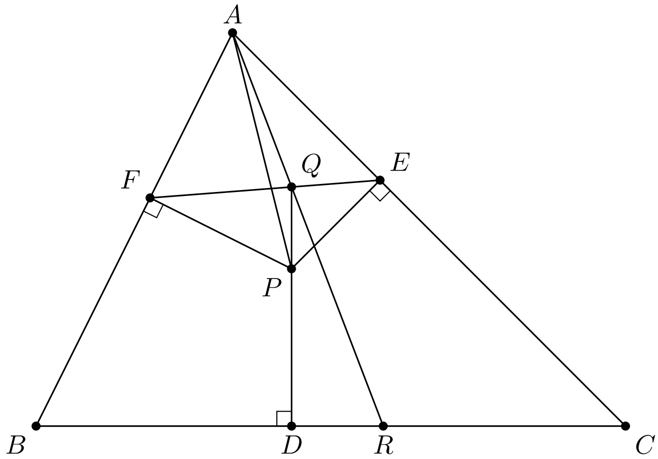
$$(7), (8), (9) \implies \frac{\sin(\angle RAN)}{\sin(\angle QAN)} = \frac{\sin(\angle B)}{\sin(\angle C)} = \frac{AC}{AB}$$

$$\implies AB\sin(\angle BAM) = AC\sin(\angle CAM) \implies AN \text{ is the median}$$

6. Let P be an arbitrary point inside $\triangle ABC$. D, E, and F are the foots of perpendiculars from P to BC, CA, and AB respectively. Suppose the extension of PD intersects EF at Q and the extension of AQ intersects BC at R. Prove that

$$\frac{BR}{CR} = \frac{\tan(\angle BAP)}{\tan(\angle CAP)}$$

Proof:



In $\triangle ABC$ we have

$$\frac{BR}{CR} = \frac{AB\sin(\angle BAR)}{AC\sin(\angle CAR)} \tag{10}$$

In $\triangle AFQ$ we have

$$\frac{FQ}{QE} = \frac{AF\sin(\angle BAR)}{AE\sin(\angle CAR)} \tag{11}$$

Also in $\triangle PFQ$ we have

$$\frac{FQ}{QE} = \frac{PF\sin(\angle B)}{PE\sin(\angle C)} \tag{12}$$

Thus

$$(11),(12) \implies \frac{\sin(\angle BAR)}{\sin(\angle CAR)} = \frac{PF.AE\sin(\angle B)}{PE.AF.\sin(\angle C)}$$

$$\overset{(10)}{\Longrightarrow} \frac{BR}{CR} = \frac{AB.PF.AE\sin(\angle B)}{AC.PE.AF\sin(\angle C)}$$

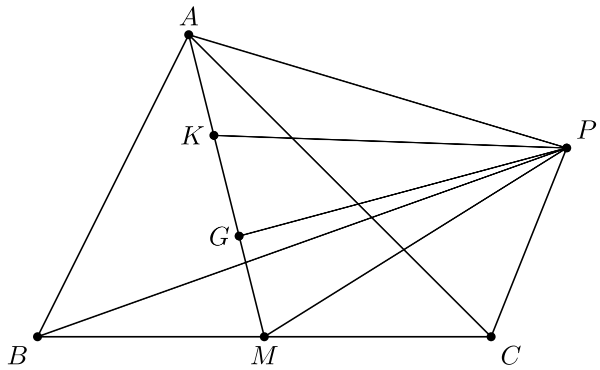
Clearly $AB\sin(\angle B) = AC\sin(\angle C)$ so

$$\frac{BR}{CR} = \frac{PF.AE}{PE.AF} = \frac{\frac{PF}{AF}}{\frac{PE}{AE}} = \frac{\tan(\angle BAP)}{\tan(\angle CAP)}$$

7. Suppose G is the centroid of $\triangle ABC$. Let P be an arbitrary point in the plane except A, B, C, and G. Prove that

$$PG^2 = \frac{1}{3}\left(PA^2 + PB^2 + PC^2\right) - \frac{1}{9}\left(a^2 + b^2 + c^2\right)$$

Proof:



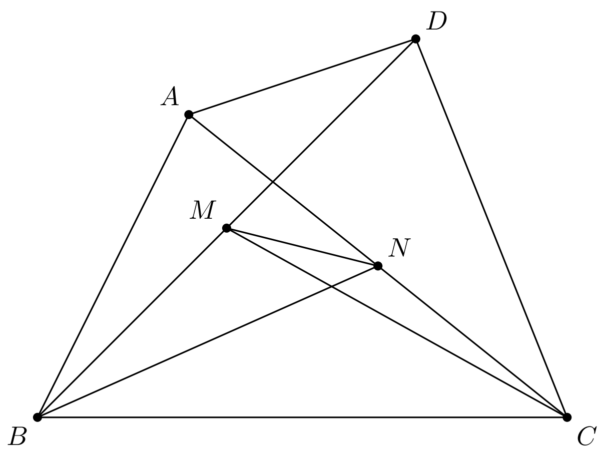
$$\begin{aligned} PG^2 &= \frac{1}{2}PK^2 + \frac{1}{2}PM^2 - \frac{1}{4}KM^2 \\ &= \frac{1}{2}\left(\frac{1}{2}PA^2 + \frac{1}{2}PG^2 - \frac{1}{4}AG^2\right) + \frac{1}{2}\left(\frac{1}{2}PB^2 + \frac{1}{2}PC^2 - \frac{1}{4}a^2\right) - \frac{1}{4}KM^2 \\ &= \frac{1}{4}PA^2 + \frac{1}{4}PG^2 - \frac{1}{8}\left(\frac{4}{9}\left(\frac{1}{2}(b^2+c^2) - \frac{1}{4}a^2\right)\right) + \frac{1}{4}PB^2 - \frac{1}{8}a^2 - \frac{1}{4}\left(\frac{4}{9}\left(\frac{1}{2}(b^2+c^2) - \frac{1}{4}a^2\right)\right) \end{aligned}$$

$$\implies \frac{3}{4}PG^2 = \frac{1}{4}\left(PA^2+PB^2+PC^2\right) - \frac{1}{12}\left(a^2+b^2+c^2\right) \implies PG^2 = \frac{1}{3}\left(PA^2+PB^2+PC^2\right) - \frac{1}{9}\left(a^2+b^2+c^2\right)$$

8. Suppose $ABCD$ is an arbitrary quadrilateral. Let M and N be the midpoint of BD and AC respectively. Prove that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$$

Proof:



In $\triangle BND$ we have

$$2MN^2 = ND^2 + BN^2 - \frac{1}{2}BD^2 \tag{13}$$

Also in $\triangle AMC$ we have

$$2MN^2 = MA^2 + MC^2 - \frac{1}{2}AC^2 \tag{14}$$

Also in $\triangle BMC$, $\triangle BNC$, $\triangle AMD$, and $\triangle AND$ we can write

$$\begin{cases} DN^2 = \frac{1}{2}AD^2 + \frac{1}{2}CD^2 - \frac{1}{4}AC^2 \\ BN^2 = \frac{1}{2}AB^2 + \frac{1}{2}BC^2 - \frac{1}{4}AC^2 \\ AM^2 = \frac{1}{2}AD^2 + \frac{1}{2}AB^2 - \frac{1}{4}BD^2 \\ CM^2 = \frac{1}{2}DC^2 + \frac{1}{2}BC^2 - \frac{1}{4}BD^2 \end{cases} \tag{15}$$

By adding (13) and (14), and using (15) we have

$$4MN^2 = AB^2 + BC^2 + CD^2 + DA^2 - AC^2 - BD^2 \implies AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$$