Session 6

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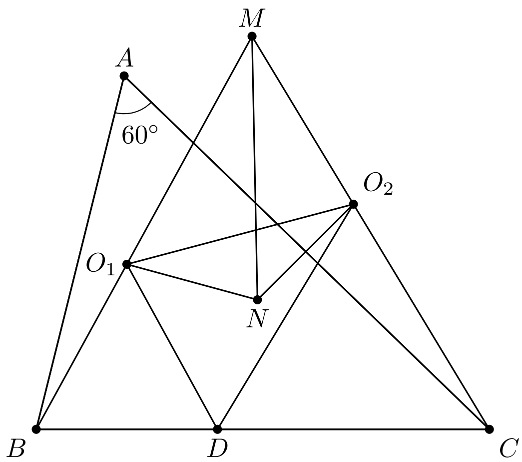
1. Let $\angle A = 60^\circ$ in $\triangle ABC$ and D be a variant point on BC. Suppose O_1 and O_2 is the circumcircle of ABD and ACD. BO_1 and BO_2 meet in M and N is the circumcircle of DO_1O_2. Prove that MN passes through an invariant point.

Proof:



$$\left.\begin{aligned}\angle O_1BD = \angle O_1DB = 90^\circ - \angle BAD\\ \angle O_2DB = \angle O_2BD = 90^\circ - \angle CAD\end{aligned}\right\} \implies \left\{\begin{aligned}&\angle O_1DO_2 = 180^\circ - \angle O_1DB - \angle O_2DB = 60^\circ\\ &\angle BMC = 180^\circ - \angle O_1BD - \angle O_2CD = 60^\circ\end{aligned}\right.$$

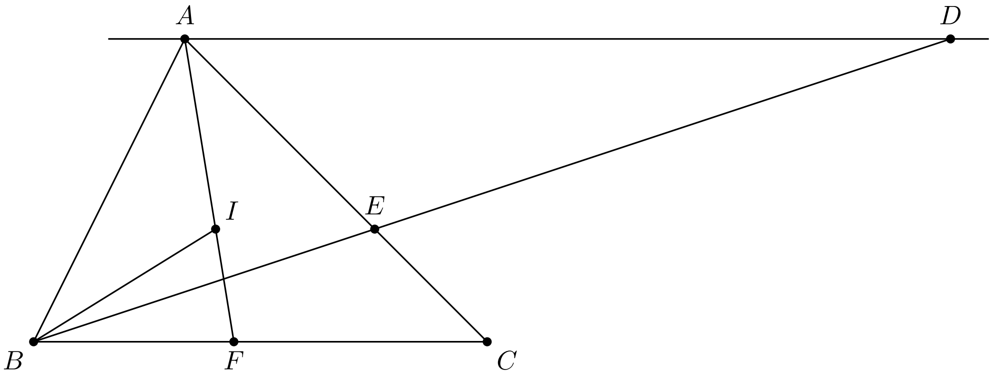
Thus M lie on the circumcircle of $\triangle ABC$. Also since N is the circumcircle of $\triangle DO_1O_2$, $\angle O_1NO_2 = 2\angle O_1DO_2 = 120^\circ$. Which implies that O_1MO_2N is cyclic. Note that $\triangle NO_1O_2$ is isosceles so

$$\left.\begin{aligned}\angle NO_1O_2 = \angle NMO_2\\ \angle NO_2O_1 = \angle NMO_1\\ \angle NO_1O_2 = \angle NO_2O_1\end{aligned}\right\} \implies \angle NMO_1 = \angle NMO_2$$

Thus MN always passes through the midpoint of $\overset{\frown}{BC}$.

2. Let l be a line passing through A and parallel to BC of $\triangle ABC$. Choose D on l such that $AD = AC + AB$. if DB intersects AC at E, then prove $EI \parallel BC$, where I is the incenter.

Proof:



$$\left.\begin{array}{l}\angle EAD = \angle ECB \\ \angle EDA = \angle EBC\end{array}\right\} \implies \triangle EAD \sim \triangle ECB \implies \frac{AE}{EC} = \frac{AD}{BC} = \frac{b+c}{a} \tag{1}$$

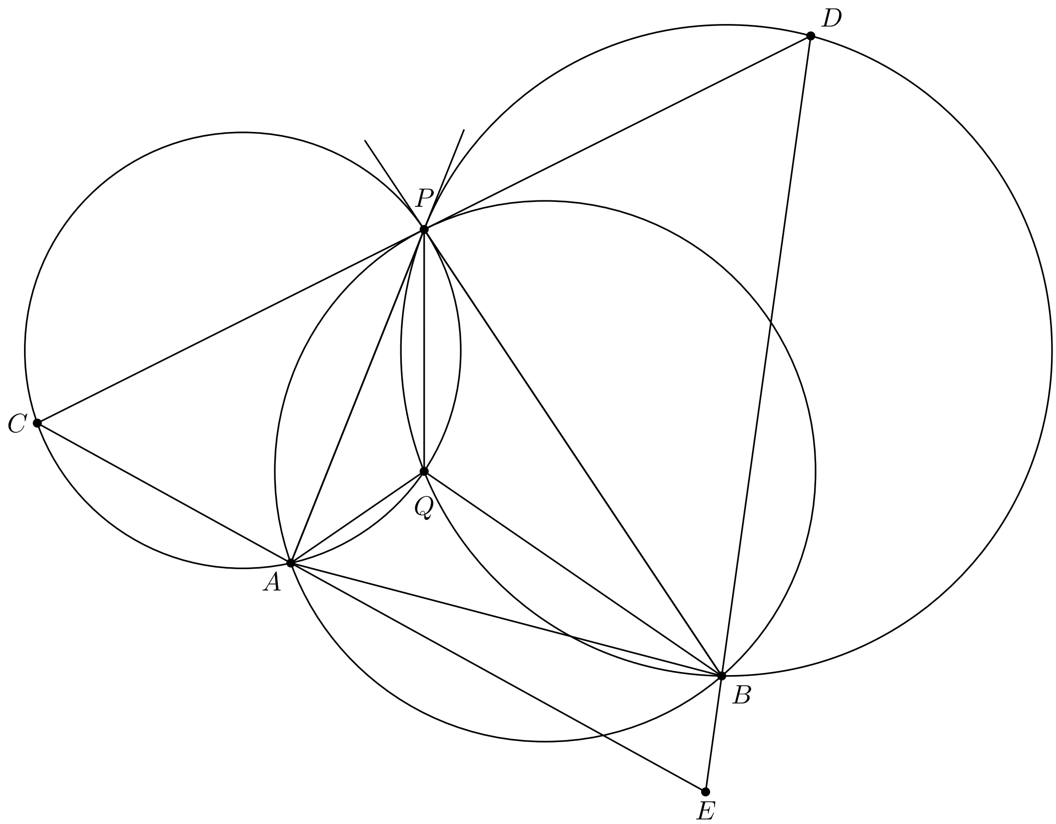
Since BI is bisector in $\triangle ABF$, we have

$$\frac{AI}{IE} = \frac{AB}{BF} = \frac{c}{\frac{ac}{b+c}} = \frac{b+c}{a} \tag{2}$$

$$(1), (2) \implies \frac{AE}{EC} = \frac{AI}{IE} \implies EI \parallel AD$$

3. Let ω_1 and ω_2 be two circles intersecting each other at P and Q. Suppose the tangent line to ω_1 from P intersects ω_2 at B and the tangent line to ω_2 from P intersects ω_1 at A. Assume that the tangent line from P to the circumcircle of $\triangle PAB$ intersects ω_1 and ω_2 at C and D. Prove that $PC = PD$.

Proof 1:



$$\left.\begin{aligned} \angle QAP = \angle QPB = \frac{\widehat{PQ}}{2} \\ \angle QPA = \angle QBP = \frac{\widehat{PQ}}{2} \end{aligned}\right\} \implies \angle BQP = \angle AQP \implies \angle PDB = \angle PCA \tag{3}$$

Suppose CA and DB meet at E, then we have

$$\left.\begin{aligned} \angle PBD = \angle CPA \\ \angle PBA = \angle CPA \end{aligned}\right\} \implies \angle PBA = \angle PBD$$

Thus PB is the external bisector of $\triangle EAB$. By the same way AP is the external bisector of $\triangle EAB$, too. Since AP and BP meet in P, then PE is the internal bisector of $\triangle ABE$. Now in isosceles triangle $\triangle CED$, PE is the bisector, so it's median too. Which means $PC = PD$.

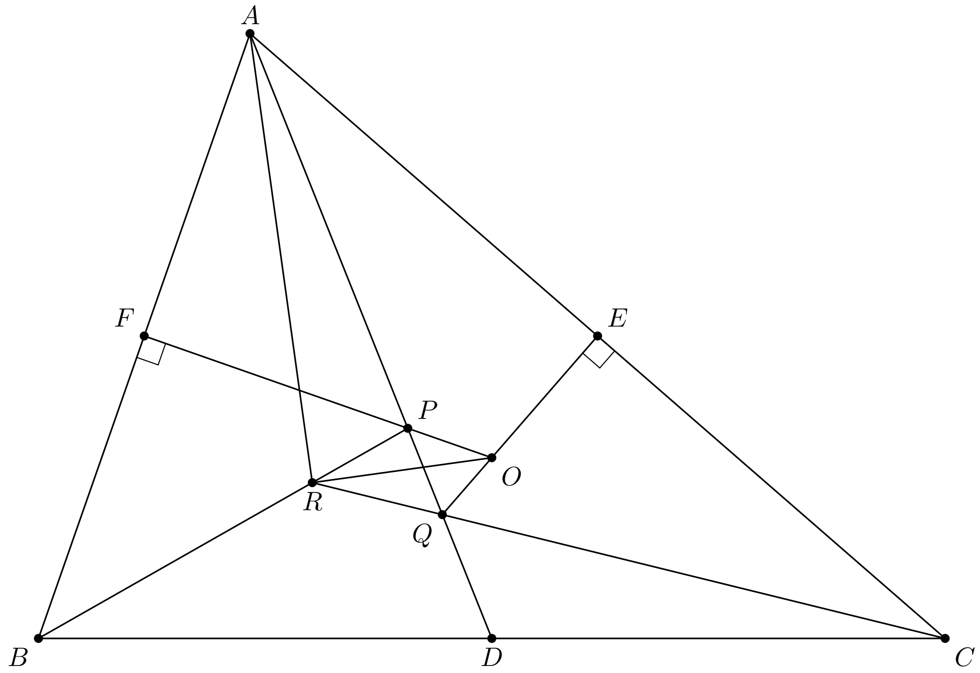
Proof 2:

$$\left.\begin{aligned} \angle PBA = \angle CPA = \angle BDP \\ \angle CAP = \angle PAB = \angle DPB \end{aligned}\right\} \implies \triangle ACP \sim \triangle APB \sim \triangle PDB$$

$$\left.\begin{aligned} \triangle PDB \sim \triangle APB \implies \frac{PD}{AP} = \frac{PB}{AB} \implies PD = \frac{PB.AP}{AB} \\ \triangle ACP \sim \triangle APB \implies \frac{PC}{PB} = \frac{AP}{AB} \implies PC = \frac{PB.AP}{AB} \end{aligned}\right\} \implies PD = PC$$

4. Let AD be the median and O the circumcenter of $\triangle ABC$. Suppose the perpendicular bisectors of AB and AC intersect AD at P and Q respectively. BP and CQ meet at R, Prove $\angle ARO = 90°$.

Proof:



Let $\angle BAP = \alpha$ and $\angle CAP = \beta$. Since FP and QE are perpendicular bisectors, $\angle ABP = \alpha$ and $\angle ACP = \beta$. Clearly PF and QE are angle bisectors in $\triangle APB$ and $\triangle ARC$ respectively. Thus in $\triangle RPQ$, PO and QO are external angle bisectors. Which implies that O is the excenter of $\triangle RPQ$ and RO is the internal angle bisector. Now instead of proving $\angle ARO = 90°$, we prove RA is the external angle bisector. By the law of sines in $\triangle RPQ$ we have

$$\frac{RP}{RQ} = \frac{\sin(2\beta)}{\sin(2\alpha)} \tag{4}$$

Again by the law of sines in $\triangle APB$ we have

$$\frac{AP}{AB} = \frac{\sin(\alpha)}{\sin(180° - 2\alpha)} = \frac{\sin(\alpha)}{\sin(2\alpha)} \implies AP = \frac{AB\sin(\alpha)}{\sin(2\alpha)} \tag{5}$$

By the same way in $\triangle AQC$ we have

$$AQ = \frac{AC\sin(\beta)}{\sin(2\beta)} \tag{6}$$

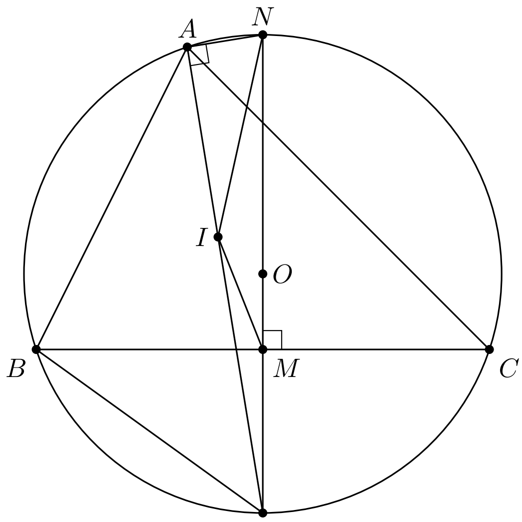
Also since AD is the median, $AB\sin(\alpha) = AC\sin(\beta)$, using this equation and (4),(5),(6) we have

$$\frac{AP}{AQ} = \frac{\sin(2\alpha)}{\sin(2\beta)} = \frac{RP}{RQ}$$

Thus AR is the external bisector.

5. Suppose I is the incenter of $\triangle ABC$ ($AC > AB$). Let N be the midpoint of $\overset{\frown}{BAC}$ and M be the midpoint of BC. Prove that $\angle IMB = \angle INA$.

Proof:



Clearly AI and NM meet at P, which is the midpoint of $\overset{\frown}{BC}$. Thus PN is the diameter and $\angle PAN = 90°$.

$$PB^2 = PM^2 + BM^2 = PM^2 + BM.MC = PM^2 + PM.MN \implies PB^2 = PM.PN$$

Also from the previous session we know that $PB = PI$ thus

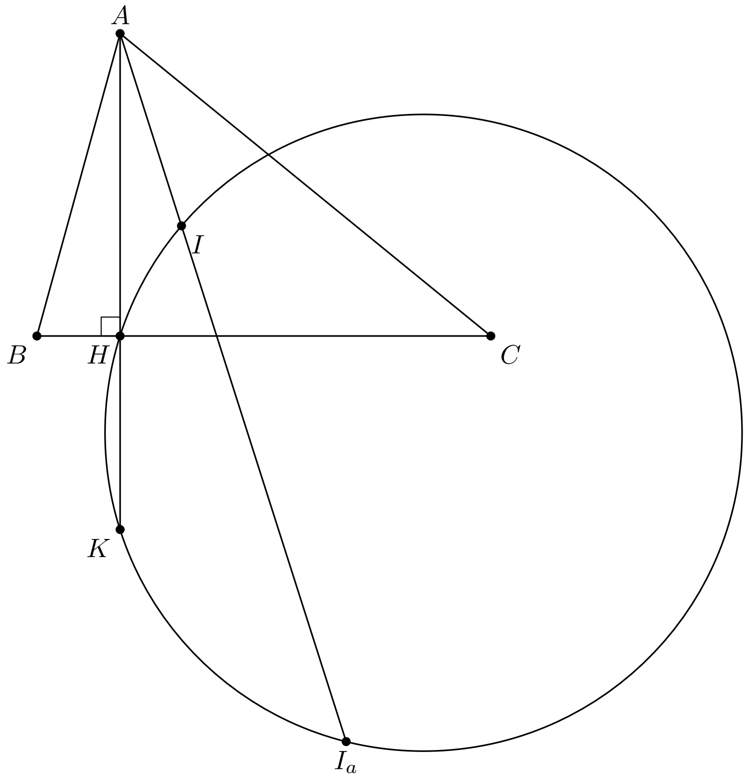
$$PI^2 = PM.PN$$

Which means that PI is tangent to the circumcircle of $\triangle MIN$. So $\angle AIN = \angle NMI$. Now we have

$$\left.\begin{array}{l}\angle AIN = \angle NMI \\ \angle IMB = 90° - \angle NMI \\ \angle INA = 90° - \angle AIN\end{array}\right\} \implies \angle IMB = \angle INA$$

6. Let AH be the Altitude of $\triangle ABC$. Suppose AH intersects the circumcircle of $\triangle HII_a$ at K. Prove that

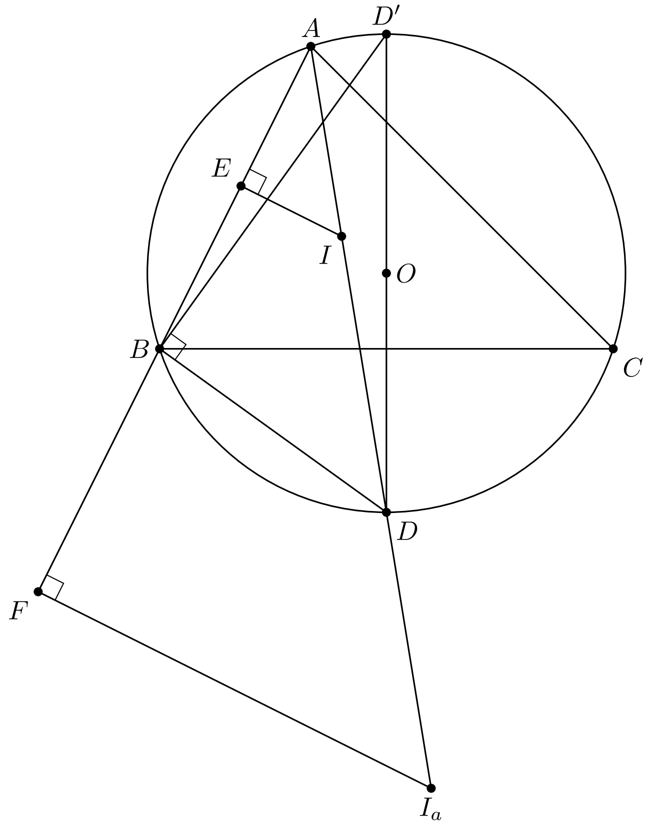
Proof:



Clearly $AH.AK = AI.AI_a$ and from the previous session we know that $AI.AI_a = bc$ thus $AK = \frac{bc}{h_a}$. Also from the previous session $S = \frac{abc}{4R} = \frac{ah_a}{2R}$ thus $AK = \frac{bc}{h_a} = 2R$

7. Prove that
    1. $OI^2 = R^2 - 2Rr$
    2. $OI_a^2 = R^2 + 2Rr_a$

Proof:



1.

$$\left.\begin{array}{l}\angle EAI = \angle BD'D = \frac{\angle A}{2}\\ \angle IEA = \angle DAD' = 90^\circ\end{array}\right\} \implies \triangle EAI \sim \triangle BD'D \implies \frac{AI}{IE} = \frac{D'D}{DB}$$

Since $DB = DI$ and $D'D = 2R$, then we have

$$AI.ID = 2Rr$$

Also the power of point I with respect to the circumcircle of $\triangle ABC$ is

$$AI.ID = R^2 - OI^2 = 2Rr \implies OI^2 = R^2 - 2Rr$$

2.

$$\left.\begin{array}{l}\angle FAI_a = \angle BDD' = \frac{\angle A}{2}\\ \angle I_aFA = \angle IEA = 90^\circ\end{array}\right\} \implies \triangle FAI_a \sim \triangle BD'D \implies \frac{AI_a}{IE} = \frac{D'D}{DB}$$

Again since $DB = DI$ and $D'D = 2R$, then we have

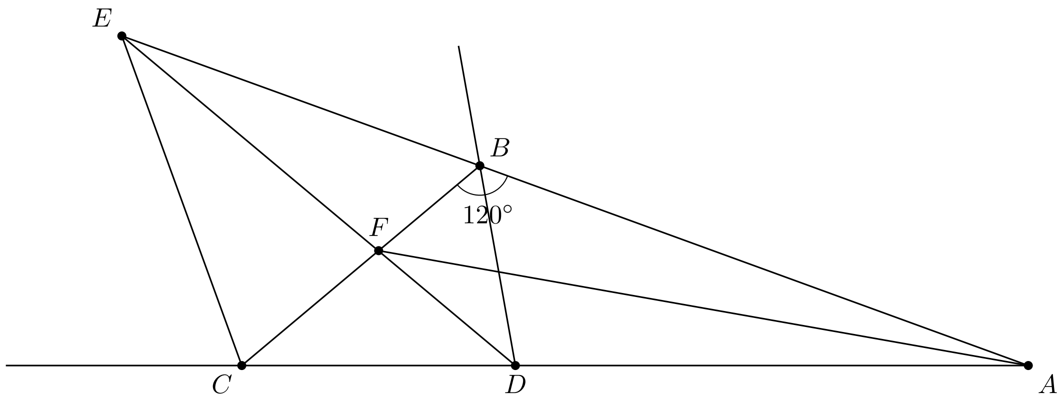
$$I_aA.ID = 2Rr_a$$

Also the power of point I_a with respect to the circumcircle of $\triangle ABC$ is

$$AI_a.ID = OI_a^2 - R^2 \implies OI_a^2 = R^2 + 2Rr_a$$

8. Suppose $\angle B = 120°$ in $\triangle ABC$. Assume that the internal angle bisector of $\angle B$ intersects CA at D and the external angle bisector of $\angle C$ intersects BA at E. If ED intersects BC at F, Prove that $\angle AFD = \angle FEC$.

Proof:



By the Menelaus theorem for D, E, and F we have

$$\frac{AD}{DC} \times \frac{CF}{FB} \times \frac{EB}{EA} = 1$$

ALso since BD and CE are angle bisectors, we have

$$\frac{AD}{DC} = \frac{c}{a}, \frac{EB}{EA} = \frac{a}{b}$$

Which implies that

$$\frac{CF}{FB} = \frac{b}{c}$$

Thus AF is the angle bisector. Now since $\angle B = 120°$, BE is the external angle bisector of $\angle B$. So since BE and CE are both external angle bisectors, E is the excenter of $\triangle DBC$. Thus

$$\angle FEC = \frac{\angle CBD}{2} = \frac{\angle B}{4} = 30°$$

Also DE is the external angle bisector of $\angle ADB$, and since AF is the angle bisector, F is the excenter of $\triangle ABD$. Thus

$$\angle AFD = \frac{\angle DBA}{2} = \frac{\angle B}{4} = 30°$$

So $\angle AFD = \angle FEC = 30°$.