Session 3

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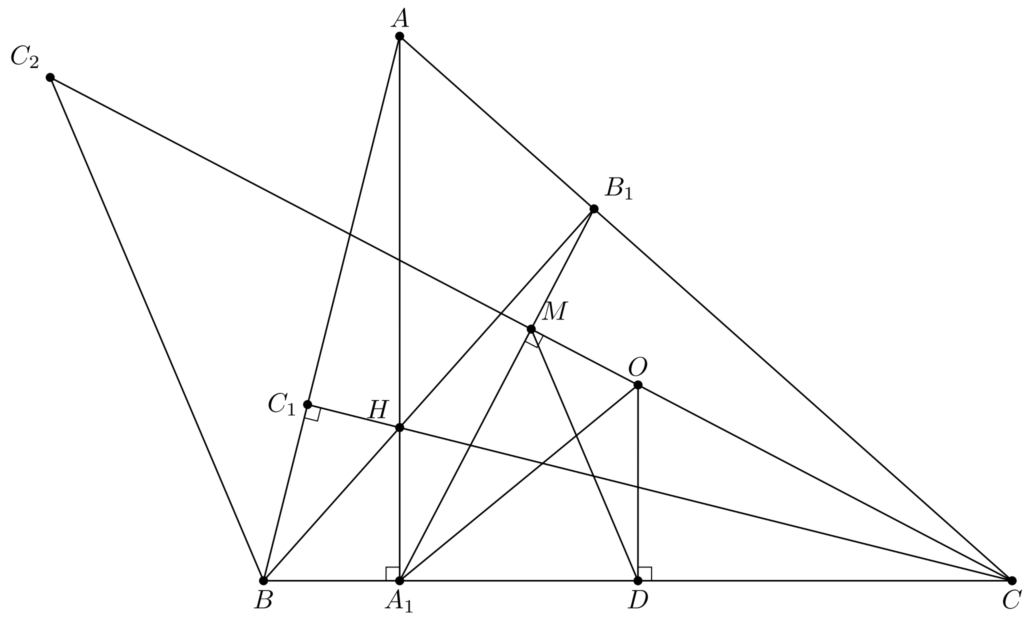
1. Let H and O be the orthocenter and circumcircle of the acute triangle $\triangle ABC$ respectively. Also AA_1, BB_1, and CC_1 are the altitudes of $\triangle ABC$. Suppose C_2 is the reflection of C over A_1B_1. Prove that O, H, C_1 and C_2 lie on a circle.

Proof:



In order to show OHC_1C_2 is cyclic we need to show

$$CO.CC_2 = CH.CC_1 \tag{1}$$

We already know C_1HA_1B is cyclic thus $CA_1.CB = CH.CC_1$. So instead of proving (1) we will show $CO.CC_2 = CB.CA_1$ or in other words C_2BA_1O is cyclic. Suppose D is the midpoint of BC, then

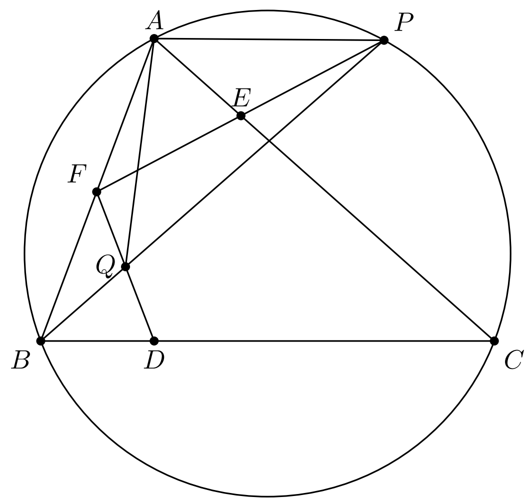
$$\angle OMB_1 = \angle ODC = 90^\circ \implies ODA_1M \text{ is cyclic} \implies \angle DMO = \angle OA_1D \tag{2}$$

$$\left.\begin{matrix} CM = MC_2 \\ CD = DB \end{matrix}\right\} \implies BC_2 \parallel DM \implies \angle BC_2C = \angle DMO \tag{3}$$

$$(2), (3) \implies \angle BC_2C = \angle OA_1D \implies C_2BA_1O \text{ is cyclic}$$

2. Let AD, BE and CF be the altitudes of $\triangle ABC$. Suppose P is one of the intersections of EF and the circumcircle of $\triangle ABC$, also FD intersects BP at Q. Prove that $AQ = AP$.

Proof:



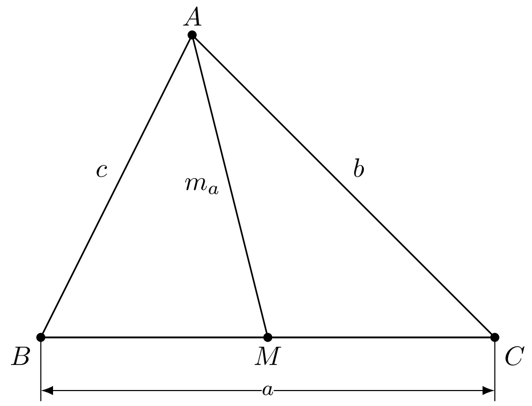
$$\left.\begin{array}{r}\angle BFD = \angle C\\ \angle APB = \angle C\end{array}\right\} \implies AFQP \text{ is cyclic}$$

$$\implies \angle AQP = \angle AFP = \angle C \implies \angle AQP = \angle APQ \implies AP = AQ$$

Let m_a be the median of $\triangle ABC$ from A. Prove that

$$m_a^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2$$

Proof 1:

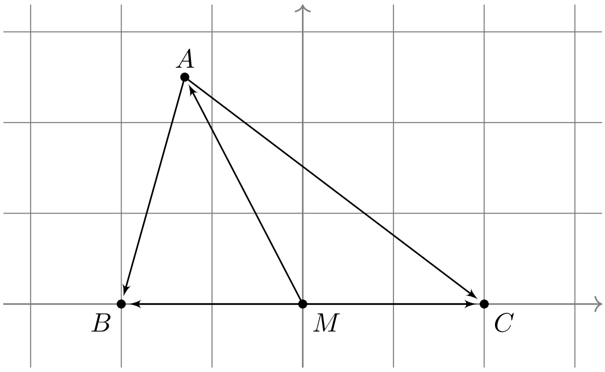


By the law of cosine in $\triangle AMB$ and $\triangle AMC$ we have

$$\left.\begin{array}{l} c^2 = \frac{1}{4}a^2 + m_a^2 - am_a\cos(\angle AMB) \\ b^2 = \frac{1}{4}a^2 + m_a^2 - am_a\cos(\angle AMC) \\ \angle AMB + \angle AMC = 180^\circ \end{array}\right\} \implies b^2 + c^2 = 2m_a^2 + \frac{1}{2}a^2$$

$$\implies m_a^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2$$

Proof 2:



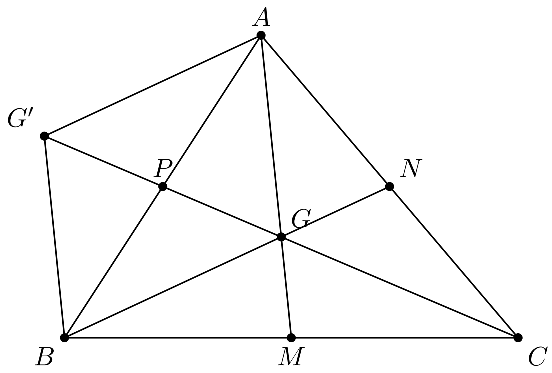
Suppose M be the center thus

$$\left.\begin{aligned} a^2 &= \left|\overrightarrow{BC}\right|^2 = \left(\vec{b}-\vec{c}\right)^2 \\ b^2 &= \left|\overrightarrow{AC}\right|^2 = \left(\vec{c}-\vec{a}\right)^2 \\ c^2 &= \left|\overrightarrow{AB}\right|^2 = \left(\vec{a}-\vec{b}\right)^2 \\ m_a^2 &= \left|\overrightarrow{MA}\right|^2 = \left|\vec{a}\right|^2 \\ \vec{m} &= 0 \iff \vec{b}+\vec{c}=0 \end{aligned}\right\} \iff 4\left|\vec{a}\right|^2 = 2\left(\vec{c}-\vec{a}\right)^2 + 2\left(\vec{a}-\vec{b}\right)^2 - \left(\vec{b}-\vec{c}\right)^2$$

$$\implies 4\left|\vec{a}\right|^2 = 2\left(\vec{a}+\vec{b}\right)^2 + 2\left(\vec{a}-\vec{b}\right)^2 - 4\left|\vec{b}\right|^2 \iff 2\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 = \left(\vec{a}+\vec{b}\right)^2 + \left(\vec{a}-\vec{b}\right)^2$$

4. Let m_a, m_b, and m_c be the medians of $\triangle ABC$. Consider $\triangle DEF$ such that $DE = m_a$, $EF = m_b$, and $FD = m_c$. Prove that $S_{DEF} = \frac{3}{4} S_{ABC}$.

Proof:



Let G' be the reflection of G with respect to P.

$$\left.\begin{array}{r} PG = PG' \\ PB = PC \end{array}\right\} \implies AGBG' \text{ is parallelogram} \implies \left\{\begin{array}{l} AG' = \frac{2}{3}BN \\ BG' = \frac{2}{3}AM \end{array}\right.$$

Also $GG' = \frac{2}{3}CP$ thus $\triangle GG'A \sim \triangle DEF$ and $S_{DEF} = \frac{9}{4} S_{GG'A}$. Also we already know that $S_{APG} = S_{APG'} = \frac{1}{6} S_{ABC}$ thus

$$S_{DEF} = \frac{9}{4} S_{GG'A} = \frac{9}{4}\big(S_{APG} + S_{APG'}\big) = \frac{9}{4}\left(\frac{1}{3} S_{ABC}\right) = \frac{3}{4} S_{ABC}$$

Corollary: We can calculate the area of a triangle by the length of its median. Suppose $2p = m_a + m_b + m_c$

$$\begin{aligned} S_{ABC} &= \frac{4}{3}\sqrt{p(p-m_a)(p-m_b)(p-m_c)} \\ &= \frac{4}{3}\sqrt{(m_a+m_b+m_c)(m_a+m_b-m_c)(m_b+m_c-m_a)(m_c+m_b-m_a)} \end{aligned}$$