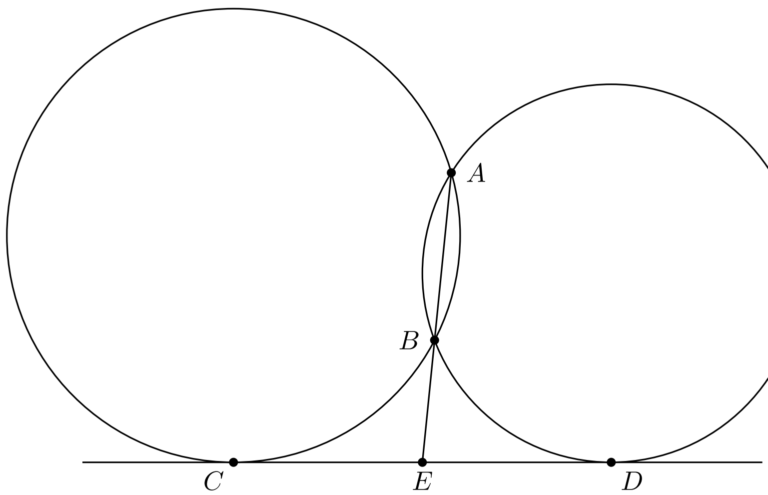
Session 5

Aria Afrooz

September 2021

1. Let D be the foot of angle bisector of $\angle A$ in $\triangle ABC$. Suppose Γ is a circle pass through A and D also BC is tangent to Γ at D. AC intersects Γ for the second time at M. BM intersects Γ at P. Prove that AP is the median of $\triangle ABD$.

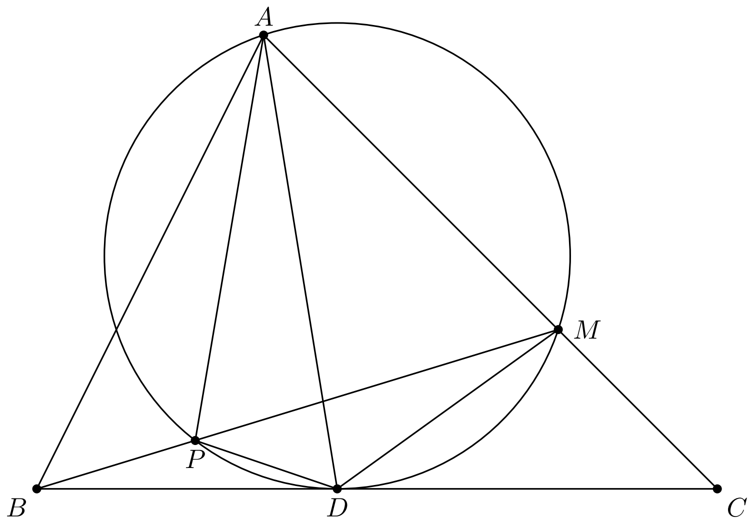
Lemma 1: Suppose two circles intersect each other at A and B. Prove that AB passes through the common tangent of two circles.



Suppose the common tangent of two circles intersects the circles at C and D respectively, and the extension of AB intersects CD at E. Thus

$$\left.\begin{aligned} ED^2 = EB.EA \\ EC^2 = EB.EA \end{aligned}\right\} \implies EC^2 = ED^2 \implies EC = ED$$

Proof 1:



$$AMDP \text{ is cyclic} \implies \angle DPM = \angle DAM \tag{1}$$

$$BC \text{ is tangent} \implies \angle PDB = \angle PAD \tag{2}$$

$$(1),(2) \implies \angle PBD = \angle PAB \implies BC \text{ is tangent to the circumcircle of } \triangle APB$$

The circumcircle of APB and the circumcircle of APD intersect each other at A and P, also BD is the common tangent of two circles. Thus by *Lemma 1* AP passes through the midpoint of BD.

Proof 2:

$$\angle DAP = \angle PDB = \frac{\overset{\frown}{PD}}{2}$$

$$\angle PBD = \angle MPD - \angle PDB = \frac{\angle A}{2} - \angle DAP = \angle BAP$$

$$MAPD \text{ is cyclic} \implies \angle DMB = \angle DAP = \angle PDB$$

$$\left.\begin{aligned} \angle PBD = \angle DBM \\ \angle DMB = \angle PDB \end{aligned}\right\} \implies \triangle PBD \sim \triangle DBM \implies \frac{PB}{PD} = \frac{DB}{DM}$$

$$\left.\begin{aligned} &\angle CDM = \tfrac{\angle A}{2} = \tfrac{\overset{\frown}{MD}}{2} \\ &\angle CDA = \angle DBA + \angle BAD = \angle B + \tfrac{\angle A}{2} \end{aligned}\right\} \implies \angle MDA = \angle CDA - \angle CDM = \angle B$$

$$\left.\begin{aligned} \angle MDA = \angle DBA = \angle B \\ \angle MAD = \angle DAB = \tfrac{\angle A}{2} \end{aligned}\right\} \implies \triangle ABD \sim \triangle ADM \implies \frac{AB}{AD} = \frac{DB}{DM}$$

$$\left.\begin{aligned} \tfrac{PB}{PD} = \tfrac{DB}{DM} \\ \tfrac{AB}{AD} = \tfrac{DB}{DM} \end{aligned}\right\} \implies \frac{PB}{PD} = \frac{AB}{AD}$$

By the law of sines in $\triangle PBD$ we have

$$\frac{PB}{PD} = \frac{\sin(\angle PDB)}{\sin(\angle PBD)} = \frac{\sin(\angle DAP)}{\sin(\angle BAP)}$$

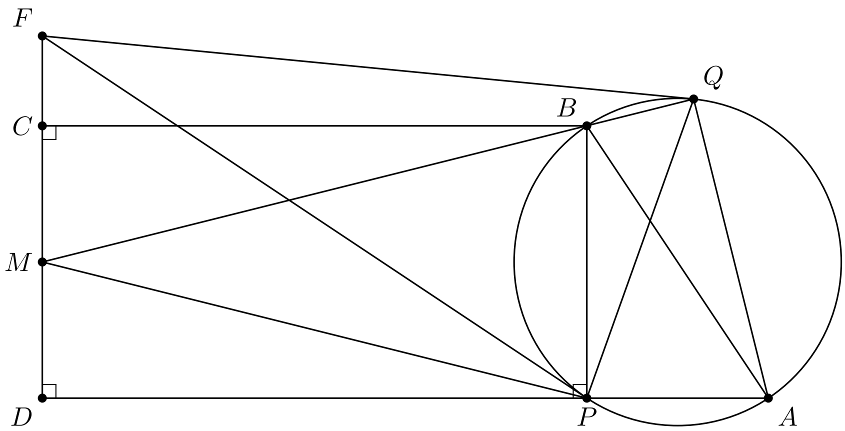
Thus

$$\frac{AB}{AD} = \frac{\sin(\angle DAP)}{\sin(\angle BAP)} \implies AB\sin(\angle BAP) = AD\sin(\angle DAP)$$

So AP passes through the midpoint of BD.

2. Let $ABCD$ be a trapezoid such that $\angle BCD = \angle CDA = 90°$. Suppose AB is the diameter of circle Γ. Suppose AD intersects Γ at P. Tangent line of Γ from P intersects the extension of CD at Q. Let QM be the second tangent line of Γ from Q. Prove that BM passes through the midpoint of CD.

Proof:



Let the extension of BQ intersects CD at M. Clearly $CBPD$ is a rectangle, so instead of proving $CM = MD$, we can porve $MP = PB$ or $\angle MBP = \angle MPB$.

$$BQAP \text{ is cyclic} \implies \angle MBP = \angle QAP \tag{3}$$

$$BP \parallel FD \implies \angle DFP = \angle BPF = \frac{\overset{\frown}{PB}}{2} = \angle MQP$$

$$\implies FQPM \text{ is cyclic} \implies \angle FPM = \angle FQM = \frac{\overset{\frown}{BQ}}{2} = \angle BAQ \tag{4}$$

$$\angle BPA = \frac{\overset{\frown}{PB}}{2} = \angle BAP \tag{5}$$

$$(4),(5) \implies \angle MPB = \angle FPM + \angle FPB = \angle BAQ + \angle MQP = \angle QAP \tag{6}$$

$$(3),(6) \implies \angle MBP = \angle MPB$$

3. Prove the equations below

   1. $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$
   2. $r_a r_b + r_b r_c + r_c r_a = p^2$
   3. $S = \frac{abc}{4R}$
   4. $S^2 = r r_a r_b r_c$
   5. $bc - \frac{a^2 bc}{(b+c)^2} = \left(\frac{2bc\cos\left(\frac{\angle A}{2}\right)}{b+c} \right)^2$

Proof:

1.

$$\begin{aligned}\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} &= \frac{p-a}{S} + \frac{p-b}{S} + \frac{p-c}{S} \\ &= \frac{3p - (a+b+c)}{S} \\ &= \frac{p}{S} \\ &= \frac{1}{r}\end{aligned}$$

2.

$$\begin{aligned}r_a r_b + r_b r_c + r_c r_a &= \frac{S^2}{(p-a)(p-b)} + \frac{S^2}{(p-b)(p-c)} + \frac{S^2}{(p-b)(p-c)} \\ &= S^2 \times \frac{(p-c) + (p-a) + (p-b)}{(p-a)(p-b)(p-c)} \\ &= S^2 \times \frac{p^2}{p(p-a)(p-b)(p-c)} \\ &= p^2\end{aligned}$$

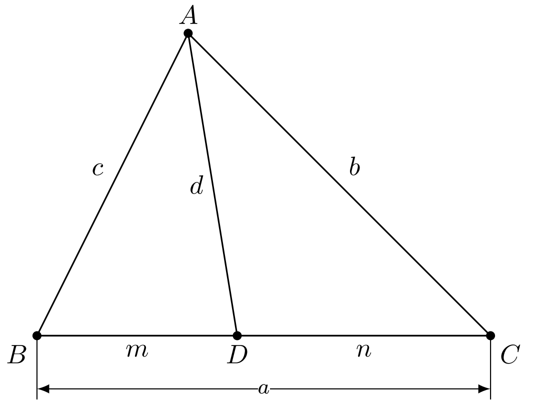
3.

$$\begin{aligned}S &= \frac{1}{2} ab \sin(\angle C) \\ &= \frac{1}{2} ab \frac{c}{2R} \\ &= \frac{abc}{4R}\end{aligned}$$

4.

$$\begin{aligned}r r_a r_b r_c &= \frac{S}{p} \times \frac{S}{p-a} \times \frac{S}{p-b} \times \frac{S}{p-c} \\ &= \frac{S^4}{p(p-a)(p-b)(p-c)} \\ &= S^2\end{aligned}$$

5. We will calculate the length of the angle bisector with two approaches



By Stewart's theorem we have

$$a\big(d^2 + mn\big) = b^2 m + c^2 n$$

Also we know that $m = \frac{ac}{b+c}$ and $n = \frac{ab}{b+c}$. Thus

$$a\left(d^2 + \frac{a^2bc}{(b+c)^2}\right) = \frac{ab^2c + abc^2}{b+c} \implies d^2 + \frac{a^2bc}{(b+c)^2} = bc\left(\frac{c}{b+c} + \frac{b}{b+c}\right) = bc$$

$$\implies d^2 = bc - \frac{a^2bc}{(b+c)^2} \tag{7}$$

Also we have

$$\frac{BD}{BC} = \frac{AD\sin(\angle BAD)}{AC\sin(\angle BAC)} \implies \frac{\frac{ac}{b+c}}{a} = \frac{d\sin\left(\frac{\angle A}{2}\right)}{b\sin(\angle A)} \implies \frac{c}{b+c} = \frac{d\sin\left(\frac{\angle A}{2}\right)}{2b\sin\left(\frac{\angle A}{2}\right)\cos\left(\frac{\angle A}{2}\right)}$$

$$\implies d = \frac{2bc\cos\left(\frac{\angle A}{2}\right)}{b+c} \tag{8}$$

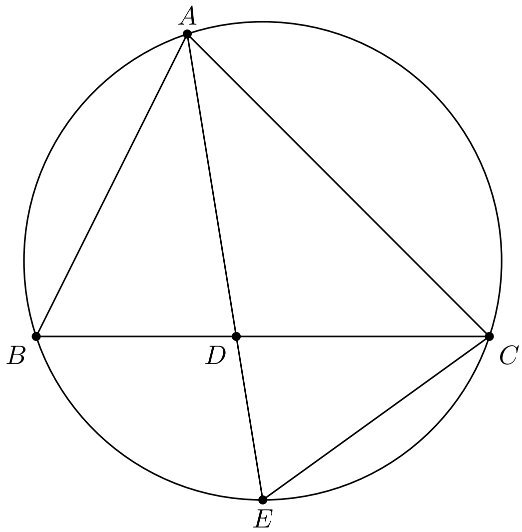
Thus by (7) and (8) we have

$$bc - \frac{a^2bc}{(b+c)^2} = \left(\frac{2bc\cos\left(\frac{\angle A}{2}\right)}{b+c}\right)^2$$

4. Let I be the incenter and I_a be the excenter of $\triangle ABC$. Also suppose AI_a intersects the circumcircle at E and intersects BC at D. Prove that

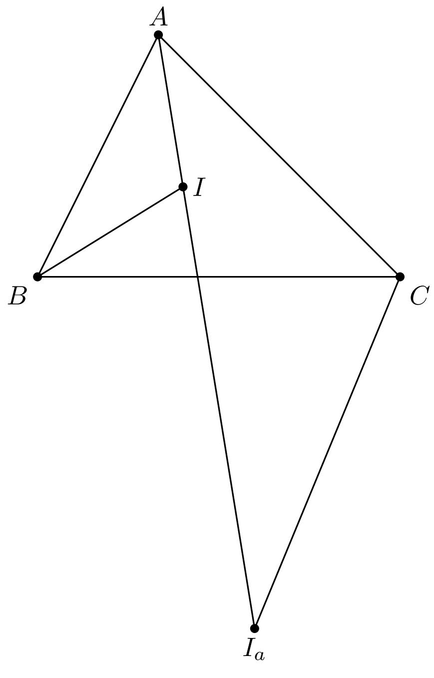
$$AE.AD = AI.AI_a = bc$$

Proof: First we will prove $AD.AE = bc$

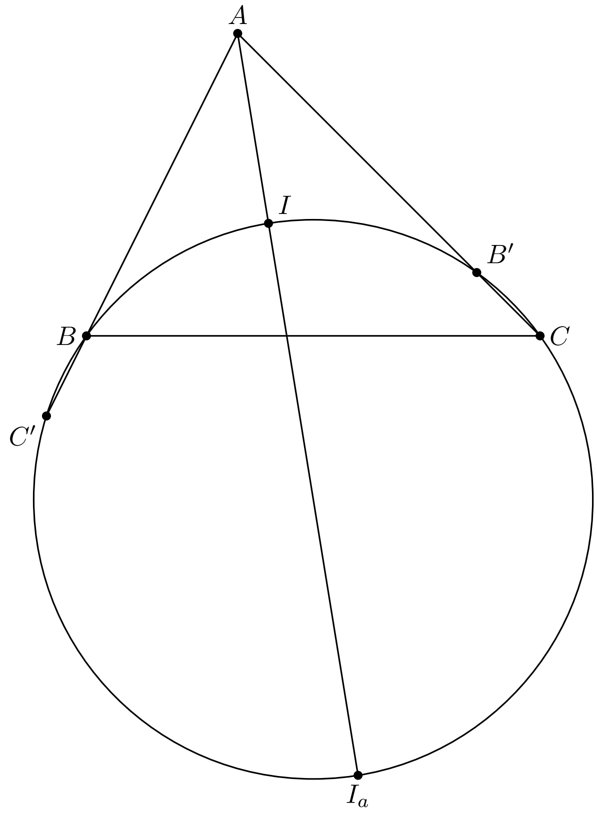


$$\left.\begin{array}{r}\angle BAD = \angle EAC \\ \angle ABC = \angle AEC\end{array}\right\} \implies \triangle ABD \sim \triangle AEC \implies \frac{AB}{AE} = \frac{AD}{AC} \implies AE.AD = bc$$

Now we prove $AI.AI_a = bc$



$$\left.\begin{array}{r}\angle BAI = \angle CAI_a = \frac{\angle A}{2} \\ \angle IBA = \angle CI_aA = \frac{\angle B}{2}\end{array}\right\} \implies \triangle ABI \sim \triangle AI_aC \implies \frac{AB}{AI_a} = \frac{AI}{AC} \implies AI.AI_a = bc$$



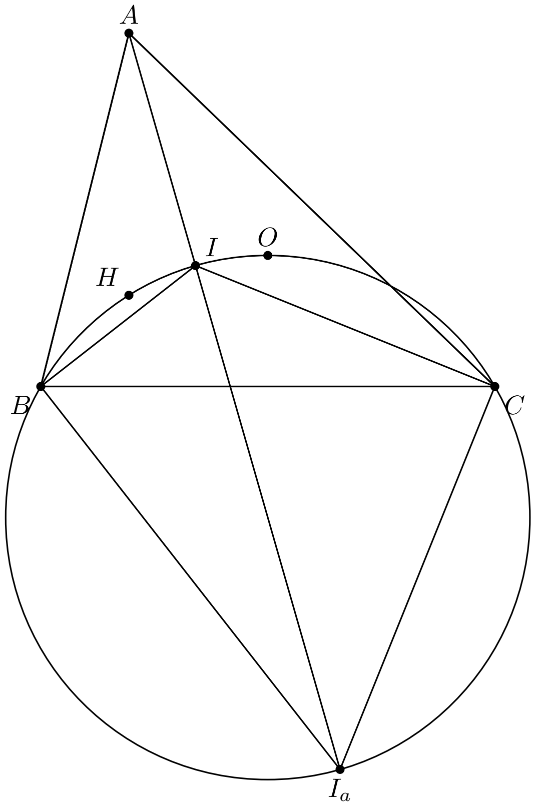
For another proof of $AI.AI_a = bc$, suppose AB and AC intersects the circumcircle of $\triangle BIC$ at C' and B' respectively. Cleary I_a lie on the circle and also II_a is the diameter. Now since AI_a passes through the center and $\angle BAI = \angle CAI$, we can easily see that $AB' = AB$ and $AC' = AC$. Thus

$$AI.AI_a = AB.AC' \xRightarrow{AC'=AC} AI.AI_a = bc$$

B' and C' are two important point in a triangle.

5. Suppose B, C, I, and H lie on a circle. Prove that $\angle A = 60°$.

Proof:

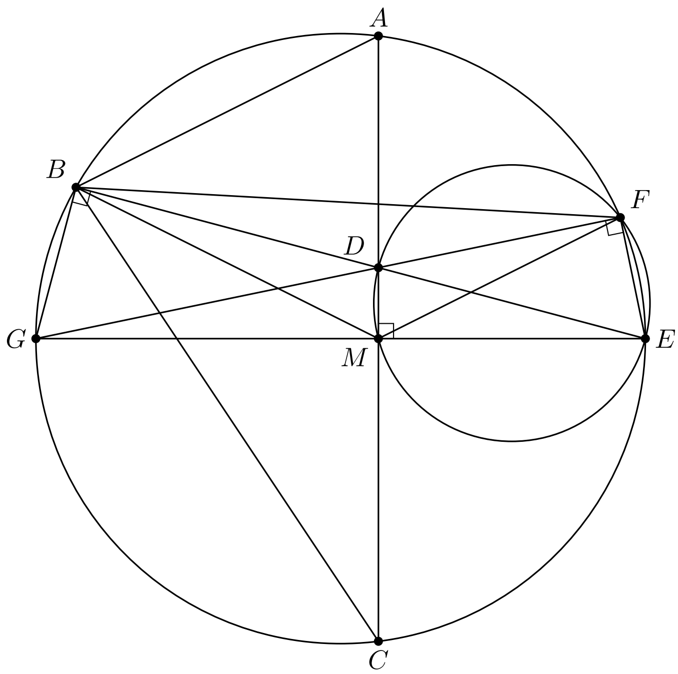


$$\left.\begin{aligned} &\angle BIC = 90° + \tfrac{\angle A}{2} \\ &\angle BHC = 180° - \angle A \\ &\angle BHC = \angle BIC \end{aligned}\right\} \implies 90° + \frac{\angle A}{2} = 180° - \angle A \implies \angle A = 60°$$

Note that O, B', C', I_a and Fermat point also lie on the same circle.

6. Let M be the midpoint of AC in $\triangle ABC$. Suppose the bisector of $\angle ABC$ intersects AC at D and the circumcircle at E. A circle with diameter DE intersects the circumcircle at F. Prove that D is the incenter of $\triangle MBF$.

Proof:



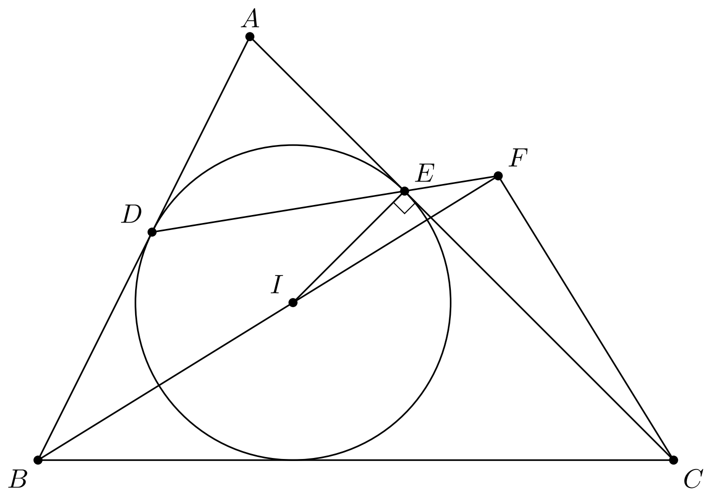
Since BE is the angle bisector, $AE = CE$. Also since M is the midpoint of AC, $AM = AC$. Thus $EM \perp AC$ which implies that $DFEM$ is cyclic. Also EM passes through the center of the circumcircle of $\triangle ABC$, So if the extension of EM intersects the circumcircle of $\triangle ABC$ at G, then GE is the diameter. Also since $\angle DFE = 90°$, the extension of DF passes through G. Also since GE is the diameter, $\angle GBE = 90°$. Which implies that $BDMG$ is cyclic. So we have three useful cyclic quadrilateral, $BFEG$, $BDMG$, $FDME$.

$$\left.\begin{array}{l} BDMG \text{ is cyclic} \implies \angle MBD = \angle MGD \\ BFEG \text{ is cyclic} \implies \angle FGE = \angle FBE \end{array}\right\} \implies \angle MBD = \angle DBF$$

Thus BD is the angle bisector. By the same way FD and MD are also angle bisectors. Thus D is the incenter of $\triangle MBF$

7. Let M and N be the midpoint of BC and AB of $\triangle ABC$ respectively. Also the inscribed circle touches BC and AC at D and E respectively. Prove that MN, DE, and the bisector of $\angle A$ concur.

Lemma 1: Suppose I is the incenter of $\triangle ABC$ and the inscribed circle touches AB and AC at D and E. If AI intersects DE at F, then $BFC = 90°$.

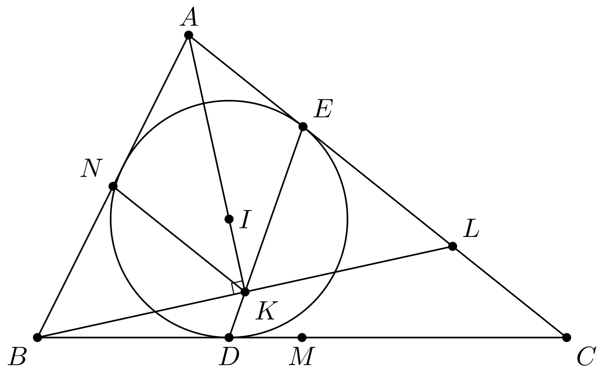


Since AD are AE are tangent line, $\angle ADB = \angle AED = 90° - \frac{\angle A}{2}$.

$$\angle DFB = \angle ADF - \angle DBF = 90° - \frac{\angle A}{2} - \frac{\angle B}{2} = \frac{\angle C}{2}$$

Thus $\angle EFB = \angle ECI$ which implies that $EFCI$ is cyclic, so $\angle IFC = \angle IEC = 90°$.

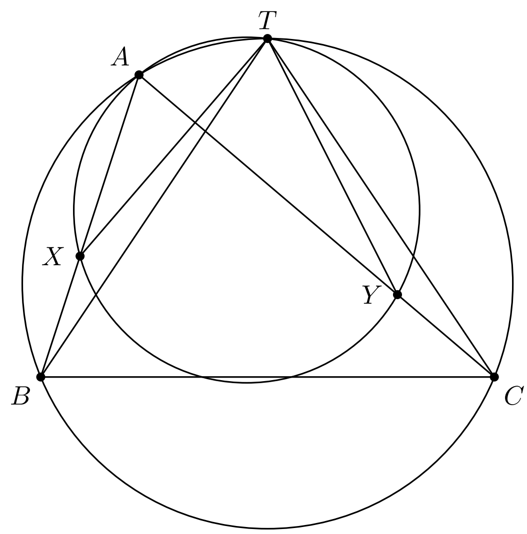
Proof:



Suppose AI intersects DE at K. By *Lemma 1* $\angle AKB = 90°$. Now suppose the extenstion of BK intersects AC at L. In $\triangle ABL$, AK is angle bisector and altitude, thus $\triangle ABL$ is isoscales and AK is also median which implies that K is the midpoint of BL. So $NK \parallel AC$, also $NM \parallel AC$ thus K lies on NM.

8. Choose variant points X and Y on AB and AC in $\triangle ABC$ ($AB \neq AC$), such that $BX = CY$. Prove that the circumcircle $\triangle AXY$ passes through an invariant point (other than A).

Proof:



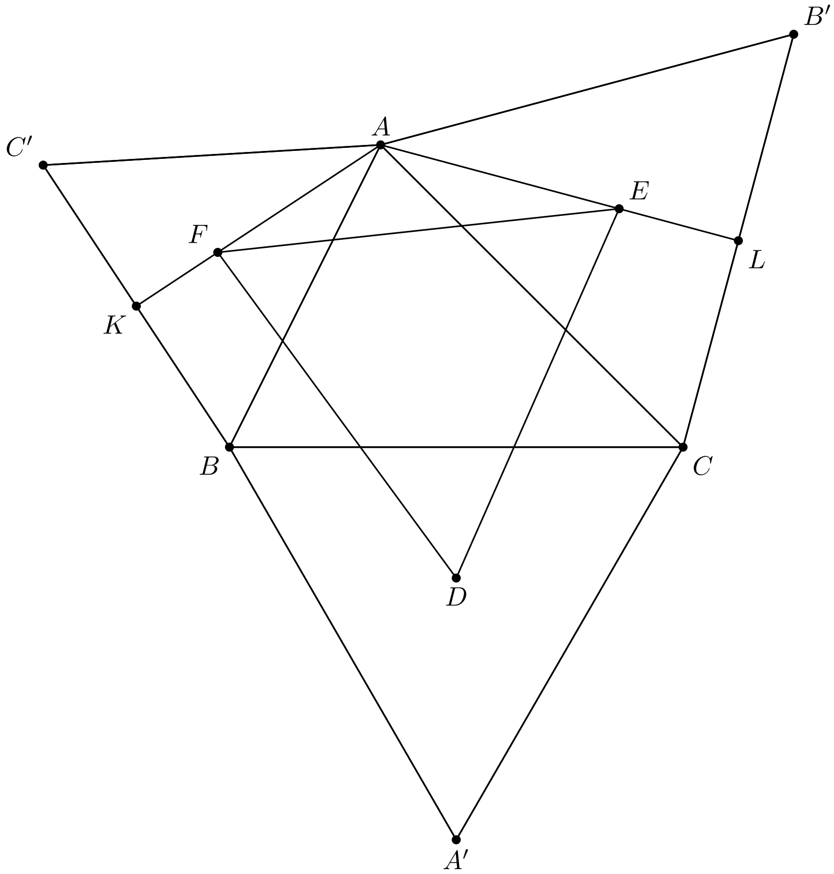
Let T be the other intersection of the circumcircle of $\triangle ABC$ and the circumcircle of $\triangle AXY$. We will prove T is on the perpendicular bisector of BC, thus it's invariant.
Since $BATC$ is cyclic, $\angle ABT = \angle ACT$. Also since $XATY$ is cyclic, $\angle AXT = \angle AYT$ which implies that $\angle TXB = \angle TYC$, thus

$$\left.\begin{aligned} &\angle ABT = \angle ACT \\ &\angle TXB = \angle TYC \\ &BX = CY \end{aligned}\right\} \implies \triangle BXT \cong \triangle CYT \implies BT = CT$$

So T is on the perpendicular bisector of BC.

9. (Napoleon's Theorem) On each side of a triangle, create an equilateral triangle, lying exterior to the original triangle. Then the segments connecting the centroids of the three equilateral triangles themselves form an equilateral triangle.

Proof:



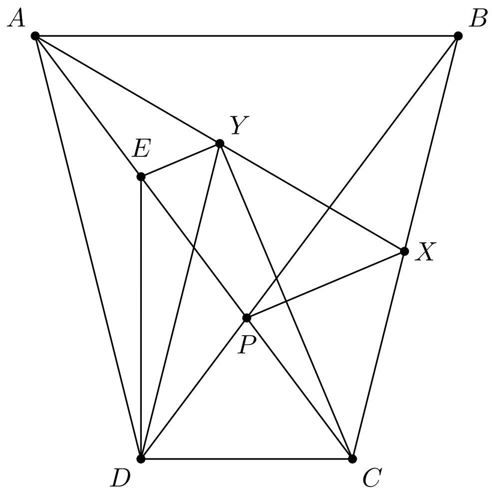
By the law of cosine in $\triangle ADE$ we have

$$\begin{aligned}
DE^2 &= AD^2 + AE^2 - 2AD.AE\cos(\angle DAE)\\
&= \frac{4}{9}AK^2 + \frac{4}{9}AL^2 - \frac{8}{9}AK.AL\cos(\angle A + 60^\circ)\\
&= \frac{1}{3}AB^2 + \frac{1}{3}AC^2 - \frac{2}{3}AB.AC\big(\cos(\angle A)\cos(60^\circ) - \sin(\angle A)\sin(60^\circ)\big)\\
&= \frac{1}{3}c^2 + \frac{1}{3}b^2 - \frac{1}{3}bc\cos(\angle A) + \frac{\sqrt{3}}{3}bc\sin(\angle A)\\
&= \frac{1}{6}c^2 + \frac{1}{6}b^2 + \left(\frac{1}{6}c^2 + \frac{1}{6}b^2 - \frac{1}{3}bc\cos(\angle A)\right) + \frac{2\sqrt{3}}{3}S\\
&= \frac{1}{6}c^2 + \frac{1}{6}b^2 + \frac{1}{6}\left(c^2 + b^2 - 2bc\cos(\angle A)\right) + \frac{2\sqrt{3}}{3}S\\
&= \frac{1}{6}\left(a^2 + b^2 + c^2\right) + \frac{2\sqrt{3}}{3}S
\end{aligned}$$

The expression is symmetric thus $DE = EF = FD$.
Further information about Fermat point and Napoleon's Theorem

10. Let P be the intersection of AC and BD in the isosceles trapezoid $ABCD(AB \parallel CD, BC = AD, AB > DC)$. The circumcircle of APB intersects BC for the second time at X. Point Y lies on AX such that $DY \parallel BC$. Prove that $\angle YDA = \angle 2YCA$.

Proof:



Suppose DE is the angle bisector of $\angle ADY$.

$$\left.\begin{array}{l} ABCD \text{ is cyclic} \implies \angle DAC = \angle DBC \\ ABXP \text{ is cyclic} \implies \angle PAX = \angle DBC \end{array}\right\} \implies \angle DAE = \angle YAE$$

So AE is the angle bisector of $\angle DAY$ and E is the incenter of $\triangle DAY$. Thus YE is the angle bisector of $\angle AYD$.

$$\left.\begin{array}{l} ABXP \text{ is cyclic} \implies \angle AXP = \angle ABP \\ ABXP \text{ is cyclic} \implies \angle PXC = \angle PAB \\ ABCD \text{ is isosceles} \implies \angle PAB = \angle ABP \end{array}\right\} \implies \angle AXP = \angle PXC$$

So XP is the angle bisector of $\angle AXC$.

$$\left.\begin{array}{l} YD \parallel XC \implies \angle AYD = \angle AXC \\ \angle AYE = \angle AYD = \frac{1}{2}\angle AYD \\ \angle AXP = \angle PXC = \frac{1}{2}\angle AXC \\ \angle APX = \angle ABP = \angle ACD \end{array}\right\} \implies \angle EYD = \angle ECD$$

Thus $EYCD$ is cyclic and $\angle YCA = \angle YDE = \frac{1}{2}\angle YDA$.