Session 2

Aria Afrooz

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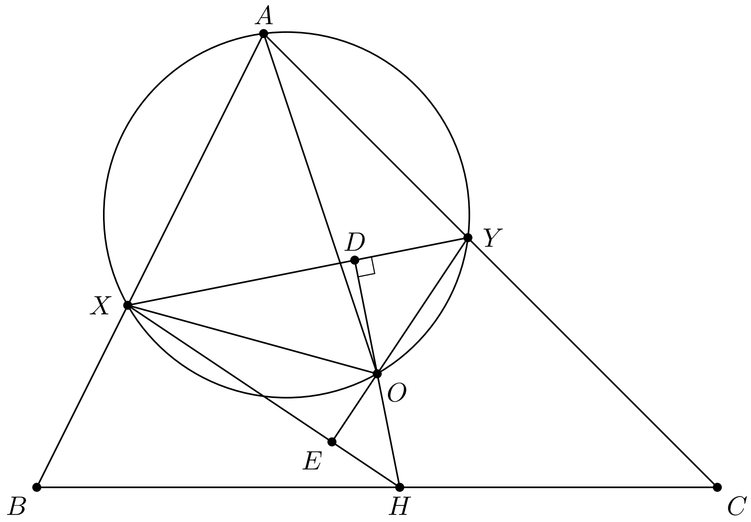
1. Let h_a, h_b, h_c be the length of altitude from A, B, C respectively and r be the radius of incircle of $\triangle ABC$. Prove that $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$.

$$\left.\begin{array}{l} ah_a = bh_b = ch_c = 2S \\ r = \frac{S}{p} \end{array}\right\} \implies \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{a+b+c}{2S} = \frac{2p}{2S} = \frac{1}{r}$$

2. Let O be the circumcenter of $\triangle ABC$. A circle passing through A, O intersects AB and AC in X and Y respectively. Prove that the orthocenter of OXY lie on BC.

Proof:



Suppose OD is the altitude of $\triangle OXY$ and it intersects BC at H. Also suppose OY intersects XH at E, We will show $\angle OEX = 90°$ which means H is the orthocenter.

$$AXOY \text{ is cyclic} \implies \begin{cases} \angle XOE = \angle A \\ \angle OXY = \angle OAY = 90° - \angle B \end{cases}$$

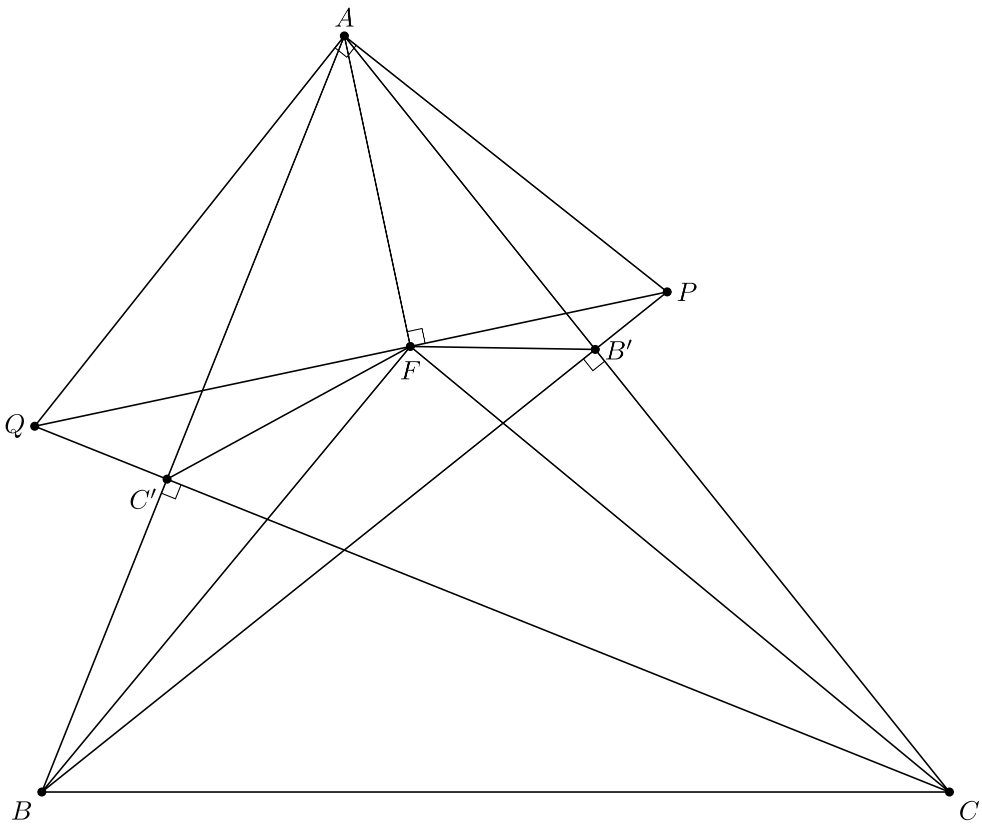
$$\left.\begin{aligned} &\angle OXD = 90° - \angle B \\ &\angle XDO = 90° \end{aligned}\right\} \implies \angle XOD = \angle B$$

$$\implies XOHB \text{ is cyclic} \implies \angle HXO = 90° - \angle A$$

$$\left.\begin{aligned} &\angle XOE = \angle A \\ &\angle HXO = 90° - \angle A \end{aligned}\right\} \implies \angle OEX = 90°$$

3. Let BB' and CC' be the altitudes form B and C of $\triangle ABC$. Suppose P and Q are two points lie in the extention of BB' and CC' respectively such that $\angle PAQ = 90°$. If F be the foot of altitue from A of $\triangle QAP$, prove that $\angle BFC = 90°$.

Proof:



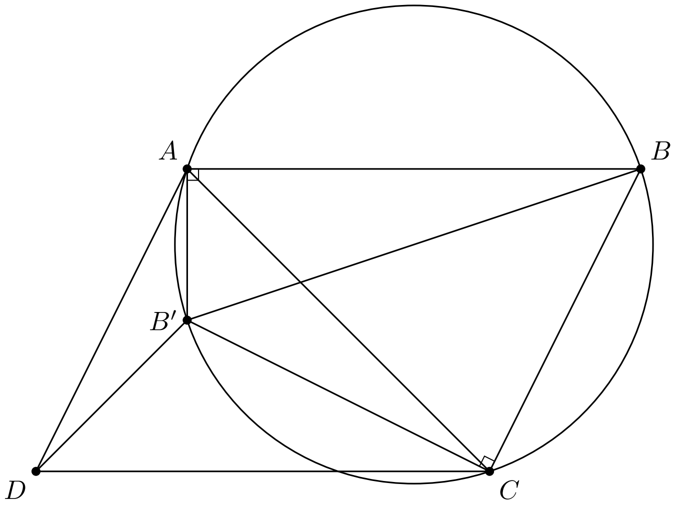
$$\angle AFP = \angle AB'P = 90° \implies AFB'P \text{ is cyclic} \implies \angle FAP = \angle BB'F \quad (1)$$

$$\angle AFQ = \angle AC'Q = 90° \implies AFC'Q \text{ is cyclc} \implies \angle FC'Q = \angle FQA \quad (2)$$

$$\left.\begin{array}{l} \angle FQA + \angle FAQ = 90° \\ \angle FPA + \angle FAP = 90° \\ \angle FAQ + \angle FAP = 90° \end{array}\right\} \implies \angle FQA + \angle FPA = 90° \quad (3)$$

4. Let $ABCD$ be a parallelogram. Suppose Γ is the circumcircle of $\triangle ABC$ and BB' is a diameter of Γ. Prove that $DB' \perp AC$.

Proof:



$$\left.\begin{array}{l} B'A \perp AB \xRightarrow{AB \| CD} B'A \perp CD \\ B'C \perp BC \xRightarrow{BC \| AD} B'C \perp AD \end{array}\right\} \implies B' \text{ is the orthocenter of } \triangle ACD$$

$$\implies B'D \perp AC$$

Proof 2: In order to show $B'D \perp AC$, we prove

$$B'A^2 + DC^2 = B'C^2 + DA^2$$

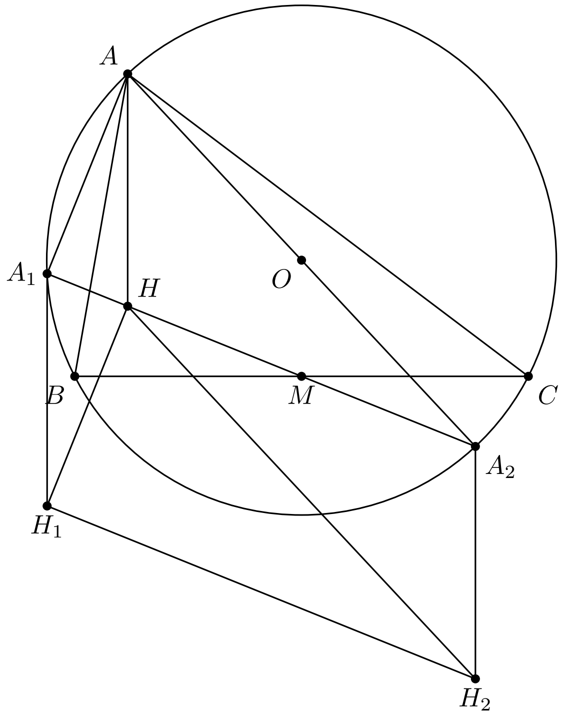
Since $AD = BC]$ and $AB = CD$, we have

$$\left.\begin{array}{l} B'A^2 + DC^2 = B'A^2 + AB^2 = BB'^2 \\ B'C^2 + DA^2 = B'C^2 + BC^2 = BB'^2 \end{array}\right\} \implies B'A^2 + DC^2 = B'C^2 + DA^2$$

$$\implies B'D \perp AC$$

5. Let H be the orthocenter of $\triangle ABC$ and M be the midpoint of BC. The extenstion of HM intersects the cicumcircle of $\triangle ABC$ at A_1 and A_2. Prove that the orthocenter of $\triangle ABC$, $\triangle A_1BC$ and $\triangle A_2BC$ form a right triangle.

Proof:



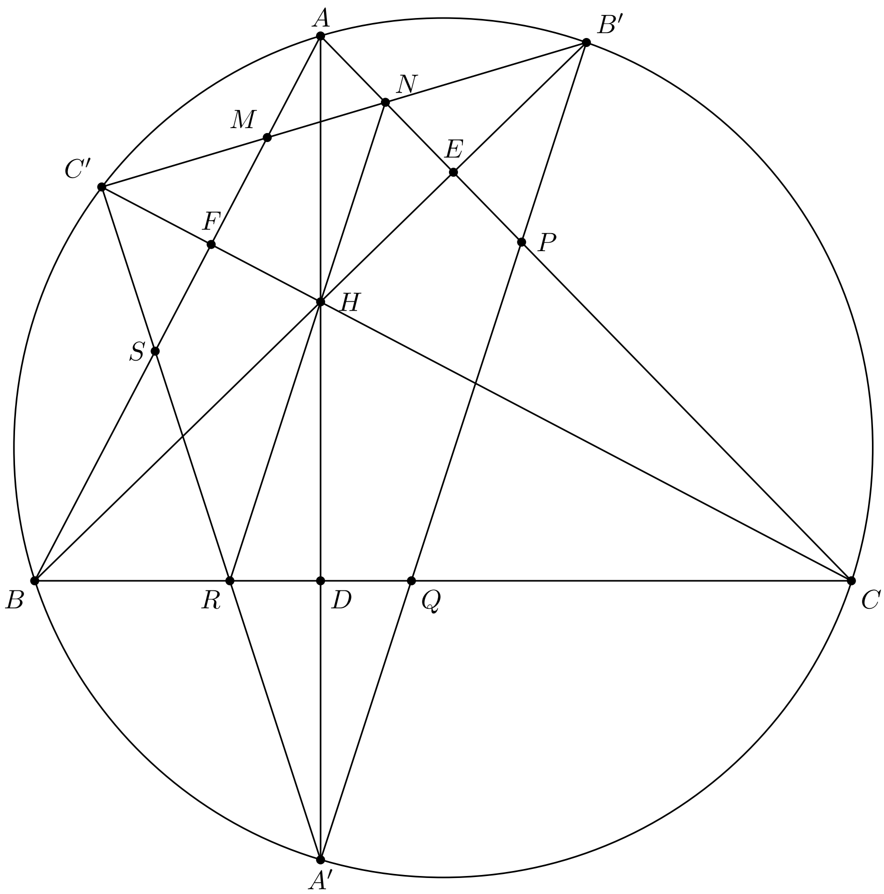
Last session we prove that AA_1H_1H and AA_2H_2H are parallelogram thus $HH_1 = AA_1$ and $HH_2 = AA_2$. Also we can easily show that $\angle A_1AA_2 = \angle H_1HH_2$ thus

$$\left.\begin{array}{l} HH_1 = AA_1 \\ HH_2 = AA_2 \\ \angle A_1AA_2 = \angle H_1HH_2 \end{array}\right\} \implies \triangle A_1AA_2 \cong \triangle H_1HH_2$$

$$\implies \angle HH_1H_2 = \angle AA_1A_2 = 90^\circ$$

6. The altitudes of $\triangle ABC$ intersects the circumcircle of $\triangle ABC$ in A', B', C' respectively. Also the sides of $\triangle ABC$ and $\triangle A'B'C'$ intersects each other at M, N, P, Q, R and S respectively. Prove that MQ, NR and PS intersects each other in the orthocenter of $\triangle ABC$.

Proof:



We will show R, H, and N lie on a line, then by the same way we can see that MQ, NR, and PS intersects each other at H.

$$\left.\begin{aligned} &\angle RHD = \angle RA'H \\ &\angle RA'H = \angle HCF = \frac{\overset{\frown}{C'A}}{2} \end{aligned}\right\} \implies \angle RHD = \angle HCF$$

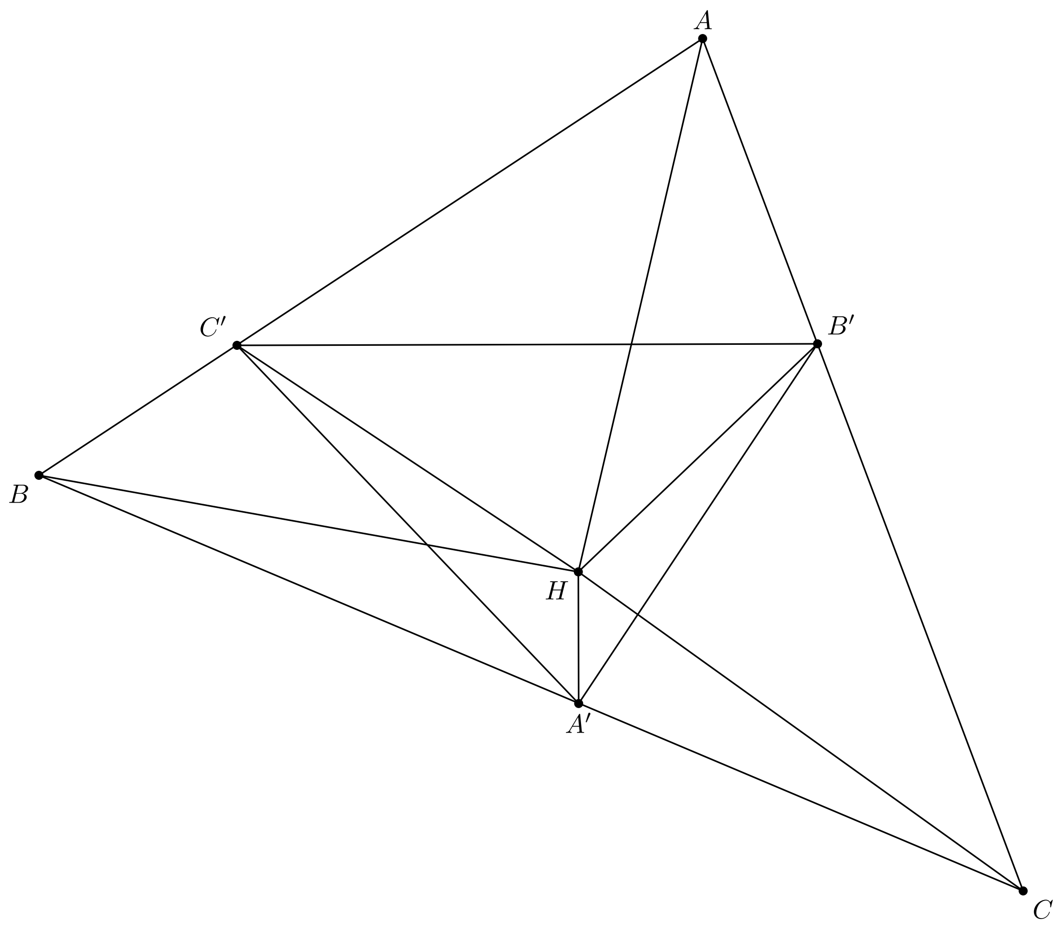
By the same way we have $\angle NHF = \angle HCD$. Also $DHFC$ is cyclic thus $\angle DHF = 180^\circ - \angle DCF$

$$\left.\begin{aligned} &\angle RHD = \angle HCF \\ &\angle NHF = \angle HCD \\ &\angle DCF = \angle HCF + \angle HCD \\ &\angle DHF = 180^\circ - \angle DCF \end{aligned}\right\} \implies \angle RHD + \angle DHF + \angle FHN = 180^\circ$$

Thus R, H, and N lie on a line.

7. Let A', B' and C' be points which lie on BC, CA and AB of $\triangle ABC$ such that $ABC \sim A'B'C'$. Prove that the orthocenter of $\triangle A'B'C'$ is the cicumcenter of ABC.

Proof:



Suppose H is the orthocenter of $\triangle A'B'C'$, we will show H is also the circumcenter of $\triangle ABC$.

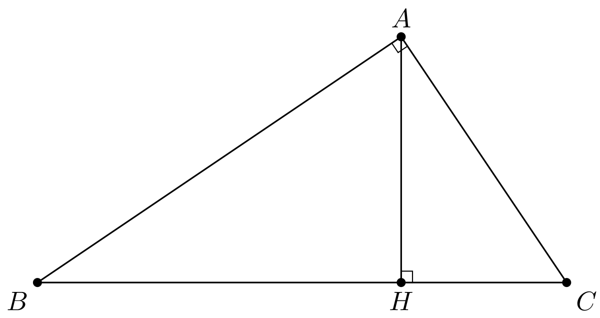
$$\angle B'HC' = 180^\circ - \angle B'A'C' = 180^\circ = \angle A \implies AB'HC' \text{ is cyclic}$$

$$\implies \begin{cases} \angle HC'B' = \angle HAB' = 90^\circ - \angle C'B'A' = 90^\circ - \angle B' \\ \angle HB'C' = \angle HAC' = 90^\circ - \angle B'C'A' = 90^\circ - \angle C \end{cases}$$

By the same way we have $\angle HBA = 90^\circ - \angle C$ and $HCA = 90^\circ - \angle B$. So $\triangle HAC$ and $\triangle HAB$ are both isosceles, thus $HA = HB = HC$ which means H is the cicumcircle of $\triangle ABC$.

8. Let BD and CE be the altitudes of $\triangle ABC$. Suppose Γ_1 is a semicircle outside of $\triangle ABC$ with the diameter AC. Also Γ_2 is a semicircle outside of $\triangle ABC$ with the diameter AB. BD intersects Γ_1 in B' and CE intersects Γ_2 in C'. Prove that $AB' = AC'$.

Lemma : Suppose $\triangle ABC$ is a right triangle ($\angle A = 90°$), and AH is the altitude. Then we have $AB^2 = BH.BC$ and $AC^2 = CH.CB$.
Proof:

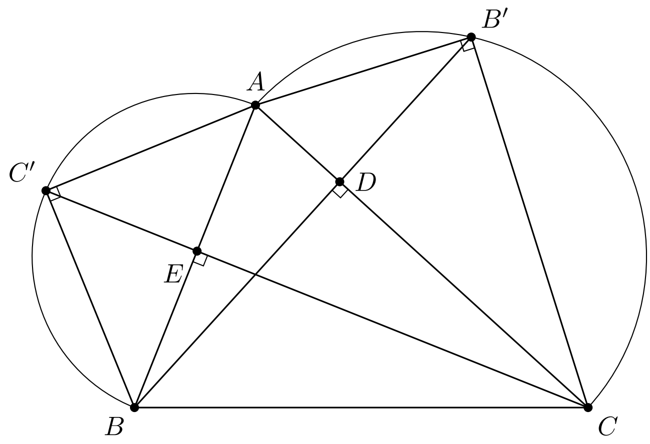


$$\left.\begin{aligned} \angle ABH &= \angle ABC \\ \angle AHB &= \angle BAC = 90° \end{aligned}\right\} \Longrightarrow \triangle ABH \sim \triangle BCA$$

$$\Longrightarrow \frac{AB}{BC} = \frac{BH}{CB} \Longrightarrow AB^2 = BH.BC$$

By the same way $\triangle CHA \sim \triangle CAB$ So
$$AC^2 = CH.CB$$

Proof of the problem:



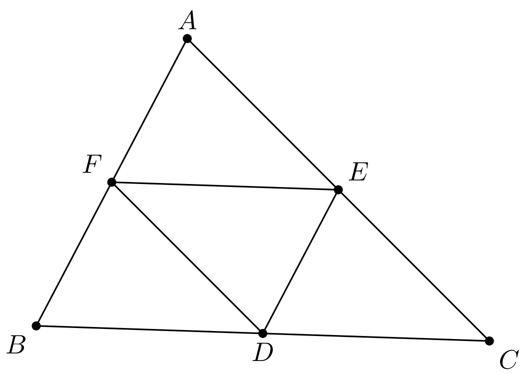
Since AB and AC are diameters, $\angle BC'A = \angle AB'C = 90°$. Also $C'E$ and $B'D$ are the altitudes, thus by the Lemma we have

$$AC'^2 = AE.AB, AB'^2 = AD.AC$$

Now since $\angle BEC = \angle BDC = 90°$, $BEDC$ is cyclic and $AE.AB = AD.AC$. Thus $AB' = AC'$.

9. Suppose D, E, and F lies on BC, CA, and AB of $\triangle ABC$ respectively. Prove that if $\triangle ABC \sim \triangle AFE \sim \triangle EDC \sim \triangle FBD \triangle DEF$, then AD, BE, and CF are median.

Proof:



It's easy to see $\{\angle AFE, \angle EFD, \angle DFB\} = \{\angle A, \angle B, \angle C\}$. Same for E and F. If $\angle AEF = \angle C$, then clearly D, E, and F are midpoints. So suppose $\angle AEF \neq \angle C$ thus $\angle AEF = \angle B$. By a little bit of calculation we can see that $DE \parallel AB$ and $DF \parallel AC$ but $EF \nparallel BC$ which is impossible. Thus D, E, and F are midpoints.