The Helstrom Bound

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Quantum state discrimination between two wave functions on a ring is considered. The optimal minimum-error probability is known to be given by the Helstrom bound. A new strategy is introduced by inserting instantaneously two impenetrable barriers dividing the ring into two chambers. In the process, the candidate wave functions, as the insertion points become nodes, get entangled with the barriers and can, if judiciously chosen, be distinguished with smaller error probability. As a consequence, the Helstrom bound under idealised conditions can be violated.

I. INTRODUCTION

Experimental design and data analysis are common challenges in science, and particular acute in quantum mechanics. In the literature different approaches are discussed. Here the Bayes procedure for state discrimination with the aim to minimise the expected cost is employed. The existence of a prior associated with the states to be distinguished is assumed.

Two disparate concepts are combined in the paper. These are quantum state discrimination and the modification of the quantum potential resulting in a transformation of the wave function. Bayesian hypothesis testing, the particular form of quantum state discrimination investigated here, was developed by Helstrom and others [1-3]. For a recent paper on the topic of state discrimination between two possible states with given prior and transition probability see Brody et al. [4]. It is generally accepted, but will be challenged in the paper, that the optimal Bayes cost in the binary case, is given by the Helstrom bound, which only depends on the prior and the transition probability between the states and can be written in a simple closed form. The second concept is the ability to modify wave functions in a beneficial way, if one inserts impenetrable barriers corresponding to a potential spike in a simple configuration space, here chosen to be a ring. In earlier papers [5, 6] these two ideas were applied to study the insertion of one barrier into a ring or a one-dimensional infinite square well both adiabatically and instantaneously. The current approach is simpler by exploiting the existing nodes of the candidate wave functions and by focusing on the link between the energy required to insert the barrier and the candidate wave functions.

Next, a description of the setup and the procedure to be analysed. The following decision problem is presented. With equal probability, prior of 1/2, one of two quantum states is put into the configuration space - a ring. Our challenge is to determine with the smallest possible error, which state has been selected. Two strategies for calculating the Bayes cost are proposed. In the first strategy, the combination of prior and transition probability be-

tween the two quantum states alone is sufficient to calculate the conventional optimal minimum error probability, i.e. the Helstrom bound, prior to the insertion of any barrier. The standard procedure, reliant on the optimal POVM as described in the book by Helstrom[1], results in the following binary decision cost

$$\frac{1}{2} - \frac{1}{2}\sqrt{1 - \cos^2(\alpha)},$$

where $\cos^2(\alpha)$ is the transition probability between the two candidate states, and cost 1 is assigned to an incorrect and cost 0 for a correct decision.

The second strategy for calculating the error probability is novel. The wave function is first modified by the simultaneous insertion of two barriers breaking the symmetry of the configuration space. The insertion of the barrier can be carried out with different speeds. We consider an extreme case: instantaneous insertion. This insertion can require energy and can modify the wave function, since its amplitude at the impenetrable barrier location will be zero and the expansion in the new bases to reproduce the original amplitude, except at the insertion points, is accompanied with a change of energy. The modified candidate wave function, now entangled with the barriers, is probed and the new binary decision cost estimated. It is shown that the extended wave functions incorporating the barriers can be orthogonal, even if the original overlap was non-zero.

Two motivations for this research stand out. On the one hand it sheds some light on foundational issues in quantum measurement theory, and on the other hand a plethora of problems in quantum information theory depend on optimal state discrimination.

Quantum mechanics on the ring is specified by the Hamiltonian and boundary conditions. The Hamiltonian of a particle trapped on a ring of radius one is

$$H = -\frac{\hbar^2}{2M} \frac{d^2}{d\theta^2}$$

with energy eigenvalues $E_n = \frac{\hbar^2}{2M} n^2$ for $n \in \mathbb{N}$. The wave function is defined for $\theta \in [0, 2\pi]$.

The structure of the rest of paper is as follows. In section II the impact of an instantaneous insertion of a barrier on a ring is studied. In section III the simultaneous insertion of two barriers is considered. The binary

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choice problem between two quantum states is tackled at the end of the section. In the conclusion the result is briefly reviewed and some general comments added.

II. INSTANTANEOUS INSERTION OF A BARRIER

In this section the barrier insertion, considered to be instantaneous, at both nodal and non-nodal points is reviewed. The nodal point insertion is dealt with first. This is easier, since the particle wave function and energy is left unchanged - for background material see section II of Bender $et\ al.\ [7]$, where a series of results for a particle in a one dimensional box, directly applicable to quantum mechanics on a ring, were established. As an aside, we only call a point a node, or more correctly a 'fixed node', if the amplitudes at this point stays zero at all times. Wave functions that are superposition of eigenfunctions of H can have zero amplitude points that change with time. These we do not consider, since an insertion at a 'transitory node' can require energy.

The situation is more intricate for an insertion at a non-nodal point. Energy is needed to modify the wave function. In the idealised setting considered here the required energy is infinite. The energy localised in the barrier point inserted at t=0 propagates through the system at t>0 and increases the energy on the ring. The result is a fractal wave function. The details of the calculation can be found in sections IV & VI of Bender et al. [7] or in section II of [5] & [6].

III. THE INSTANTANEOUS INSERTION OF TWO IMPENETRABLE BARRIERS ON A RING

The case of two simultaneous insertions changing the ring into two separate infinite square wells is considered in this section. In the following paragraphs the cost is evaluated before and after the insertion of two barriers for distinguishing the following two candidate wave functions defined, at t=0, as

$$\phi(\theta) := \frac{1}{\sqrt{\pi}} \sin(\theta),$$

$$\psi(\theta) := \frac{1}{\sqrt{\pi}} \sin(\theta - \alpha),$$

where $\theta \in [0, 2\pi]$, and $\alpha \in (0, \pi/2)$. The initial overlap of the wave functions is

$$|\langle \phi | \psi \rangle|^2 = \frac{1}{\pi^2} \left| \int_0^{2\pi} d\theta \sin(\theta) \sin(\theta + \alpha) \right|^2 = \cos^2(\alpha),$$

and the standard Helstrom cost is given by

$$\frac{1}{2} - \frac{1}{2}\sqrt{1 - \cos^2(\alpha)}.$$

Due to the pre-insertion symmetry both candidate wave functions are eigenfunctions of the Hamiltonian of a particle on a ring. The symmetry is only broken by the barriers. The barriers are inserted at the point 0 and α at time t=0. The barrier inserted at point 0 leaves

 ϕ unchanged, but the second barrier at point α hits a non-nodal point. The situation is the reverse for ψ , since it lacks a node at α , but has a node at 0. As explained before, at nodal points no energy transfer occurs, but a barrier at a non-nodal point is associated with a change of energy.

An instantaneous insertion requires, due to the change of the configuration space, an expansion of the original wave functions into the energy basis of the two separate one dimensional infinite wells. This will be carried out next. The first expansion is in the interval $(0,\alpha)$ with the discrete energy levels $E_n^{\alpha} = \frac{n^2\pi^2\hbar^2}{2M\alpha^2}$ and the second expansion is for the interval $(\alpha,2\pi)$ with the discrete energy levels $E_n^{2\pi-\alpha} = \frac{n^2\pi^2\hbar^2}{2M(2\pi-\alpha)^2}$ such that the first candidate wave function has directly after the insertion the following form

$$\phi_{after}(\theta) := \begin{cases} \sqrt{\frac{1}{\pi}} \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi\theta}{\alpha}\right) & 0 < \theta < \alpha, \\ \sqrt{\frac{1}{\pi}} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi\theta}{2\pi - \alpha}\right) & \alpha < \theta < 2\pi, \end{cases}$$

where

$$a_n := \frac{1}{\pi} \int_0^{\alpha} d\theta \sin(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right)$$
$$= (-1)^n \frac{\alpha n}{\alpha^2 - \pi^2 n^2} \sin(\alpha),$$

and

$$b_n := \frac{1}{\pi} \int_a^{2\pi} d\theta \sin(\theta) \sin\left(\frac{n\pi\theta}{2\pi - \alpha}\right)$$
$$= -\frac{(2\pi - \alpha)n}{(\alpha - (n+2)\pi)(\alpha + (n-2)\pi)} \sin(a).$$

Similarly, the transition probability of ϕ to the combination of the eigenfunctions of energy E_n^{α} and $E_m^{2\pi-\alpha}$ is $|a_nb_m|^2$, and the energy transfer from the barrier inserted at the non-nodal point α is

$$\Delta E_{nm}^{\phi} := \frac{\pi^2 \hbar^2}{2M} \left(\frac{n^2}{\alpha^2} + \frac{m^2}{(2\pi - \alpha)^2} - \frac{1}{4\pi^2} \right).$$

By an appropriate choice of α , e.g. $\alpha = \pi/4$, the energy change ΔE_{nm}^{ϕ} is always positive. Similarly, the second candidate wave function can be expanded into

$$\psi_{after}(\theta) := \begin{cases} \sqrt{\frac{1}{\pi}} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi\theta}{\alpha}\right) & 0 < \theta < \alpha, \\ \sqrt{\frac{1}{\pi}} \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi\theta}{2\pi - \alpha}\right) & \alpha < \theta < 2\pi, \end{cases}$$

where

$$c_n := \frac{1}{\pi} \int_0^{\alpha} d\theta \sin(\theta - \alpha) \sin\left(\frac{n\pi\theta}{\alpha}\right)$$
$$= \frac{\alpha n}{\alpha^2 - \pi^2 n^2} \sin(\alpha)$$

and

$$d_n := \frac{1}{\pi} \int_{\alpha}^{2\pi} d\theta \sin(\theta - \alpha) \sin\left(\frac{n\pi\theta}{2\pi - \alpha}\right)$$
$$= (-1)^n \frac{(2\pi - \alpha)n}{(\alpha - (n+2)\pi)(\alpha + (n-2)\pi)} \sin(\alpha).$$

The transition probability of ψ to the combination of the eigenfunctions of E_n^a and $E_m^{2\pi-a}$ is $|c_n d_m|^2$, and the energy transfer from the barrier inserted at the non-nodal point 0 is again ΔE_{nm}^{ψ} .

Different elements in the expansion are associated to different energies. Due to energy conservation there has to be a source for this change of energy, which will be always positive for a value of α of $\pi/4$. This additional energy can only come from the barrier inserted at the non-nodal point. A laser beam could be a possible realisation of the barrier. The photons of the laser beam, or any other realisation of the barrier, have an energy dependent entanglement with the expanded wave function on the ring, where each combination of energy levels in the two chambers is associated with a complementary state for the barriers to achieve energy conservation. The energy transfer is to ϕ from the barrier at α and to ψ from the barrier at 0, since each candidate wave function has its energy only modified through one specific barrier corresponding to the initial non-nodal point.

The extended wave function including the barrier can be written before the insertion as either

$$\Phi_{before} = \psi \otimes \omega_0^{before}(0) \otimes \omega_0^{before}(\alpha),$$

or

$$\Psi_{before} = \phi \otimes \omega_0^{before}(0) \otimes \omega_0^{before}(\alpha),$$

where $\omega_0^{before}(0)$ and $\omega_0^{before}(\alpha)$ correspond to the wave functions associated with the pre-insertion barriers at the points 0 and α respectively. Directly after the insertion the extended wave functions are transformed into

$$\Phi_{after} = \sum_{n,m=1}^{\infty} a_n b_m \sin\left(\frac{n\pi x}{\alpha}\right)$$
$$\otimes \sin\left(\frac{m\pi y}{2\pi - \alpha}\right) \otimes \omega_0^{after}(0) \otimes \omega_{n,m}^{after}(\alpha)$$

and

$$\Psi_{after} = \sum_{n,m=1}^{\infty} c_n d_m \sin\left(\frac{n\pi x}{\alpha}\right)$$
$$\otimes \sin\left(\frac{m\pi y}{2\pi - \alpha}\right) \otimes \omega_{n,m}^{after}(0) \otimes \omega_0^{after}(\alpha)$$

where $x \in (0, \alpha)$ and $y \in (\alpha, 2\pi)$, and $\omega_0^{after}(0)$ & $\omega_0^{after}(\alpha)$ are the barrier wave functions, if inserted at either a nodal point at 0 or α . $\omega_{n,m}^{after}(\alpha)$ and $\omega_{n,m}^{after}(0)$ correspond to barriers that transferred ΔE_{nm} to the candidate wave functions Φ_{after} and Ψ_{after} respectively. The transition probability for α taken to be $\pi/4$ is therefore altered during the insertion processes and has the form

$$\begin{split} &|\langle \Phi_{after} | \Psi_{after} \rangle|^2 \\ &= |\sum_{n,m=1}^{\infty} a_n c_n b_m d_m \langle \omega_0^{after}(0) | \omega_{nm}^{after}(0) \rangle \\ &\langle \omega_{nm}^{after}(\pi/4) | \omega_0^{after}(\pi/4) \rangle|^2 = 0, \end{split}$$

because both $\langle \omega_0^{after}(0)|\omega_{nm}^{after}(0)\rangle$ and $\langle \omega_{nm}^{after}(\pi/4)|\omega_0^{after}(\pi/4)\rangle$ are zero for all n and m. It follows from the association of the different barrier states, e.g. photon states representing a laser beam, with different energies.

After the insertion, the entanglement of the two candidate wave functions with the respective barriers results in two states, which are orthogonal. As a consequence, as an improvement from the conventional Helstrom bound, the associated binary decision cost reduces to zero. In the upcoming conclusion the result is reviewed and some comments added.

IV. CONCLUSION

The aim of the paper was to show that the Helstrom bound in the binary quantum discrimination setting can be breached. Inserting a barrier instantaneously in a ring at a non-nodal point always requires energy, whereas an insertion at a nodal point leaves the energy unchanged. If two barriers are inserted, one at a node and one at a nonnodal point, then there is only energy-level dependent entanglement of the wave function on the ring with the barrier at the non-nodal point. Furthermore, by having the node of one candidate wave function to be the nonnodal point of the other, one can ensure that the energy transfer in the two cases is to barriers at distinct points. This can be used to help distinguish between different states. An extension beyond an instantaneous insertion should be possible, since any non-adiabatic insertion at a non-nodal point needs energy. Nevertheless, a careful analysis of the finite speed case is required. The exact cost reduction depends on how precisely one can place the barrier, the width of the barrier, the speed of insertion, i.e. three issues related to time evolution of the potential associated with the barrier. If each of these points can be addressed satisfactorily, then not only in the idealised situation can one obtain an improvement of the optimal cost beyond the Helstrom bound.

Naturally, one can criticise the failure to provide a realistic time evolution; only the infinitely fast insertion was examined. In defence, one can point to the idealised nature of the proposal and the statement that more realistic examples can be viewed as an extrapolation of the procedure under consideration.

The following Gedanken experiment might be instructive, because it shows that it is possible to construct an example were no information is transferred to the experimenter, when the potential representing the barrier is altered. Imagine an experimenter doing a fixed amount of work per unit of time to insert the barrier, i.e. pushes in the barrier with constant power. Dependent on the test wave function the change of the potential is either larger or smaller. The potential takes on different shapes and affect the states in different ways, ergo can change the distance between the states, without direct, energy based, leakage of information to the outside.

Questions about the possibility of superluminal communication can be raised, but one should keep in mind that the Schrödinger equation, a diffusion equation without a propagation speed limit, has its limitations.

Almost without fail quantum algorithms can be viewed as procedures to distinguish between different states. The method described above works for states with arbitrary overlap and should find application in the area of quantum algorithm.

Experimental implementation, for example in the area of Bose-Einstein condensate or ion traps, with a laser beam as a barrier, should be of interest.

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Appendix A: An Example involving an intermediate measurement

In the appendix, the zero-one cost is determined for a discrimnation strategy that employs as a first step a POVM to update the input state. The probability weighted output of this process is then probed using the conventional Helstrom method. The combined new error probability is calculated. It can break the straightforward Helstrom bound, which only depends on the prior and the transition probability, if the candidate states and the components of the intermediate POVM are judiciously chosen. The measurement induced updating of the density matrix involves a normalization[1] and gives the process a non-linear twist.

Notation is next introduced conducive to the problem at hand. The candidate states are $|\psi\rangle = \sin\alpha|0\rangle + \cos\alpha|1\rangle$ and $|\phi\rangle = \sin\beta|0\rangle + \cos\beta|1\rangle$, more conveniently written as $|\phi\rangle = \cos\delta|\psi\rangle + \sin\delta|\psi_{\perp}\rangle$ with

$$\cos^2 \delta = (\sin \alpha \sin \beta + \cos \alpha \cos \beta)^2 = \cos^2(\alpha - \beta).$$

The density matrices are $\rho = |\phi\rangle\langle\phi|$ and $\sigma = |\psi\rangle\langle\psi|$ with the respective priors of ξ and $1 - \xi$. The 'in-between measurement' has a POVM associated with it, which is a resolution of the identity of the form $\mathbf{I} = \sum_{i=1}^{N} \mathbf{\Pi}_{i}$, with the $\mathbf{\Pi}_{i}$ being positive semi-definite matrices. The candidate density matrix χ , which can be ρ or σ , is first updated using the formula

$$\chi_i = \frac{\pi_i \chi \pi_i^{\dagger}}{tr(\pi_i \chi \pi_i^{\dagger})}$$

with $\Pi_i = \pi_i^{\dagger} \pi_i$. The particular decomposition of Π_i is of no concern, since in subsequent calculations π_i^{\dagger} and π_i always appear as a product. The new cost function for a standard optimal measurement following the POVM is

$$\sum_{i=1}^{N} \operatorname{Tr} \left(\mathbf{\Pi}_{i} \left(\xi \rho + (1-\xi)\sigma \right) \right) \left(\frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\xi_{i} (1-\xi_{i}) \cos^{2} \delta_{i}} \right)$$

with

$$\xi_i(1-\xi_i)\cos^2\delta_i := \xi(1-\xi)\frac{\mathrm{Tr}(\mathbf{\Pi}_i\rho)\mathrm{Tr}(\mathbf{\Pi}_i\sigma)}{(\mathrm{Tr}(\xi\mathbf{\Pi}_i\sigma + (1-\xi)\mathbf{\Pi}_i\rho))^2}\frac{\mathrm{Tr}(\mathbf{\Pi}_i\rho\mathbf{\Pi}_i\sigma)}{\mathrm{Tr}(\mathbf{\Pi}_i\rho)\mathrm{Tr}(\mathbf{\Pi}_i\sigma)}.$$

As a comparison, the Helstrom cost is

$$\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\xi(1 - \xi)\cos^2\delta}.$$

As a reminder, all positive semi-definite matrices $\mathbf{A} \& \mathbf{B}$ satisfy $\text{Tr}(\mathbf{A}\mathbf{B}) \leq \text{Tr}(\mathbf{A})\text{Tr}(\mathbf{B})$. In addition, $(\text{Tr}(\mathbf{A}) - \text{Tr}(\mathbf{B}))^2 \geq 0$, and therefore

$$(\operatorname{Tr}(\mathbf{\Pi}_i \rho) + \operatorname{Tr}(\mathbf{\Pi}_i \sigma))^2 \ge 4\operatorname{Tr}(\mathbf{\Pi}_i \rho)\operatorname{Tr}(\mathbf{\Pi}_i \sigma).$$

For the subsequent calculation ξ is fixed at 1/2. Under this condition, there exists ρ, σ , and a set $\{\Pi_i\}$ such that for all i the difference

$$\operatorname{Tr}(\rho\sigma)\operatorname{Tr}(\mathbf{\Pi}_i\rho)\operatorname{Tr}(\mathbf{\Pi}_i\sigma) - \operatorname{Tr}(\mathbf{\Pi}_i\rho\mathbf{\Pi}_i\sigma)$$

is strictly positive[2]. This is all one needs to breach the Helstrom bound.

Should linearity be taken for granted in quantum mechanics, is a question intimately linked with the method employed. Does it hold sway unconditionally? Might not linearity be only another good approximation? If it is acknowledged that linearity is a 'god that failed', then a wholesale reassessment of quantum mechanics, and with it state discrimination, follows.

A speculative comment about a possible interpretation of quantum mechanics rounds of the appendix. It is related to the long-running dispute between views associated with Heraclitus and Parmenides. The claim is that change and therefore experience is due to the modification of the boundary between objects[3]. It could be termed the 'Red Queen interpretation' of quantum mechanics, since the Red Queen in Lewis Carroll's 'Through the Looking-Glass' commands Alice to perform "all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast". Unitary evolution keeps relative distances of states constant, and only the ceaseless modification of the boundary between system and environment leads to experience.

- [1] C.W. Helstrom, Quantum Detection and Estimation Theory (Academic Press, New York, 1976).
- [2] A.S. Holevo, Jour. Multivar. Anal. 3, 337 (1973).
- [3] H.P. Yuen, R.S. Kennedy, and M. Lax, IEEE Trans. Inform. Theory IT-21, 125 (1975).
- [4] D.C. Brody & B.K. Meister, Phys. Rev. Lett. 76 1-5 (1996), (arXiv:quant-ph/9507008).
- [5] B.K. Meister, arXiv:1106.5196 [quant-ph].
- [6] B.K. Meister, arXiv:1110.5284 [quant-ph].
- [7] C.M. Bender, D.C. Brody, & B.K. Meister, *Proceedings of the Royal Society London* A461, 733-753 (2005), (arXiv:quant-ph/0309119).
- [1] The normalisation adds non-linearity to the process, since $\frac{\pi^{\dagger}(\rho+\sigma)\pi}{\text{Tr}(\pi^{\dagger}(\sigma+\rho)\pi)}$ is not necessarily equal to the probability weighted sum of $\frac{\pi^{\dagger}\rho\pi}{\text{Tr}(\pi^{\dagger}\rho\pi)}$ and $\frac{\pi^{\dagger}\sigma\pi}{\text{Tr}(\pi^{\dagger}\sigma\pi)}$. [2] Heuristic argument: Take each Π_i to be constructed out of M, with $M \ll N$, one dimensional orthogonal subspaces
- [2] Heuristic argument: Take each Π_i to be constructed out of M, with $M \ll N$, one dimensional orthogonal subspaces $|e_{ki}\rangle\langle e_{ki}|$ with $k \in \{1,...,M\}$ with the inner-product of $\langle e_{ji}|e_{ki}\rangle = 0$ for k not equal to j. The quadratic in M array of real parts of the products $\langle e_{ji}|\phi\rangle\langle\phi|e_{ki}\rangle\langle e_{ki}|\psi\rangle\langle\psi|e_{ji}\rangle$ can be chosen to have positive and negative outcomes. If, for example, the individual signs of the products are randomly distributed and the absolute value of the products are all of similar size, then the sum would grow sub-linearly with the number of elements in the array. As a consequence, for a specially chosen set $\{\Pi_i\}$, $\text{Tr}(\Pi_i\rho\Pi_i\sigma)$ for each i can be smaller than $\text{Tr}(\Pi_i\rho)\text{Tr}(\Pi_i\sigma)$ times a selected 'constant'.
- [3] A curious example of a shift of boundary can be found in human development. An infant until a few months old, it is claimed, does not perceive the feet to be under its conscious control. The boundary eventually shifts and envelops the limbs.