Table of Contents

Pràctical 8

```
clear; close all;
format long G;
```

(a) The ODE

If we let $u(y,t) = G(y)F(t)$, when we compute the EDO resulting from the previous equation, we have:

$\frac{d^2F}{dt^2}G = g\frac{d}{dy}\{yF\frac{dG}{dy}\} = gF\{\frac{dG}{dt} + y\frac{d^2G}{dt^2}\}$

This can be written as:

$\frac{1}{F}\frac{d^2F}{dt^2} = \frac{g}{G}\{y\frac{d^2G}{dy^2} + \frac{dG}{dy}\}$

This will only have a solution if both sides of the equation are equal to a constant k. Now, depending on the sign of k, we'll have different solutions. Now if we take the temporal component of the wave function we have $\frac{1}{F(t)}\frac{d^2F(t)}{dt^2} = k$

- For $k < 0$, let $k = -\lambda^2$ (with $\lambda \in R$). We have that $F(t) = Ae^{i\lambda t} + Be^{-i\lambda t}$.

- For $k = 0$, we have $F(t) = Ct + D$.

- For $k > 0$, let $k = \lambda^2$ (with $\lambda \in R$). We then have that $F(t) = Fe^{\lambda t} + Ge^{\lambda t}$.

The former two options ($k \geq 0$) are not bounded $\forall t$, since for $t \to \infty$ both solutions tend to ∞ as well. Therefore, they will not be suitable solutions to our problem. However, the solution for $k < 0$ is feasible. Then, let us rewrite the resulting equation for $G(y)$ by using $k = -\lambda^2$.

$y\frac{d^2G}{dy^2} + \frac{dG}{dy} + \frac{\lambda^2}{g}G = 0$

which is exactly the equation we were looking for.

(b) Eigenvalues & Eigenfunctions

From the statement of the practical we can define a the boundary condition for $y = l$: $u(l,t) = 0$, which means $G(l) = 0$. However, even though we know that $u(y,t)$ must be bounded for $y = 0$, we have no defined value a concrete boundary condition for $y = 0$ (it is the free end of the chain).

```
% Define the parameters of the problem
n = 26;
[D,x] = chebdiff(n); D2  = D*D; % 1st and 2nd differentiation matrix
y = (x+1)/2; % map the chebyshev nodes to the actual domain of y
```

Now, from the equation given in (a), we have that $y\frac{d^2G}{dy^2} + \frac{dG}{dy} = -\frac{\lambda^2}{g}G$. Let us write $L = y\frac{d^2}{dy^2} + \frac{d}{dy}$ and $\mu = -\frac{\lambda^2}{g}$.

Since there is only one boundary condition for G and it is $G(l) = 0$, we would have $c_{11} = 1$ and $c_{12} = c_{13} = c_{21} = c_{22} = c_{23} = 0$. Therefore, it doesn't really make sense to compute the operators M_1, M_2 and M_3 since they will be of no use. Now, we'll want to solve $LG = \mu G$ by finding the eigenvalues μ and eigenvectors G of the ODE.

```
L = 4*diag(y)*D2 + 2*D; % Define L operator
L = L(2:end, 2:end); % Take out the column corresponding to y=l

% Find eigenvectors and eigenvalues of the operator
[V, E] = eig(L); E = diag(E);
[E, Perm] = sort(E);
V = V(:, Perm);
lambda = sqrt(-E); lambdafirst = lambda(end-2:end);

disp(lambdafirst); % Display the smallest 3 eigenvectors
```

```
4.32686395645551
2.76003905514316
 1.2024127788479
```

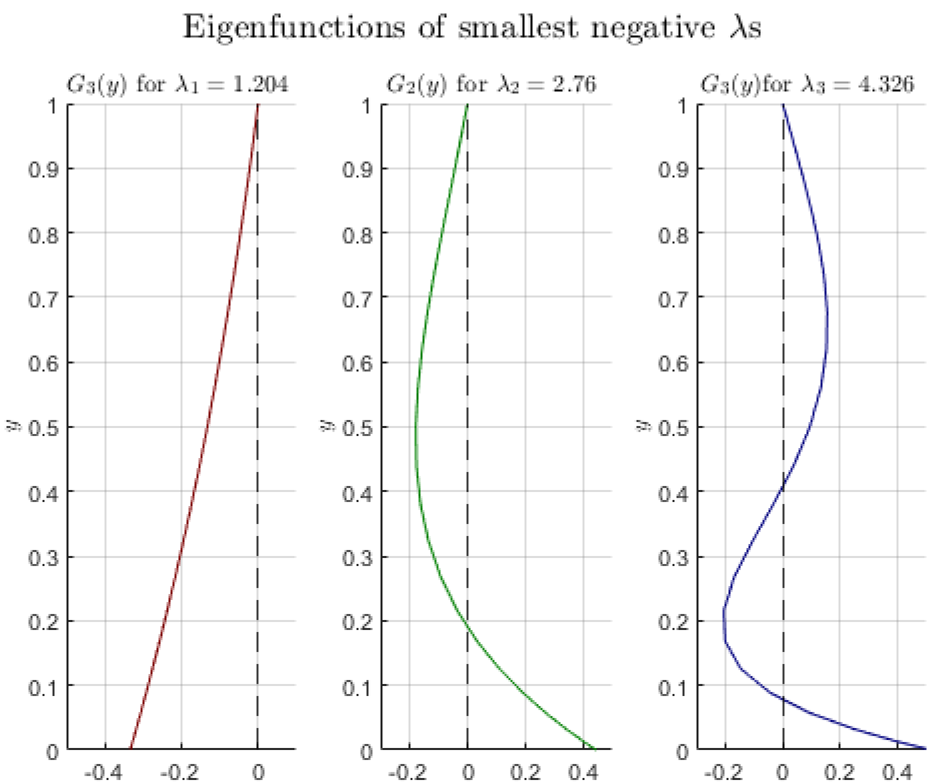
Plot the three eigenfunctions of the smallest eigenvalues

```
figure(1)
subplot(1, 3, 1)
hold on
plot([0; V(:, end)], y, 'color', [0.5 0 0])
xline(0, '--k');
hold off
grid on
xlim([-0.5 0.1])
title('$G_3(y)$ for $\lambda_1=1.204$ ', 'Interpreter', 'latex')
ylabel('$y$', 'Interpreter','latex')

subplot(1, 3, 2)
hold on
plot([0; V(:, end-1)], y, 'color', [0 0.5 0])
xline(0, '--k');
hold off
grid on
xlim([-0.3 0.5])
title('$G_2(y)$ for $\lambda_2=2.76$ ', 'Interpreter', 'latex')
ylabel('$y$', 'Interpreter','latex')
```

```
subplot(1, 3, 3)
hold on
plot([0; V(:, end-2)], y, 'color', [0 0 0.5])
xline(0, '--k');
hold off
grid on
xlim([-0.3 0.5])
title('$G_3(y)$for $\lambda_3=4.326$ ', 'Interpreter', 'latex')
ylabel('$y$', 'Interpreter','latex')

sgtitle('Eigenfunctions of smallest negative $\lambda
$s','Interpreter', 'latex')
```



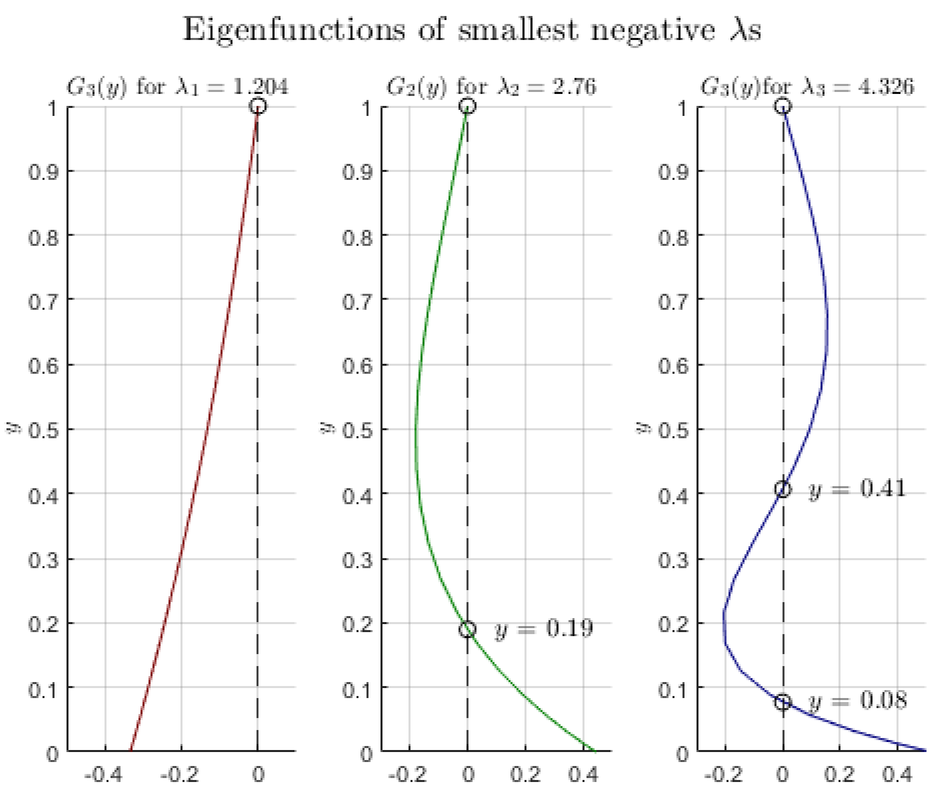
(c) Bessel functions & nodes

The nodes of the wave are those values of y for which $u(y,t) = 0$. We also know that $J_0(2\lambda_k) = 0$. Then, we may be able to find the nodes of each oscillation mode by looking for $2\lambda_i\sqrt(y) = 2\lambda_k$. This means we will be able to compute the nodes for the oscillation mode k with $y = \left(\frac{\lambda_i}{\lambda_k}\right)^2$ for $i \leq k$. We'll plot this positions in their respective plots of $G(y)$.

```
for kk = 1:3
    for ii = 1:kk
        zero = (lambdafirst(kk)/lambdafirst(ii))^2; % compute zero
```

```
        zero_round = round(zero,2); % just for graphical purposes

        figure(1)
        subplot(1, 3, (4-ii))
        if abs(zero -1) > eps
            text(0,zero,['\quad $y =$ '
 num2str(zero_round)], 'Interpreter', 'latex')
        end
        hold on
        plot(0, zero, 'ok');
        hold off
    end
end
```



Auxiliar codes

```
% Code 5B: Chebyshev Differentiation matrix
% Input: n
% Output: differentiation matrix D and Chebyshev nodes
function [D,x] = chebdiff(n)
    x =cos([0:n]'*pi/n); d = [.5 ;ones(n-1,1);.5];
    D = zeros(n+1,n+1);
    for ii = 0:n
     for jj = 0:n
      ir = ii + 1 ; jc = jj + 1;
      if ii == jj
```

```
      kk = [0:ii-1 ii+1:n]'; num = (-1).^kk.*d(kk+1) ;
      D(ir,jc) =((-1)^(ir)/d(ir))*sum(num./(x(ir)-x(kk+1)));
     else
      D(ir,jc) = d(jc)*(-1)^(ii+jj)/((x(ir)-x(jc))*d(ir));
     end
    end
   end
end
```

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