Let A be the real 2×2 matrix $A = \begin{bmatrix} a & 2b \\ 0 & a \end{bmatrix}$, with $a, b \in \mathbb{R}$, positive or negative.

(a) Compute $\|A\|_1$, $\|A\|_\infty$ and the Frobenius norm $\|A\|_F$ in terms of a and b.

(1.0 p.)

(b) Compute $\|A\|_2$ in terms of a and b.

(1.0 p.)

(c) For $a = 4$ and $b = -3$, compute a unitary vector $\mathbf{u}$ such that $\|A\mathbf{u}\|_2 = \|A\|_2$.

(1.0 p.)

Problems

Problem 1 (3.0 pts.) Consider the matrix $\mathbf{A} = \begin{bmatrix} 7 & 10 \\ 15 & 23 \end{bmatrix}$.

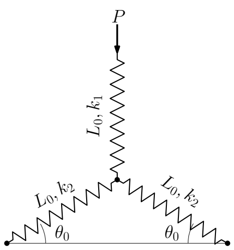
(a) Knowing that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$, compute a matrix (or matrices) $\mathbf{X} \in \mathbb{M}_2(\mathbb{R})$ such that $\mathbf{X}^2 = \mathbf{A}$. Detail your calculations, the equations you have solved, methodology, strategies, etc.

(1.5 p.)

(b) Find a matrix $\mathbf{X} \in \mathbb{M}_2(\mathbb{R})$ such that $\mathbf{X}^3 = \begin{bmatrix} 37 & 54 \\ 81 & 119 \end{bmatrix}$. Provide the pertinent explanations.

(1.5 p.)

Problem 2 (4.0 pts.) Consider a system composed by 3 springs of the same length (L_0) in the configuration shown in the figure below.



When a force P is applied to the first spring in the vertical direction (pointig downwards), the equilibrium vertical displacements of the upermost extreme of the vertical spring (u_1) and of the vertex where the 3 spring meet (u_2) are described by the following non-linear system of equations:

$$\left(\frac{1}{\sqrt{1 + x_2^2 - 2x_2 \sin\theta_0}} - 1 \right) (\sin\theta_0 - x_1) - \omega(x_1 - x_2) = 0$$

$$w(x_1 - x_2) - \lambda = 0$$

where we have defined the non-dimensional parameters $\lambda = \frac{P}{k_2 L_0}$, $\omega = \frac{k_1}{k_2}$ and variables $x_1 = \frac{u_1}{L_0}$, $x_2 = \frac{u_2}{L_0}$. Take the values $\omega = 1$ and $\theta_0 = \pi/4$.

(a) Write a Matlab function that returns the result of the non-linear equations as a function of the normalized displacements and the applied load (x_1, x_2, λ). Find the equilibrium position (x_1, x_2) for $\lambda = 1$.

(1.0 p.)

(b) When the load is increased from $\lambda = 1 \times 10^{-3}$ to $\lambda = 2 \times 10^{-1}$, try to find numerically the equilibrium solutions and represent the obtained values as points in a graph (λ, u_2).

(1.5 p.)

(c) Use a continuation method for calculating the full branch of solutions in the same range of λ. Represent the obtained curve.

(1.5 p.)