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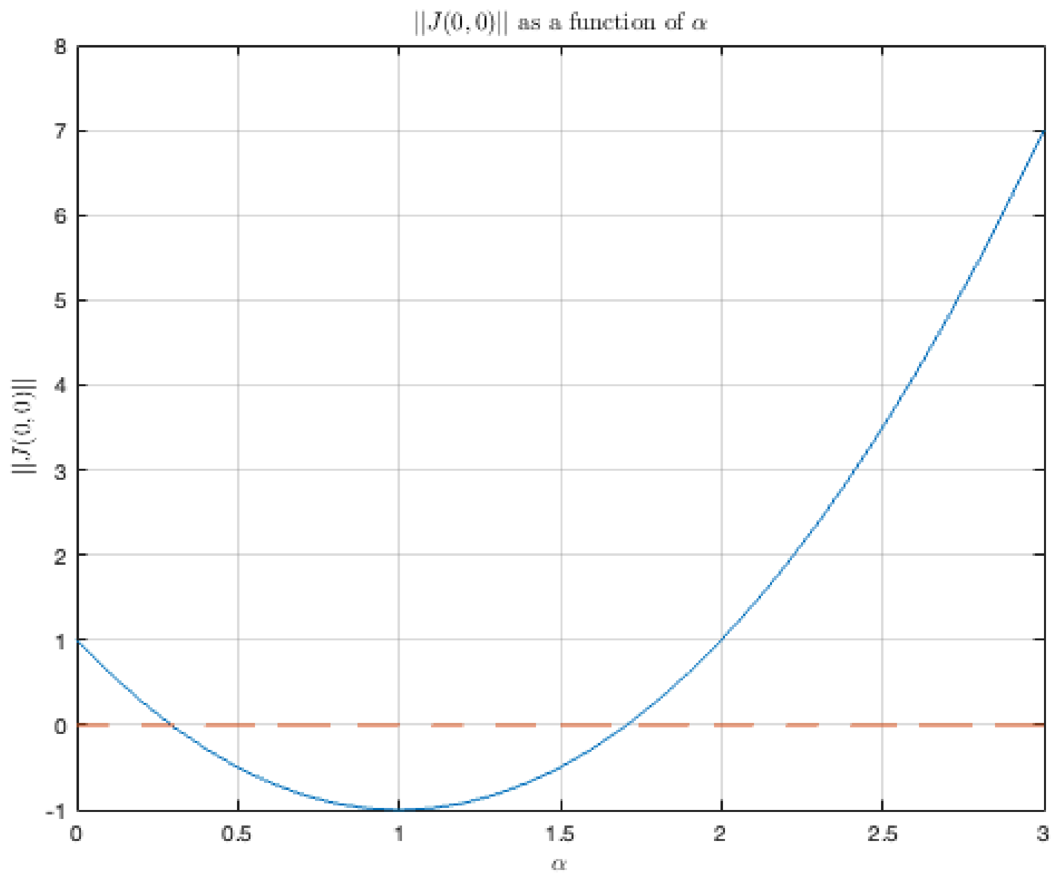
Practical 6 - Cortés & García

```
clear all
close all
format long

% Definition of the function we want to optimize
% phi = [phi1; phi2] -> column vector with the two angles phi1 and phi2
% alpha = l*omega^2/(2g) -> as said in the statement
f = @(phi, alpha) [tan(phi(1)) - alpha*(2*sin(phi(1)) + sin(phi(2))) ; ...
    tan(phi(2)) - 2*alpha*(sin(phi(1)) + sin(phi(2)))];
```

(a) Branches of equilibrium

```
alVec0 = 0:0.1:3;   % Values of alpha
detJVec = alVec0*0; % Store the determinant of the Jacobian at (0,0) for
                    % each value of alpha

for i = 1:length(alVec0)
    f_alpha = @(x) f(x, alVec0(i));
    detJVec(i) = det(jac(f_alpha, [0; 0]));
end

figure(1)
plot(alVec0, detJVec, "LineWidth", 0.9)
hold on
plot(alVec0, zeros(size(alVec0)), "--")
grid on
xlabel("$\alpha$", "Interpreter","latex")
ylabel("$||J(0,0)||$", "Interpreter","latex")
title("$||J(0,0)||$ as a function of $\alpha$", "Interpreter","latex")
hold off
```



For α other than the critical values where the determinant of the Jacobian matrix is not 0, we will expect the system to have a unique solution. For those critical values of alpha that cancel the determinant, we may expect new branch solutions to emerge. Graphically speaking, this values of α are approximately $\alpha_1 = 0.3$ and $\alpha_2 = 1.7$ (the values where the determinant of the Jacobian crosses the horizontal line (red discontinuous line in the plot) at $y = 0$).

Anallitically, we have that the jacobian of this function at (0, 0) is

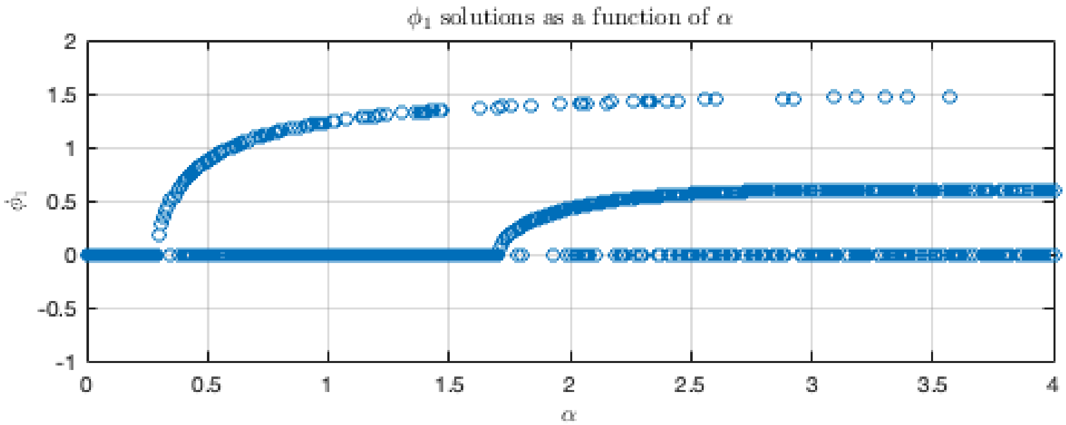
$$J_\alpha(0,0) = \begin{pmatrix} 1-2\alpha & -\alpha \\ -2\alpha & 1-2\alpha \end{pmatrix}$$

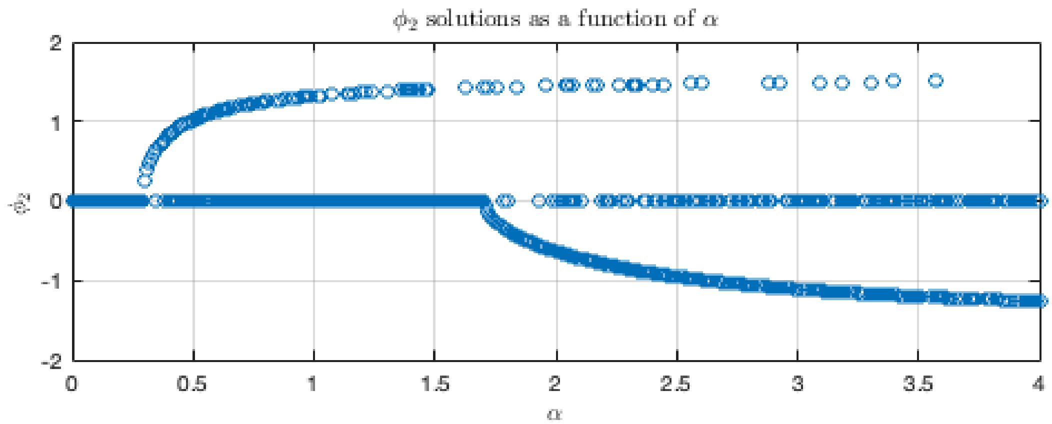
We can clearly see that the determinant of this matrix is $|J_\alpha(0,0)| = (1-2\alpha)^2 - 2\alpha = 2\alpha^2 - 4\alpha + 1$. The cases in which this determinant will be 0 are the solutions $\alpha = \frac{2\pm\sqrt{2}}{2}$. We then have that $\alpha_1 = \frac{2-\sqrt{2}}{2} = 0.2928...$ (Solution 1) and $\alpha_2 = \frac{2+\sqrt{2}}{2} = 1.7071...$ (Solution 2) (approximately the values we observed graphically).

(b) Newton

```
alVec = [];
XkVec = [];

for al = 0:0.01:4
    % For each value of alpha, we will do 10 iterations, to find all
```

```
    % possible solutions
    for it = 1:10
        x0 = [(pi/2)*rand(); pi*rand()-pi/2]; % Initial random guess

        f_alpha = @(x) f(x, al);

        % Solution to which this x0 converges by Newton method
        [Xk, ~, ~] = newtonn(x0, eps, 50, f_alpha);
        xk = Xk(:, end);

        % Discard those solutions outside the wanted domain
        if xk(1) >= 0 && xk(1) < pi/2 && xk(2) > -pi/2 && xk(2) < pi/2
            XkVec = [XkVec , xk];
            alVec = [alVec al];
        end
    end
end

figure(2)
subplot(2, 1, 1)
plot(alVec, XkVec(1,:), 'o')
grid on
xlabel("$\alpha$", "Interpreter","latex")
ylabel("$\phi_1$", "Interpreter","latex")
title("$\phi_1$ solutions as a function of $\alpha$", "Interpreter","latex")
axis([0 4 -1 2])
subplot(2, 1, 2)
plot(alVec, XkVec(2,:), 'o')
grid on
xlabel("$\alpha$", "Interpreter","latex")
ylabel("$\phi_2$ ", "Interpreter","latex")
title("$\phi_2$ solutions as a function of $\alpha$", "Interpreter","latex")
axis([0 4 -2 2])
hold off
```





To reach the goal of observing what happens through various values of α for various randomly chosen initial conditions, we are setting a random initial guess in the domains of both ϕ_1 and ϕ_2 10 times and we then apply Newton's Method to try and find an equilibrium point.

Graphically, we observe that when $\alpha < \alpha_1$, there is only one solution: the equilibrium position is the trivial solution $\phi_1 = \phi_2 = 0$. However, when $\alpha_1 \leq \alpha < \alpha_2$, the equilibrium position will depend on the initial guess and there is another possibility aside from the trivial solution. When $\alpha_2 \leq \alpha$ there appears another branch and there are actually 3 possible solutions (the trivial one and 2 more). Therefore, we can see graphical proof that the 2 branches we predicted appear.

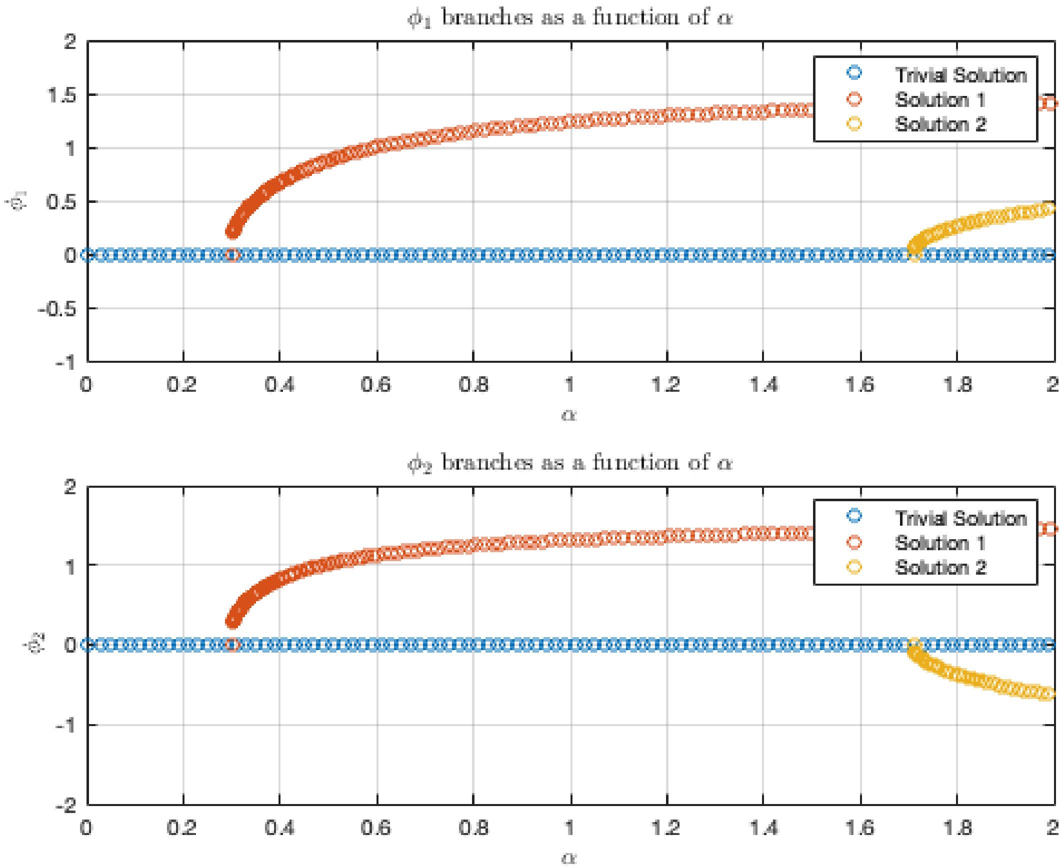
(c) Continuation

Now we want to plot both $\phi_1(\alpha)$ and $\phi_2(\alpha)$ for the 3 solution branches found. When it comes to the Trivial Solution, it will be easy: it is only needed to take 2 initial close solutions ($\phi_1(0) = \phi_2(0) = \phi_1(0.01) = \phi_2(0.01) = 0$) and see how the equilibrium positions evolve with α.

The same will need to be done for the other 2 solutions, but starting at α_1 and α_2, where each branch starts respectively. To do this, we'll benefit from the previous results: we'll want to find the values α closest to the values α_1 and α_2 we obtained analitically. In the previous loop, we used a step between αs of 0.01, so that is the maximum difference we'll want to look at. Then, we'll take the solutions for ϕ_1 and ϕ_2 for those values of α, and start the continuation step from there. We'll iterate the continuation enough to obtain a clear picture of how it evolves with α.

```
% Define the funcion
f_cont = @(x) f([x(1) ; x(2)], x(3));

% Trivial Solution
x00 = [0; 0; 0];
x10 = [0; 0; 0.01];

% Represent the solutions for this branch using contstep function
alVecCont0 = zeros(1, 152);
alVecCont0(1) = x00(3); alVecCont0(2) = x10(3);
XkVecCont0 = zeros(2, 152);
XkVecCont0(:, 1) = x00(1:2); XkVecCont0(:, 2) = x10(1:2);

for it = 0:150
    s = .02/norm(x10-x00);
    [y, iconv] = contstep(f_cont, x00, x10, s, eps, 50);

    x00 = x10;
    x10 = y;

    alVecCont0(it+2) = y(3);
    XkVecCont0(:, it+2) = [y(1) ; y(2)];
end

% Repeat the process for Solution 1
idx1 = find(abs(alVec - (2-sqrt(2))/2) < 1e-2);
x01 = [0; 0; alVec(idx1(end-1))];
x11 = [XkVec(:, idx1(end)) ; alVec(idx1(end))];

alVecCont1 = zeros(1, 202);
alVecCont1(1) = x01(3); alVecCont1(2) = x11(3);
XkVecCont1 = zeros(2, 202);
XkVecCont1(:, 1) = x01(1:2); XkVecCont1(:, 2) = x11(1:2);

for it = 0:200
    s = .02/norm(x11-x01);
    [y, iconv] = contstep(f_cont, x01, x11, s, eps, 50);

    x01 = x11;
    x11 = y;

    alVecCont1(it+2) = y(3);
    XkVecCont1(:, it+2) = [y(1) ; y(2)];
end

% Repeat the process for Solution 2
idx2 = find(abs(alVec - (2+sqrt(2))/2) < 1e-2);
x02 = [0; 0; alVec(idx2(end-1))];
x12 = [XkVec(:, idx2(end)) ; alVec(idx2(end))];

alVecCont2 = zeros(1, 102);
alVecCont2(1) = x02(3); alVecCont2(2) = x12(3);
XkVecCont2 = zeros(2, 102);
XkVecCont2(:, 1) = x02(1:2); XkVecCont2(:, 2) = x12(1:2);
```

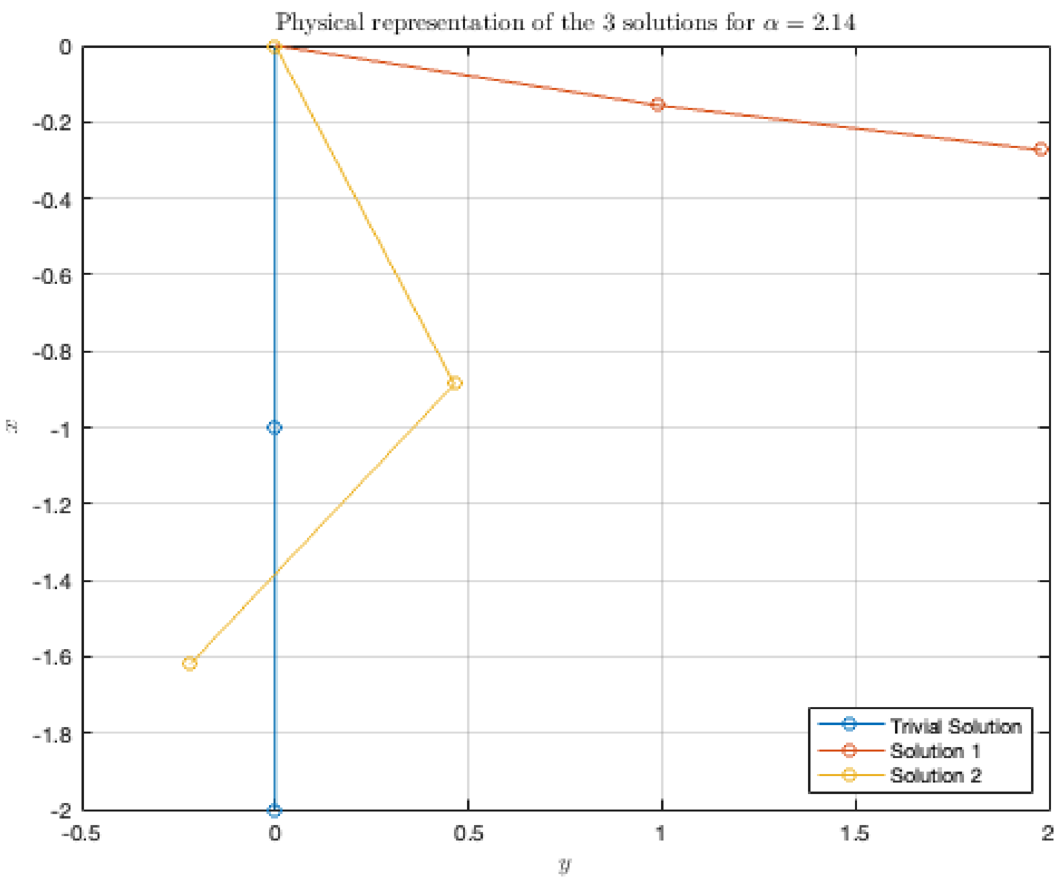
```
for it = 0:100
    s = .02/norm(x12-x02);
    [y, iconv] = contstep(f_cont, x02, x12, s, eps, 50);

    x02 = x12;
    x12 = y;

    alVecCont2(it+2) = y(3);
    XkVecCont2(:, it+2) = [y(1) ; y(2)];
end

figure(3)
subplot(2, 1, 1) % phi1
plot(alVecCont0, XkVecCont0(1,:), 'o')
hold on
plot(alVecCont1, XkVecCont1(1,:), 'o')
plot(alVecCont2, XkVecCont2(1,:), 'o')
axis([0 2 -1 2])
xlabel("$\alpha$", "Interpreter", "latex")
ylabel("$\phi_1$", "Interpreter", "latex")
title("$\phi_1$ branches as a function of $\alpha$", "Interpreter","latex")
legend('Trivial Solution', 'Solution 1', 'Solution 2')
grid on
hold off
subplot(2, 1, 2) % phi2
plot(alVecCont0, XkVecCont0(2,:), 'o')
hold on
plot(alVecCont1, XkVecCont1(2,:), 'o')
plot(alVecCont2, XkVecCont2(2,:), 'o')
axis([0 2 -2 2])
xlabel("$\alpha$", "Interpreter", "latex")
ylabel("$\phi_2$", "Interpreter", "latex")
title("$\phi_2$ branches as a function of $\alpha$", "Interpreter","latex")
legend('Trivial Solution', 'Solution 1', 'Solution 2')
grid on
hold off
```



We can now see clearly how ϕ_1 and ϕ_2 evolve as a function of α. We can see that these new solutions start at α_1 and α_2 and they correspond to the ones we had anticipated. We can also see that for a large enough α (from $\alpha = 3$ approximately) both angles seem to converge to a value each, which would physically mean that when ω is large enough (the pendulum is turning fast enough) the equilibrium position will remain the same when increasing the angular velocity depending on the initial conditions of the problem.

```
idxx0 = find(abs(alVecCont0 - 2.14) < 1e-2);
idxx1 = find(abs(alVecCont1 - 2.14) < 1e-2);
idxx2 = find(abs(alVecCont2 - 2.14) < 1e-2);

phi10 = XkVecCont0(1, idxx0(1));
phi20 = XkVecCont0(2, idxx0(1));
phi11 = XkVecCont1(1, idxx1(1));
phi21 = XkVecCont1(2, idxx1(1));
phi12 = XkVecCont2(1, idxx2(1));
phi22 = XkVecCont2(2, idxx2(1));


l = 1; % Assume length of the pendulums of l=1
Sol0 = [[0; 0] , ...
    [l*sin(phi10); -l*cos(phi10)] , ...
    [l*sin(phi10)+l*sin(phi20); -l*cos(phi10)-l*cos(phi20)]];
Sol1 = [[0; 0] , ...
```

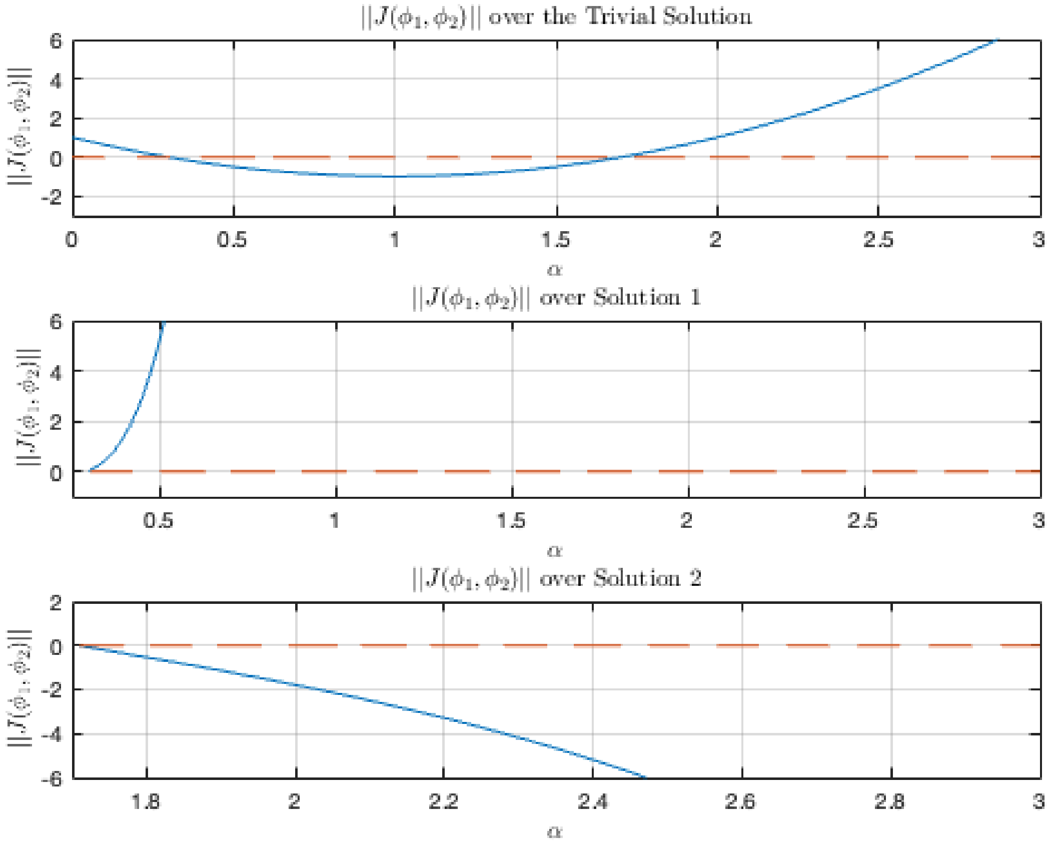
```
    [l*sin(phi11); -l*cos(phi11)] , ...
    [l*sin(phi11)+l*sin(phi21); -l*cos(phi11)-l*cos(phi21)]];
Sol2 = [[0; 0] , ...
    [l*sin(phi12); -l*cos(phi12)] , ...
    [l*sin(phi12)+l*sin(phi22); -l*cos(phi12)-l*cos(phi22)]];

% Plot the positions of the 2 pendulums
figure(5)
plot(Sol0(1,:), Sol0(2,:), '-o')
hold on
plot(Sol1(1,:), Sol1(2,:), '-o')
plot(Sol2(1,:), Sol2(2,:), '-o')
title("Physical representation of the 3 solutions for $\alpha=2.14$", ...
    "Interpreter","latex")
xlabel("$y$", "Interpreter","latex")
ylabel("$x$", "Interpreter","latex")
legend('Trivial Solution', 'Solution 1', 'Solution 2', 'Location', ...
    'southeast')
grid on
hold off
```



For $\alpha = 2.14$, this solutions of the actual pendulum are represented in the previous figure.

For each solution branch, we will evaluate the determinat over the branch to see if the determinant cancels out at some point, which would make new branches appear.

```
% Solution 1
detJVec1 = alVecCont1*0;

for i = 1:length(alVecCont1)
    f_alpha = @(x) f(x, alVecCont1(i));
    detJVec1(i) = det(jac(f_alpha, XkVecCont1(1:2,i)));
end

% Solution 2
detJVec2 = alVecCont2*0;

for i = 1:length(alVecCont2)
    f_alpha = @(x) f(x, alVecCont2(i));
    detJVec2(i) = det(jac(f_alpha, XkVecCont2(1:2,i)));
end

figure(6)
subplot(3, 1, 1)
plot(alVec0, detJVec, "LineWidth", 0.8)
hold on
plot(alVec0, zeros(size(alVec0)), "--")
hold off
xlabel("$\alpha$", "Interpreter","latex")
ylabel("$||J(\phi_1,\phi_2)||$", "Interpreter","latex")
title("$||J(\phi_1, \phi_2)||$ over the Trivial Solution", ...
    "Interpreter","latex")
grid on
axis([0 3 -3 6])

subplot(3, 1, 2)
plot(alVecCont1, detJVec1, "LineWidth", 0.8)
hold on
plot(alVecCont1, zeros(size(alVecCont1)), "--")
hold off
xlabel("$\alpha$", "Interpreter","latex")
ylabel("$||J(\phi_1,\phi_2)||$", "Interpreter","latex")
title("$||J(\phi_1, \phi_2)||$ over Solution 1", "Interpreter","latex")
grid on
axis([0.25 3 -1 6])

subplot(3, 1, 3)
plot(alVecCont2, detJVec2, "LineWidth", 0.8)
hold on
plot(alVecCont2, zeros(size(alVecCont2)), "--")
xlabel("$\alpha$", "Interpreter","latex")
ylabel("$||J(\phi_1,\phi_2)||$", "Interpreter","latex")
title("$||J(\phi_1, \phi_2)||$ over Solution 2", "Interpreter","latex")
axis([1.7 3 -6 2])
grid on
hold off
```



When we evaluate the determinant of the Jacobian matrix for each branch, it looks like it won't be 0 again for any of them, which leads us to think that there will not be more new branches. More specifically, for the branch that starts at α_1, the determinant of the jacobian is incrases with α and for the one that starts at α_2, it decreases with α.

Auxiliar Codes

```
% Code 20: Newton’s method for n-dimensional systems
% Input: x0 - initial guess (column vector)
%        tol - tolerance so that ||x_{k+1} - x_{k} || < tol
%        itmax - max number of iterations
%        fun - function’s name
% Output:   XK - iterated
%           resd: resulting residuals of iteration: ||F_k||
%           it:   number of required iterations to satisfy tolerance
function [XK,resd,it] = newtonn(x0,tol,itmax,fun)
    xk = [x0];
    resd = [norm(feval(fun,xk))];
    XK = [x0];
    it = 1;

    tolk = 1.0;
    n = length(x0);
```

```
    while it < itmax && tolk > tol
        Fk = feval(fun, xk);

        DFk = jac(fun, xk);
        [P,L,U] = pplu(DFk);

        dxk = plusolve(L,U,P,-Fk);

        xk = xk + dxk;
        XK = [XK xk];
        resd = [resd norm(Fk)];
        tolk = norm(XK(:, end)-XK(:, end-1));
        it = it + 1;
    end
end

% Code 21: secant continuation step Input:    y0 and y1 (two close column
% vectors)
%           s: pseudo-arclength parameter tol - Newton’s tolerance:
%           ||y_[k+1] - y_[k] || < tol % itmax - max number of iterations
%           fun - function’s name: f(y_1,y_2,...,y_n,y_n+1)
% Output:   y - next point along curve f = 0
%           y belongs to plane orth. to y1-y0 passing through secant
%           predictor y1 + s(y1-y0) iconv (0 if y is convergenced to
%           desired tol.)
function [y,iconv] = contstep(fun,y0,y1,s,tol,itmax)
    tolk = 1.0;
    it = 0;
    n = length(y0)-1;

    v = y1-y0;
    yp = y1+s*v;
    xk = yp;

    while tolk > tol && it < itmax
        Fk = [feval(fun,xk); v'*(xk-yp)];
        DFk = [jac(fun,xk); v'];

        [P,L,U] = pplu(DFk);
        dxk = plusolve(L,U,P,-Fk);

        xk = xk + dxk;
        tolk = norm(dxk);
        it = it + 1;
    end

    y = xk;
    if it <= itmax && tolk < tol
        iconv = 0;
    else
        iconv = 1;
    end
end
```

```
% Code 19: Computation of the Jacobian J Input:   F(x) : R^m ---> R^n
%           x : (m x 1)-vector ; F: (n x 1)-vector
% Output: DF(x) (n x m) Jacobian matrix at x
function DF = jac(F,x)
    f1 = feval(F,x);
    n = length(f1);
    m = length(x);

    DF = zeros(n,m);
    H = sqrt(eps)*eye(m);

    for j = 1:m
        f2 = feval(F,x+H(:,j));
        DF(:,j) = (f2 - f1)/H(j,j);
    end
end

% Code 13: PA = LU factorization (partial pivoting) Input: A (non-singular
% square matrix) Output: L (unit lower triangular matrix)
%          U (upper triangular matrix) P (reordering vector)
function [P, L, U] = pplu(A)
    [m,n] = size(A);

    if m~=n
           error('not square matrix');
    end

    U = A;
    L = eye(n);

    P = [1:n]';

    for k = 1:n-1
        [~, imax] = max(abs(U(k:end,k)));
        imax = imax+k-1;
        i1 = [k, imax];
        i2 = [imax, k];

        U(i1,:) = U(i2,:); % Column k will be column imax and column imax will
 be column k
        P(k) = imax;

        L(i1,1:k-1) = L(i2, 1:k-1);

        for jj = [k+1:n]
            L(jj, k) = U(jj, k)/U(k, k);
            U(jj, k:n) = U(jj, k:n) - L(jj, k)*U(k,k:n);
        end
    end
end

% Code 14: PA = LU (Solver for Ax = b) Input:    L (unit lower triangular
% matrix)
%            U (upper triangular matrix) P (reordering vector) b (right-hand
```

```
%           side)
% Output:   solution x
function x = plusolve(L, U, P, b)
    n = length(b);
    for k = 1:n-1
        b([k P(k)]) = b([P(k) k]);
    end
    y = fs(L, b);
    x = bs(U, y);
end

% Code 11: Forward Substitution for Lower Triangular Systems Input:    L:
% Low Triangular non-singular square matrix
%           b: column right-hand side
% Output:   x: solution of Lx=b
function x = fs(L, b)
    x = 0*b;
    n = length(b);
    x(1) = b(1)/L(1,1);

    for ii = 2:n
        x(ii) = (b(ii)-L(ii, 1:ii-1)*x(1:ii-1))/L(ii,ii);
    end
end

% Code 12: Backward Substitution for Upper Triangular Systems Input:    U:
% Upp. Triangular non-singular square matrix
%           b: column right-hand side
% Output:   x: solution of Ux=b
function x = bs(U, b)
    x = 0*b;
    n = length(b);
    x(n) = b(n)/U(n,n);

    for ii = n-1:-1:1
        x(ii) = (b(ii)-U(ii, ii+1:n)*x(ii+1:n))/U(ii,ii);
    end
end
```

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