C1 - Single-Joint Control

AuSRoS 2024
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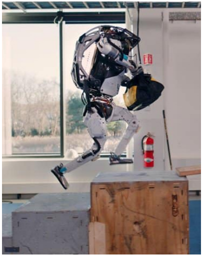
Section 1

Motivation and Overview

Motivation: High-Performance Humanoid Robots



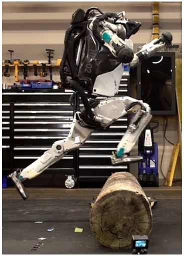
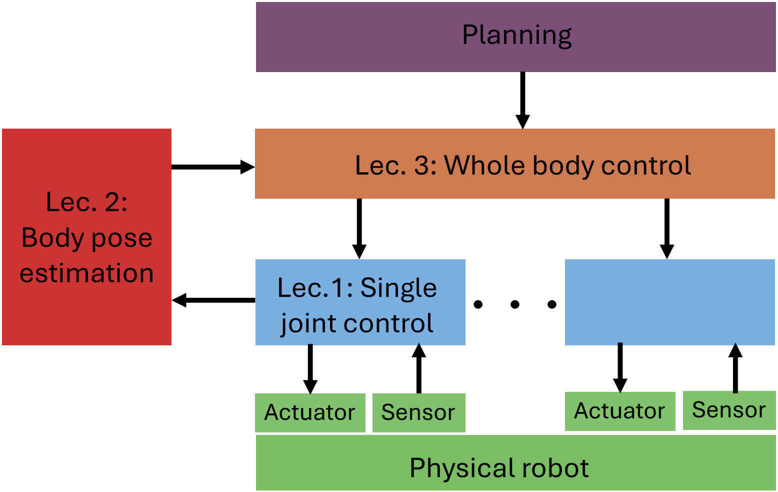
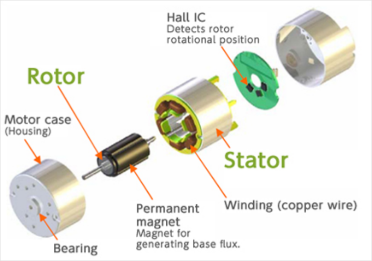


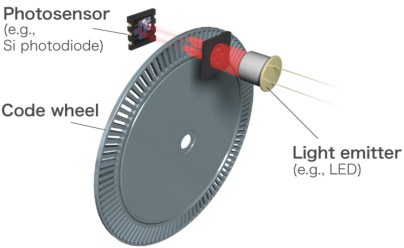
Image: bostondynamics.com

Hierarchy of Control System



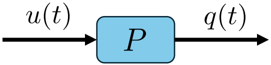
Actuators and Sensors





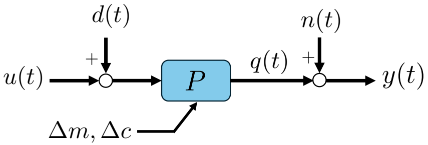
Control of Single-Joint System

- Single joint dynamics:



$$P: \quad m\ddot{q} + c\dot{q} + \sin q = u$$

- Aim of Control:
  - Find $u(t)$ s.t. $q(t)$ tracks the target angle $r(t)$.
- **Challenges:**
  - disturbance $d(t)$
  - measure noise $n(t)$
  - parameter perturbation $\Delta m, \Delta c$



Section 2

Why Feedback?

Open-loop v.s. Closed-loop Control

- Open-loop (feedforward) control $u(t) = C(r(t))$

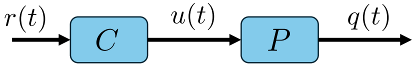


Figure: Open-loop system block diagram

The **feedforward** control cannot handle the uncertainties as it only depends on the reference.

- Closed-loop (feedback) control: $u(t) = C(r(t) - y(t))$

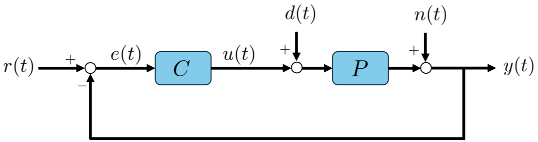


Figure: Closed-loop system block diagram

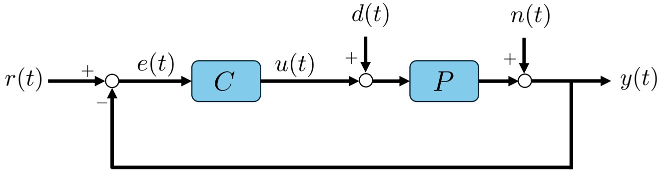
The **feedback** control allows us to compensate the effects of uncertainties by using the measurement $y(t)$.

Section 3

Single-Joint Control: PID

PID controllers

Proportional + integral + derivative (PID) controllers are by far the most common form of controller in industry.



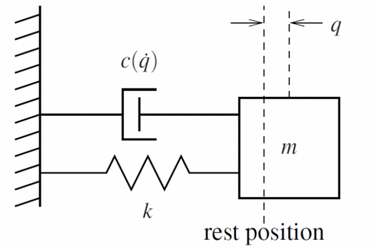
The control signal is calculated as:

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_d \dot{e}(t)$$

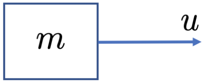
where $e(t) = r(t) - y(t)$ is the tracking error.

- P, PI, and PD controllers have only one or two terms.
- Rate feedback is a common variant in robotics: derivative term is $-k_d \dot{y}$. I.e. reference not differentiated.
- Derivative often has filtering.

PD Controller



$$m\ddot{q} = u$$

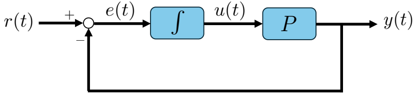


$$u = -K_P q - K_D \dot{q}$$

- PD Controller creates virtual spring and damper
- The gains can be tuned to adjust the spring stiffness and damping

Integral Controller

- Integral control can be interpreted in many ways.
- Simple case (no dynamics): consider some unknown but monotone nonlinear function $P : y = f(u)$ with $\frac{df}{du} > 0$.



- Integral control acts like a continuous Newton method for root finding:
$$\dot{u} = k_i(r - y) = k_i(r - f(u))$$
- Alternatively, consider the gradient flow for the nonlinear least-squares cost $J(u) = \frac{1}{2}(r - f(u))^2$:
$$\dot{u} = -\nabla J(u) = k_i(r - f(u))\frac{df}{du}.$$

Since $\frac{df}{du} > 0$, integral control follows rescaled gradient flow.

Example: robot joint with flexible transmission

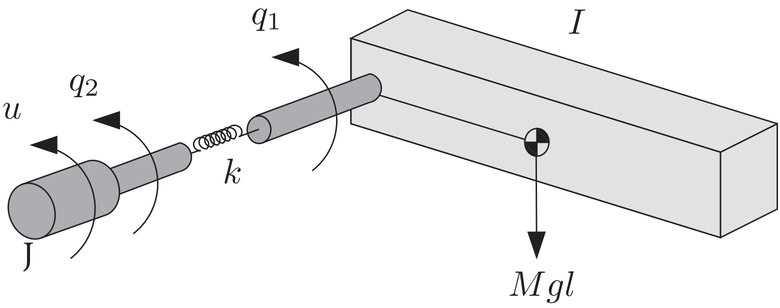


Image: Spong, Vidyasagar, Hutchinson (2020), Robot Modelling & Control, 2nd Ed.

Section 4

Loop Shaping and Fundamental Limitations

Frequency Domain Analysis for Linear Control System

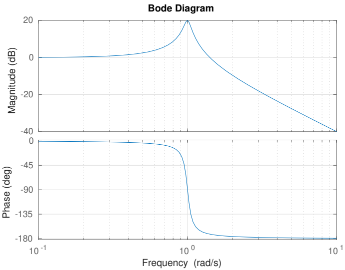
- Ordinal differential equation (ODE) in time domain:

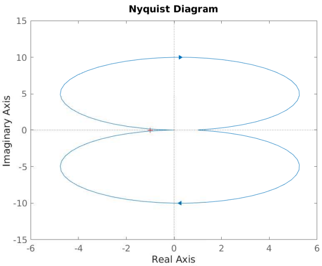
$$P: \; m\frac{d^2}{dt^2}q(t) + c\frac{d}{dt}q(t) + kq(t) = u(t) \tag{1}$$

- **Transfer function** (applying Laplace transformation to (1))

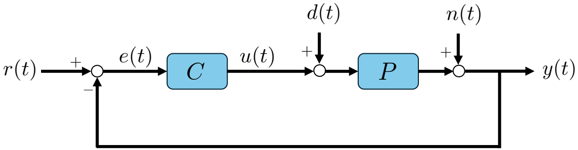
$$P(s) = \frac{q(s)}{u(s)} = \frac{1}{ms^2 + cs + k} \tag{2}$$

- Frequency response: $s = j\omega$





The "Gang of Four"



By introducing

- Loop transfer function $L = PC$
- Sensitivity function $S = \frac{1}{1+PC} = \frac{1}{1+L}$
- Complementary sensitivity function $T = \frac{PC}{1+PC} = \frac{L}{1+L}$

we have the relations:

$$\begin{bmatrix} q \\ e \\ u \end{bmatrix} = \begin{bmatrix} T & -PS & -T \\ S & -S & -T \\ CS & -T & -CS \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

The transfer functions S, T, PS, CS define all the internal signal relationships of the loop, and are known as the "gang of four".

The Sensitivity Function and Disturbance Response

For a stable plant P without control, the disturbance response (assuming $r = n = 0$) is

$$q = S(s)P(s)d$$

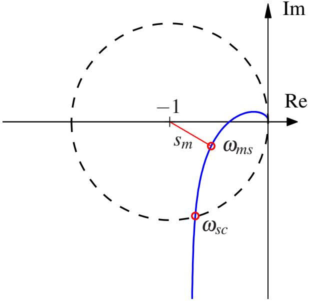
- If $|S(j\omega)| < 1$ then the control system *attenuates disturbances* at that frequency ω;
- If $|S(j\omega)| = 0$, then disturbances at that frequency are *eliminated*. E.g. an integrator does this for zero frequency.
- But if $|S(j\omega)| > 1$ the feedback system actually *makes things worse* than they would have been in open loop.

Interpreting Sensitivity: Distance to Instability

Notice that

$$|S(j\omega)| = \frac{1}{|1 + L(j\omega)|} = \frac{1}{|L(j\omega) - (-1)|}$$

hence the magnitude of the sensitivity function can be interpreted as the inverse of the distance to the -1 point on the Nyquist plot.



The red line is to the peak of the sensitivity magnitude. All the points inside the dotted circle have $|S(j\omega)| > 1$.

Image: K.J. Åström & R.M. Murray (2009) Feedback Systems: An Introduction for Scientists and Engineers

First Limitation: Complementary Sensitivity Function

- The effect of all external signals on tracking error:

$$e = Sr - SPd + Tn$$

  So, ideally, both S and T are small.

- But since $S = \frac{1}{1+L}$ and $T = \frac{L}{1+L}$, we see that[1]

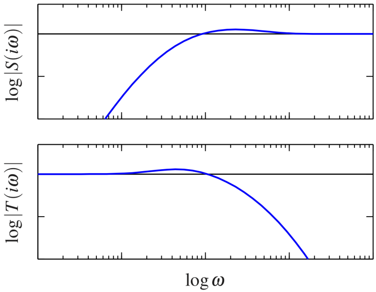
$$S + T = \frac{1+L}{1+L} = 1.$$

The interpretation is that we cannot have **both** $|S(j\omega)|$ and $|T(j\omega)|$ small for all frequencies. Hence we must choose which frequencies should have $|S(j\omega)|$ small, and which should have $|T(j\omega)|$ small. This trade-off can be interpreted as choosing the level of *trust* in measurements.

[1]Because of this relationship, we call S the *sensitivity function* and T the *complementary sensitivity function*.

Shaping S and T

- For many real systems, reference signals and disturbances have fairly low frequency content (i.e. change infrequently or slowly), while measurement noise tends to be high-frequency.
- For such systems, it makes sense to have $S(j\omega)$ small at low frequencies, and $T(j\omega)$ small at high frequencies.



This can be achieved by having $L(j\omega) \gg 1$ at low frequencies, and $L(j\omega) \ll 1$ at high frequencies.

Image: K.J. Åström & R.M. Murray (2009) Feedback Systems: An Introduction for Scientists and Engineers

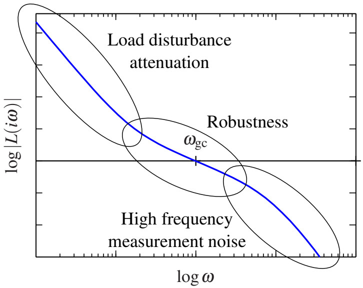
Second Limitation: Bode's gain-phase relation

- We might want the loop gain $|L(j\omega)|$ to decrease *very* rapidly just above the expected bandwidth of the reference and disturbances.
- Unfortunately, Bode's gain-phase relation says this is impossible:

$$\angle L(j\omega_{gc}) \approx 90^\circ \times \frac{d \log |L(j\omega_{gc})|}{d \log \omega_{gc}} \tag{3}$$

- I.e. the phase is approximately proportional to the derivative of the gain, on a Bode plot. Each $\pm$ 20 dB/dec slope corresponds to a $\pm 90^\circ$ phase shift.
- This means a rapid loop-gain roll off will induce a large phase lag, which will have a damaging effect on the phase margin and can lead to instability

Desired Loop Shape



The loop transfer function $L = PC$ should have

- Large gain at low frequency for disturbance rejection and reference tracking, since $S = \frac{1}{1+L}$.
- Low gain at high frequency, for rejection of measurement noise, since $T = \frac{L}{1+L}$.
- Moderate slope at crossover frequency ω_{gc} for phase margin, since Bode's relation says $\angle G(j\omega) \approx \frac{\pi}{2} \frac{d \log|G(j\omega)|}{d \log \omega}$

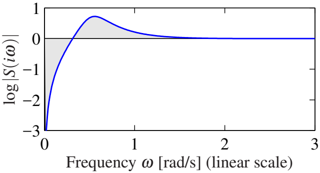
Image: K.J. Åström & R.M. Murray (2009) Feedback Systems: An Introduction for Scientists and Engineers

Third Limitation: Bode's sensitivity integral

Consider a stable open-loop system, then

$$\int_0^\infty \log|S(j\omega)| d\omega = 0$$

If $\log|S(j\omega)| < 0$ for some frequencies, i.e. $|S(j\omega)| < 1$, then we must have $\log|S(j\omega)| > 0$, i.e. $|S(j\omega)| > 1$ for others.

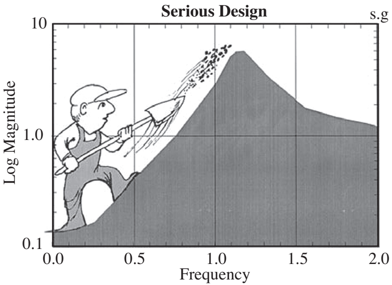


This is known as the **waterbed effect**: if sensitivty is "pushed down" somewhere, it always "pops up" somewhere else, like a waterbed. Worse for unstable systems since the integral is positive.

Image: K.J. Åström & R.M. Murray (2009) Feedback Systems: An Introduction for Scientists and Engineers

Conservation of Dirt

Stein interpreted this as a conservation law: “conservation of dirt”. The designer reduces sensitivity in some frequency band, but is always adding sensitivity in another..



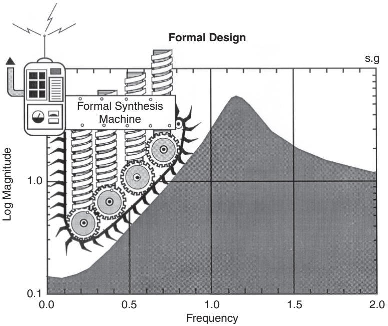
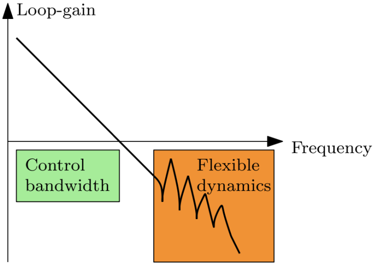
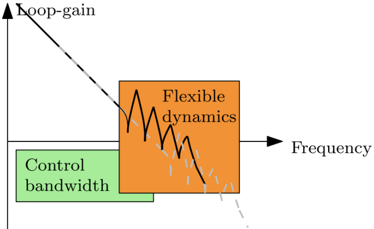
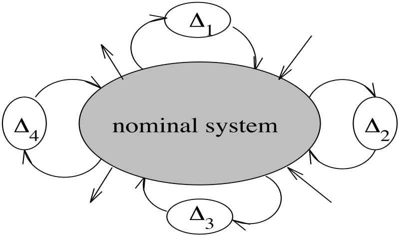


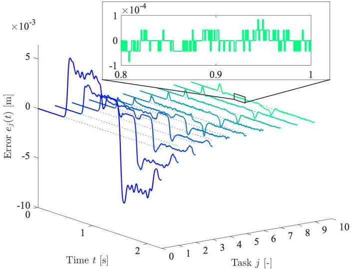
Image: G. Stein (2003). Respect the Unstable, *IEEE Control Syst. Mag.* (lecture on Youtube)

Advanced Methods









Images: T. Oomen. Advanced Motion Control for Precision Mechatronics, *IEEJ J. Industry Appl.*, and Zhou, Doyle, & Glover, Robust & Optimal Control.