Calculus II

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5.1 The Natural Logarithmic Function: Differentiation

Definition of the Natural Logarithmic Function

The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.

Properties of the Natural Logarithmic Function

The natural logarithmic function has three important properties.

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

Logarithmic Properties

If a and b are positive numbers and n is rational, then the four properties below are true.

$\ln 1 = 0$

$\ln(a^n) = n \ln a$

$\ln(ab) = \ln a + \ln b$

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Definition of e

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$$

Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

$$\frac{d}{dx}\left[\ln x\right] = \frac{1}{x}, \qquad x > 0$$

$$\frac{d}{dx}\left[\ln u\right] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \qquad u > 0$$

Derivative Involving Absolute Value

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}\left[\ln |u|\right] = \frac{u'}{u}.$$

5.2 The Natural Logarithmic Function: Integration

Log Rule for Integration

Let u be a differentiable function of x.

$$\int \frac{1}{x}\, dx \;=\; \ln |x| + C$$

$$\int \frac{1}{u}\, du \;=\; \ln |u| + C$$

Alternative Form of the Log Rule

Since $\dfrac{du}{dx} = u'$, we can write the Log Rule in the following useful form:

$$\int \frac{u'}{u}\, dx \;=\; \ln |u| + C.$$

Guidelines for Integration

1. Learn a basic list of integration formulas.
2. Find an integration formula that resembles all or part of the integrand and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. When you cannot find a u-substitution that works, try altering the integrand.

   You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division.
4. When a graphing utility that finds antiderivatives symbolically is available, use it.
5. Check your result by differentiating to obtain the original integrand.

Integrals of the Six Basic Trigonometric Functions

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

5.3 Inverse Functions

Definition of Inverse Function

A function g is the **inverse function** of the function f when

$$f(g(x)) = x \quad \text{for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \quad \text{for each } x \text{ in the domain of } f.$$

The function g is denoted by f^{-1}.

Reflective Property of Inverse Functions

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Guidelines for Finding an Inverse Function

1. Use Theorem 5.7 to determine whether the function $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \qquad f'(g(x)) \neq 0.$$

5.4 Exponential Functions: Differentiation and Integration

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x.$$

That is, $y = e^x$ if and only if $x = \ln y$.

Operations with Exponential Functions

Let a and b be any real numbers.

$$e^a e^b = e^{a+b}$$

$$\frac{e^a}{e^b} = e^{a-b}$$

Properties of the Natural Exponential Function

1. The domain of $f(x) = e^x$ is
$$(-\infty, \infty)$$
and the range is
$$(0, \infty).$$

2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.

3. The graph of $f(x) = e^x$ is concave upward on its entire domain.

4. $$\lim_{x \to -\infty} e^x = 0$$

5. $$\lim_{x \to \infty} e^x = \infty$$

Derivatives of the Natural Exponential Function

Let u be a differentiable function of x.

$$\frac{d}{dx}\left[e^x\right] = e^x$$

$$\frac{d}{dx}\left[e^u\right] = e^u \frac{du}{dx}$$

Integration Rules for Exponential Functions

Let u be a differentiable function of x.

$$\int e^x \, dx = e^x + C$$

$$\int e^u \, du = e^u + C$$

5.5 Bases Other than e and Applications

Definition of Exponential Function to Base a

If a is a positive real number ($a \neq 1$) and x is any real number, then the **exponential function to the base** a is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

Definition of Logarithmic Function to Base a

If a is a positive real number ($a \neq 1$) and x is any positive real number, then the **logarithmic function to the base** a is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

Properties of Inverse Functions

1. $y = a^x$ if and only if $x = \log_a y$
2. $a^{\log_a x} = x$, for $x > 0$
3. $\log_a a^x = x$, for all x

Derivatives for Bases Other than e

Let a be a positive real number ($a \neq 1$), and let u be a differentiable function of x.

$$\frac{d}{dx}\left[a^x\right] = (\ln a)a^x \qquad \frac{d}{dx}\left[\log_a x\right] = \frac{1}{(\ln a)x}$$

$$\frac{d}{dx}\left[a^u\right] = (\ln a)a^u \frac{du}{dx} \qquad \frac{d}{dx}\left[\log_a u\right] = \frac{1}{(\ln a)u}\frac{du}{dx}$$

The Power Rule for Real Exponents

Let n be any real number, and let u be a differentiable function of x.

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1} \qquad \frac{d}{dx}\left[u^n\right] = nu^{n-1}\frac{du}{dx}$$

A Limit Involving e

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

Summary of Compound Interest Formulas

In the formulas below, P is the amount deposited, t is the number of years, A is the balance after t years, r is the annual interest rate (in decimal form), and n is the number of compoundings per year.

1. Compounded n times per year:
$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

2. Compounded continuously:
$$A = Pe^{rt}$$

5.6 Indeterminate Forms and L'Hôpital's Rule

The Extended Mean Value Theorem

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

5.7 Inverse Trigonometric Functions: Differentiation

Definitions of Inverse Trigonometric Functions

Function	**Domain**	**Range**
$y = \arcsin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$
$y = \operatorname{arccot} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \operatorname{arcsec} x$ iff $\sec y = x$	$\lvert x\rvert \ge 1$	$0 \le y \le \pi, \quad y \ne \dfrac{\pi}{2}$
$y = \operatorname{arccsc} x$ iff $\csc y = x$	$\lvert x\rvert \ge 1$	$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}, \quad y \ne 0$

Properties of Inverse Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \qquad \text{and} \qquad \arcsin(\sin y) = y.$$

If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \qquad \text{and} \qquad \arctan(\tan y) = y.$$

If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\operatorname{arcsec} x) = x \qquad \text{and} \qquad \operatorname{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}\left[\arcsin u\right] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\left[\operatorname{arccot} u\right] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}\left[\arccos u\right] = \frac{-u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\left[\operatorname{arcsec} u\right] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}\left[\arctan u\right] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}\left[\operatorname{arccsc} u\right] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Basic Differentiation Rules for Elementary Functions

$$\frac{d}{dx}[cu] = cu'$$

$$\frac{d}{dx}[u \pm v] = u' \pm v'$$

$$\frac{d}{dx}[uv] = uv' + vu'$$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[u^n] = nu^{n-1}u'$$

$$\frac{d}{dx}[x] = 1$$

$$\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$$

$$\frac{d}{dx}[\ln u] = \frac{u'}{u}$$

$$\frac{d}{dx}[e^u] = e^u u'$$

$$\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$$

$$\frac{d}{dx}[a^u] = (\ln a)a^u u'$$

$$\frac{d}{dx}[\sin u] = (\cos u)u'$$

$$\frac{d}{dx}[\cos u] = -(\sin u)u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)u'$$

$$\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$$

$$\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$$

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

5.8 Inverse Trigonometric Functions: Integration

Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$$

Basic Integration Rules ($a > 0$)

$$\int kf(u)\,du = k\int f(u)\,du$$

$$\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$$

$$\int du = u + C$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int e^u\,du = e^u + C$$

$$\int a^u\,du = \left(\frac{1}{\ln a}\right) a^u + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\,du = -\ln|\csc u + \cot u| + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$$

5.9 Hyperbolic Functions

Definitions of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}, \quad x \neq 0$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\coth x = \frac{1}{\tanh x}, \quad x \neq 0$$

Hyperbolic Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\coth^2 x - \operatorname{csch}^2 x = 1$$

$$\sinh^2 x = \frac{-1 + \cosh 2x}{2}$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\cosh^2 x = \frac{1 + \cosh 2x}{2}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

Derivatives and Integrals of Hyperbolic Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh u] = (\cosh u)u' \qquad \int \cosh u \, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = (\sinh u)u' \qquad \int \sinh u \, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u' \qquad \int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u' \qquad \int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \, \tanh u)u' \qquad \int \operatorname{sech} u \, \tanh u \, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \, \coth u)u' \qquad \int \operatorname{csch} u \, \coth u \, du = -\operatorname{csch} u + C$$

Inverse Hyperbolic Functions

Function	**Domain**
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$	$[1, \infty)$
$\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$	$(-1, 1)$
$\coth^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{x+1}{x-1}\right)$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$	$(0, 1]$
$\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty, 0) \cup (0, \infty)$

Differentiation and Integration Involving Inverse Hyperbolic Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh^{-1} u] = \frac{u'}{\sqrt{u^2+1}} \qquad \frac{d}{dx}[\coth^{-1} u] = \frac{u'}{1-u^2}$$

$$\frac{d}{dx}[\cosh^{-1} u] = \frac{u'}{\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{sech}^{-1} u] = \frac{-u'}{u\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\tanh^{-1} u] = \frac{u'}{1-u^2} \qquad \frac{d}{dx}[\operatorname{csch}^{-1} u] = \frac{-u'}{\lvert u\rvert\sqrt{1+u^2}}$$

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln\left|\frac{a+u}{a-u}\right| + C$$

$$\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a} \ln\left(\frac{a + \sqrt{a^2 \pm u^2}}{\lvert u\rvert}\right) + C$$