1 Chapter 1

2 Chapter 2

2.1 Definitions

Spaces:

$\mathbb{R}^n$	Euclidean space
$D^n = \{x \in \mathbb{R}^n \| \|x\| \leq 1\}$	n-disk
$S^{n-1} = \{x \in \mathbb{D}^n \| \|x\| = 1\}$	$n-1$-sphere
$E^n = D^n - S^{n-1}$	n-cell
$I^n = \{x \in \mathbb{R}^n \| 0 \leq x_i \leq 1\}$	n-cube
$\partial I^n = \{x \in \mathbb{I}^n \| \exists i, x_i = 0, 1\}$	boundary of I^n
$\Delta^n = \Delta[n] = \{x \in \mathbb{R}^{n+1} \| x_i \geq 0, \sum_i x_i = 1\}$	n-simplex
$\partial \Delta^n = \{x \in \Delta^n \| \exists i, x_i = 0\}$	Boundary of n-simplex

Path: $u : [a, b] \to X$ from $x = u(a)$ to $y = u(b)$ (usually reparametrized to $[0, 1] \to X$)
Inverse path: $u^- : t \to u(1-t)$ from y to x
Product path: $u * v : t \to \begin{cases} u(2t) & t \leq \frac{1}{2} \\ v(2t-1) & \frac{1}{2} \leq t \end{cases}$
Constant path: $k_x : t \to x$
$\pi_0 : \text{TOP} \to \text{SET}$

- $\pi_0(X)$: Set of path connected components of X
- $\pi_0(f)([x]) = [f(x)]$

$W : \text{TOP} \to \text{CAT}$

- $W(X)$: Paths $u : [0, a] \to X$ and composition is defined on $[0, a+b]$ for associativity
- $W(f)(x) = f(x), W(f)(u) = f \circ u$

2.2 Homotopy notions

Homotopy: $H_t : X \times [0, 1] \to Y$ from $f = H_0 : X \to Y$ to $g = H_1 : X \to Y$; $H : f \simeq g$ (composition/inverse immediate)
Homotopy $H_t : X \to Y$ relative to $A \subset X$ if $H_t : A \to Y$ is independent of t
Homotopy between f and a constant map is a null homotopy
Null homotopy of $\text{id}_X : X \to X$ is a contraction
Path category $W(X, Y)$

- Objects: $f : X \to Y$
- Morphisms: Homotopy $H_t : [0, a] \times X \to Y$ between f and g

hTOP is TOP quotiented by the homotopy relation.

hTOP	TOP
Isomorphic	Homotopy equivalent/Same homotopy type
Isomorphic to $\{*\}$	Contractible
Isomorphism	h-equivalence
Constant map	Null homotopic

Hom functors in hTOP of $f : X \to Y$:

$$f_* : [Z, X] \to [Z, Y], g \to fg \quad f^* : [Y, Z] \to [X, Z], h \to hf$$

Remark. *Generally lower index for covariant and upper index for contravariant*

TOP^0: Category of pointed spaces
hTOP^0: Quotient of TOP^0 by homotopy
Forgetful functor $\mathrm{TOP}^0 \to \mathrm{TOP}$ has a left adjoint, $X \to (X + \{*\}, *)$

Remark. *The smash product $A \wedge B = \frac{A\times B}{A\vee B}$ is always compatible with homotopies and is a tensor product in some appropriate subcategory, i.e. compactly generated spaces*

TOP(2): Pairs of topological spaces $A \subset X$
Note that the product we use here is not the categorical product, instead it is defined as

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$$

so that $(I^m, \partial I^m) \times (I^n, \partial I^n) = (I^{m+n}, \partial I^{m+n})$
TOP(3): Pairs of topological spaces $A \subset B \subset X$
TOP_B: Slice category, objects are morphisms $x : X \to B$ and morphisms are $f : X \to Y$ such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x"'] & & Y \arrow[dl, "y"] \\ & B & \end{tikzcd}$$

- A morphism from $\mathrm{id}_B : B \to B$ to $p : E \to B$ is a section of p
- If $p \cong \mathrm{id}_B$ in hTOP_B, then it is shrinkable

TOP^K: Coslice category, objects are morphisms $a : K \to A$ and morphisms are $f : A \to B$ such that

$$\begin{tikzcd} & K \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

- A morphism from $i : K \to X$ to $\mathrm{id}_K : K \to K$ is a retraction of i and i is an embedding
- $i : K \subset X$, then K is a retract of X
- If $i \cong \mathrm{id}_K$ in hTOP^K, then it is a deformation retract

Note that $\mathrm{TOP}_{\{*\}} \cong \mathrm{TOP}$ and $\mathrm{TOP}^{\{*\}} \cong \mathrm{TOP}^0$
$H_t : A \to B$ is a homotopy in the (co)slice category if each $H_t, t \in [0, 1]$ is a morphism in the (co)slice category, hence we get the quotient categories $\mathrm{hTOP}^K, \mathrm{hTOP}_B$.

2.3 Internal hom objects

Let Y^X or $F(X,Y)$ be the set of continuous maps from X to Y with the compact open topology. Suppose that X is locally compact, then Y^X is the exponential object, i.e.

$$\begin{tikzcd}
X \times Y \arrow[d, dotted, "f^\wedge \times \mathrm{id}_Y"'] \arrow[dr, dash, "f"] & \\
Z^Y \times Y \arrow[r, "e_{Y,Z}"'] & Z
\end{tikzcd}$$

f induces $f^\wedge$ and $f^\wedge$ induces f, alternatively

$$\operatorname{Hom}(- \times Y, Z) \cong \operatorname{Hom}\left(-, Z^Y\right)$$

which also tells us the functors $-^Y$ is a right adjoint to $- \times Y$.
Unfortunately in categories with zero objects, i.e. TOP^0, then exponential objects generally dont exist unless the category is trivial as if Y^X exists, we have

$$\operatorname{Hom}(X,Y) \cong \operatorname{Hom}(0 \times X, Y) \cong \operatorname{Hom}\left(0, Y^X\right) \cong \{*\}$$

However, we may have some form of tensor-hom adjuncation.
In the category TOP^0, we define $F^0(X,Y)$ as the subspace of pointed maps of $F(X,Y)$ and the constant map is the basepoint. Any pointed map $X \times Y \to Z$ induces a pointed map $X \to F^0(X,Y)$ if it sends $X \times y \cup x \times Y$ to z, hence it corresponds to maps from $X \wedge Y \to Z$. The adjuncation in this case reduces to

$$F^0(X \wedge Y, Z) \cong F^0\left(X, F^0(Y,Z)\right)$$

when X, Y are locally compact. This gives us our tensor-hom adjuncation.
If we quotient by homotopy and assume X is locally compact and $e^0_{X,Y}$ is continuous, then we get

$$[X \wedge Y, Z]^0 \cong \left[X, F^0(X,Y)\right]^0$$

2.4 Fundamental groupoid

$\Pi : \mathrm{TOP} \to \mathrm{GRPd}$

- $\Pi(X)$: Quotient of $W(X)$ by homotopy

Somewhat cleaner way to state van-Kampen theorem:

Theorem (Seifert–Van Kampen [May, Thm 2.7]). *Suppose that $\mathcal{U}$ is a covering of X such that if $U_1, U_2 \in \mathcal{U}$, then $U_1 \cap U_2 \in \mathcal{U}$. This turns $\mathcal{U}$ into a category where morphisms are inclusions, then*

$$\Pi(X) \cong \operatorname{colim}_{U \in \mathcal{U}} \Pi(U)$$

Choosing a base point, we get the functor $\pi_1 : \mathrm{TOP}^0 \to \mathrm{GRP}$.

Remark. *Proposition 2.7.3 of tom Dieck that the fundamental group of a monoid in* TOP^0 *is commutative and agrees with the monoid operation comes from a more general theorem, the Eckmann–Hilton argument*

Theorem (Eckmann-Hilton argument)**.** *If* $\cdot, *$ *are unital binary operations on* X *with units* $1.$ *and* 1_* *such that*

$$(a \cdot b) * (c \cdot d) = (a * b) \cdot (c * d)$$

Then $\cdot, *$ *coincide, are associative and commutative*

2.5 Enriching TOP

We give a groupoid structure, $\Pi(X, Y)$ to each hom set $\mathrm{Hom}_{\mathrm{TOP}}(X, Y)$ with homotopy as morphisms. This provides us with a 2-category, i.e.

$$\begin{tikzcd}
X \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=60, "h"'{name=H}] & Y \arrow[r, bend left=60, "u"{name=U}] \arrow[r, "v"{name=V, description}] \arrow[r, bend right=60, "w"'{name=W}] & Z
\arrow[Rightarrow, from=F, to=G, "\alpha"]
\arrow[Rightarrow, from=G, to=H, "\beta"]
\arrow[Rightarrow, from=U, to=V, "\gamma"]
\arrow[Rightarrow, from=V, to=W, "\delta"]
\end{tikzcd}$$

such that all the compositions makes sense, i.e.

$$(\delta\gamma)\,(\beta\alpha) = (\delta\beta)\,(\gamma\alpha)$$

We can enrich similar categories like TOP^0

3 Chapter 3

3.1 Definitions

Suppose $p : E \to B$ is surjective and $U \subset B$ is open

- **Trivialization** of p over U is a homeomorphism $p^{-1}(U) \to U \times F$
- p is **locally trivial** if a open covering $\mathcal{U}$ exists where a trivialization of p over $U \in \mathcal{U}$ exists for all U
- $\mathcal{U}$ is a **bundle chart**
- F is the **typical fibre**
- p is **trivial over** U if a bundle chart over U exists
- **Bundles/Fibre bundles** are locally trivial maps

Covering space/Covering of B is a locally trivial trivial map $p : E \to B$ with discrete fibres

- If $\phi_U : p^{-1}(U) \to U \times F$ is a trivialization, then $\phi_U^{-1}(U \times \{*\})$ are the **sheets** over U
- If $|F| = n$, then p is a n-fold covering
- A **trivial covering** is the covering $p : B \times F \to B$
- U is **admissible** or **evenly covered** if a trivialization exists
- E is the **total space** and B is the **base space**

3.2 Coverings with group actions

A **left** G**-principal covering** is a covering $p : E \to B$ and a properly discontinuous group action G on E such that $p(gx) = p(x)$ and the action on fibres are transitive
$\alpha \in \mathrm{Aut}(p)$ if $\alpha : p \to p$ is a morphism in TOP_B. These are **deck transformations**
The map $x \to gx$ gives a map $G \to \mathrm{Aut}(p)$

Theorem (Galois correspondence)**.** *Let $p : E \to B$ be a covering, then*

- *If E is connected,* $\mathrm{Aut}(p)$ *is a properly discontinuous action on E*
- *If B is locally path connected, H subgroup of* $\mathrm{Aut}(p)$*, then $E/H \to B$ is a covering*

A **right** G**-principal covering** is a covering $p : E \to B$ and a properly discontinuous group action G on E such that $p(xg) = p(x)$ and the action on fibres are transitive.
Let F be a set with a left G action, then the space $E \times_G F$ constructed by quotienting $E \times F$ by $(xg, f) = (x, gf)$ is an **associated covering**.

Remark. *Seems like for this part we need to assume that that G is a free action of the fibres as well and F is given the discrete topology*

Theorem. *The map $p_F : E \times_G F \to B, (x, f) \to p(f)$ is a covering with typical fibre F*

Proof. Suppose that $\mathcal{F}$ is the typical fibre of p.
First we show the typical fibre of p_F is F. It's immediate that the typical fibre is given by $\frac{\mathcal{F}\times F}{\sim}$ immediately showing discreteness. Next, notice that $\{(\mathfrak{f}, f)\,|\,f \in F\}$ are the representatives of $\frac{\mathcal{F}\times F}{\sim}$ for some arbitrary $\mathfrak{f}$ as supose $\mathfrak{f}' = \mathfrak{f}g$, then $(\mathfrak{f}', f) = (\mathfrak{f}, gf)$ and $(\mathfrak{f}, f) = (\mathfrak{f}, f')$ implies that either $\mathfrak{f}$ has a nontrivial stabalizer or $f = f'$, hence we need to assume the action is free on $\mathcal{F}$.
Next we show that this is indeed a covering. Suppose U has a trivialization, i.e. $p^{-1}(U) \cong U \times \mathcal{F}$. Then $\frac{p^{-1}(U)\times F}{\sim} \cong \frac{U\times\mathcal{F}\times F}{\sim} \cong U \times F$. Hence p_F is a covering with typical fibre F. $\square$

This gives us the functor

$$A(p) : G\text{-}\mathrm{SET} \to \mathrm{COV}_B$$

from the category of sets with a left G action to the category of covering spaces over B (a subcategory of TOP_B).
If $A(p)$ is an equivalence of categories, then p is the universal cover

3.3 Lifting

$F : X \to E$ is a **lifting** of $f : X \to B$ along $p : E \to B$ if $pF = f$, i.e. a morphism in TOP_B
If X is connected and p is a covering, liftings that agree somewhere are unique.
A map $p : E \to B$ has **homotopy lifting property** (HLP) for a space X if for each homotopy h_t and initial condition H_0, we can extend H_0 to the homotopy H_t such that the diagram commutes:

$$\begin{tikzcd} X \arrow[rr, dotted, "H_t"] \arrow[dr, "h_t"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

H is a lifting of h with initial conditions a. p is a **fibration** if it has HLP for all spaces

Theorem. *Coverings $p : E \to B$ are fibrations*

Proof. First show that projection maps $U \times F \to U$ are fibrations, then glue these projection maps and use uniqueness of liftings $\square$

As liftings along coverings are unique, the diagram below is a pullback:

$$\begin{tikzcd} E^I \arrow[r, "p^I"] \arrow[d, "e^0_E"'] & B^I \arrow[d, "e^0_B"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

Let $p : E \to B$ be a map with HLP for I and $F_b = p^{-1}(b)$.
For every map $[v] \in \Pi(B)$, we obtain a well defined map $v_\sharp : \pi_0(F_b) \to \pi_0(F_c)$. Suppose $V : I \to E$ is a lifting of v with $V(0) = x$, then $v_\sharp[x] = [V(1)]$.
With this we obtain the **transport functor** $T_p : \Pi(B) \to \mathrm{SET}$

- $b \to \pi_0(B)$
- $[v] \to v_\sharp$

Let $p(x) = b$ and let $\partial_x : \pi_1(B, b) \to \pi_0(F_b, x), [v] \to v_\sharp(x)$ and $i : F_b \subset E$, then we have the exact sequence

$$
\begin{tikzcd}
\pi_1(F_b, x) \arrow[r, "i_*"] & \pi_1(E, x) \arrow[r, "p_*"] & \pi_1(B, b) \arrow[dll, "\partial_x"] \\
\pi_0(F_b, x) \arrow[r, "i_*"] & \pi_0(E, x) \arrow[r, "p_*"] & \pi_0(B, b)
\end{tikzcd}
$$

as well as the isomorphisms of sets $\partial_x : \frac{\pi_1(B,b)}{p_*\pi_1(E,x)} \cong \pi_0(F_b, x), i_* : \frac{\pi_0(F_b,x)}{\pi_1(B,b)} \cong \pi_0(E, x)$
For a covering $p : E \to B$ with B path connected, the exact sequence simplifies to

$$
\begin{tikzcd}
1 \arrow[r] & \pi_1(E, x) \arrow[r, "p_*"] & \pi_1(B, b) \arrow[dll, "\partial_x"] \\
\pi_0(F_b, x) \arrow[r, "i_*"] & \pi_0(E, x) \arrow[r] & \{*\}
\end{tikzcd}
$$

Furthermore suppose that $p : E \to B$ is a right G-principal covering with E path connceted, then we get the exact sequence

$$
1 \longrightarrow \pi_1(E, x) \xrightarrow{p_*} \pi_1(B, b) \xrightarrow{\delta_x} G \longrightarrow 1
$$

and the image of p_* is normal.

3.4 Coverings

Outline:

1. Construct the inverse X of $T : \mathrm{TRA}_B \to \mathrm{COV}_B$ that exists and is an equivalence of categories for sufficiently nice B
2. Construct the functor $\epsilon_B : \mathrm{TRA}_B \to \pi_b\text{-}\mathrm{SET}$ and the inverse η_b
3. Hence $A(p) : G\text{-}\mathrm{SET} \to \mathrm{COV}_B$ is an equivalence of categories iff the total space of p is simply connected

Let $\mathrm{TRA}_B = [\Pi(B), \mathrm{SET}]$, the transport functor in the previous section yields the functor $T : \mathrm{COV}_B \to \mathrm{TRA}_B$.
If B is path connected and T is an equivalence of categories, then B is a **transport space**.
A set $U \in B$ is **transport simple** if any paths in U between identical points are homotopic in B.
B is **semi-locally simply connected** if it has an open covering with transport simple sets.
B is **transport local** if it is path connected, locally path connected and semi-locally simply connected.

Theorem. *If B is then B is a transport space*

Proof. We need to construct the inverse of T, $X : \mathrm{TRA}_B \to \mathrm{COV}_B$. Let $\Phi : \Pi(B) \to \mathrm{SET}$ be some functor, we will construct a covering $p : X(\Phi) \to B$. As a set $X(\Phi) = \coprod_{b \in B} \Phi(b)$. To get a reasonable topology on it, we consider a covering $\mathcal{U}$ of B by transport simple path connected open sets. For every $b \in U \in \mathcal{U}$, we define $\phi_{U,b} : U \times \Phi(b) \to p^{-1}(U)$ and by gluing these maps together, we obtain a topology on $X(\Phi)$ and a covering p.
Verification of functoriality and inverse are somewhat direct from definition. $\square$

With this, consider the hom functor $\mathrm{Hom}_{\Pi(B)}(b, -) \in \mathrm{TRA}_B$ and let $p^b : E^b \to B$ be its associated covering. Then E^b is simply connected right $\mathrm{Hom}_{\Pi(B)}(b, b)$-principal covering.
Suppose that B is path connected, then $\Pi = \Pi(B)$ is a connected groupoid. Let $\Pi(x, y) = \mathrm{Hom}_{\Pi(B)}(x, y)$ and $\pi_b = \Pi(b, b)$
For a functor $F : \Pi \to \mathrm{SET}$, we have the left π_B-set $F(b)$ giving us the functor $\epsilon_b : \mathrm{TRA}_B \to \pi_b\text{-}\mathrm{SET}$.
For a left π_B-set A, we define the functor $\Pi(b, -) \times_{\pi_B} A : \Pi \to \mathrm{SET}$ where $A \times_G B$ is the set $A \times B$ quotiented by $(ag, b) = (a, gb)$. This gives us the functor $\eta_b : \pi_b\text{-}\mathrm{SET} \to \mathrm{TRA}_B$, the inverse of ϵ_b.
Finally we have the following categories and functors:

$$G\text{-}\mathrm{SET} \xrightarrow{A(p)} \mathrm{COV}_B \underset{X}{\overset{T}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathrm{TRA}_B \underset{\eta_b}{\overset{\epsilon_b}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \pi_b\text{-}\mathrm{SET}$$

where X exists if B is transport-local and ϵ_B, η_B require B to be path connected to exist.
Finally we have

Theorem. *The following are equivalent:*

- *B is a transport space, i.e. T is an equivalence of categories*
- *B has a universal right G-principal covering $p : E \to B$, i.e. $A(p)$ is an equivalence of categories*

Note that the exact sequences above imply that E is simply connected.
Define the **orbit category** $\mathrm{Or}(G)$ consisting of homogenous G-sets ($\frac{G}{H}$ for any subgroup H) and G-maps. This category is a strict subcategory of $G\text{-}\mathrm{SET}$, consisting only the transitive sets.

For a covering $p : E \to B$, we obtain the injective map $p_* : \pi_1(E, x) \to \pi_1(B, p(x))$ and the image is called the **characteristic subgroup** of p wrt x.
Let $p : E \to B$ be a simply connected covering, then the subcategory $A(p)\,(\mathrm{Or}(\pi_b))$ of COV_B is equivalent to the subcategory consisting of connected coverings. This tells us that the connected coverings of a transport space is determined by subgroups of the fundamental group.

Theorem. *Let B be a transport space and $p : E \to B$ a covering.*

- *The action of* $\mathrm{Aut}(p)$ *on E makes it a left-*$\mathrm{Aut}(p)$ *principal covering*
- *A simply connected covering is a universal covering*
- *Universal coverings are unique up to isomorphism*
- *The automorphism group of a universal cover is isomorphic to* $\pi_1(B, b)$
- *E^b is simply connected*
- *We have a Galois correspondence between isomorphism classes of connected coverings and subgroups of* $\pi_1(B, b)$

If B is not a transport space but is path connected and locally path connected, then we have a similar result where coverings by path connected total spaces are isomorphic iff the characteristic subgroups are conjugate in $\pi_1(B, b)$.
Suppose we have coverings $p : E \to B$ and $f : Z \to B$, then a covering $\Phi : Z \to E$ such that

$$\begin{tikzcd} Z \arrow[rr, dashed, "\Phi"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

exists iff $f_*\pi_1(Z, z) \subset p_*\pi_1(E, x)$ for some $f(z) = p(x)$.
If X is a topological group with identity x and $p : E \to X$ is a covering with E path connected and locally path connected, then for each $e \in p^{-1}(b)$, there exists a unique group structure on E such that e is the identity and p a homomorphism.

4 Chapter 4

4.1 Mapping cylinders

For a map $f : X \to Y$ **mapping cylinder** $Z(f)$ is constructed by the pushout

$$\begin{tikzcd}
X+X \arrow[r, "\mathrm{id}+f"] \arrow[d, "\langle i_0, i_1\rangle"'] & X+Y \arrow[d, dashed, "\langle j, J\rangle"] \arrow[dr, bend left, "\langle f, \mathrm{id}\rangle"] & \\
X \times I \arrow[r, dashed, "a"'] \arrow[rr, bend right, "f"'] & Z(f) \arrow[r, dashed, "q"] & Y
\end{tikzcd}$$

We also have $Jq \cong \mathrm{id}$ as $X \times I \cong X$ and $f = qj$, j a closed immersion and q a homotopy equivalence. We see that Z has nice functorial properties. Suppose we have the homotopy commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\
X' \arrow[r, "f'"] \arrow[d, "\alpha'"'] & Y' \arrow[d, "\beta'"] \\
X'' \arrow[r, "f''"] & Y''
\end{tikzcd}$$

and let the homotopy equivalences be $\Phi : f'\alpha \cong \beta f, \Phi' : f''\alpha' \cong \beta' f'$
This induces the following homotopy commutative diagram

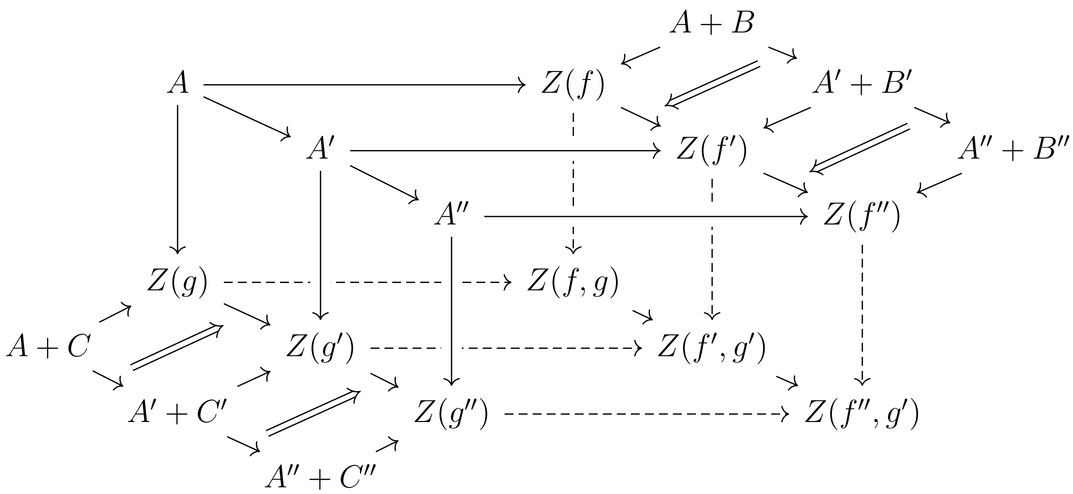
$$\begin{tikzcd}
X+Y \arrow[r] \arrow[d, "\alpha+\beta"'] & Z(f) \arrow[d, "Z(\alpha,\beta,\Phi)"] \\
X'+Y' \arrow[r] \arrow[d, "\alpha'+\beta'"'] & Z(f') \arrow[d, "Z(\alpha',\beta',\Phi'')"] \\
X''+Y'' \arrow[r] & Z(f'')
\end{tikzcd}$$

where each small square commutes in TOP and the whole diagram commutes in hTOP.
Given maps $f : A \to B$ and $g : A \to C$, we can construct the double mapping cylinder by either of the two pushouts:

$$\begin{tikzcd}
A+A \arrow[r, "f+g"] \arrow[d, "\langle i_0, i_1\rangle"'] & B+C \arrow[d, dashed, "\langle j_0, j_1\rangle"] \\
A \times I \arrow[r, dashed] & Z(f,g)
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "j^B"] \arrow[d, "j^C"'] & Z(f) \arrow[d, dashed] \\
Z(g) \arrow[r, dashed] & Z(f,g)
\end{tikzcd}$$

The functorality of the double mapping cylinder can be seem from the following commutative homotopy diagram:



Suppose we have the homotopy commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_+"] \arrow[d, "f_-"'] & X_+ \arrow[d, "j_+"] \arrow[ddr, dashed, bend left] & \\
X_- \arrow[r, "j_-"'] \arrow[drr, dashed, bend right] & X & \\
 & & Z\left(f_-, f_+\right) \arrow[ul, dashed, "\phi"]
\end{tikzcd}$$

The square is called a **homotopy pushout** or **homotopy cocartesian** if ϕ is a homotopy equivalence (i.e. a pushout in the category hTOP)

Let $f_\pm, j_\pm$ be inclusions and $X = X_+ \cup X_-$, then the diagram is a pushout in TOP.

Let $N\left(X_-, X_+\right) = X_- \times 0 \cup X_0 \times I \times X_+ \times 1$ be a subspace of $X \times I$ and let $p_N : N\left(X_-, X_+\right) \to X$ be the projection map.

The covering $X_\pm$ is **numerable** if p_N has a section. With this, we have the following condition to determine if the diagram above is a pushout in hTOP:

Theorem 4.1. *$Z\left(f_-, f_+\right) \cong X$ if the covering $X_\pm$ is numerable*

Given the projection maps $X \xleftarrow{f} X \times Y \xrightarrow{g} Y$, the double mapping cylinder $Z(f,g) = X \star Y$ is known as the **join** of X and Y

4.2 Suspensions and loops

Here we work in pointed categories
The **suspension** functor $\Sigma : \mathrm{TOP}^0 \to \mathrm{TOP}^0$ is given by

- $\Sigma X = S^1 \wedge X = \frac{X \times I}{X \times \partial I \cup \{x\} \times I}$
- $S\Sigma_* [X, Y]^0 = [\Sigma X, \Sigma Y]^0$

Note that Σ_* is a homomorphism if $X = \Sigma A$ and any pointed homotopy $H_t : X \to Y$ corresponds uniquely to a pointed map $\overline{K} : \Sigma X \to Y$
The set $[\Sigma X, Y]^0$ has a natural group structure by composition of homotopies. Furthermore as $[\Sigma^n X, Y]^0$ has n natural composition laws by composin the homotopies at the the ith coordinate and these satisfies the assumptions of Eckmann-Hilton argument, they are equivalent. This also tells us that the higher homotopy groups, $\pi_n(X) = [S^n, X]^0$, are commutative groups.
We can dualize everything above:
The **loop** space of X is $\Omega X = F^0 (S^1, X)$. This consists of loops in X with basepoint x. This is naturally a topological group with the product of loops.
There is a natural group structure on $\mathrm{Hom}_{\mathrm{TOP}^0} (X, \Omega Y)$ given by $[f] +_m [g] = [f][g]$.
As S^1 is locally compact, we have the tensor-hom adjuncation

$$[\Sigma X, Y]^0 \cong [X, \Omega Y]^0$$

and this commutes with the group operation.
In the set $[\Sigma X, \Omega Y]^0$, by the Eckmann-Hilton argument, the group operations on these two sets coincide and are commutative.

4.3 Group objects

Perhaps a nicer way of looking at group/any objects is via Yoneda's lemma, see [Wat] for more details:

Theorem 4.2 (Yoneda). *For any category C, we have a full and faithful functor, the Yoneda embedding $y_C : C \to [C^{\mathrm{op}}, \mathrm{SET}]$, given by $y_C(c) \to \mathrm{Hom}_C (-, c)$*

An object $g \in C$ is a **group object** if the functor $y_C(g)$ factors through GRP, i.e. the following diagram commutes where GRP $\to$ SET is the forgetful functor

$$\begin{tikzcd} & \mathrm{GRP} \arrow[dr, hook] & \\ C^{\mathrm{op}} \arrow[ur, dashed, "G"] \arrow[rr, "y_C(g)"'] & & \mathrm{SET} \end{tikzcd}$$

To give an explicit construction, recall that for a group G, we need to have a unit element, an inverse operation and a group operation, given by

$$\begin{aligned} e_G &: 1 \to G \\ \mathrm{inv}_G &: G \to G^{\mathrm{op}} \\ \cdot_G &: G \times G \to G \end{aligned}$$

such that the following diagrams commute to ensure associativity, unit and inverse holds:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id}_G \times \cdot_G"] \arrow[d, "\cdot_G \times \mathrm{id}_G"'] & G \times G \arrow[d, "\cdot_G"] \\ G \times G \arrow[r, "\cdot_G"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(e, \mathrm{id}_G)"] \arrow[d, "(\mathrm{id}_G, e)"'] \arrow[dr, equal] & G \times G \arrow[d, "\cdot_G"] \\ G \times G \arrow[r, "\cdot_G"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\mathrm{inv}_G, \mathrm{id}_G)"] \arrow[d, "(\mathrm{id}_G, \mathrm{inv}_G)"'] \arrow[dr, "e"] & G \times G \arrow[d, "\cdot_G"] \\ G \times G \arrow[r, "\cdot_G"'] & G \end{tikzcd}$$

where $e : G \to G$ is the composite morphism $G \to 1 \xrightarrow{e_G} G$

The maps can immediately be constructed by Yoneda's lemma, take for instance the product map. If g is a group object with $G : C^{\mathrm{op}} \to \mathrm{GRP}$ as the functor to group and $f : c \to d$ is a morphism in C, then the following diagram commutes:

$$\begin{tikzcd} G(c) \times G(c) \arrow[r, "\cdot_{G(c)}"] & G(c) \\ G(d) \times G(d) \arrow[r, "\cdot_{G(d)}"'] \arrow[u, "G(f) \times G(f)"] & G(d) \arrow[u, "G(f)"'] \end{tikzcd}$$

telling us we have a natural transformation $\cdot_G : G \times G \to G$. This gives us the morphism $\cdot_g : g \times g \to g$ by Yoneda's lemma and as the Yoneda embedding is full and faithful, the commutativity of the diagrams above defining a group is immediate.

Similarly one defines cogroups as group objects in the opposite category. With this view, it is immediate that if $c \in C$ is a cogroup, then $\mathrm{Hom}_C(c, -)$ is a functor to GRP.

As examples in hTOP^0, ΣX is a cogroup as $[\Sigma X, Y]^0$ is a group and ΩY is a group as $[X, \Omega Y]^0$ is a group.

4.4 Fibre sequence

Again here we work in pointed spaces. A map $f : X \to Y$ induces a map $f^* : [Y, B]^0 \to [X, B]^0$. The kernels of f is any element that gets sent to the basepoint of Y and the kernel of f^* is any element that gets sent to a nullhomotopic element. This allows us to define exact sequences of topological spaces.

A sequence of spaces $U \xrightarrow{f} V \xrightarrow{g} W$ is **h-coexact** if for all spaces B, the sequence

$$[U, B]^0 \xleftarrow{f^*} [V, B]^0 \xleftarrow{g^*} [W, B]^0$$

is exact.
The **cylinder** $XI = \frac{X\times I}{*\times I}$ describes homotopies in TOP^0 (morphisms $XI \to Y$ are homotopies).
The **cone** $CX = \frac{X\times I}{X\times 0\cup *\times I} = X \wedge I$ describes homotopies starting from constant maps in TOP^0.
Define $C(f)$ as the pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "f_1"] \\
CX \arrow[r, "j"'] & C(f)
\end{tikzcd}$$

and by its universal property, we have the h-coexact sequence $X \xrightarrow{f} Y \xrightarrow{f_1} C(f)$ which can be iterated to create a long h-coexact sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{f_2} C\left(f_1\right) \xrightarrow{f_3} \ldots$$

Furthermore we have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "f"'] & CX \arrow[r, "p"] \arrow[d, "j"] & Y/i_1X & = \Sigma X \\
Y \arrow[r, "f_1"] \arrow[d, "i_1"'] & C(f) \arrow[r, "p(f)"] \arrow[d, "f_2"] & C(f)/f_1Y \arrow[u, phantom] \arrow[d] & = \Sigma X \\
CY \arrow[r, "j_1"'] & C\left(f_1\right) \arrow[r, "q(f)"'] & C\left(f_1\right)/j_1Y & = \Sigma X
\end{tikzcd}$$

and $q(f)$ is a homotopy equivalence.
Applying this to itself, we obtain

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & CX \arrow[d] & \\
Y \arrow[r, "f_1"] \arrow[d] & C(f) \arrow[r] \arrow[d, "f_2"'] & \Sigma X \arrow[dd, "\Sigma f\circ\iota"] \\
CY \arrow[r] & C\left(f_1\right) \arrow[ur, "q(f)"'] \arrow[dr, "p\left(f_1\right)"] \arrow[d, "f_3"'] & \\
 & C\left(f_2\right) \arrow[r, "q\left(f_1\right)"'] & \Sigma Y
\end{tikzcd}$$

where $\iota : (x, t) \to (x, 1-t)$ to ensure commtuativity.
As $q(f), q\left(f_1\right)$ are homotopy equivalences and a sequence remains h-coexact if we replace elements with h-equivalent ones, we obtain the h-coexact sequence

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y$$

And we can apply this sequence to itself iteratively and noting that Σ and C commutes to obtain the **Puppe-sequence** or the **cofibre sequence** of f:

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y \to \Sigma C(f) \to \Sigma C(f) \to \Sigma^2 X \to \Sigma^2 Y \dots$$

as $\Sigma f : \Sigma X \to \Sigma Y$

Applying the functor $[-, B]^0$, we see that from the 4th place onwards these are groups and from the 7th place onwards these are abelian groups.

For any map $f : X \to Y$, let $\mu : C(f) \to \Sigma X \vee C(f)$ be defined as

$$\mu(x,t) = \begin{cases} ((x, 2t), *) & 2t \leq 1 \\ (*, (x, 2t-1)) & 2t \geq 1 \end{cases}$$

and $\mu(y) = y$. This map is a **h-coaction** of the h-cogroup ΣX on $C(f)$ as we have

$$[\Sigma X, B]^0 \times [C(f), B]^0 \cong [\Sigma X \vee C(f), B]^0 \to [C(f), B]$$

where the last map is induced by μ.

With the map $Y \xrightarrow{f_1} C(f) \xrightarrow{p(f)} \Sigma X$ and any maps $\alpha_1, \alpha_2 : \Sigma X \to B$, this group action satisfies $(\alpha_1)\left(p(f)^* \alpha_2\right) = p(f)^* \left(\alpha_1 \alpha_2\right)$ and f_1^* is an injective map on orbits of this action.

We can dualize everything as always.

A sequence of spaces $U \xrightarrow{f} V \xrightarrow{g} W$ is **h-exact** if for all spaces B, the sequence

$$[B, U]^0 \xrightarrow{f^*} [B, V]^0 \xrightarrow{g^*} [B, W]^0$$

is exact.

To dualize the cone, we use the exponential object adjuncation $[X \wedge I, Y]^0 \cong [X, F^0(Y, I)]^0$ and define $FY = F^0(Y, I)$. We then define $F(f)$ as the pullback

$$\begin{tikzcd} F(f) \arrow[r, dashed, "q"] \arrow[d, dashed, "f^1"'] & FY \arrow[d, "e^1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and by its universal property, we have the h-exact sequence $F(f) \xrightarrow{f_1} X \xrightarrow{f} Y$ which can be iterated to create a long h-exact sequence

$$\dots \xrightarrow{f^4} F\left(f^2\right) \xrightarrow{f^3} F\left(f^1\right) \xrightarrow{f^2} F(f) \xrightarrow{f^1} X \xrightarrow{f} Y$$

We can dualize the diagrams above to obtain

$$\begin{tikzcd}
Y & FY \arrow[l, "e^1"'] & \Omega Y \arrow[l, "i"'] \\
X \arrow[u, "f"] & F(f) \arrow[l, "f^1"'] \arrow[u, "q"'] & \Omega Y \arrow[l, "i(f)"'] \arrow[u] \\
FX \arrow[u, "e^1"] & F\left(f^1\right) \arrow[l, "q^1"] \arrow[u, "f^2"'] & \Omega Y \arrow[l, "j(f)"] \arrow[u]
\end{tikzcd}$$

where $j(f)$ is a h-equivalence and using this on itself, we obtain

$$\begin{tikzcd}
Y & FY \arrow[l] & \\
X \arrow[u, "f"] & F(f) \arrow[l, "f^1"'] \arrow[u] & \Omega Y \arrow[l, "i(f)"'] \arrow[dl, "j(f)"] \\
FX \arrow[u] & F\left(f^1\right) \arrow[l] \arrow[u, "f^2"] & \\
& F\left(f^2\right) \arrow[u, "f^3"] & \Omega X \arrow[l, "j\left(f^1\right)"] \arrow[ul, "i\left(f^1\right)"'] \arrow[uu, "\iota \circ \Omega f"']
\end{tikzcd}$$

and finally we obtain the dual long h-exact sequence

$$\Omega X \to \Omega Y \to F(f) \to X \to Y$$

and repeating this on the map $\Omega f : \Omega X \to \Omega Y$, we obtain the long h-exact sequence

$$\ldots \Omega^2 X \to \Omega^2 Y \to \Omega F(f) \to \Omega X \to \Omega Y \to F(f) \to X \to Y$$

known as the **fibre seqence** of f. Similarly applying the functor $[B, -]^0$, we see that from the 4th place onwards these are groups and from the 7th place onwards these are abelian groups.

Finally to dualize the group action, again let $f : X \to Y$ be any map. We have the h-action $m : \Omega Y \times F(f) \to F(f)$ defined as

$$m\left([f(t), (x, g(t))]\right) = \begin{cases} (x, f(2t)) & 2t \leq 1 \\ (x, g(2t-1)) & 2t \geq 1 \end{cases}$$

This map induces the map

$$[B, \Omega Y]^0 \times [B, F(f)]^0 \cong [B, \Omega Y \times F(f)]^0 \to [B, F(f)]^0$$

Furthermore with the maps $\Omega Y \overset{i(f)}{\to} F(f) \overset{f^1}{\to} X$ and any maps $\alpha_1, \alpha_2 : B \to \Omega Y$, this group action satisfies $(i(f)_* \alpha_1)(\alpha_2) = i(f)_*(\alpha_1 \alpha_2)$ and f^1_* is an injective map on the orbits of the action.

Note that these sequences can be used to prove the long exact sequence of homotopy groups and the Mayer Vietoris exact sequence.

5 Chapter 5

5.1 Cofibration properties

$i : A \to X$ has the **homotopy extension property** (HEP) for Y if for every h, f, there exists some H such that the (equivalent) diagrams commutes:

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "e^0"] \\
X \arrow[r, "f"'] \arrow[ur, dashed, "H"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dr, "i_0^X"] \arrow[drr, bend left, "f"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_0^A"'] & & X \times I \arrow[r, dashed, "H"] & Y \\
 & A \times I \arrow[ur, "i\times\mathrm{id}"'] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}
$$

H is an **extension** of h with **initial conditions** f. If i has HEP for every space Y, then it is a **cofibration**. This is somewhat similar to being a monomorphism with a cokernel as this allows the factorization of any nullhomotopic map fi through X/A, although the factorization may not be unique here. We can determine quickly if a map is a cofibration by studying the mapping cylinder due to the following diagram:

$$
\begin{tikzcd}
A \arrow[rrr, "i"] \arrow[dd, "i_0^A"'] & & & X \arrow[dll, "i_0^X"'] \arrow[dd, "b"] \arrow[dddr, bend left, "y"] & \\
 & X \times I \arrow[drr, dashed, "r"] & & & \\
A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[rrr, "k"'] \arrow[drrrr, bend right, "h"'] & & & Z(i) \arrow[ull, dotted, "s"] \arrow[dr, dotted, "\sigma"] & \\
 & & & & Y
\end{tikzcd}
$$

We have the equivalent statements for a map $i : A \to X$:

- i is a cofibration
- i has HEP for $Z(i)$
- $s : Z(i) \to X \times I$ has a retraction

This tells us that cofibrations must be embeddings as $ki_1 = ri_1^X i : A \to Z(i)$ is an embedding. One can further show for Hausdorff spaces that this is a closed embedding.
(X, x) is **well-pointed** and x is **nondegenerate** if $x \in X$ is a cofibration.
Cofibrations are preserved under products of locally compact spaces:

$$
\begin{tikzcd}
A \arrow[r, "h^\wedge"] \arrow[d, "i"'] & \left(Z^Y\right)^I \cong \left(Z^I\right)^Y \arrow[d, "e^0"] \\
X \arrow[r, "f^\wedge"'] \arrow[ur, dashed, "H^\wedge"] & Z^Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \times Y \arrow[r, "h"] \arrow[d, "i\times\mathrm{id}"'] & Z^I \arrow[d, "e^0"] \\
X \times Y \arrow[r, "f"'] \arrow[ur, dashed, "H"] & Z
\end{tikzcd}
$$

A useful lemma for proofs is that if $A \times I \in X \times I$ has HEP for a space Y, then since we have the homeomorphism of pairs $(I \times I, I \times 0 \cup \partial I \times I) \cong (I \times I, I \times 0)$, the maps $\phi : A \times I \times I \to Y$ and $\alpha : X \times (I \times 0 \cup \partial I \times I)$ induces a map $\Phi : X \times I \times I \to Y$ as long as $\alpha = \phi$ on $A \times (I \times 0 \cup \partial I \times I)$.

We can show HEP is preserved under pushouts by arrow chasing:

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, dashed, "hf"] \arrow[ddd, "j"'] & & B \arrow[dl, dashed, "h"'] \arrow[ddd, "J"] \\
& Z^I \arrow[d, dashed, "e^0"] & \\
& Z & \\
X \arrow[rr, "F"'] \arrow[uur, dashed, "K"] \arrow[ur, dashed, "\phi F"] & & Y \arrow[uul, dotted, "H"'] \arrow[ul, dashed, "\phi"']
\end{tikzcd}
$$

If j is a cofibration, then J is the cofibration **induced** from j via **cobase change** along f. For every cofibration j, this associates every map from $A \to B$ with a cofibration J. Furthermore, if we have maps $f, g : A \to B$, a homotopy $\phi : f \to g$ and the induced cofibrations j_f, j_g, then we get a morphism $\kappa : j_g \to j_f$ in TOP^B:

$$
\begin{tikzcd}
A \arrow[r, bend left, "g"] \arrow[r, dashed, "\phi_t" description] \arrow[r, dashed, bend right, "f"'] \arrow[d, "j"'] & B \arrow[d, "j_f"] \arrow[ddr, bend left, "j_g"] & \\
X \arrow[r, bend left, "\Phi_1"] \arrow[r, dashed, "\Phi_t" description] \arrow[r, dashed, bend right, "F"'] \arrow[drr, bend right, "G"'] & Y_f & \\
& & Y_g \arrow[ul, dashed, "\kappa"]
\end{tikzcd}
$$

Although κ may not be unique, it turns out that the homotopy class $[\kappa]^B$ only depends $[\phi]$. Let $h\text{-}\mathrm{COF}^B$ be the full subcategory of $h\text{-}\mathrm{TOP}^B$ of cofibrations under B, then we have the contravariant functor $\Pi(A, B) \to h\text{-}\mathrm{COF}^B$ with the construction above. Since $\Pi(A, B)$ is a groupoid, we obtain the **homotopy theorem for cofibrations** stating that $[\kappa]^B$ is an isomorphism in $h\text{-}\mathrm{TOP}^B$.

5.2 Cofibration transport

Let $i : K \to A$ be a cofibration and $\phi_t : K \to X$ be a homotopy, this induces the map

$$\phi^\sharp : [(A,i),(X,\phi_0)]^K \to [(A,i),(X,\phi_1)]^K$$

defined by $\phi^\sharp [\Phi_0] = \Phi_1$ for the extension $\Phi_t : A \to X$ of ϕ with initial conditions Φ_0. This gives us the **transport functor** of a cofibration $i : K \to A$ from $\Pi(K,X) \to \mathrm{SET}$ sending $\phi_0 \to [i,\phi_0]^K$ and $[\phi] \to \phi^\sharp$.
This functor tells us the difference between being homotopic in TOP and in TOP^K in the sense that if we have morphisms $g, g' : K \to X$ in TOP and $f, f' : i \to g, g'$ in TOP^K, then $[f] = [f']$ in h-TOP iff there exists some $\phi \in \Pi(K,A)$ such that $[f']^K = \phi^\sharp [f]^K$

5.3 Replacing maps by cofibrations

We relook at the construction of the mapping cylinder (here the unit interval is flipped from the previous chapter):

$$\begin{tikzcd}
X + X \arrow[r, "f+\mathrm{id}"] \arrow[d, "\langle i_0, i_1\rangle"'] & Y + X \arrow[d, "\langle s, j\rangle"] \\
X \times I \arrow[r, "a"'] & Z(f)
\end{tikzcd}$$

as i_0, i_1 are cofibrations, s, j are cofibrations as well. This gives us the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "p"] \arrow[d, shift left, "s"] \arrow[dr, "f_1"] & Y/X \\
X \arrow[r, "j"'] & Z(f) \arrow[r, "P"'] \arrow[u, shift left, "q"] & C(f) \arrow[u, dashed, "r"']
\end{tikzcd}$$

where

- j, s, f_1 are cofibrations
- s is a deformation retraction with inverse q
- $f = qj$, q a homotopy equivalence and j a cofibration
- If f is a cofibration, then q is a homotopy equivalence under X and r the induced homotopy equivalence

This factorization is unique in the sense if $f = qj = q'j' : X \to Y$, $X \xrightarrow{j} Z \xrightarrow{q} Y$, $X \xrightarrow{j'} Z' \xrightarrow{q'} Y$, then q, q' and j, j' are homotopic and Z, Z' are homotopy equivalent. Define $Z/j(X)$ as the (homotopical) **cofibre** of f, then this implies this cofibre is unique up to homotopy. This gives us an immediate result about homotopy pushouts:

For a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "J"] \\ X \arrow[r, "F"'] & Y \end{tikzcd}$$

with j a cofibration, then this diagram is a homotopy pushout.

5.4 Characterization of cofibration

First we note an equivalent condition for cofibrations:

Theorem ([Strom, Thm 2]). *$i : A \in X$ is a cofibration iff $X \times \{0\} \cup A \times I$ is a retract of $X \times I$*

If the inclusion is closed, this is immediate as we have the homeomorphism $X \times \{0\} \cup A \times I \cong Z(i)$. Otherwise the identity map is generally not a homeomorphism, even for inclusions like $(0, 1) \subset [0, 1]$.
The condition that $X \times \{0\} \cup A \times I$ is a retract of $X \times I$ is equivalent to the existance of a homotopy $\psi_t : X \to X$ relative to A and a map $u : X \to I$ exists such that

- $\psi_0 = \mathrm{id}_X$
- $A \subset v^{-1}(0)$
- $\psi_t(x) \in A$ for $t > v(x)$

For a closed inclusion, we define the pair (X, A) a **neighbourhood deformation retract** (NDR) if we have a homotopy $\psi_t : X \to X$ relative to A and a map $u : X \to I$ exists such that

- $\psi_0 = \mathrm{id}_X$
- $A = v^{-1}(0)$
- $\psi_1(x) \in A$ for $1 > v(x)$

It follows that (X, A) is a closed cofibration iff it's a NDR.
This also tells us that if $(X, A), (X, B)$ are closed cofibrations, then $(X, A \cup B)$ is a closed cofibration. If $(X, A), (Y, B)$ are cofibrations and A is closed, then $(X, A) \times (Y, B)$ is a cofibration.

5.5 Fibration properties

We can dualize the theory of cofibrations to obtain fibrations.
$p : E \to B$ has the **homotopy lifting property** (HLP) for X if for every h, a, there exists some H such that the (equivalent) diagrams commutes:

$$\begin{tikzcd}
B & X \times I \arrow[l, "h"'] \arrow[dl, dashed, "H"] \\
E \arrow[u, "p"] & X \arrow[l, "a"] \arrow[u, "i_0^X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& E \arrow[dl, "p"'] & & \\
B & & E^I \arrow[ul, "e_E^0"'] \arrow[dl, "p^I"] & X \arrow[l, dashed, "H"'] \arrow[ull, bend right, "a"'] \arrow[dll, bend left, "h"] \\
& B^I \arrow[ul, "e_B^0"] & &
\end{tikzcd}$$

H is an **lifting** of h with **initial conditions** a. If p has HLP for every space X, then it is a **fibration**. Similarly, fibrations is somewhat like epimorphisms with kernels, We can determine quickly if a map is a fibration by dualizing the mapping cylinder, the space $W(p)$ defined as the pullback

$$\begin{tikzcd}
B & E \arrow[l, "p"'] \\
B^I \arrow[u, "e_B^0"] & W(p) \arrow[l, "k"] \arrow[u, "b"']
\end{tikzcd}$$

then by considering the commutative diagram

$$\begin{tikzcd}
B & & E \arrow[ll, "p"'] & \\
& E^I \arrow[ur, "e_E^0"] \arrow[dl, "p^I"'] \arrow[dr, dotted, "r"'] & & \\
B^I \arrow[uu, "e_B^0"] & & W(p) \arrow[ll, "k"] \arrow[uu, "b"'] \arrow[ul, dashed, bend right, "s"'] & \\
& & & X \arrow[ul, dotted, "\rho"] \arrow[uuul, bend right, "a"'] \arrow[ulll, bend left, "h"]
\end{tikzcd}$$

We have the equivalent statements for a map $p : E \to B$:

- i is a cofibration
- i has HEP for $Z(i)$
- $s : Z(i) \to X \times I$ has a retraction

Fibrations are preserved under $-^Y$ for Y locally compact:

$$\begin{tikzcd}
B & X\times Y\times I \arrow[l, "h"'] \arrow[dl, dashed, "H"'] \\
E \arrow[u, "p"] & X\times Y \arrow[l, "a"] \arrow[u, "i_0^{X\times Y}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B^Y & X\times I \arrow[l, "h^\wedge"'] \arrow[dl, dashed, "H^\wedge"'] \\
E^Y \arrow[u, "p^Y"] & X \arrow[l, "a^\wedge"] \arrow[u, "i_0^{X}"']
\end{tikzcd}$$

We can show HLP is preserved under pullbacks by arrow chasing:

$$\begin{tikzcd}
B & & C \arrow[ll, "f"'] \\
& X\times I \arrow[ul, "fh"'] \arrow[ur, "h"] \arrow[ddl, dashed, "K"'] \arrow[ddr, dotted, "H"] & \\
& X \arrow[u, dashed, "i_0^X"] & \\
E \arrow[uuu, "p"] \arrow[ur, "aF"] & & F \arrow[uuu, "P"'] \arrow[ul, "a"'] \arrow[ll, "F"]
\end{tikzcd}$$

If p is a fibration, then P is the fibration **induced** from p via **base change** along f. For every fibration p, this associates every map from $B \to C$ with a fibration P. Furthermore, if we have maps $f, g : B \to C$, a homotopy $\phi : f \to g$ and the induced fibrations p_f, p_g, then we get a morphism $\kappa : p_f \to p_g$ in TOP_C:

$$\begin{tikzcd}
B & C \arrow[l, bend right, "g"'] \arrow[l, dashed, "\phi_t" description] \arrow[l, dashed, bend left, "f"] & \\
E \arrow[u, "\rho"] & F_f \arrow[u, "\rho_f"'] \arrow[l, bend right, "\Phi_1"'] \arrow[l, dashed, "\Phi_t" description] \arrow[l, dashed, bend left, "F"] \arrow[dr, "\kappa"] & \\
& & F_g \arrow[uul, bend right, "\rho_g"'] \arrow[ull, bend left, "G"]
\end{tikzcd}$$

Although κ may not be unique, it turns out that the homotopy class $[\kappa]_C$ only depends $[\phi]$. Let $h\text{-}\mathrm{FIB}_C$ be the full subcategory of $h\text{-}\mathrm{TOP}_C$ of fibrations over C, then we have the covariant functor $\Pi(C, B) \to h\text{-}\mathrm{FIB}_C$ with the construction above. This generalizes the previous section on fibre transport of coverings. Since $\Pi(C, B)$ is a groupoid, we obtain the **homotopy theorem for fibrations** stating that $[\kappa]_C$ is an isomorphism in $h\text{-}\mathrm{TOP}_C$.

5.6 Fibration transport

Let $p : E \to B$ be a fibration and $\phi_t : Y \to B$ be a homotopy, this induces the map

$$\phi^\sharp : [(Y,\phi_0),(E,p)]_B \to [(Y,\phi_0),(E,p)]_B$$

defined by $\phi^\sharp[\Phi_0] = \Phi_1$ for the lifting $\Phi_t : Y \to E$ of ϕ with initial conditions Φ_0. This gives us the **transport functor** of a fibration $p : E \to B$ from $\Pi(Y,B) \to \mathrm{SET}$ sending $\phi_0 \to [\phi_0, p]_B$ and $[\phi] \to \phi^\sharp$.
This functor tells us the difference between being homotopic in TOP and in TOP_B in the sense that if we have morphisms $g, g' : Y \to B$ in TOP and $f, f' : g, g' \to p$ in TOP_B, then $[f] = [f']$ in h-TOP iff there exists some $\phi \in \Pi(Y,B)$ such that $[f']_B = \phi^\sharp[f]_B$

5.7 Replacing maps by fibrations

We relook at the construction of $W(f)$ for an arbitrary map $f : X \to Y$:

$$\begin{tikzcd} Y \times Y & X \times Y \arrow[l, "f\times\mathrm{id}"'] \\ Y^I \arrow[u, "(e^0,e^1)"] & W(f) \arrow[l] \arrow[u, "(q,p)"'] \end{tikzcd}$$

as e_0, e_1 are fibrations, p, q are fibrations as well. This gives us the commutative diagram

$$\begin{tikzcd} Y & X \arrow[l, "f"'] \arrow[d, shift right, "s"'] & F = f^{-1}(*) \arrow[l, "j"'] \\ Y \arrow[u, equal] & W(f) \arrow[l, "p"] \arrow[u, shift right, "q"'] & F(f) = p^{-1}(*) \arrow[l, "J"'] \arrow[u, dashed, "r"'] \arrow[ul, "f^1"] \end{tikzcd}$$

where the last column exists for pointed maps and

- p, q, f^1 are fibrations
- s is a shrinkable map with inverse q
- $f = ps$, s a homotopy equivalence and p a fibration
- If f is a fibration, then q is a homotopy equivalence over Y and r the induced homotopy equivalence

This factorization is unique in the sense if $f = ps = p's' : X \to Y$, $X \xrightarrow{s} W \xrightarrow{p} Y$, $X \xrightarrow{s'} Z' \xrightarrow{p'} Y$, then s, s' and p, p' are homotopic and W, W' are homotopy equivalent. Define $p^{-1}(*)$ as the (homotopical) **fibre** of f, then this implies this fibre is unique up to homotopy.
This gives us an immediate result about homotopy pushouts:
For a pullback diagram

$$\begin{tikzcd} A & B \arrow[l, "f"'] \\ X \arrow[u, "p"] & Y \arrow[l, "F"] \arrow[u, "P"'] \end{tikzcd}$$

with p a fibration, then this diagram is a homotopy pullback. (Dualize everything in section 4.2)

5.8 Fibrations and cofibrations

Finally we note now fibrations and cofibrations behave together.
Let $p : E \to B$ has HLP for X and $i : A \subset X$ is a cofibration and h-equivalence. Suppose we are given $f : X \to B$ and $a : A \to E$ such that $pa = fi$, then a lifting F of f extending a exists:

$$
\begin{tikzcd}
E \arrow[d, "p"] & A \arrow[l, "a"'] \arrow[d, "i"] \\
B & X \arrow[l, "f"'] \arrow[dl, "i_0^X"] \\
X \times I \arrow[u, dashed, "f\Phi"'] \arrow[uu, dashed, bend left, "F"] \arrow[ur, dashed, bend right, "\Phi"']
\end{tikzcd}
$$

In the diagram the map Φ comes from studying section 5.4 defined as $\Phi_t(x) = \psi_{tv(x)^{-1}}(x)$ and HLP gives F from $f\Phi$.
If $i : A \subset B$ is a closed cofibration of locally compact spaces, the restriction map $p : Z^B \to Z^A$ is a fibration.
Let $p : E \to B$ be a fibration, and $B_0 \subset B$ be a cofibration, then $E_0 = p^{-1}(B_0) \subset E$ is a cofibration.

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