1 Chapter 1

2 Chapter 2

2.1 Definitions

Spaces:

Spaces.	
\mathbb{R}^n	Euclidean space
$D^n = \{x \in \mathbb{R}^n x \le 1\}$	n-disk
$S^{n-1} = \{ x \in \mathbb{D}^n x = 1 \}$	n-1-sphere
$E^n = D^n - S^{n-1}$	n-cell
$I^n = \{ x \in \mathbb{R}^n 0 \le x_i \le 1 \}$	n-cube
$\partial I^n = \{ x \in \mathbb{I}^n \exists i, x_i = 0, 1 \}$	boundary of I^n
$\Delta^n = \Delta[n] = \{x \in \mathbb{R}^{n+1} x_i \ge 0, \sum_i x_i = 1\}$	n-simplex
$\partial \Delta^n = \{ x \in \Delta^n \exists i, x_i = 0 \}$	Boundary of n -simplex

Path: $u:[a,b] \to X$ from x=u(a) to y=u(b) (usually reparametrized to $[0,1] \to X$)

Inverse path: $u^-: t \to u(1-t)$ from y to x

Product path:
$$u * v : t \to \begin{cases} u(2t) & t \le \frac{1}{2} \\ v(2t-1) & \frac{1}{2} \le t \end{cases}$$

Constant path: $k_x: t \to x$

 $\pi_0: \text{TOP} \to \text{SET}$

- $\pi_0(X)$: Set of path connected components of X
- $\pi_0(f)([x]) = [f(x)]$

 $W: \mathrm{TOP} \to \mathrm{CAT}$

- W(X): Paths $u:[0,a]\to X$ and composition is defined on [0,a+b] for associativity
- $W(f)(x) = f(x), W(f)(u) = f \circ u$

2.2 Homotopy notions

Homotopy: $H_t: X \times [0,1] \to Y$ from $f = H_0: X \to Y$ to $g = H_1: X \to Y$; $H: f \simeq g$ (composition/inverse immediate)

1

Homotopy $H_t: X \to Y$ relative to $A \subset X$ if $H_t: A \to Y$ is independent of t

Homotopy between f and a constant map is a null homotopy

Null homotopy of $id_X: X \to X$ is a contraction

Path category W(X,Y)

- Objects: $f: X \to Y$
- Morphisms: Homotopy $H_t: [0,a] \times X \to Y$ between f and g

hTOP is TOP quotiented by the homotopy relation.

hTOP	TOP
Isomorphic	Homotopy equivalent/Same homotopy type
Isomorphic to {*}	Contractible
Isomorphism	h-equivalence
Constant map	Null homotopic

Hom functors in hTOP of $f: X \to Y$:

$$f_*: [Z, X] \to [Z, Y], g \to fg \quad f^*: [Y, Z] \to [X, Z], h \to hf$$

Remark. Generally lower index for covariant and upper index for contravariant

TOP⁰: Category of pointed spaces

hTOP⁰: Quotient of TOP⁰ by homotopy

Forgetful functor $TOP^0 \to TOP$ has a left adjoint, $X \to (X + \{*\}, *)$

Remark. The smash product $A \wedge B = \frac{A \times B}{A \vee B}$ is always compatible with homotopies and is a tensor product in some appropriate subcategory, i.e. compactly generated spaces

TOP(2): Pairs of topological spaces $A \subset X$

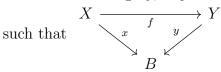
Note that the product we use here is not the categorical product, instead it is defined as

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$$

so that $(I^m, \partial I^m) \times (I^n, \partial I^n) = (I^{m+n}, \partial I^{m+n})$

TOP(3): Pairs of topological spaces $A \subset B \subset X$

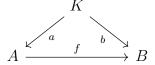
TOP_B: Slice category, objects are morphisms $x: X \to B$ and morphisms are $f: X \to Y$



- A morphism from $id_B: B \to B$ to $p: E \to B$ is a section of p
- If $p \cong id_B$ in hTOP_B, then it is shrinkable

 TOP^K : Coslice category, objects are morphisms $a: K \to A$ and morphisms are $f: A \to B$

such that



- A morphism from $i: K \to X$ to $id_K: K \to K$ is a retraction of i and i is an embedding
- $i: K \subset X$, then K is a retract of X
- If $i \cong id_K$ in hTOP^K, then it is a deformation retract

Note that $\mathsf{TOP}_{\{*\}} \cong \mathsf{TOP}$ and $\mathsf{TOP}^{\{*\}} \cong \mathsf{TOP}^0$

 $H_t: A \to B$ is a homotopy in the (co)slice category if each $H_t, t \in [0,1]$ is a morphism in the (co)slice category, hence we get the quotient categories $hTOP^K$, $hTOP_B$.

2.3 Internal hom objects

Let Y^X or F(X,Y) be the set of continuous maps from X to Y with the compact open topology. Suppose that X is locally compact, then Y^X is the exponential object, i.e.

$$X \times Y$$

$$f^{\wedge} \times id_{Y} \qquad f$$

$$Z^{Y} \times Y \xrightarrow{e_{Y,Z}} Z$$

f induces f^{\wedge} and f^{\wedge} induces f, alternatively

$$\operatorname{Hom}(-\times Y, Z) \cong \operatorname{Hom}(-, Z^Y)$$

which also tells us the functors $-^{Y}$ is a right adjoint to $-\times Y$.

Unfortunately in categories with zero objects, i.e. TOP^0 , then exponential objects generally dont exist unless the category is trivial as if Y^X exists, we have

$$\operatorname{Hom}(X,Y) \cong \operatorname{Hom}(0 \times X,Y) \cong \operatorname{Hom}(0,Y^X) \cong \{*\}$$

However, we may have some form of tensor-hom adjuncation.

In the category TOP^0 , we define $F^0(X,Y)$ as the subspace of pointed maps of F(X,Y) and the constant map is the basepoint. Any pointed map $X \times Y \to Z$ induces a pointed map $X \to F^0(X,Y)$ if it sends $X \times y \cup x \times Y$ to z, hence it corresponds to maps from $X \wedge Y \to Z$. The adjuncation in this case reduces to

$$F^{0}\left(X\wedge Y,Z\right)\cong F^{0}\left(X,F^{0}\left(Y,Z\right)\right)$$

when X, Y are locally compact. This gives us our tensor-hom adjuncation.

If we quotient by homotopy and assume X is locally compact and $e_{X,Y}^0$ is continuous, then we get

$$[X \wedge Y, Z]^0 \cong [X, F^0(X, Y)]^0$$

2.4 Fundamental groupoid

 $\Pi: TOP \to GRPd$

• $\Pi(X)$: Quotient of W(X) by homotopy

Somewhat cleaner way to state van-Kampen theorem:

Theorem (Seifert-Van Kampen [May, Thm 2.7]). Suppose that \mathcal{U} is a covering of X such that if $U_1, U_2 \in \mathcal{U}$, then $U_1 \cap U_2 \in \mathcal{U}$. This turns \mathcal{U} into a category where morphisms are inclusions, then

$$\Pi(X) \cong \operatorname{colim}_{U \in \mathcal{U}} \Pi(U)$$

Choosing a base point, we get the functor $\pi_1 : \text{TOP}^0 \to \text{GRP}$.

Remark. Proposition 2.7.3 of tom Dieck that the fundamental group of a monoid in TOP⁰ is commutative and agrees with the monoid operation comes from a more general theorem, the Eckmann–Hilton argument

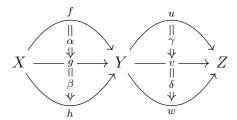
Theorem (Eckmann-Hilton argument). If \cdot , * are unital binary operations on X with units 1. and 1_* such that

$$(a \cdot b) * (c \cdot d) = (a * b) \cdot (c * d)$$

Then \cdot , * coincide, are associative and commutative

2.5 Enriching TOP

We give a groupoid structure, $\Pi(X,Y)$ to each hom set $\operatorname{Hom}_{\operatorname{TOP}}(X,Y)$ with homotopy as morphisms. This provides us with a 2-category, i.e.



such that all the compositions makes sense, i.e.

$$(\delta\gamma)(\beta\alpha) = (\delta\beta)(\gamma\alpha)$$

We can enrich similar categories like TOP^0

3 Chapter 3

3.1 Definitions

Suppose $p: E \to B$ is surjective and $U \subset B$ is open

- Trivialization of p over U is a homeomorphism $p^{-1}(U) \to U \times F$
- p is **locally trivial** if a open covering \mathcal{U} exists where a trivialization of p over $U \in \mathcal{U}$ exists for all U
- \mathcal{U} is a bundle chart
- F is the typical fibre
- p is **trivial over** U if a bundle chart over U exists

• Bundles/Fibre bundles are locally trivial maps

Covering space/Covering of B is a locally trivial trivial map $p: E \to B$ with discrete fibres

- If $\phi_U: p^{-1}(U) \to U \times F$ is a trivialization, then $\phi_U^{-1}(U \times \{*\})$ are the **sheets** over U
- If |F| = n, then p is a n-fold covering
- A **trivial covering** is the covering $p: B \times F \to B$
- U is admissible or evenly covered if a trivialization exists
- E is the total space and B is the base space

3.2 Coverings with group actions

A left G-principal covering is a covering $p: E \to B$ and a properly discontinuous group action G on E such that p(gx) = p(x) and the action on fibres are transitive $\alpha \in \operatorname{Aut}(p)$ if $\alpha: p \to p$ is a morphism in TOP_B . These are deck transformations The map $x \to gx$ gives a map $G \to \operatorname{Aut}(p)$

Theorem (Galois correspondence). Let $p: E \to B$ be a covering, then

- If E is connected, Aut(p) is a properly discontinuous action on E
- If B is locally path connected, H subgroup of Aut(p), then $E/H \to B$ is a covering

A **right** G-**principal covering** is a covering $p: E \to B$ and a properly discontinuous group action G on E such that p(xg) = p(x) and the action on fibres are transitive. Let F be a set with a left G action, then the space $E \times_G F$ constructed by quotienting $E \times F$ by (xg, f) = (x, gf) is an **associated covering**.

Remark. Seems like for this part we need to assume that that G is a free action of the fibres as well and F is given the discrete topology

Theorem. The map $p_F: E \times_G F \to B, (x, f) \to p(f)$ is a covering with typical fibre F

Proof. Suppose that \mathcal{F} is the typical fibre of p.

First we show the typical fibre of p_F is F. It's immediate that the typical fibre is given by $\frac{\mathcal{F} \times F}{\sim}$ immediately showing discreteness. Next, notice that $\{(\mathfrak{f}, f) | f \in F\}$ are the representatives of $\frac{\mathcal{F} \times F}{\sim}$ for some arbitrary \mathfrak{f} as supose $\mathfrak{f}' = \mathfrak{f}g$, then $(\mathfrak{f}', f) = (\mathfrak{f}, gf)$ and $(\mathfrak{f}, f) = (\mathfrak{f}, f')$ implies that either \mathfrak{f} has a nontrivial stabalizer or f = f', hence we need to assume the action is free on \mathcal{F} .

Next we show that this is indeed a covering. Suppose U has a trivialization, i.e. $p^{-1}(U) \cong U \times \mathcal{F}$. Then $\frac{p^{-1}(U) \times F}{\sim} \cong U \times \mathcal{F}$. Hence p_F is a covering with typical fibre F. \square

This gives us the functor

$$A(p): G\operatorname{-}\operatorname{SET} \to \operatorname{COV}_B$$

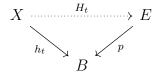
from the category of sets with a left G action to the category of covering spaces over B (a subcategory of TOP_B).

If A(p) is an equivalence of categories, then p is the universal cover

3.3 Lifting

 $F: X \to E$ is a **lifting** of $f: X \to B$ along $p: E \to B$ if pF = f, i.e. a morphism in TOP_B If X is connected and p is a covering, liftings that agree somewhere are unique.

A map $p: E \to B$ has **homotopy lifting property** (HLP) for a space X if for each homotopy h_t and initial condition H_0 , we can extend H_0 to the homotopy H_t such that the diagram commutes:



H is a lifting of h with initial conditions a. p is a **fibration** if it has HLP for all spaces

Theorem. Coverings $p: E \to B$ are fibrations

Proof. First show that projection maps $U \times F \to U$ are fibrations, then glue these projection maps and use uniqueness of liftings

As liftings along coverings are unique, the diagram below is a pullback:

$$E^{I} \xrightarrow{p^{I}} B^{I}$$

$$e_{E}^{0} \downarrow \qquad \qquad \downarrow e_{B}^{0}$$

$$E \xrightarrow{p} B$$

Let $p: E \to B$ be a map with HLP for I and $F_b = p^{-1}(b)$.

For every map $[v] \in \Pi(B)$, we obtain a well defined map $v_{\sharp} : \pi_0(F_b) \to \pi_0(F_c)$. Suppose $V : I \to E$ is a lifting of v with V(0) = x, then $v_{\sharp}[x] = [V(1)]$.

With this we obtain the **transport functor** $T_p:\Pi(B)\to \operatorname{SET}$

- $b \to \pi_0(B)$
- $[v] \rightarrow v_{\sharp}$

Let p(x) = b and let $\partial_x : \pi_1(B, b) \to \pi_0(F_b, x)$, $[v] \to v_{\sharp}(x)$ and $i : F_b \subset E$, then we have the exact sequence

as well as the isomorphisms of sets $\partial_x: \frac{\pi_1(B,b)}{p_*\pi_1(E,x)} \cong \pi_0\left(F_b,x\right), i_*: \frac{\pi_0(F_b,x)}{\pi_1(B,b)} \cong \pi_0(E,x)$ For a covering $p: E \to B$ with B path connected, the exact sequence simplifies to

Furthermore suppose that $p: E \to B$ is a right G-principal covering with E path connected, then we get the exact sequence

$$1 \longrightarrow \pi_1(E, x) \xrightarrow{p_*} \pi_1(B, b) \xrightarrow{\delta_x} G \longrightarrow 1$$

and the image of p_* is normal.

3.4 Coverings

Outline:

- 1. Construct the inverse X of $T: TRA_B \to COV_B$ that exists and is an equivalence of categories for sufficiently nice B
- 2. Construct the functor $\epsilon_B : \text{TRA}_B \to \pi_b \text{-SET}$ and the inverse η_b
- 3. Hence $A(p):G\operatorname{-}\mathrm{SET}\to\mathrm{COV}_B$ is an equivalence of categories iff the total space of p is simply connected

Let $TRA_B = [\Pi(B), SET]$, the transport functor in the previous section yields the functor $T : COV_B \to TRA_B$.

If B is path connected and T is an equivalence of categories, then B is a **transport space**. A set $U \in B$ is **transport simple** if any paths in U between identical points are homotopic in B.

B is **semi-locally simply connected** if it has an open covering with transport simple sets. B is **transport local** if it is path connected, locally path connected and semi-locally simply connected.

Theorem. If B is then B is a transport space

Proof. We need to construct the inverse of $T, X : TRA_B \to COV_B$. Let $\Phi : \Pi(B) \to SET$ be some functor, we will construct a covering $p : X(\Phi) \to B$. As a set $X(\Phi) = \coprod_{b \in B} \Phi(b)$. To get a reasonable topology on it, we consider a covering \mathcal{U} of B by transport simple path connected open sets. For every $b \in \mathcal{U} \in \mathcal{U}$, we define $\phi_{U,b} : U \times \Phi(b) \to p^{-1}(U)$ and by gluing these maps together, we obtain a topology on $X(\Phi)$ and a covering p.

Verification of functoriality and inverse are somewhat direct from definition. \Box

With this, consider the hom functor $\operatorname{Hom}_{\Pi(B)}(b,-) \in \operatorname{TRA}_B$ and let $p^b : E^b \to B$ be its associated covering. Then E^b is simply connected right $\operatorname{Hom}_{\Pi(B)}(b,b)$ -principal covering. Suppose that B is path connected, then $\Pi = \Pi(B)$ is a connected groupoid. Let $\Pi(x,y) = \operatorname{Hom}_{\Pi(B)}(x,y)$ and $\pi_b = \Pi(b,b)$

For a functor $F: \Pi \to \text{SET}$, we have the left π_B -set F(b) giving us the functor $\epsilon_b: \text{TRA}_B \to \pi_b$ -SET.

For a left π_B -set A, we define the functor $\Pi(b,-)\times_{\pi_B}A:\Pi\to \operatorname{SET}$ where $A\times_G B$ is the set $A\times B$ quotiented by (ag,b)=(a,gb). This gives us the functor $\eta_b:\pi_b\operatorname{-SET}\to\operatorname{TRA}_B$, the inverse of ϵ_b .

Finally we have the following categories and functors:

$$G \operatorname{-SET} \xrightarrow{A(p)} \operatorname{COV}_B \xrightarrow{\stackrel{T}{\longleftarrow}} \operatorname{TRA}_B \xrightarrow{\stackrel{\epsilon_b}{\longleftarrow}} \pi_b \operatorname{-SET}$$

where X exists if B is transport-local and ϵ_B , η_B require B to be path connected to exist. Finally we have

Theorem. The following are equivalent:

- B is a transport space, i.e. T is an equivalence of categories
- B has a universal right G-principal covering $p: E \to B$, i.e. A(p) is an equivalence of categories

Note that the exact sequences above imply that E is simply connected.

Define the **orbit category** Or(G) consisting of homogenous G-sets ($\frac{G}{H}$ for any subgroup H) and G-maps. This category is a strict subcategory of G-SET, consisting only the transitive sets.

For a covering $p: E \to B$, we obtain the injective map $p_*: \pi_1(E, x) \to \pi_1(B, p(x))$ and the image is called the **characteristic subgroup** of p wrt x.

Let $p: E \to B$ be a simply connected covering, then the subcategory A(p) (Or (π_b)) of COV_B is equivalent to the subcategory consisting of connected coverings. This tells us that the connected coverings of a transport space is determined by subgroups of the fundamental group.

Theorem. Let B be a transport space and $p: E \to B$ a covering.

• The action of Aut(p) on E makes it a left-Aut(p) principal covering

- A simply connected covering is a universal covering
- Universal coverings are unique up to isomorphism
- The automorphism group of a universal cover is isomorphic to $\pi_1(B,b)$
- E^b is simply connected
- We have a Galois correspondence between isomorphism classes of connected coverings and subgroups of $\pi_1(B,b)$

If B is not a transport space but is path connected and locally path connected, then we have a similar result where coverings by path connected total spaces are isomorphic iff the characteristic subgroups are conjugate in $\pi_1(B, b)$.

Suppose we have coverings $p: E \to B$ and $f: Z \to B$, then a covering $\Phi: Z \to E$ such that

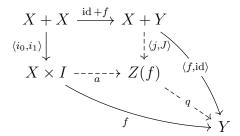
$$Z \xrightarrow{f} P$$
 exists iff $f_*\pi_1(Z,z) \subset p_*\pi_1(E,x)$ for some $f(z) = p(x)$.

If X is a topological group with identity x and $p: E \to X$ is a covering with E path connected and locally path connected, then for each $e \in p^{-1}(b)$, there exists a unique group structure on E such that e is the identity and p a homomorphism.

4 Chapter 4

4.1 Mapping cylinders

For a map $f: X \to Y$ mapping cylinder Z(f) is constructed by the pushout



We also have $Jq \cong \operatorname{id}$ as $X \times I \cong X$ and f = qj, j a closed immersion and q a homotopy equivalence. We see that Z has nice functorial properties. Suppose we have the homotopy commutative diagram

$$X \xrightarrow{f} Y$$

$$\alpha \downarrow \qquad \qquad \downarrow \beta$$

$$X' - f' \to Y'$$

$$\alpha' \downarrow \qquad \qquad \downarrow \beta'$$

$$X'' - f'' \to Y''$$

and let the homotopy equivalences be $\Phi: f'\alpha \cong \beta f, \Phi': f''\alpha' \cong \beta' f'$ This induces the following homotopy commutative diagram

$$X + Y \longrightarrow Z(f)$$

$$\alpha + \beta \downarrow \qquad \qquad \downarrow Z(\alpha, \beta, \Phi)$$

$$X' + Y' \longrightarrow Z(f')$$

$$\alpha' + \beta' \downarrow \qquad \qquad \downarrow Z(\alpha', \beta', \Phi'')$$

$$X'' + Y'' \longrightarrow Z(f'')$$

where each small square commutes in TOP and the whole diagram commutes in hTOP. Given maps $f: A \to B$ and $g: A \to C$, we can construct the double mapping cylinder by either of the two pushouts:

$$A + A \xrightarrow{f+g} B + C$$

$$\langle i_0, i_1 \rangle \downarrow \qquad \qquad \downarrow \langle j_0, j_1 \rangle$$

$$A \times I \xrightarrow{j^B} Z(f, g)$$

$$A \xrightarrow{j^B} Z(f)$$

$$\downarrow \downarrow$$

$$Z(g) \xrightarrow{f+g} B + C$$

$$\downarrow \langle j_0, j_1 \rangle \downarrow$$

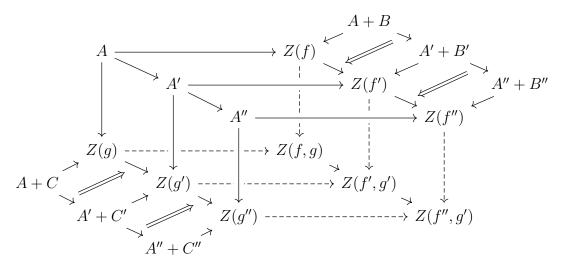
$$\downarrow \downarrow$$

$$Z(g) \xrightarrow{f+g} B + C$$

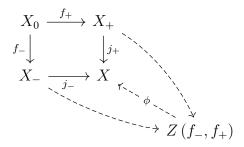
$$\downarrow \langle j_0, j_1 \rangle \downarrow$$

$$\downarrow \downarrow$$

The functorality of the double mapping cylinder can be seem from the following commutative homotopy diagram:



Suppose we have the homotopy commutative diagram



The square is called a **homotopy pushout** or **homotopy cocartesian** if ϕ is a homotopy equivalence (i.e. a pushout in the category hTOP)

Let f_{\pm}, j_{\pm} be inclusions and $X = X_{+} \cup X_{-}$, then the diagram is a pushout in TOP.

Let $N(X_-, X_+) = X_- \times 0 \cup X_0 \times I \times X_+ \times 1$ be a subspace of $X \times I$ and let $p_N : N(X_-, X_+) \to X$ be the projection map.

The covering X_{\pm} is **numerable** if p_N has a section. With this, we have the following condition to determine if the diagram above is a pushout in hTOP:

Theorem 4.1. $Z(f_{-}, f_{+}) \cong X$ if the covering X_{\pm} is numerable

Given the projection maps $X \stackrel{f}{\leftarrow} X \times Y \stackrel{g}{\rightarrow} Y$, the double mapping cylinder $Z(f,g) = X \star Y$ is known as the **join** of X and Y

4.2 Suspensions and loops

Here we work in pointed categories

The suspension functor $\Sigma : TOP^0 \to TOP^0$ is given by

•
$$\Sigma X = S^1 \wedge X = \frac{X \times I}{X \times \partial I \cup \{x\} \times I}$$

•
$$S\Sigma_* [X, Y]^0 = [\Sigma X, \Sigma Y]^0$$

Note that Σ_* is a homomorphism if $X = \Sigma A$ and any pointed homotopy $H_t: X \to Y$ corresponds uniquely to a pointed map $\overline{K}: \Sigma X \to Y$

The set $[\Sigma X, Y]^0$ has a natural group structure by composition of homotopies. Furthermore as $[\Sigma^n X, Y]^0$ has n natural composition laws by composin the homotopies at the the ith coordinate and these satisfies the assumptions of Eckmann-Hilton argument, they are equivalent. This also tells us that the higher homotopy groups, $\pi_n(X) = [S^n, X]^0$, are commutative groups.

We can dualize everything above:

The **loop** space of X is $\Omega X = F^0(S^1, X)$. This consists of loops in X with basepoint x. This is naturally a topological group with the product of loops.

There is a natural group structure on $\operatorname{Hom}_{\operatorname{TOP}^0}(X, \Omega Y)$ given by $[f] +_m [g] = [f][g]$. As S^1 is locally compact, we have the tensor-hom adjuncation

$$[\Sigma X, Y]^0 \cong [X, \Omega Y]^0$$

and this commutes with the group operation.

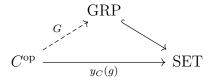
In the set $[\Sigma X, \Omega Y]^0$, by the Eckmann-Hilton argument, the group operations on these two sets coincide and are commutative.

4.3 Group objects

Perhaps a nicer way of looking at group/any objects is via Yoneda's lemma, see [Wat] for more details:

Theorem 4.2 (Yoneda). For any category C, we have a full and faithful functor, the Yoneda embedding $y_C: C \to [C^{op}, SET]$, given by $y_C(c) \to \text{Hom}_C(-, c)$

An object $g \in C$ is a **group object** if the functor $y_C(g)$ factors through GRP, i.e. the following diagram commutes where GRP \rightarrow SET is the forgetful functor



To give an explicit construction, recall that for a group G, we need to have a unit element, an inverse operation and a group operation, given by

$$e_G: 1 \to G$$

 $\operatorname{inv}_G: G \to G^{\operatorname{op}}$
 $\cdot_G: G \times G \to G$

such that the following diagrams commute to ensure associativity, unit and inverse holds:

where $e: G \to G$ is the composite morphism $G \to 1 \stackrel{e_G}{\to} G$

The maps can immediately be constructed by Yoneda's lemma, take for instance the product map. If g is a group object with $G: C^{op} \to GRP$ as the functor to group and $f: c \to d$ is a morphism in C, then the following diagram commutes:

$$G(c) \times G(c) \xrightarrow{\cdot G(c)} G(c)$$

$$G(f) \times G(f) \uparrow \qquad \qquad \uparrow G(f)$$

$$G(d) \times G(d) \xrightarrow{\cdot G(d)} G(d)$$

telling us we have a natural transformation $\cdot_G: G \times G \to G$. This gives us the morphism $g : g \times g \to g$ by Yoneda's lemma and as the Yoneda embedding is full and faithful, the commutativity of the diagrams above defining a group is immediate.

Similarly one defines cogroups as group objects in the opposite category. With this view, it

is immediate that if $c \in C$ is a cogroup, then $\operatorname{Hom}_C(c,-)$ is a functor to GRP. As examples in hTOP^0 , ΣX is a cogroup as $[\Sigma X,Y]^0$ is a group and ΩY is a group as $[X, \Omega Y]^0$ is a group.

4.4 Fibre sequence

Again here we work in pointed spaces. A map $f: X \to Y$ induces a map $f^*: [Y, B]^0 \to Y$ $[X,B]^0$. The kernels of f is any element that gets sent to the basepoint of Y and the kernel of f^* is any element that gets sent to a nullhomotopic element. This allows us to define exact sequences of topological spaces.

A sequence of spaces $U \xrightarrow{f} V \xrightarrow{g} W$ is **h-coexact** if for all spaces B, the sequence

$$[U,B]^0 \overset{f^*}{\leftarrow} [V,B]^0 \overset{g^*}{\leftarrow} [W,B]^0$$

is exact.

The **cylinder** $XI = \frac{X \times I}{* \times I}$ describes homotopies in TOP⁰ (morphisms $XI \to Y$ are homotopies).

The **cone** $CX = \frac{X \times I}{X \times 0 \cup * \times I} = X \wedge I$ describes homotopies starting from constant maps in TOP^0 .

Define C(f) as the pushout

$$X \xrightarrow{f} Y$$

$$\downarrow_{i_1} \downarrow f_1$$

$$CX \xrightarrow{j} C(f)$$

and by its universal property, we have the h-coexact sequence $X \xrightarrow{f} Y \xrightarrow{f_1} C(f)$ which can be iterated to create a long h-coexact sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{f_2} C(f_1) \xrightarrow{f_3} \dots$$

Furthermore we have the commutative diagram

$$X \xrightarrow{i_{1}} CX \xrightarrow{p} Y/i_{1}X = \Sigma X$$

$$f \downarrow \qquad \qquad \downarrow j \qquad \qquad \downarrow$$

$$Y \xrightarrow{f_{1}} C(f) \xrightarrow{p(f)} C(f)/f_{1}Y = \Sigma X$$

$$i_{1} \downarrow \qquad \qquad \downarrow f_{2} \qquad \qquad \downarrow$$

$$CY \xrightarrow{j_{1}} C(f_{1}) \xrightarrow{q(f)} C(f_{1})/j_{1}Y = \Sigma X$$

Applying this to itself, we obtain

where $\iota:(x,t)\to(x,1-t)$ to ensure commutativity.

As q(f), $q(f_1)$ are homotopy equivalences and a sequence remains h-coexact if we replace elements with h-equivalent ones, we obtain the h-coexact sequence

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y$$

And we can apply this sequence to itself iteratively and noting that Σ and C commutes to obtain the **Puppe-sequence** or the **cofibre sequence** of f:

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y \to \Sigma C(f) \to \Sigma C(f) \to \Sigma^2 X \to \Sigma^2 Y \dots$$

as $\Sigma f: \Sigma X \to \Sigma Y$

Applying the functor $[-, B]^0$, we see that from the 4th place onwards these are groups and from the 7th place onwards these are abelian groups.

For any map $f: X \to Y$, let $\mu: C(f) \to \Sigma X \vee C(f)$ be defined as

$$\mu(x,t) = \begin{cases} ((x,2t),*) & 2t \le 1\\ (*,(x,2t-1)) & 2t \ge 1 \end{cases}$$

and $\mu(y) = y$. This map is a **h-coaction** of the h-cogroup ΣX on C(f) as we have

$$[\Sigma X, B]^0 \times [C(f), B]^0 \cong [\Sigma X \vee C(f), B]^0 \to [C(f), B]$$

where the last map is induced by μ .

With the map $Y \xrightarrow{f_1} C(f) \xrightarrow{p(f)} \Sigma X$ and any maps $\alpha_1, \alpha_2 : \Sigma X \to B$, this group action satisfies $(\alpha_1) (p(f)^* \alpha_2) = p(f)^* (\alpha_1 \alpha_2)$ and f_1^* is an injective map on orbits of this action. We can dualize everything as always.

A sequence of spaces $U \xrightarrow{f} V \xrightarrow{g} W$ is **h-exact** if for all spaces B, the sequence

$$[B,U]^0 \xrightarrow{f^*} [B,V]^0 \xrightarrow{g^*} [B,W]^0$$

is exact.

To dualize the cone, we use the exponential object adjuncation $[X \wedge I, Y]^0 \cong [X, F^0(Y, I)]^0$ and define $FY = F^0(Y, I)$. We then define F(f) as the pullback

$$F(f) \xrightarrow{q} FY$$

$$f^{1} \downarrow \qquad \qquad \downarrow e^{1}$$

$$X \xrightarrow{f} Y$$

and by its universal property, we have the h-exact sequence $F(f) \xrightarrow{f_1} X \xrightarrow{f} Y$ which can be iterated to create a long h-exact sequence

$$\dots \xrightarrow{f^4} F\left(f^2\right) \xrightarrow{f^3} F\left(f^1\right) \xrightarrow{f^2} F(f) \xrightarrow{f^1} X \xrightarrow{f} Y$$

We can dualize the diagrams above to obtain

$$Y \xleftarrow{e^{1}} FY \xleftarrow{i} \Omega Y$$

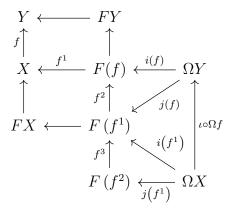
$$f \uparrow \qquad \uparrow^{q} \qquad \uparrow$$

$$X \leftarrow f^{1} - F(f) \leftarrow i(f) - \Omega Y$$

$$e^{1} \uparrow \qquad \uparrow^{f^{2}} \qquad \uparrow$$

$$FX \xleftarrow{q^{1}} F(f^{1}) \xleftarrow{j(f)} \Omega Y$$

where j(f) is a h-equivalence and using this on itself, we obtain



and finally we obtain the dual long h-exact sequence

$$\Omega X \to \Omega Y \to F(f) \to X \to Y$$

and repeating this on the map $\Omega f: \Omega X \to \Omega Y$, we obtain the long h-exact sequence

$$\dots \Omega^2 X \to \Omega^2 Y \to \Omega F(f) \to \Omega X \to \Omega Y \to F(f) \to X \to Y$$

known as the **fibre sequence** of f. Similarly applying the functor $[B, -]^0$, we see that from the 4th place onwards these are groups and from the 7th place onwards these are abelian groups.

Finally to dualize the group action, again let $f: X \to Y$ be any map. We have the h-action $m: \Omega Y \times F(f) \to F(f)$ defined as

$$m([f(t), (x, g(t))]) = \begin{cases} (x, f(2t)) & 2t \le 1\\ (x, g(2t-1)) & 2t \ge 1 \end{cases}$$

This map induces the map

$$[B, \Omega Y]^0 \times [B, F(f)]^0 \cong [B, \Omega Y \times F(f)]^0 \rightarrow [B, F(f)]^0$$

Furthermore with the maps $\Omega Y \stackrel{i(f)}{\to} F(f) \stackrel{f^1}{\to} X$ and any maps $\alpha_1, \alpha_2 : B \to \Omega Y$, this group action satisfies $(i(f)_*\alpha_1)(\alpha_2) = i(f)_*(\alpha_1\alpha_2)$ and f^1_* is an injective map on the orbits of the action.

5 Chapter 5

References

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