1 Chapter 1

2 Chapter 2

2.1 Definitions

Spaces:

$\mathbb{R}^n$	Euclidean space
$D^n = \{x \in \mathbb{R}^n \| \|x\| \leq 1\}$	n-disk
$S^{n-1} = \{x \in \mathbb{D}^n \| \|x\| = 1\}$	$n-1$-sphere
$E^n = D^n - S^{n-1}$	n-cell
$I^n = \{x \in \mathbb{R}^n \| 0 \leq x_i \leq 1\}$	n-cube
$\partial I^n = \{x \in \mathbb{I}^n \| \exists i, x_i = 0, 1\}$	boundary of I^n
$\Delta^n = \Delta[n] = \{x \in \mathbb{R}^{n+1} \| x_i \geq 0, \sum_i x_i = 1\}$	n-simplex
$\partial \Delta^n = \{x \in \Delta^n \| \exists i, x_i = 0\}$	Boundary of n-simplex

Path: $u : [a, b] \to X$ from $x = u(a)$ to $y = u(b)$ (usually reparametrized to $[0, 1] \to X$)
Inverse path: $u^- : t \to u(1-t)$ from y to x
Product path: $u * v : t \to \begin{cases} u(2t) & t \leq \frac{1}{2} \\ v(2t-1) & \frac{1}{2} \leq t \end{cases}$
Constant path: $k_x : t \to x$
$\pi_0 : \text{TOP} \to \text{SET}$

- $\pi_0(X)$: Set of path connected components of X
- $\pi_0(f)([x]) = [f(x)]$

$W : \text{TOP} \to \text{CAT}$

- $W(X)$: Paths $u : [0, a] \to X$ and composition is defined on $[0, a + b]$ for associativity
- $W(f)(x) = f(x), W(f)(u) = f \circ u$

2.2 Homotopy notions

Homotopy: $H_t : X \times [0, 1] \to Y$ from $f = H_0 : X \to Y$ to $g = H_1 : X \to Y$; $H : f \simeq g$ (composition/inverse immediate)
Homotopy $H_t : X \to Y$ relative to $A \subset X$ if $H_t : A \to Y$ is independent of t
Homotopy between f and a constant map is a null homotopy
Null homotopy of $\text{id}_X : X \to X$ is a contraction
Path category $W(X, Y)$

- Objects: $f : X \to Y$
- Morphisms: Homotopy $H_t : [0, a] \times X \to Y$ between f and g

hTOP is TOP quotiented by the homotopy relation.

hTOP	TOP
Isomorphic	Homotopy equivalent/Same homotopy type
Isomorphic to $\{*\}$	Contractible
Isomorphism	h-equivalence
Constant map	Null homotopic

Hom functors in hTOP of $f : X \to Y$:

$$f_* : [Z, X] \to [Z, Y], g \to fg \quad f^* : [Y, Z] \to [X, Z], h \to hf$$

Remark. *Generally lower index for covariant and upper index for contravariant*

TOP^0: Category of pointed spaces
hTOP^0: Quotient of TOP^0 by homotopy
Forgetful functor $\text{TOP}^0 \to \text{TOP}$ has a left adjoint, $X \to (X + \{*\}, *)$

Remark. *The smash product $A \wedge B = \frac{A \times B}{A \vee B}$ is always compatible with homotopies and is a tensor product in some appropriate subcategory, i.e. compactly generated spaces*

TOP(2): Pairs of topological spaces $A \subset X$
Note that the product we use here is not the categorical product, instead it is defined as

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$$

so that $(I^m, \partial I^m) \times (I^n, \partial I^n) = (I^{m+n}, \partial I^{m+n})$
TOP(3): Pairs of topological spaces $A \subset B \subset X$
TOP_B: Slice category, objects are morphisms $x : X \to B$ and morphisms are $f : X \to Y$ such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x"'] & & Y \arrow[dl, "y"] \\ & B & \end{tikzcd}$$

- A morphism from $\text{id}_B : B \to B$ to $p : E \to B$ is a section of p
- If $p \cong \text{id}_B$ in hTOP_B, then it is shrinkable

TOP^K: Coslice category, objects are morphisms $a : K \to A$ and morphisms are $f : A \to B$ such that

$$\begin{tikzcd} & K \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

- A morphism from $i : K \to X$ to $\text{id}_K : K \to K$ is a retraction of i and i is an embedding
- $i : K \subset X$, then K is a retract of X
- If $i \cong \text{id}_K$ in hTOP^K, then it is a deformation retract

Note that $\text{TOP}_{\{*\}} \cong \text{TOP}$ and $\text{TOP}^{\{*\}} \cong \text{TOP}^0$
$H_t : A \to B$ is a homotopy in the (co)slice category if each $H_t, t \in [0, 1]$ is a morphism in the (co)slice category, hence we get the quotient categories $\text{hTOP}^K, \text{hTOP}_B$.

2.3 Internal hom objects

Let Y^X or $F(X,Y)$ be the set of continuous maps from X to Y with the compact open topology. Suppose that X is locally compact, then Y^X is the exponential object, i.e.

$$\begin{tikzcd} X \times Y \arrow[d, dashed, "f^\wedge \times \mathrm{id}_Y"'] \arrow[dr, "f"] & \\ Z^Y \times Y \arrow[r, "e_{Y,Z}"'] & Z \end{tikzcd}$$

f induces $f^\wedge$ and $f^\wedge$ induces f, alternatively

$$\mathrm{Hom}\left(- \times Y, Z\right) \cong \mathrm{Hom}\left(-, Z^Y\right)$$

which also tells us the functors $-^Y$ is a right adjoint to $- \times Y$.
Unfortunately in categories with zero objects, i.e. TOP^0, then exponential objects generally dont exist unless the category is trivial as if Y^X exists, we have

$$\mathrm{Hom}(X,Y) \cong \mathrm{Hom}\left(0 \times X, Y\right) \cong \mathrm{Hom}\left(0, Y^X\right) \cong \{*\}$$

However, we may have some form of tensor-hom adjuncation.
In the category TOP^0, we define $F^0(X,Y)$ as the subspace of pointed maps of $F(X,Y)$ and the constant map is the basepoint. Any pointed map $X \times Y \to Z$ induces a pointed map $X \to F^0(X,Y)$ if it sends $X \times y \cup x \times Y$ to z, hence it corresponds to maps from $X \wedge Y \to Z$. The adjuncation in this case reduces to

$$F^0\left(X \wedge Y, Z\right) \cong F^0\left(X, F^0\left(Y, Z\right)\right)$$

when X, Y are locally compact. This gives us our tensor-hom adjuncation.
If we quotient by homotopy and assume X is locally compact and $e^0_{X,Y}$ is continuous, then we get

$$[X \wedge Y, Z]^0 \cong \left[X, F^0(X,Y)\right]^0$$

2.4 Fundamental groupoid

$\Pi : \mathrm{TOP} \to \mathrm{GRPd}$

- $\Pi(X)$: Quotient of $W(X)$ by homotopy

Somewhat cleaner way to state van-Kampen theorem:

Theorem (Seifert–Van Kampen [**?**, Thm 2.7])**.** *Suppose that $\mathcal{U}$ is a covering of X such that if $U_1, U_2 \in \mathcal{U}$, then $U_1 \cap U_2 \in \mathcal{U}$. This turns $\mathcal{U}$ into a category where morphisms are inclusions, then*

$$\Pi(X) \cong \mathrm{colim}_{U \in \mathcal{U}} \Pi(U)$$

Choosing a base point, we get the functor $\pi_1 : \mathrm{TOP}^0 \to \mathrm{GRP}$.

Remark. *Proposition 2.7.3 of tom Dieck that the fundamental group of a monoid in* TOP^0 *is commutative and agrees with the monoid operation comes from a more general theorem, the Eckmann–Hilton argument*

Theorem (Eckmann-Hilton argument)**.** *If* $\cdot, *$ *are unital binary operations on* X *with units* $1.$ *and* 1_* *such that*

$$(a \cdot b) * (c \cdot d) = (a * b) \cdot (c * d)$$

Then $\cdot, *$ *coincide, are associative and commutative*

2.5 Enriching TOP

We give a groupoid structure, $\Pi(X, Y)$ to each hom set $\mathrm{Hom}_{\mathrm{TOP}}(X, Y)$ with homotopy as morphisms. This provides us with a 2-category, i.e.

$$\begin{tikzcd}
X \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & Y \arrow[r, bend left=60, "u"] \arrow[r, "v" description] \arrow[r, bend right=60, "w"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3ex, "\Downarrow\delta"] & Z
\end{tikzcd}$$

such that all the compositions makes sense, i.e.

$$(\delta\gamma)\,(\beta\alpha) = (\delta\beta)\,(\gamma\alpha)$$

We can enrich similar categories like TOP^0

3 Chapter 3

3.1 Definitions

Suppose $p : E \to B$ is surjective and $U \subset B$ is open

- **Trivialization** of p over U is a homeomorphism $p^{-1}(U) \to U \times F$
- p is **locally trivial** if a open covering $\mathcal{U}$ exists where a trivialization of p over $U \in \mathcal{U}$ exists for all U
- $\mathcal{U}$ is a **bundle chart**
- F is the **typical fibre**
- p is **trivial over** U if a bundle chart over U exists
- **Bundles/Fibre bundles** are locally trivial maps

Covering space/Covering of B is a locally trivial trivial map $p : E \to B$ with discrete fibres

- If $\phi_U : p^{-1}(U) \to U \times F$ is a trivialization, then $\phi_U^{-1}(U \times \{*\})$ are the **sheets** over U
- If $|F| = n$, then p is a n-fold covering
- A **trivial covering** is the covering $p : B \times F \to B$
- U is **admissible** or **evenly covered** if a trivialization exists
- E is the **total space** and B is the **base space**

3.2 Coverings with group actions

A **left G-principal covering** is a covering $p : E \to B$ and a properly discontinuous group action G on E such that $p(gx) = p(x)$ and the action on fibres are transitive
$\alpha \in \mathrm{Aut}(p)$ if $\alpha : p \to p$ is a morphism in TOP_B. These are **deck transformations**
The map $x \to gx$ gives a map $G \to \mathrm{Aut}(p)$

Theorem (Galois correspondence). *Let $p : E \to B$ be a covering, then*

- *If E is connected,* $\mathrm{Aut}(p)$ *is a properly discontinuous action on E*
- *If B is locally path connected, H subgroup of* $\mathrm{Aut}(p)$*, then $E/H \to B$ is a covering*

A **right G-principal covering** is a covering $p : E \to B$ and a properly discontinuous group action G on E such that $p(xg) = p(x)$ and the action on fibres are transitive.
Let F be a set with a left G action, then the space $E \times_G F$ constructed by quotienting $E \times F$ by $(xg, f) = (x, gf)$ is an **associated covering**.

Remark. *Seems like for this part we need to assume that that G is a free action of the fibres as well and F is given the discrete topology*

Theorem. *The map $p_F : E \times_G F \to B, (x, f) \to p(f)$ is a covering with typical fibre F*

Proof. Suppose that $\mathcal{F}$ is the typical fibre of p.
First we show the typical fibre of p_F is F. It's immediate that the typical fibre is given by $\frac{\mathcal{F}\times F}{\sim}$ immediately showing discreteness. Next, notice that $\{(\mathfrak{f}, f) \,|\, f \in F\}$ are the representatives of $\frac{\mathcal{F}\times F}{\sim}$ for some arbitrary $\mathfrak{f}$ as supose $\mathfrak{f}' = \mathfrak{f}g$, then $(\mathfrak{f}', f) = (\mathfrak{f}, gf)$ and $(\mathfrak{f}, f) = (\mathfrak{f}, f')$ implies that either $\mathfrak{f}$ has a nontrivial stabalizer or $f = f'$, hence we need to assume the action is free on $\mathcal{F}$.
Next we show that this is indeed a covering. Suppose U has a trivialization, i.e. $p^{-1}(U) \cong U \times \mathcal{F}$. Then $\frac{p^{-1}(U)\times F}{\sim} \cong \frac{U\times\mathcal{F}\times F}{\sim} \cong U \times F$. Hence p_F is a covering with typical fibre F. □

This gives us the functor

$$A(p) : G\text{-}\mathrm{SET} \to \mathrm{COV}_B$$

from the category of sets with a left G action to the category of covering spaces over B (a subcategory of TOP_B).
If $A(p)$ is an equivalence of categories, then p is the universal cover

3.3 Lifting

$F : X \to E$ is a **lifting** of $f : X \to B$ along $p : E \to B$ if $pF = f$, i.e. a morphism in TOP_B
If X is connected and p is a covering, liftings that agree somewhere are unique.
A map $p : E \to B$ has **homotopy lifting property** (HLP) for a space X if for each homotopy h_t and initial condition H_0, we can extend H_0 to the homotopy H_t such that the diagram commutes:

$$\begin{tikzcd} X \arrow[rr, dotted, "H_t"] \arrow[dr, "h_t"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

H is a lifting of h with initial conditions a. p is a **fibration** if it has HLP for all spaces

Theorem. *Coverings $p : E \to B$ are fibrations*

Proof. First show that projection maps $U \times F \to U$ are fibrations, then glue these projection maps and use uniqueness of liftings □

As liftings along coverings are unique, the diagram below is a pullback:

$$\begin{tikzcd} E^I \arrow[r, "p^I"] \arrow[d, "e^0_E"'] & B^I \arrow[d, "e^0_B"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

Let $p : E \to B$ be a map with HLP for I and $F_b = p^{-1}(b)$.
For every map $[v] \in \Pi(B)$, we obtain a well defined map $v_\sharp : \pi_0\left(F_b\right) \to \pi_0\left(F_c\right)$. Suppose $V : I \to E$ is a lifting of v with $V(0) = x$, then $v_\sharp[x] = [V(1)]$.
With this we obtain the **transport functor** $T_p : \Pi(B) \to \mathrm{SET}$

- $b \to \pi_0(B)$
- $[v] \to v_\sharp$

Let $p(x) = b$ and let $\partial_x : \pi_1\left(B, b\right) \to \pi_0\left(F_b, x\right), [v] \to v_\sharp(x)$ and $i : F_b \subset E$, then we have the exact sequence

$$\begin{tikzcd}
\pi_1\left(F_b, x\right) \arrow[r, "i_*"] & \pi_1\left(E, x\right) \arrow[r, "p_*"] & \pi_1(B, b) \arrow[dll, rounded corners, "\partial_x"] \\
\pi_0\left(F_b, x\right) \arrow[r, "i_*"] & \pi_0\left(E, x\right) \arrow[r, "p_*"] & \pi_0(B, b)
\end{tikzcd}$$

as well as the isomorphisms of sets $\partial_x : \frac{\pi_1(B,b)}{p_*\pi_1(E,x)} \cong \pi_0\left(F_b, x\right), i_* : \frac{\pi_0(F_b,x)}{\pi_1(B,b)} \cong \pi_0(E, x)$
For a covering $p : E \to B$ with B path connected, the exact sequence simplifies to

$$\begin{tikzcd}
1 \arrow[r] & \pi_1\left(E, x\right) \arrow[r, "p_*"] & \pi_1(B, b) \arrow[dll, rounded corners, "\partial_x"] \\
\pi_0\left(F_b, x\right) \arrow[r, "i_*"] & \pi_0\left(E, x\right) \arrow[r] & \{*\}
\end{tikzcd}$$

Furthermore suppose that $p : E \to B$ is a right G-principal covering with E path connceted, then we get the exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \pi_1\left(E, x\right) \arrow[r, "p_*"] & \pi_1(B, b) \arrow[r, "\delta_x"] & G \arrow[r] & 1
\end{tikzcd}$$

and the image of p_* is normal.

3.4 Coverings

Outline:

1. Construct the inverse X of $T : \mathrm{TRA}_B \to \mathrm{COV}_B$ that exists and is an equivalence of categories for sufficiently nice B
2. Construct the functor $\epsilon_B : \mathrm{TRA}_B \to \pi_b\text{-}\mathrm{SET}$ and the inverse η_b
3. Hence $A(p) : G\text{-}\mathrm{SET} \to \mathrm{COV}_B$ is an equivalence of categories iff the total space of p is simply connected

Let $\mathrm{TRA}_B = [\Pi(B), \mathrm{SET}]$, the transport functor in the previous section yields the functor $T : \mathrm{COV}_B \to \mathrm{TRA}_B$.
If B is path connected and T is an equivalence of categories, then B is a **transport space**.
A set $U \in B$ is **transport simple** if any paths in U between identical points are homotopic in B.
B is **semi-locally simply connected** if it has an open covering with transport simple sets.
B is **transport local** if it is path connected, locally path connected and semi-locally simply connected.

Theorem. *If B is then B is a transport space*

Proof. We need to construct the inverse of T, $X : \mathrm{TRA}_B \to \mathrm{COV}_B$. Let $\Phi : \Pi(B) \to \mathrm{SET}$ be some functor, we will construct a covering $p : X(\Phi) \to B$. As a set $X(\Phi) = \coprod_{b \in B} \Phi(b)$. To get a reasonable topology on it, we consider a covering $\mathcal{U}$ of B by transport simple path connected open sets. For every $b \in U \in \mathcal{U}$, we define $\phi_{U,b} : U \times \Phi(b) \to p^{-1}(U)$ and by gluing these maps together, we obtain a topology on $X(\Phi)$ and a covering p.
Verification of functoriality and inverse are somewhat direct from definition. □

With this, consider the hom functor $\mathrm{Hom}_{\Pi(B)}(b, -) \in \mathrm{TRA}_B$ and let $p^b : E^b \to B$ be its associated covering. Then E^b is simply connected right $\mathrm{Hom}_{\Pi(B)}(b, b)$-principal covering.
Suppose that B is path connected, then $\Pi = \Pi(B)$ is a connected groupoid. Let $\Pi(x, y) = \mathrm{Hom}_{\Pi(B)}(x, y)$ and $\pi_b = \Pi(b, b)$
For a functor $F : \Pi \to \mathrm{SET}$, we have the left π_B-set $F(b)$ giving us the functor $\epsilon_b : \mathrm{TRA}_B \to \pi_b\text{-}\mathrm{SET}$.
For a left π_B-set A, we define the functor $\Pi(b, -) \times_{\pi_B} A : \Pi \to \mathrm{SET}$ where $A \times_G B$ is the set $A \times B$ quotiented by $(ag, b) = (a, gb)$. This gives us the functor $\eta_b : \pi_b\text{-}\mathrm{SET} \to \mathrm{TRA}_B$, the inverse of ϵ_b.
Finally we have the following categories and functors:

$$G\text{-}\mathrm{SET} \xrightarrow{A(p)} \mathrm{COV}_B \underset{X}{\overset{T}{\rightleftarrows}} \mathrm{TRA}_B \underset{\eta_b}{\overset{\epsilon_b}{\rightleftarrows}} \pi_b\text{-}\mathrm{SET}$$

where X exists if B is transport-local and ϵ_B, η_B require B to be path connected to exist.
Finally we have

Theorem. *The following are equivalent:*

- *B is a transport space, i.e. T is an equivalence of categories*
- *B has a universal right G-principal covering $p : E \to B$, i.e. $A(p)$ is an equivalence of categories*

Note that the exact sequences above imply that E is simply connected.
Define the **orbit category** $\mathrm{Or}(G)$ consisting of homogenous G-sets ($\frac{G}{H}$ for any subgroup H) and G-maps. This category is a strict subcategory of $G\text{-}\mathrm{SET}$, consisting only the transitive sets.

For a covering $p : E \to B$, we obtain the injective map $p_* : \pi_1(E, x) \to \pi_1(B, p(x))$ and the image is called the **characteristic subgroup** of p wrt x.
Let $p : E \to B$ be a simply connected covering, then the subcategory $A(p)\,(\mathrm{Or}(\pi_b))$ of COV_B is equivalent to the subcategory consisting of connected coverings. This tells us that the connected coverings of a transport space is determined by subgroups of the fundamental group.

Theorem. *Let B be a transport space and $p : E \to B$ a covering.*

- *The action of* $\mathrm{Aut}(p)$ *on E makes it a left-*$\mathrm{Aut}(p)$ *principal covering*
- *A simply connected covering is a universal covering*
- *Universal coverings are unique up to isomorphism*
- *The automorphism group of a universal cover is isomorphic to* $\pi_1(B, b)$
- E^b *is simply connected*
- *We have a Galois correspondence between isomorphism classes of connected coverings and subgroups of* $\pi_1(B, b)$

If B is not a transport space but is path connected and locally path connected, then we have a similar result where coverings by path connected total spaces are isomorphic iff the characteristic subgroups are conjugate in $\pi_1(B, b)$.
Suppose we have coverings $p : E \to B$ and $f : Z \to B$, then a covering $\Phi : Z \to E$ such that

$$\begin{tikzcd} Z \arrow[rr, dashed, "\Phi"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

exists iff $f_*\pi_1(Z, z) \subset p_*\pi_1(E, x)$ for some $f(z) = p(x)$.

If X is a topological group with identity x and $p : E \to X$ is a covering with E path connected and locally path connected, then for each $e \in p^{-1}(b)$, there exists a unique group structure on E such that e is the identity and p a homomorphism.

4 Chapter 4

4.1 Mapping cylinders

For a map $f : X \to Y$ **mapping cylinder** $Z(f)$ is constructed by the pushout

$$\begin{tikzcd}
X + X \arrow[r, "\mathrm{id} + f"] \arrow[d, "\langle i_0, i_1 \rangle"'] & X + Y \arrow[d, dashed, "\langle j, J \rangle"] \arrow[ddr, bend left, "\langle f, \mathrm{id} \rangle"] & \\
X \times I \arrow[r, dashed, "a"'] \arrow[drr, bend right, "f"'] & Z(f) \arrow[dr, dashed, "q"] & \\
 & & Y
\end{tikzcd}$$

We also have $Jq \cong \mathrm{id}$ as $X \times I \cong X$ and $f = qj$, j a closed immersion and q a homotopy equivalence. We see that Z has nice functorial properties. Suppose we have the homotopy commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\
X' \arrow[r, "f'"] \arrow[d, "\alpha'"'] & Y' \arrow[d, "\beta'"] \\
X'' \arrow[r, "f''"] & Y''
\end{tikzcd}$$

and let the homotopy equivalences be $\Phi : f'\alpha \cong \beta f, \Phi' : f''\alpha' \cong \beta' f'$
This induces the following homotopy commutative diagram

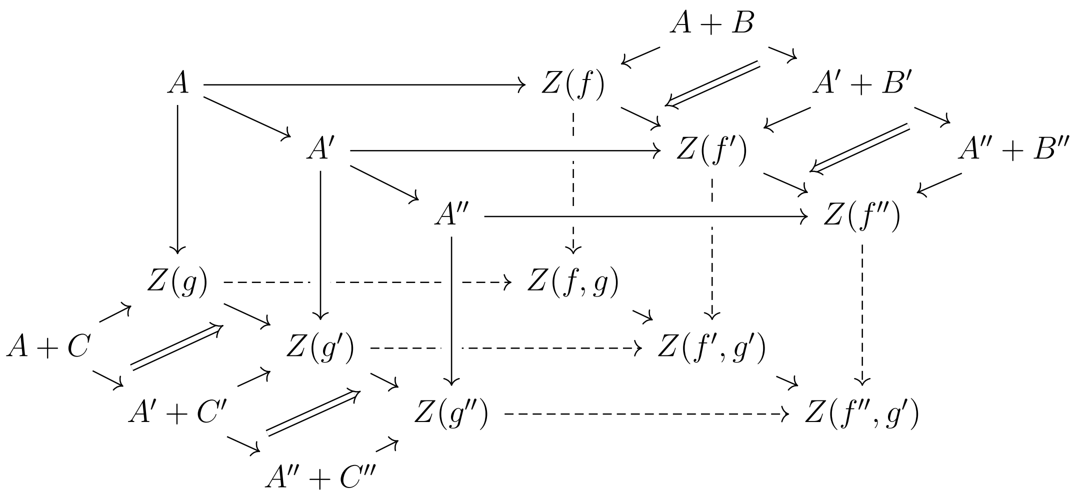
$$\begin{tikzcd}
X + Y \arrow[r] \arrow[d, "\alpha+\beta"'] & Z(f) \arrow[d, "Z(\alpha,\beta,\Phi)"] \\
X' + Y' \arrow[r] \arrow[d, "\alpha'+\beta'"'] & Z(f') \arrow[d, "Z(\alpha',\beta',\Phi'')"] \\
X'' + Y'' \arrow[r] & Z(f'')
\end{tikzcd}$$

where each small square commutes in TOP and the whole diagram commutes in hTOP.
Given maps $f : A \to B$ and $g : A \to C$, we can construct the double mapping cylinder by either of the two pushouts:

$$\begin{tikzcd}
A + A \arrow[r, "f+g"] \arrow[d, "\langle i_0, i_1 \rangle"'] & B + C \arrow[d, dashed, "\langle j_0, j_1 \rangle"] \\
A \times I \arrow[r, dashed] & Z(f, g)
\end{tikzcd}$$

$$
\begin{tikzcd}
A \arrow[r, "j^B"] \arrow[d, "j^C"'] & Z(f) \arrow[d, dashed] \\
Z(g) \arrow[r, dashed] & Z(f,g)
\end{tikzcd}
$$

The functorality of the double mapping cylinder can be seem from the following commutative homotopy diagram:



Suppose we have the homotopy commutative diagram

$$
\begin{tikzcd}
X_0 \arrow[r, "f_+"] \arrow[d, "f_-"'] & X_+ \arrow[d, "j_+"] \arrow[ddr, dashed, bend left] & \\
X_- \arrow[r, "j_-"'] \arrow[drr, dashed, bend right] & X & \\
 & & Z\left(f_-, f_+\right) \arrow[ul, dashed, "\phi"']
\end{tikzcd}
$$

The square is called a **homotopy pushout** or **homotopy cocartesian** if ϕ is a homotopy equivalence (i.e. a pushout in the category hTOP)

Let $f_\pm, j_\pm$ be inclusions and $X = X_+ \cup X_-$, then the diagram is a pushout in TOP.

Let $N\left(X_-, X_+\right) = X_- \times 0 \cup X_0 \times I \times X_+ \times 1$ be a subspace of $X \times I$ and let $p_N : N\left(X_-, X_+\right) \to X$ be the projection map.

The covering $X_\pm$ is **numerable** if p_N has a section. With this, we have the following condition to determine if the diagram above is a pushout in hTOP:

Theorem 4.1. *$Z\left(f_-, f_+\right) \cong X$ if the covering $X_\pm$ is numerable*

Given the projection maps $X \xleftarrow{f} X \times Y \xrightarrow{g} Y$, the double mapping cylinder $Z(f,g) = X \star Y$ is known as the **join** of X and Y

4.2 Suspensions and loops

Here we work in pointed categories
The **suspension** functor $\Sigma : \mathrm{TOP}^0 \to \mathrm{TOP}^0$ is given by

- $\Sigma X = S^1 \wedge X = \frac{X \times I}{X \times \partial I \cup \{x\} \times I}$
- $S\Sigma_* [X, Y]^0 = [\Sigma X, \Sigma Y]^0$

Note that Σ_* is a homomorphism if $X = \Sigma A$ and any pointed homotopy $H_t : X \to Y$ corresponds uniquely to a pointed map $\overline{K} : \Sigma X \to Y$
The set $[\Sigma X, Y]^0$ has a natural group structure by composition of homotopies. Furthermore as $[\Sigma^n X, Y]^0$ has n natural composition laws by composin the homotopies at the the ith coordinate and these satisfies the assumptions of Eckmann-Hilton argument, they are equivalent. This also tells us that the higher homotopy groups, $\pi_n(X) = [S^n, X]^0$, are commutative groups.
We can dualize everything above:
The **loop** space of X is $\Omega X = F^0 (S^1, X)$. This consists of loops in X with basepoint x. This is naturally a topological group with the product of loops.
There is a natural group structure on $\mathrm{Hom}_{\mathrm{TOP}^0} (X, \Omega Y)$ given by $[f] +_m [g] = [f][g]$.
As S^1 is locally compact, we have the tensor-hom adjuncation

$$[\Sigma X, Y]^0 \cong [X, \Omega Y]^0$$

and this commutes with the group operation.
In the set $[\Sigma X, \Omega Y]^0$, by the Eckmann-Hilton argument, the group operations on these two sets coincide and are commutative.

4.3 Group objects

Perhaps a nicer way of looking at group/any objects is via Yoneda's lemma, see [**?**] for more details:

Theorem 4.2 (Yoneda)**.** *For any category C, we have a full and faithful functor, the Yoneda embedding $y_C : C \to [C^{\mathrm{op}}, \mathrm{SET}]$, given by $y_C(c) \to \mathrm{Hom}_C(-, c)$*

An object $g \in C$ is a **group object** if the functor $y_C(g)$ factors through GRP, i.e. the following diagram commutes where GRP $\to$ SET is the forgetful functor

$$\begin{tikzcd}
& \mathrm{GRP} \arrow[dr, hook] & \\
C^{\mathrm{op}} \arrow[ur, dashed, "G"] \arrow[rr, "y_C(g)"'] & & \mathrm{SET}
\end{tikzcd}$$

To give an explicit construction, recall that for a group G, we need to have a unit element, an inverse operation and a group operation, given by

$$\begin{aligned} e_G &: 1 \to G \\ \mathrm{inv}_G &: G \to G^{\mathrm{op}} \\ \cdot_G &: G \times G \to G \end{aligned}$$

such that the following diagrams commute to ensure associativity, unit and inverse holds:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id}_G \times \cdot_G"] \arrow[d, "\cdot_G \times \mathrm{id}_G"'] & G \times G \arrow[d, "\cdot_G"] \\ G \times G \arrow[r, "\cdot_G"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "{(e,\mathrm{id}_G)}"] \arrow[d, "{(\mathrm{id}_G,e)}"'] \arrow[dr, equal] & G \times G \arrow[d, "\cdot_G"] \\ G \times G \arrow[r, "\cdot_G"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "{(\mathrm{inv}_G,\mathrm{id}_G)}"] \arrow[d, "{(\mathrm{id}_G,\mathrm{inv}_G)}"'] \arrow[dr, "e"] & G \times G \arrow[d, "\cdot_G"] \\ G \times G \arrow[r, "\cdot_G"'] & G \end{tikzcd}$$

where $e : G \to G$ is the composite morphism $G \to 1 \xrightarrow{e_G} G$

The maps can immediately be constructed by Yoneda's lemma, take for instance the product map. If g is a group object with $G : C^{\mathrm{op}} \to \mathrm{GRP}$ as the functor to group and $f : c \to d$ is a morphism in C, then the following diagram commutes:

$$\begin{tikzcd} G(c) \times G(c) \arrow[r, "\cdot_{G(c)}"] & G(c) \\ G(d) \times G(d) \arrow[r, "\cdot_{G(d)}"'] \arrow[u, "G(f) \times G(f)"] & G(d) \arrow[u, "G(f)"'] \end{tikzcd}$$

telling us we have a natural transformation $\cdot_G : G \times G \to G$. This gives us the morphism $\cdot_g : g \times g \to g$ by Yoneda's lemma and as the Yoneda embedding is full and faithful, the commutativity of the diagrams above defining a group is immediate.

Similarly one defines cogroups as group objects in the opposite category. With this view, it is immediate that if $c \in C$ is a cogroup, then $\mathrm{Hom}_C(c, -)$ is a functor to GRP.

As examples in hTOP^0, ΣX is a cogroup as $[\Sigma X, Y]^0$ is a group and ΩY is a group as $[X, \Omega Y]^0$ is a group.

4.4 Fibre sequence

Again here we work in pointed spaces. A map $f : X \to Y$ induces a map $f^* : [Y, B]^0 \to [X, B]^0$. The kernels of f is any element that gets sent to the basepoint of Y and the kernel of f^* is any element that gets sent to a nullhomotopic element. This allows us to define exact sequences of topological spaces.

A sequence of spaces $U \xrightarrow{f} V \xrightarrow{g} W$ is **h-coexact** if for all spaces B, the sequence

$$[U, B]^0 \xleftarrow{f^*} [V, B]^0 \xleftarrow{g^*} [W, B]^0$$

is exact.
The **cylinder** $XI = \frac{X \times I}{* \times I}$ describes homotopies in TOP^0 (morphisms $XI \to Y$ are homotopies).
The **cone** $CX = \frac{X \times I}{X \times 0 \cup * \times I} = X \wedge I$ describes homotopies starting from constant maps in TOP^0.
Define $C(f)$ as the pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "f_1"] \\
CX \arrow[r, "j"'] & C(f)
\end{tikzcd}$$

and by its universal property, we have the h-coexact sequence $X \xrightarrow{f} Y \xrightarrow{f_1} C(f)$ which can be iterated to create a long h-coexact sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{f_2} C\left(f_1\right) \xrightarrow{f_3} \ldots$$

Furthermore we have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "f"'] & CX \arrow[r, "p"] \arrow[d, "j"] & Y/i_1X \arrow[d] & = \Sigma X \\
Y \arrow[r, "f_1"] \arrow[d, "i_1"'] & C(f) \arrow[r, "p(f)"] \arrow[d, "f_2"] & C(f)/f_1Y \arrow[d] & = \Sigma X \\
CY \arrow[r, "j_1"'] & C\left(f_1\right) \arrow[r, "q(f)"'] & C\left(f_1\right)/j_1Y & = \Sigma X
\end{tikzcd}$$

and $q(f)$ is a homotopy equivalence.
Applying this to itself, we obtain

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & CX \arrow[d] & \\
Y \arrow[r, "f_1"] \arrow[d] & C(f) \arrow[r] \arrow[d, "f_2"'] & \Sigma X \arrow[dd, "\Sigma f \circ \iota"] \\
CY \arrow[r] & C\left(f_1\right) \arrow[ur, "q(f)"'] \arrow[dr, "p\left(f_1\right)"] \arrow[d, "f_3"'] & \\
& C\left(f_2\right) \arrow[r, "q\left(f_1\right)"'] & \Sigma Y
\end{tikzcd}$$

where $\iota : (x, t) \to (x, 1 - t)$ to ensure commtuativity.
As $q(f), q\left(f_1\right)$ are homotopy equivalences and a sequence remains h-coexact if we replace elements with h-equivalent ones, we obtain the h-coexact sequence

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y$$

And we can apply this sequence to itself iteratively and noting that Σ and C commutes to obtain the **Puppe-sequence** or the **cofibre sequence** of f:

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y \to \Sigma C(f) \to \Sigma C(f) \to \Sigma^2 X \to \Sigma^2 Y \ldots$$

as $\Sigma f : \Sigma X \to \Sigma Y$
Applying the functor $[-, B]^0$, we see that from the 4th place onwards these are groups and from the 7th place onwards these are abelian groups.
For any map $f : X \to Y$, let $\mu : C(f) \to \Sigma X \vee C(f)$ be defined as

$$\mu(x,t) = \begin{cases} ((x,2t),*) & 2t \leq 1 \\ (*,(x,2t-1)) & 2t \geq 1 \end{cases}$$

and $\mu(y) = y$. This map is a **h-coaction** of the h-cogroup ΣX on $C(f)$ as we have

$$[\Sigma X, B]^0 \times [C(f), B]^0 \cong [\Sigma X \vee C(f), B]^0 \to [C(f), B]$$

where the last map is induced by μ.
With the map $Y \xrightarrow{f_1} C(f) \xrightarrow{p(f)} \Sigma X$ and any maps $\alpha_1, \alpha_2 : \Sigma X \to B$, this group action satisfies $(\alpha_1)\left(p(f)^* \alpha_2\right) = p(f)^* \left(\alpha_1 \alpha_2\right)$ and f_1^* is an injective map on orbits of this action.
We can dualize everything as always.
A sequence of spaces $U \xrightarrow{f} V \xrightarrow{g} W$ is **h-exact** if for all spaces B, the sequence

$$[B, U]^0 \xrightarrow{f^*} [B, V]^0 \xrightarrow{g^*} [B, W]^0$$

is exact.
To dualize the cone, we use the exponential object adjuncation $[X \wedge I, Y]^0 \cong [X, F^0(Y, I)]^0$ and define $FY = F^0(Y, I)$. We then define $F(f)$ as the pullback

$$\begin{tikzcd} F(f) \arrow[r, dashed, "q"] \arrow[d, dashed, "f^1"'] & FY \arrow[d, "e^1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and by its universal property, we have the h-exact sequence $F(f) \xrightarrow{f_1} X \xrightarrow{f} Y$ which can be iterated to create a long h-exact sequence

$$\ldots \xrightarrow{f^4} F\left(f^2\right) \xrightarrow{f^3} F\left(f^1\right) \xrightarrow{f^2} F(f) \xrightarrow{f^1} X \xrightarrow{f} Y$$

We can dualize the diagrams above to obtain

$$
\begin{tikzcd}
Y & FY \arrow[l, "e^1"'] & \Omega Y \arrow[l, "i"'] \\
X \arrow[u, "f"] & F(f) \arrow[u, "q"'] \arrow[l, "f^1"'] & \Omega Y \arrow[u] \arrow[l, "i(f)"'] \\
FX \arrow[u, "e^1"] & F\left(f^1\right) \arrow[u, "f^2"'] \arrow[l, "q^1"] & \Omega Y \arrow[u] \arrow[l, "j(f)"]
\end{tikzcd}
$$

where $j(f)$ is a h-equivalence and using this on itself, we obtain

$$
\begin{tikzcd}
Y & FY \arrow[l] & \\
X \arrow[u, "f"] & F(f) \arrow[u] \arrow[l, "f^1"'] & \Omega Y \arrow[l, "i(f)"'] \arrow[dl, "j(f)"] \\
FX \arrow[u] & F\left(f^1\right) \arrow[u, "f^2"] \arrow[l] & \\
& F\left(f^2\right) \arrow[u, "f^3"] & \Omega X \arrow[l, "j\left(f^1\right)"] \arrow[ul, "i\left(f^1\right)"'] \arrow[uu, "\iota\circ\Omega f"']
\end{tikzcd}
$$

and finally we obtain the dual long h-exact sequence

$$\Omega X \to \Omega Y \to F(f) \to X \to Y$$

and repeating this on the map $\Omega f : \Omega X \to \Omega Y$, we obtain the long h-exact sequence

$$\ldots \Omega^2 X \to \Omega^2 Y \to \Omega F(f) \to \Omega X \to \Omega Y \to F(f) \to X \to Y$$

known as the **fibre seqence** of f. Similarly applying the functor $[B, -]^0$, we see that from the 4th place onwards these are groups and from the 7th place onwards these are abelian groups.

Finally to dualize the group action, again let $f : X \to Y$ be any map. We have the h-action $m : \Omega Y \times F(f) \to F(f)$ defined as

$$m\left([f(t), (x, g(t))]\right) = \begin{cases} (x, f(2t)) & 2t \leq 1 \\ (x, g(2t-1)) & 2t \geq 1 \end{cases}$$

This map induces the map

$$[B, \Omega Y]^0 \times [B, F(f)]^0 \cong [B, \Omega Y \times F(f)]^0 \to [B, F(f)]^0$$

Furthermore with the maps $\Omega Y \overset{i(f)}{\to} F(f) \overset{f^1}{\to} X$ and any maps $\alpha_1, \alpha_2 : B \to \Omega Y$, this group action satisfies $\left(i(f)_* \alpha_1\right)(\alpha_2) = i(f)_* \left(\alpha_1 \alpha_2\right)$ and f^1_* is an injective map on the orbits of the action.

Note that these sequences can be used to prove the long exact sequence of homotopy groups and the Mayer Vietoris exact sequence.

5 Chapter 5

5.1 Cofibration properties

$i : A \to X$ has the **homotopy extension property** (HEP) for Y if for every h, f, there exists some H such that the (equivalent) diagrams commutes:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "e^0"] \\
X \arrow[r, "f"'] \arrow[ur, dashed, "H"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dr, "i_0^X"] \arrow[drr, bend left, "f"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_0^A"'] & & X \times I \arrow[r, dashed, "H"] & Y \\
 & A \times I \arrow[ur, "i\times\mathrm{id}"'] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

H is an **extension** of h with **initial conditions** f. If i has HEP for every space Y, then it is a **cofibration**. This is somewhat similar to being a monomorphism with a cokernel as this allows the factorization of any nullhomotopic map fi through X/A, although the factorization may not be unique here. We can determine quickly if a map is a cofibration by studying the mapping cylinder due to the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dd, "i_0^A"'] & & X \arrow[dl, "i_0^X"'] \arrow[dd, "b"] \arrow[dddr, bend left, "y"] & \\
 & X \times I \arrow[dr, dashed, "r"] & & \\
A \times I \arrow[ur, "i\times\mathrm{id}"] \arrow[rr, "k"'] \arrow[drrr, bend right, "h"'] & & Z(i) \arrow[ul, dotted, "s"] \arrow[dr, dotted, "\sigma"] & \\
 & & & Y
\end{tikzcd}$$

We have the equivalent statements for a map $i : A \to X$:

- i is a cofibration
- i has HEP for $Z(i)$
- $s : Z(i) \to X \times I$ has a retraction

This tells us that cofibrations must be embeddings as $ki_1 = ri_1^X i : A \to Z(i)$ is an embedding. One can further show for Hausdorff spaces that this is a closed embedding.
(X, x) is **well-pointed** and x is **nondegenerate** if $x \in X$ is a cofibration.
Cofibrations are preserved under products of locally compact spaces:

$$
\begin{tikzcd}
A \arrow[r, "h^\wedge"] \arrow[d, "i"'] & \left(Z^Y\right)^I \cong \left(Z^I\right)^Y \arrow[d, "e^0"] \\
X \arrow[ur, dashed, "H^\wedge"] \arrow[r, "f^\wedge"'] & Z^Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times Y \arrow[r, "h"] \arrow[d, "i\times \mathrm{id}"'] & Z^I \arrow[d, "e^0"] \\
X \times Y \arrow[ur, dashed, "H"] \arrow[r, "f"'] & Z
\end{tikzcd}
$$

A useful lemma for proofs is that if $A \times I \in X \times I$ has HEP for a space Y, then since we have the homeomorphism of pairs $(I \times I, I \times 0 \cup \partial I \times I) \cong (I \times I, I \times 0)$, the maps $\phi : A \times I \times I \to Y$ and $\alpha : X \times (I \times 0 \cup \partial I \times I)$ induces a map $\Phi : X \times I \times I \to Y$ as long as $\alpha = \phi$ on $A \times (I \times 0 \cup \partial I \times I)$.
We can show HEP is preserved under pushouts by arrow chasing:

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, dashed, "hf"] \arrow[ddd, "j"'] & & B \arrow[dl, dashed, "h"'] \arrow[ddd, "J"] \\
& Z^I \arrow[d, dashed, "e^0"] & \\
& Z & \\
X \arrow[rr, "F"'] \arrow[uur, dashed, "K"] \arrow[ur, dashed, "\phi F"] & & Y \arrow[uul, dotted, "H"'] \arrow[ul, dashed, "\phi"']
\end{tikzcd}
$$

If j is a cofibration, then J is the cofibration **induced** from j via **cobase change** along f. For every cofibration j, this associates every map from $A \to B$ with a cofibration J. Furthermore, if we have maps $f, g : A \to B$, a homotopy $\phi : f \to g$ and the induced cofibrations j_f, j_g, then we get a morphism $\kappa : j_g \to j_f$ in TOP^B:

$$
\begin{tikzcd}
A \arrow[r, bend left, "g"] \arrow[r, dashed, "\phi_t" description] \arrow[r, dashed, bend right, "f"'] \arrow[d, "j"'] & B \arrow[d, "j_f"] \arrow[dr, bend left, "j_g"] & \\
X \arrow[r, bend left, "\Phi_1"] \arrow[r, dashed, "\Phi_t" description] \arrow[r, dashed, bend right, "F"'] \arrow[rr, bend right=40, "G"'] & Y_f & Y_g \arrow[l, dashed, "\kappa"']
\end{tikzcd}
$$

Although κ may not be unique, it turns out that the homotopy class $[\kappa]^B$ only depends $[\phi]$. Let $h\text{-}\mathrm{COF}^B$ be the full subcategory of $h\text{-}\mathrm{TOP}^B$ of cofibrations under B, then we have the contravariant functor $\Pi(A, B) \to h\text{-}\mathrm{COF}^B$ with the construction above. Since $\Pi(A, B)$ is a groupoid, we obtain the **homotopy theorem for cofibrations** stating that $[\kappa]^B$ is an isomorphism in $h\text{-}\mathrm{TOP}^B$.

5.2 Cofibration transport

Let $i : K \to A$ be a cofibration and $\phi_t : K \to X$ be a homotopy, this induces the map

$$\phi^\sharp : [(A, i), (X, \phi_0)]^K \to [(A, i), (X, \phi_1)]^K$$

defined by $\phi^\sharp [\Phi_0] = \Phi_1$ for the extension $\Phi_t : A \to X$ of ϕ with initial conditions Φ_0. This gives us the **transport functor** of a cofibration $i : K \to A$ from $\Pi(K, X) \to \mathrm{SET}$ sending $\phi_0 \to [i, \phi_0]^K$ and $[\phi] \to \phi^\sharp$.
This functor tells us the difference between being homotopic in TOP and in TOP^K in the sense that if we have morphisms $g, g' : K \to X$ in TOP and $f, f' : i \to g, g'$ in TOP^K, then $[f] = [f']$ in h-TOP iff there exists some $\phi \in \Pi(K, A)$ such that $[f']^K = \phi^\sharp [f]^K$

5.3 Replacing maps by cofibrations

We relook at the construction of the mapping cylinder (here the unit interval is flipped from the previous chapter):

$$\begin{tikzcd}
X + X \arrow[r, "f+\mathrm{id}"] \arrow[d, "\langle i_0, i_1 \rangle"'] & Y + X \arrow[d, "\langle s, j \rangle"] \\
X \times I \arrow[r, "a"'] & Z(f)
\end{tikzcd}$$

as i_0, i_1 are cofibrations, s, j are cofibrations as well. This gives us the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "p"] \arrow[d, shift left, "s"] \arrow[dr, "f_1"] & Y/X \\
X \arrow[r, "j"'] & Z(f) \arrow[r, "P"'] \arrow[u, shift left, "q"] & C(f) \arrow[u, dashed, "r"']
\end{tikzcd}$$

where

- j, s, f_1 are cofibrations
- s is a deformation retraction with inverse q
- $f = qj$, q a homotopy equivalence and j a cofibration
- If f is a cofibration, then q is a homotopy equivalence under X and r the induced homotopy equivalence

This factorization is unique in the sense if $f = qj = q'j' : X \to Y$, $X \xrightarrow{j} Z \xrightarrow{q} Y$, $X \xrightarrow{j'} Z' \xrightarrow{q'} Y$, then q, q' and j, j' are homotopic and Z, Z' are homotopy equivalent. Define $Z/j(X)$ as the (homotopical) **cofibre** of f, then this implies this cofibre is unique up to homotopy. This gives us an immediate result about homotopy pushouts:

For a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle J} \\ X & \xrightarrow[F]{} & Y \end{array}$$

with j a cofibration, then this diagram is a homotopy pushout.

5.4 Characterization of cofibration

First we note an equivalent condition for cofibrations:

Theorem ([**?**, Thm 2])**.** *$i : A \in X$ is a cofibration iff $X \times \{0\} \cup A \times I$ is a retract of $X \times I$*

If the inclusion is closed, this is immediate as we have the homeomorphism $X \times \{0\} \cup A \times I \cong Z(i)$. Otherwise the identity map is generally not a homeomorphism, even for inclusions like $(0,1) \subset [0,1]$.
The condition that $X \times \{0\} \cup A \times I$ is a retract of $X \times I$ is equivalent to the existance of a homotopy $\psi_t : X \to X$ relative to A and a map $u : X \to I$ exists such that

- $\psi_0 = \mathrm{id}_X$
- $A \subset v^{-1}(0)$
- $\psi_t(x) \in A$ for $t > v(x)$

For a closed inclusion, we define the pair (X, A) a **neighbourhood deformation retract** (NDR) if we have a homotopy $\psi_t : X \to X$ relative to A and a map $u : X \to I$ exists such that

- $\psi_0 = \mathrm{id}_X$
- $A = v^{-1}(0)$
- $\psi_1(x) \in A$ for $1 > v(x)$

It follows that (X, A) is a closed cofibration iff it's a NDR.
This also tells us that if $(X, A), (X, B)$ are closed cofibrations, then $(X, A \cup B)$ is a closed cofibration. If $(X, A), (Y, B)$ are cofibrations and A is closed, then $(X, A) \times (Y, B)$ is a cofibration.

5.5 Fibration properties

We can dualize the theory of cofibrations to obtain fibrations.
$p : E \to B$ has the **homotopy lifting property** (HLP) for X if for every h, a, there exists some H such that the (equivalent) diagrams commutes:

$$\begin{tikzcd}
B & X \times I \arrow[l, "h"'] \arrow[dl, dashed, "H"'] \\
E \arrow[u, "p"] & X \arrow[l, "a"] \arrow[u, "i_0^X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & E \arrow[dl, "p"'] & & \\
B & & E^I \arrow[ul, "e_E^0"'] \arrow[dl, "p^I"] & X \arrow[l, dashed, "H"'] \arrow[ull, bend right, "a"'] \arrow[dll, bend left, "h"] \\
 & B^I \arrow[ul, "e_B^0"] & &
\end{tikzcd}$$

H is an **lifting** of h with **initial conditions** a. If p has HLP for every space X, then it is a **fibration**. Similarly, fibrations is somewhat like epimorphisms with kernels, We can determine quickly if a map is a fibration by dualizing the mapping cylinder, the space $W(p)$ defined as the pullback

$$\begin{tikzcd}
B & E \arrow[l, "p"'] \\
B^I \arrow[u, "e_B^0"] & W(p) \arrow[l, "k"] \arrow[u, "b"']
\end{tikzcd}$$

then by considering the commutative diagram

$$\begin{tikzcd}
B & & E & \\
 & E^I \arrow[ur, "e_E^0"] \arrow[dl, "p^I"'] \arrow[dr, dotted, "r"'] & & \\
B^I \arrow[uu, "e_B^0"] & & W(p) \arrow[ll, "k"] \arrow[uu, "b"'] \arrow[ul, dashed, bend right, "s"'] & \\
 & & & X \arrow[ul, dotted, "\rho"] \arrow[uuul, bend right, "a"'] \arrow[ulll, bend left, "h"]
\end{tikzcd}$$

We have the equivalent statements for a map $p : E \to B$:

- i is a cofibration
- i has HEP for $Z(i)$
- $s : Z(i) \to X \times I$ has a retraction

Fibrations are preserved under -^{Y} for Y locally compact:

$$
\begin{tikzcd}
B & X \times Y \times I \arrow[l, "h"'] \arrow[dl, dashed, "H"] \\
E \arrow[u, "p"] & X \times Y \arrow[l, "a"] \arrow[u, "i_0^{X\times Y}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B^Y & X \times I \arrow[l, "h^\wedge"'] \arrow[dl, dashed, "H^\wedge"] \\
E^Y \arrow[u, "p^Y"] & X \arrow[l, "a^\wedge"] \arrow[u, "i_0^X"']
\end{tikzcd}
$$

We can show HLP is preserved under pullbacks by arrow chasing:

$$
\begin{tikzcd}
B & & C \arrow[ll, "f"'] \\
& X \times I \arrow[ul, "fh"'] \arrow[ur, "h"] \arrow[ddl, dashed, "K"'] \arrow[ddr, dotted, "H"] & \\
& X \arrow[u, dashed, "i_0^X"] & \\
E \arrow[uuu, "p"] \arrow[ur, "aF"] & & F \arrow[uuu, "P"'] \arrow[ul, "a"'] \arrow[ll, "F"]
\end{tikzcd}
$$

If p is a fibration, then P is the fibration **induced** from p via **base change** along f. For every fibration p, this associates every map from $B \to C$ with a fibration P. Furthermore, if we have maps $f, g : B \to C$, a homotopy $\phi : f \to g$ and the induced fibrations p_f, p_g, then we get a morphism $\kappa : p_f \to p_g$ in TOP_C:

$$
\begin{tikzcd}
B & C \arrow[l, bend right, "g"'] \arrow[l, dashed, "\phi_t" description] \arrow[l, dashed, bend left, "f"] & \\
E \arrow[u, "\rho"] & F_f \arrow[u, "\rho_f"'] \arrow[l, bend right, "\Phi_1"'] \arrow[l, dashed, "\Phi_t" description] \arrow[l, dashed, bend left, "F"] \arrow[dr, "\kappa"] & \\
& & F_g \arrow[uul, bend right, "\rho_g"'] \arrow[ull, bend left, "G"]
\end{tikzcd}
$$

Although κ may not be unique, it turns out that the homotopy class $[\kappa]_C$ only depends $[\phi]$. Let $h\text{-}\mathrm{FIB}_C$ be the full subcategory of $h\text{-}\mathrm{TOP}_C$ of fibrations over C, then we have the covariant functor $\Pi(C, B) \to h\text{-}\mathrm{FIB}_C$ with the construction above. This generalizes the previous section on fibre transport of coverings. Since $\Pi(C, B)$ is a groupoid, we obtain the **homotopy theorem for fibrations** stating that $[\kappa]_C$ is an isomorphism in $h\text{-}\mathrm{TOP}_C$.

5.6 Fibration transport

Let $p : E \to B$ be a fibration and $\phi_t : Y \to B$ be a homotopy, this induces the map

$$\phi^\sharp : [(Y,\phi_0),(E,p)]_B \to [(Y,\phi_0),(E,p)]_B$$

defined by $\phi^\sharp[\Phi_0] = \Phi_1$ for the lifting $\Phi_t : Y \to E$ of ϕ with initial conditions Φ_0. This gives us the **transport functor** of a fibration $p : E \to B$ from $\Pi(Y,B) \to \mathrm{SET}$ sending $\phi_0 \to [\phi_0, p]_B$ and $[\phi] \to \phi^\sharp$.
This functor tells us the difference between being homotopic in TOP and in TOP_B in the sense that if we have morphisms $g, g' : Y \to B$ in TOP and $f, f' : g, g' \to p$ in TOP_B, then $[f] = [f']$ in h-TOP iff there exists some $\phi \in \Pi(Y,B)$ such that $[f']_B = \phi^\sharp[f]_B$

5.7 Replacing maps by fibrations

We relook at the construction of $W(f)$ for an arbitrary map $f : X \to Y$:

$$\begin{tikzcd}
Y \times Y & X \times Y \arrow[l, "f\times\mathrm{id}"'] \\
Y^I \arrow[u, "(e^0,e^1)"] & W(f) \arrow[l] \arrow[u, "(q,p)"']
\end{tikzcd}$$

as e_0, e_1 are fibrations, p, q are fibrations as well. This gives us the commutative diagram

$$\begin{tikzcd}
Y & X \arrow[l, "f"'] \arrow[d, shift right, "s"'] & F = f^{-1}(*) \arrow[l, "j"'] \\
Y \arrow[u, equal] & W(f) \arrow[l, "p"] \arrow[u, shift right, "q"'] & F(f) = p^{-1}(*) \arrow[l, "J"'] \arrow[ul, "f^1"'] \arrow[u, dashed, "r"']
\end{tikzcd}$$

where the last column exists for pointed maps and

- p, q, f^1 are fibrations
- s is a shrinkable map with inverse q
- $f = ps$, s a homotopy equivalence and p a fibration
- If f is a fibration, then q is a homotopy equivalence over Y and r the induced homotopy equivalence

This factorization is unique in the sense if $f = ps = p's' : X \to Y$, $X \xrightarrow{s} W \xrightarrow{p} Y$, $X \xrightarrow{s'} Z' \xrightarrow{p'} Y$, then s, s' and p, p' are homotopic and W, W' are homotopy equivalent. Define $p^{-1}(*)$ as the (homotopical) **fibre** of f, then this implies this fibre is unique up to homotopy.
This gives us an immediate result about homotopy pushouts:
For a pullback diagram

$$\begin{tikzcd}
A & B \arrow[l, "f"'] \\
X \arrow[u, "p"] & Y \arrow[l, "F"] \arrow[u, "P"']
\end{tikzcd}$$

with p a fibration, then this diagram is a homotopy pullback. (Dualize everything in section 4.2)

5.8 Fibrations and cofibrations

Finally we note now fibrations and cofibrations behave together.
Let $p : E \to B$ has HLP for X and $i : A \subset X$ is a cofibration and h-equivalence. Suppose we are given $f : X \to B$ and $a : A \to E$ such that $pa = fi$, then a lifting F of f extending a exists:

$$\begin{tikzcd}
E \arrow[d, "p"] & A \arrow[l, "a"'] \arrow[d, hook, "i"] \\
B & X \arrow[l, "f"'] \arrow[dl, "i_0^X"] \\
X \times I \arrow[u, dashed, "f\Phi"] \arrow[uu, dashed, bend left, "F"] \arrow[ur, dashed, bend right, "\Phi"']
\end{tikzcd}$$

In the diagram the map Φ comes from studying section 5.4 defined as $\Phi_t(x) = \psi_{tv(x)^{-1}}(x)$ and HLP gives F from $f\Phi$.
If $i : A \subset B$ is a closed cofibration of locally compact spaces, the restriction map $p : Z^B \to Z^A$ is a fibration.
Let $p : E \to B$ be a fibration, and $B_0 \subset B$ be a cofibration, then $E_0 = p^{-1}(B_0) \subset E$ is a cofibration.

6 Chapter 7

6.1 Stable homotopy category

We have the suspension morphism $\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$ that may be stable after sufficient iterations, for instance for spheres. This leads us to motivate the definition of the stable homotopy category where morphisms are taken as colimits. Formally, the category ST has

- Objects: $(X, n), X \in \mathrm{TOP}^0, n \in \mathbb{Z}$
- Morphisms: $\mathrm{Hom}_{\S}((X, n), (Y, m)) = \mathrm{colim}_k \mathrm{Hom}_{\mathrm{TOP}^0}\left(\Sigma^{n+k}X, \Sigma^{m+k}Y\right)$
- Tensor: $(X, n) \otimes (Y, m) = (X \wedge Y, m + n)$

This category is symmetric monoidial if the smash product is associative, i.e. for compactly generated spaces. Although the Hom sets are groups, the category is not additive. We tend to write $(X, 0)$ as X.
(X, n) and (Y, m) are isomorphic in ST iff there exists some k such that $\Sigma^{n+k}X \cong \Sigma^{m+k}Y$.
If $X \cong Y$ in ST, then X and Y are stably homotopic.
For certain cases, we have explicit computations for the Hom groups:

$$\mathrm{Hom}_{\mathrm{ST}}(S^n, S^n) = \mathbb{Z}$$

$$\mathrm{Hom}_{\mathrm{ST}}\left(S^0, X\right) = \mathbb{Z}^{|\pi_0(X)|-1} \quad \text{if } X \text{ is well-pointed}$$

6.2 Duality

This section aims to construct the category $\mathcal{C}$ with objects $(\mathbb{R}^n, X)$ and morphisms are proper maps $f : X \to Y$ as well as the functor $D : \mathbb{C} \to \mathrm{ST}$. For convenience, we define the notation $A|B = (A, A - B)$.
The functor D sends $(\mathbb{R}^n, X)$ to $(C(\mathbb{R}^n|X), -n)$ where $C(A, B) = C(B \subset A)$ is the mapping cone. The construction Df is a bit more complicated:
Define a **scaling function** for some proper map $f : X \to Y$ as a function $\phi : Y \to (0, \infty)$ such that $\phi(f(x)) \geq \|x\|$. We can immediately see one exists by the function

$$\psi(f(x)) = \max\{\|x'\| \,|x' \in X, \|f(x')\| \leq \|f(x)\|\} + 1$$

With this function, define $M(\phi) = \{(x, y)|\phi(y) \geq \|x\|\}$ and $G(f) = \{(x, f(x))|x \in X\}$, then for any scaling function ϕ, we have the map

$$D_{1,\phi} : \mathbb{R}^{n+m}|D^n \times Y \simeq \mathbb{R}^{n+m}|M(\phi) \subset \mathbb{R}^{n+m}|G(f), \quad (x, y) \mapsto (\phi(y) \cdot x, y)$$

Given any scaling functions ϕ_1, ϕ_2, since $t\phi_1 + (1-t)\phi_2, t \in [0, 1]$ is a scaling function as well, we have a homotopy from D_{1,ϕ_1} to D_{1,ϕ_2} given by $D_{1,t\phi_1+(1-t)\phi_2}$. Let D_1 be the homotopy class of this map.

Finally by Tietze extension, we have a map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ giving us the homeomorphism

$$D_{2,\tilde{f}} : \mathbb{R}^{n+m}|G(f) \to \mathbb{R}^{n+m}|X \times 0, \quad (x,y) \mapsto \left(x - \tilde{f}(y), y\right)$$

Similarly the homotopy class of this is independent of the extension function by linear homotopy. Let D_2 be the homotopy class of this map. Finally define $D_\sharp(f) = D_2 \circ D_1 : \mathbb{R}^n|D^n \times \mathbb{R}^m|Y \to \mathbb{R}^n|X \times \mathbb{R}^m|0$.
Note that we have a pointed homotopy equivalence $\alpha : C(X,A) \wedge C(Y,B) \to C\left((X,A) \times (Y,B)\right)$ whenever the projection map $Z\left(A \times Y \supset A \times B \subset X \times B\right) \to A \times Y \cup X \times B$ is a homotopy equivalence. With this, we construct the map $D(f)$ with

$$\begin{tikzcd}
C\left(\mathbb{R}^m|Y\right) \wedge S^n \arrow[r, "(-1)^{nm}"] \arrow[d, "Df"'] & S^n \wedge C\left(\mathbb{R}^m|Y\right) \arrow[r, "\cong"] & C\left(\mathbb{R}^n|D^n\right) \wedge C\left(\mathbb{R}^m|y\right) \arrow[d, "\alpha"] \\
C\left(\mathbb{R}^n|X\right) \wedge S^m & C\left(\mathbb{R}^n|X \times \mathbb{R}^m|0\right) \arrow[l, "\alpha^{-1}"] & C\left(\mathbb{R}^n|D^n \times \mathbb{R}^m|Y\right) \arrow[l, "CD_\sharp f"]
\end{tikzcd}$$

Using this functor, one can proof that for closed subsets $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ which are homotopic with some proper map $f : X \to Y$ and $n \leq m$:

- If $\mathbb{R}^n \neq X$, then $\mathbb{R}^m \neq Y$
- If $\mathbb{R}^n \neq X$, then $(\mathbb{R}^n - X, -n) \cong (\mathbb{R}^m - Y, -m)$ in ST
- If $\mathbb{R}^n = X$, then $(S^0, -n-1) \cong (\mathbb{R}^m - Y, -m)$ in ST
- If $m = n$, then $\pi_0\left(\mathbb{R}^n - X\right) = \pi_0\left(\mathbb{R}^m - Y\right)$

We shall now show that $D\left((\mathbb{R}^n, X)\right)$ is dualizable in the category ST with dual $(X^+, 0)$.
First we define the notion of duality in a symmetric monoidial category $(\mathcal{C}, \otimes, 1)$:
A duality between objects $A, B \in \mathcal{C}$ consists of the following data:

- Evaluation map: $\epsilon : B \otimes A \to 1$
- Coevaluation map: $\eta : 1 \to A \otimes B$
- Triangle identity: $(1 \otimes \epsilon)(\eta \otimes 1) = \mathrm{id}, \quad (\epsilon \otimes 1)(1 \otimes \eta) = \mathrm{id}$

This is a duality in the sense that $- \wedge X$ and $Y \wedge -$ are adjoint.
In the case of $\mathcal{C} = \mathrm{ST}$, our tensor product is the wedge product and $1 = S^0 = (S^n, -n)$. The objects $B = D\left((\mathbb{R}^n, X)\right), A = X^+$ are dual to each other.
The evaluation map is given by

$$\epsilon : C\left(\mathbb{R}^n|K\right) \wedge C\left(K, \emptyset\right) \cong C\left(\mathbb{R}^n|K \times K|K\right) \overset{C(d)}{\to} C\left(\mathbb{R}^n|0\right) \cong S^n$$

where $d : (x,k) \to x - k$ is the difference map.

For the coevaluation map, let D be a disk that contains K and V be some open neighbourhood of D, then consider the diagram

$$\mathbb{R}^n|0 \xleftarrow{\supset} \mathbb{R}^n|D \xrightarrow{\subset} \mathbb{R}^n|X \xrightarrow{\supset} V|X \xrightarrow{\Delta} V|V \times \mathbb{R}^n|X$$

where $\Delta : x \mapsto (x, x)$ is the diagonal map. Under the mapping cone functor, the two $\supset$ becomes a homotopy equivalence, giving us a map $S^n \to V^+ \wedge C\left(\mathbb{R}^n|X\right)$. Finally composing this with a retraction $V^+ \to K^+$, we obtain our coevaluation map. Note that the retraction may not exist. If it exists, X is known as a Euclidean neighbourhood retract (ENR) and the existance is independent of the embedding. For this case, the (co)evaluation maps can be written as

- Evaluation map: $\epsilon : C\left(\mathbb{R}^n|K\right) \otimes K^+ \to S^n$
- Coevaluation map: $\eta : S^n \to K^+ \otimes C\left(\mathbb{R}^n|K\right)$

6.3 (Co)homology

A homology theory for pointed spaces is a family of functors $\tilde{h}_n : \mathrm{TOP}^0 \to R\text{-}\mathrm{MOD} . n \in \mathbb{Z}$ and natural transformations $\sigma_{(n)}$ such that $\sigma_{(n)} : \tilde{h}_n : \tilde{h}_{n+1} \circ \Sigma$ is an isomorphism and the following axioms hold:

- **Homotopy invariance** For each pointed homotopy f_t we have $\tilde{h}_n(f_0) = \tilde{h}_n(f_1)$
- **Exactness** For each pointed map $f : X \to Y$ the image of the sequence $X \to Y \to C(f)$ under $\tilde{h}_n$ is exact

This theory is called **additive** if for a family X_j of well pointed spaces, the inclusion $\oplus \tilde{h}_n(X_j) = \tilde{h}_n\left(\vee X_j\right)$ is an isomorphism.
The groups $\tilde{h}_n(S^0)$ are the coefficient groups of the theory.
Cohomology theories are similar except the functor is contravariant. The axioms remain the same.
(Co)homology theory for pairs of spaces can be constructed as $h(X, A) = \tilde{h}(C(X, A))$ and these satisfies the Eilenberg Steenrod axioms. The excision axiom comes from $C(X - U, A - U) \cong C(X, A)$ under a suitable hypothesis.
A **pre-spectrum** is a family $Z(n)$ of pointed spaces with maps and a family of maps $e_n : \Sigma Z(n) \to Z(n+1)$ of pointed maps. We shall shall prespectrums spectras as we only work with prespectras.
A **Ω-spectrum** is a spectrum with the adjoint maps $\epsilon_n : Z(n) \to \Omega Z(n+1)$ adjoint to e_n and these are pointed homotopy equivalences.
For any spectrum $Z = (Z(n), e_n)$, we construct a cohomology theory as

$$Z^k(X) = \operatorname{colim}_n \left[\Sigma^{n+k} X, Z(n)\right]^0$$

where the maps are induced by $e_n\Sigma-$. The axioms are satisfied by direct computation and the cofibre sequence.
If Z is a Ω-spectrum. then $Z^k(X) = [X, Z(k)]^0$.
We can obtain spectrums from a space Y by $\Sigma^n Y$ and if we have another spectrum $Z(n)$, we can construct the spectrum $Y \wedge Z(n)$, This gives us the cohomlogy theory

$$Z^k(X;Y) = \operatorname{colim}_n \left[\Sigma^n X, Y \wedge Z(n+k)\right]^0$$

which is a cohomology theory in X.
From here, we construct homology theory for a spectrum $Z(n)$ as $E_k(X) = Z^k\left(S^k, Y\right) = Z_{-k}\left(S^k, Y\right)$. The proof is somewhat involved as Y is 'in the wrong part' of the hom functors.
From here it's quite strightforward to obtain Alexander duality:
Suppse we have some duality with the (co)evaluation maps

- Evaluation map: $\epsilon : C\left(\mathbb{R}^n | K\right) \otimes K^+ \to S^n$
- Coevaluation map: $\eta : S^n \to K^+ \otimes C\left(\mathbb{R}^n | K\right)$

and we have the map

$$\left[A \wedge S^t, E_{k+t}\right]^0 \xrightarrow{B\wedge -} \left[B \wedge A \wedge S^t, B \wedge E_{k+t}\right]^0 \xrightarrow{\eta^*} \left[S^n \wedge S^t, B \wedge E_{k+t}\right]^0$$

and this gives us

$$D^{\cdot} : E^k(A) \to E_{n-k}(B)$$

Similarly we can get

$$D_{\cdot} : E_{n-k}(B) \to E^k(A)$$

By duality, we get

$$D_{\cdot} D^{\cdot} = (-1)^{nk}\,\mathrm{id} \quad D^{\cdot} D_{\cdot} = (-1)^{nk}\,\mathrm{id}$$

Let PE_*, Ph^* be (co)homology theories on TOP(2) given by $PE_*(X,A) = E_*\left(C(X,A)\right), Ph^*(X,A) = h^*\left(C(X,A)\right)$, then we obtain the usual Alexander duality for compact ENR spaces:

$$PE_{n-k}\left(\mathbb{R}^n, \mathbb{R}^n - X\right) = Ph^k(X, \emptyset) \quad PE_{n-k}(X, \emptyset) = Ph^k\left(\mathbb{R}^n, \mathbb{R}^n - X\right)$$

Another presentation of Alexander duality is given by $Z^k(A \wedge X; Y) = Z_{n-k}(X; B \wedge Y)$

6.4 Compactly generated spaces

References

[May] May, J. P. (1999). A concise course in algebraic topology. University of Chicago press.

[Wat] Waterhouse, W. C. (2012). Introduction to affine group schemes (Vol. 66). Springer Science & Business Media.

[Strom] Strom, A. (1968). Note on Cofibrations II. MATHEMATICA SCANDINAVICA, 22, 130-142. https://doi.org/10.7146/math.scand.a-10877