# LLL algorithm and usage in cryptography

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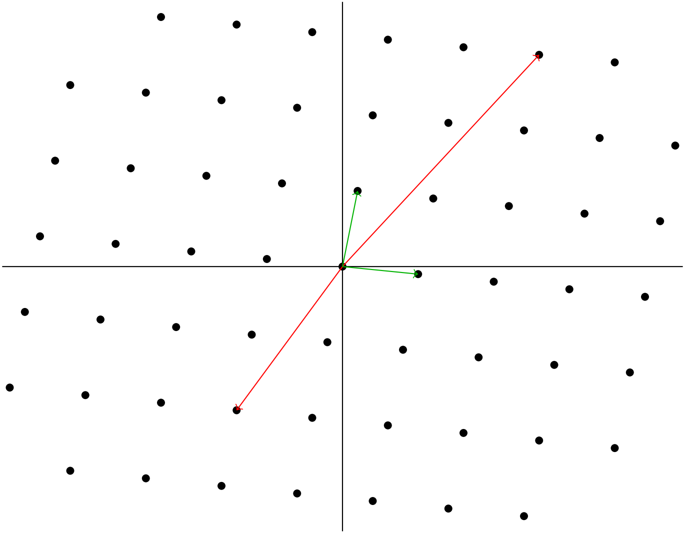
# Lattices

A lattice is a free $\mathbb{Z}$-module with $d$ generators as a subset of $\mathbb{R}^n$
Example: $\mathbb{Z}^n$ in $\mathbb{R}^n$

## Lattices

A lattice is a free $\mathbb{Z}$-module with $d$ generators as a subset of $\mathbb{R}^n$
Example: $\mathbb{Z}^n$ in $\mathbb{R}^n$
A lattice reduction algorithm is an algorithm that finds a 'short' and 'nearly orthogonal' basis

# Lattice in $\mathbb{R}^2$



# Euclidean algorithm

The Euclidean algorithm returns the gcd of $a, b$

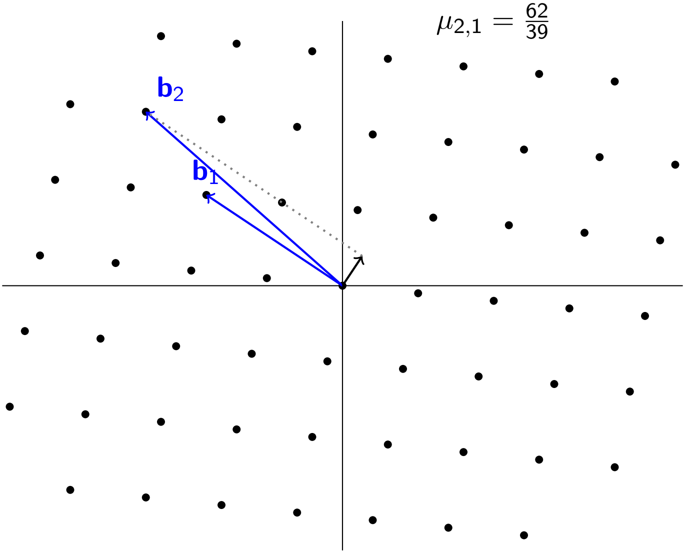
```
while $b \neq 0$ do
    if $|a| > |b|$ then
        $a, b \leftarrow b, a$
    end if
    $d \leftarrow \frac{b}{a}$
    $b \leftarrow b - \lfloor d \rceil a$
end while
return $a$
```

$a, b$ is just a lattice in $\mathbb{R}^1$ and $\gcd(a, b)$ is it's reduced lattice

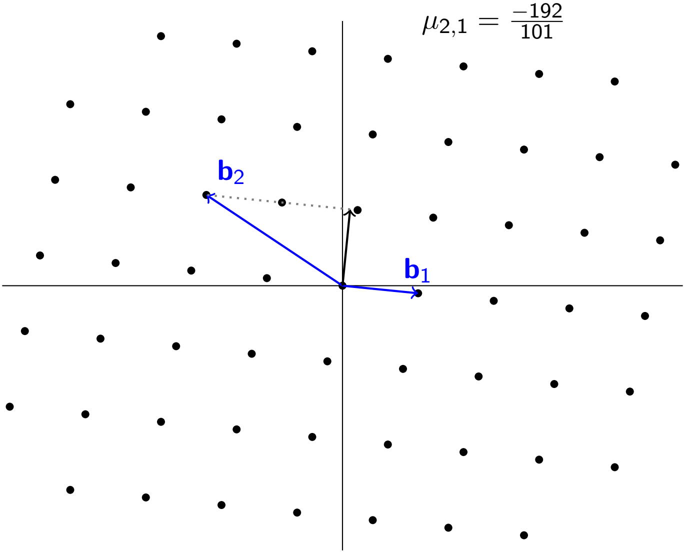
# Gaussian Lattice Reduction

```
Let $\mathbf{b}_1, \mathbf{b}_2$ be a basis
  while $\lfloor \mu_{2,1} \rceil \neq 0$ do
    if $||\mathbf{b}_1|| > ||\mathbf{b}_2||$ then
      $\mathbf{b}_1, \mathbf{b}_2 \leftarrow \mathbf{b}_2, \mathbf{b}_1$
    end if
    $\mu_{2,1} \leftarrow \frac{(\mathbf{b}_2, \mathbf{b}_1)}{||b_1||^2}$
    $\mathbf{b}_2 \leftarrow \mathbf{b}_2 - \lfloor \mu_{2,1} \rceil \mathbf{b}_1$
  end while
  return $\mathbf{b}_1, \mathbf{b}_2$
```

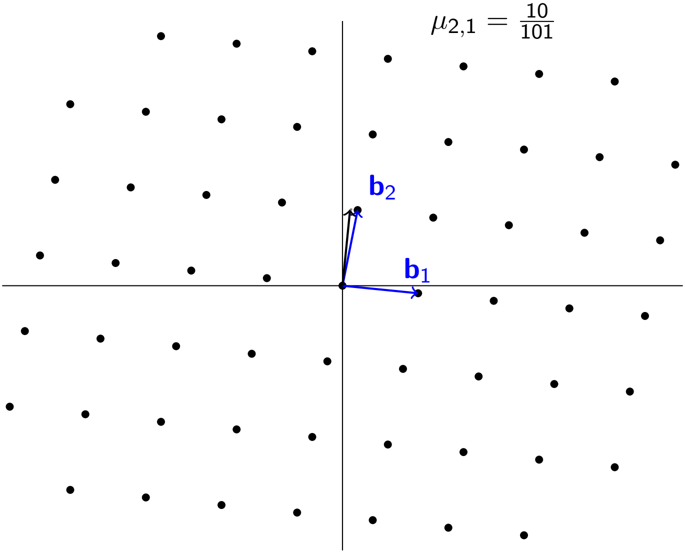
# Example



# Example



# Example



# Gram-Schmidt

For some vectors $\mathbf{b}_i \in \mathbb{R}^n$, define the orthogonal vectors $\mathbf{b}_i^*$ as

$$\mathbf{b}_i^* = \mathbf{b}_i - \sum_{j=1}^{i-1} \frac{\left(\mathbf{b}_i, \mathbf{b}_j^*\right)}{\left|\left|\mathbf{b}_j^*\right|\right|^2}\mathbf{b}_j^* = \mathbf{b}_i - \sum_{j=1}^{i-1} \mu_{j,i}\mathbf{b}_j^*$$

with $\mu_{i,j} = \frac{\left(\mathbf{b}_i, \mathbf{b}_j^*\right)}{\left|\left|\mathbf{b}_j^*\right|\right|^2}$

Then the space generated by $b_i$ and $b_i^*$ are the same

Typically we normalize the vectors but for lattice reduction purposes this is not done

# LLL-reduced

For some basis $\mathbf{b}_i$, let $\mathbf{b}_i^*$ be the Gram-Schmidt orthogonalized basis. Then the basis is LLL-reduced for $\delta \in \left(\frac{1}{4}, 1\right)$ iff:

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$$\mu_{i,j} = \frac{\left(\mathbf{b}_i, \mathbf{b}_j^*\right)}{\left|\left|\mathbf{b}_j^*\right|\right|^2}$$

1. Size reduced: $j < i, \mu_{i,j} \leq \frac{1}{2}$

# LLL-reduced

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$$\mu_{i,j} = \frac{\left(\mathbf{b}_i, \mathbf{b}_j^*\right)}{\left|\left|\mathbf{b}_j^*\right|\right|^2}$$

1. Size reduced: $j < i, \mu_{i,j} \leq \frac{1}{2}$
2. Lovász condition: $\left(\delta - \mu_{i+1,i}^2\right) ||\mathbf{b}_i^*||^2 \leq \left|\left|\mathbf{b}_{i+1}^*\right|\right|^2$

# LLL algorithm

```
$i \leftarrow 2$
while $i < n$ do
    for $j = i-1, i-2, ..., 1$ do
        if $|\mu_{i,j}| > \frac{1}{2}$ then
            $\mathbf{b}_i \leftarrow \mathbf{b}_i - \lfloor \mu_{i,j} \rceil \mathbf{b}_j$
        end if
    end for
    if $\left(\delta - \mu_{i,i-1}^2\right) ||\mathbf{b}_{i-1}^*||^2 \leq ||\mathbf{b}_i^*||^2$ then
        $i \leftarrow i + 1$
    else
        $i \leftarrow \max(i-1, 2)$
        $\mathbf{b}_{i-1}, \mathbf{b}_i \leftarrow \mathbf{b}_i, \mathbf{b}_{i-1}$
    end if
end while
```

# Example

| $\mathbf{b}_1$ | $\mathbf{b}_2$ | $\mathbf{b}_3$ |
|---|---|---|
| $(1,2,0)$ | $(1,3,2)$ | $(2,2,1)$ |
| $\mathbf{b}_1^*$ | $\mathbf{b}_2^*$ | $\mathbf{b}_3^*$ |
| $(1,2,0)$ | $\left(-\frac{2}{5},\frac{1}{5},2\right)$ | $\left(\frac{20}{21},-\frac{10}{21},\frac{5}{21}\right)$ |

$$\mu_{2,1}=\frac{7}{5}$$

# Example

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| $(1,2,0)$ | $(0,1,2)$ | $(2,2,1)$ |
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$$\mu_{2,1} = \frac{2}{5}$$

$$\left(\frac{3}{4} - \left(\frac{2}{5}\right)^2\right) ||\mathbf{b}_1^*||^2 \leq ||\mathbf{b}_2^*||^2$$

# Example

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$$\mu_{3,2} = \frac{8}{21}$$

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$$\mu_{3,1} = \frac{6}{5}$$

# Example

$$\begin{array}{ccc}
\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \\
(1,2,0) & (0,1,2) & (1,0,1) \\
\\
\mathbf{b}_1^* & \mathbf{b}_2^* & \mathbf{b}_3^* \\
(1,2,0) & \left(-\frac{2}{5},\frac{1}{5},2\right) & \left(\frac{20}{21},-\frac{10}{21},\frac{5}{21}\right)
\end{array}$$

$$\mu_{3,2} = \frac{8}{21}$$

$$\left(\frac{3}{4} - \left(\frac{8}{21}\right)^2\right) ||\mathbf{b}_2^*||^2 > ||\mathbf{b}_3^*||^2$$

# Example

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| $(1,2,0)$ | $\left(\frac{4}{5},-\frac{2}{5},1\right)$ | $\left(-\frac{10}{9},-\frac{5}{9},\frac{10}{9}\right)$ |

$$\mu_{2,1} = \frac{1}{5}$$

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$$\mu_{2,1} = \frac{1}{5}$$

$$\left(\frac{3}{4} - \left(\frac{1}{5}\right)^2\right) ||\mathbf{b}_1^*||^2 > ||\mathbf{b}_2^*||^2$$

# Example

$$\begin{array}{ccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \\ (1,0,1) & (1,2,0) & (0,1,2) \end{array}$$

$$\begin{array}{ccc} \mathbf{b}_1^* & \mathbf{b}_2^* & \mathbf{b}_3^* \\ (1,0,1) & \left(\frac{1}{2},2,-\frac{1}{2}\right) & \left(-\frac{10}{9},-\frac{5}{9},\frac{10}{9}\right) \end{array}$$

$$\mu_{2,1} = \frac{1}{2}$$

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$$\mu_{2,1} = \frac{1}{2}$$

$$\left(\frac{3}{4} - \left(\frac{1}{2}\right)^2\right) ||\mathbf{b}_1^*||^2 \leq ||\mathbf{b}_2^*||^2$$

# Example

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$$\mu_{3,2} = \frac{2}{9}$$

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$$\mu_{3,1} = 1$$

# Example

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| $(1, 0, 1)$ | $\left(\frac{1}{2}, 2, -\frac{1}{2}\right)$ | $\left(-\frac{10}{9}, -\frac{5}{9}, \frac{10}{9}\right)$ |

$$\mu_{3,2} = \frac{2}{9}$$

$$\left(\frac{3}{4} - \left(\frac{2}{9}\right)^2\right) ||\mathbf{b}_2^*||^2 > ||\mathbf{b}_3^*||^2$$

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$$\mu_{2,1} = 0$$

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$$\mu_{2,1} = 0$$

$$\left(\frac{3}{4} - (0)^2\right) ||\mathbf{b}_1^*||^2 \leq ||\mathbf{b}_2^*||^2$$

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$$\mu_{3,2} = \frac{1}{3}$$

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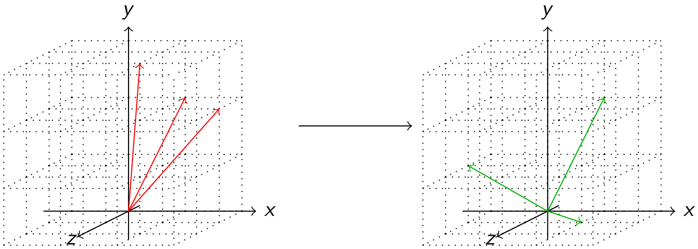
$$\mu_{3,1} = \frac{1}{2}$$

$$\left(\frac{3}{4} - \left(\frac{1}{2}\right)^2\right) ||\mathbf{b}_2^*||^2 \leq ||\mathbf{b}_3^*||^2$$

# Example

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$



# Bounds

A lattice is a free $\mathbb{Z}$-module with $d$ generators as a subset of $\mathbb{R}^n$
Some matrix $B$ generate a lattice with its rows as the basis $b_i$

$$\det(B) = \sqrt{\det\left(BB^T\right)} = \prod_i ||b_i^*||$$

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Suppose $B$ is LLL-reduced and let $\lambda_1$ be length of the shortest vector in the lattice

$$||b_1|| \leq \min\left(\left(\frac{4}{4\delta - 1}\right)^{\frac{d-1}{2}} \lambda_1, \left(\frac{4}{4\delta - 1}\right)^{\frac{d-1}{4}} \det(L)^{\frac{1}{d}}\right)$$

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$$||b_1|| \leq \min\left(\left(\frac{4}{4\delta - 1}\right)^{\frac{d-1}{2}} \lambda_1, \left(\frac{4}{4\delta - 1}\right)^{\frac{d-1}{4}} \det(L)^{\frac{1}{d}}\right)$$

For random lattices LLL usually finds $||b_1|| \lessapprox 1.02^d \det(L)^{\frac{1}{d}}$

# Rational approximation

To find a rational approximation of $x$, let $B$ be a big number.

$$\begin{pmatrix} 1 & 0 & xB \\ 0 & 1 & -B \end{pmatrix}$$

Smallest vector from LLL is of the form $(a, b, k)$ with $0 \approx \frac{k}{B} = ax - b$

# Approximate integer linear relations

Let $x_i$ be some arbitrary numbers and $B$ be a big number

$$\begin{pmatrix} 1 & 0 & \dots & 0 & x_1 B \\ 0 & 1 & \dots & 0 & x_2 B \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & x_n B \end{pmatrix}$$

Smallest vector from LLL is of the form $(c_1, c_2, \dots, c_n, x)$ with $\sum c_i x_i \approx 0$

# Algebraic number approximation

To find a algebraic approximation of $x$, let $B$ be a big number and $n$ be the degree of a polynomial

$$\begin{pmatrix} 1 & 0 & \dots & 0 & B \\ 0 & 1 & \dots & 0 & xB \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & x^n B \end{pmatrix}$$

Then the smallest vector of the LLL reduced matrix is of the form $(f_0, f_1, \dots, f_n, k)$ with $k$ small $\sum f_i x^i \approx 0$

# Howgrave Graham

Let $f(x)$ be some univariate polynomial of degree $d$. For some modulus $N$ and bound $B$:
$f\left(x_0\right) = 0 \pmod{N}$, $x_0 < B$ and $|f(x)| < N$ for all $0 < x < B$ implies $f\left(x_0\right) = 0$ over $\mathbb{R}$

# Howgrave Graham

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$f(x_0) = 0 \pmod{N}$, $x_0 < B$ and $|f(x)| < N$ for all $0 < x < B$ implies $f(x_0) = 0$ over $\mathbb{R}$
$f(x_0) = 0 \pmod{N}$ and $||f(Bx)||_2 < \frac{N}{\sqrt{d}}$ implies $f(x_0) = 0$ over $\mathbb{R}$

# Coppersmith algorithm(sketch)

If $x_0 < B$ is a root for some polynomials $f, g_i$ in $\frac{\mathbb{Z}}{N\mathbb{Z}}$, then the lattice generated by $f, g_i$ all have $x_0$ as a root in $\frac{\mathbb{Z}}{N\mathbb{Z}}$

# Coppersmith algorithm(sketch)

If $x_0 < B$ is a root for some polynomials $f, g_i$ in $\frac{\mathbb{Z}}{N\mathbb{Z}}$, then the lattice generated by $f, g_i$ all have $x_0$ as a root in $\frac{\mathbb{Z}}{N\mathbb{Z}}$

1. Construct polynomials $g_i$
2. Use $f(Bx)$ and $g_i(Bx)$ in the lattice
3. $h$ is hopefully a small vector in the lattice with $||h(x)||_2 < \frac{N}{\sqrt{d}} \implies h\left(\frac{x_0}{B}\right) = 0$ in $\mathbb{R}$

# Coppersmith algorithm

$g_i(x) = Nx^i$ is has root $x_0$ in $\frac{\mathbb{Z}}{N\mathbb{Z}}$

$$G = \begin{pmatrix} N & 0 & 0 & \dots & 0 & 0 \\ 0 & NB & 0 & \dots & 0 & 0 \\ 0 & 0 & NB^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & NB^{d-1} & 0 \\ f_0 & f_1 B & f_2 B^2 & \dots & f_{d-1} B^{d-1} & B^d \end{pmatrix}$$

$$\det(G) = N^d B^{\frac{d(d+1)}{2}} \quad \dim(G) = d+1$$

Let $\mathbf{v}$ be a short vector from LLL, then $h(x) = \sum_{i=0}^{n} v_i x^i$ possibly has a root $\frac{x_0}{B}$ over $\mathbb{R}$

# Theoretical discussion

Current lattice only ensures shortest vector of $O\left(N^{\frac{d}{d+1}}B^{\frac{d}{2}}\right)$, which must be less than $O(N)$ to work, so $B < O\left(N^{\frac{2}{d(d+1)}}\right)$

$B < N^{\frac{1}{d}}$ is a open conjectured theoretical limit for finding 'small roots' efficiently
Take $f(x) = x^2 + px \pmod{p}^2$, if $B = p^{\frac{1}{d}+\epsilon}$, number of small roots is unbounded and our polynomial over integers can't have so many s

Add more vectors in $(f(x), N)$ to decrease $\det(G)^{\frac{1}{d}}$

# Notation

Let $g_i$ be some polynomials $\sum_j g_{i,j}x^j$, then define the lattice $G$ generated from these polynomials as

$$G = \begin{pmatrix} g_{0,0} & g_{0,1} & g_{0,2} & \cdots \\ g_{1,0} & g_{1,1} & g_{1,2} & \cdots \\ g_{2,0} & g_{2,1} & g_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

# First improvement

Define $g_{0,j}(x) = Nx^j$ and $g_{1,j}(x) = f(x)x^j$, $0 \leq j < d$ and construct a lattice $G$ using coefficients of $g_{i,j}(Bx)$

$$\det(G) = N^d B^{\frac{(2d-1)2d}{2}} \quad \dim(G) = 2d$$

The shortest vector has length $O\left(N^{\frac{1}{2}} B^{\frac{2d-1}{2}}\right)$, bounded by $O(N)$ to find small roots

$$B < O\left(N^{\frac{1}{2d-1}}\right)$$

# Some motivation

$f(x)^a \pmod{N^a}$ has the same roots as $f(x) \pmod{N}$

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$f(x)^a \pmod{N^a}$ has the same roots as $f(x) \pmod{N}$
$N^a g(x) \pmod{N^{a+b}}$ has the same roots as $g(x) \pmod{N^b}$

# Some motivation

$f(x)^a \pmod{N^a}$ has the same roots as $f(x) \pmod{N}$
$N^a g(x) \pmod{N^{a+b}}$ has the same roots as $g(x) \pmod{N^b}$
Adding more vectors(strategically) decreases $\frac{N^m}{\det(L)^{\frac{1}{d}}}$, allowing for larger bounds of size of roots

# Final improvement

Define $g_{i,j}(x) = N^{h-j} f(x)^j x^i$ for some $h$, $0 \leq i < d$, $0 \leq j < h$ and construct a lattice $G$ using coefficients of $g_{i,j}(Bx)$

$$\det(G) = N^{d\frac{(h+1)h}{2}} B^{\frac{(dh-1)dh}{2}} \quad \dim(G) = dh$$

The shortest vector has length $O\left(N^{\frac{h+1}{2}} B^{\frac{dh-1}{2}}\right)$, bounded by $O\left(N^h\right)$ to find small roots

$$B < O\left(N^{\frac{h-1}{dh-1}}\right)$$

$\lim_{h\to\infty} \frac{h-1}{dh-1} = \frac{1}{d}$, can get arbitrary close to $N^{\frac{1}{d}}$

# Example

For some bound $B$, polynomial $x^3 + f_2x^2 + f_1x + f_2$ and modulus $N$
$h = 3$, $g_{i,j}(x) = N^{h-j}f(x)^jx^i$, $0 \le i < d$, $0 \le j < h$

$$\begin{pmatrix}
N^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & BN^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B^2N^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
N^2f_0 & BN^2f_1 & B^2N^2f_2 & B^3N^2 & 0 & 0 & 0 & 0 & 0 \\
0 & BN^2f_0 & B^2N^2f_1 & B^3N^2f_2 & B^4N^2 & 0 & 0 & 0 & 0 \\
0 & 0 & B^2N^2f_0 & B^3N^2f_1 & B^4N^2f_2 & B^5N^2 & 0 & 0 & 0 \\
Nf_0^2 & 2\,BNf_0f_1 & (Nf_1^2 + 2\,Nf_0f_2)B^2 & 2\,(Nf_1f_2 + Nf_0)B^3 & (Nf_2^2 + 2\,Nf_1)B^4 & 2\,B^5Nf_2 & B^6N & 0 & 0 \\
0 & BNf_0^2 & 2\,B^2Nf_0f_1 & (Nf_1^2 + 2\,Nf_0f_2)B^3 & 2\,(Nf_1f_2 + Nf_0)B^4 & (Nf_2^2 + 2\,Nf_1)B^5 & 2\,B^6Nf_2 & B^7N & 0 \\
0 & 0 & B^2Nf_0^2 & 2\,B^3Nf_0f_1 & (Nf_1^2 + 2\,Nf_0f_2)B^4 & 2\,(Nf_1f_2 + Nf_0)B^5 & (Nf_2^2 + 2\,Nf_1)B^6 & 2\,B^7Nf_2 & B^8N
\end{pmatrix}$$

# Unknown modulus

Unknown modulus $p < N^\beta$ with $p|N$

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Define $g_{i,j}(x) = N^{h-j} f(x)^j x^i$, $0 \le i < d$, $0 \le j < h$ and $g_{i,h} = f(x)^h x^i$ with $0 \le i < t$ and construct a lattice $G$ using coefficients of $g_{i,j}(Bx)$ and let $n = dh + t$ for convenience.

$$\det(G) = N^{d\frac{(h-1)h}{2}} B^{\frac{(n-1)n}{2}} \quad \dim(G) = n$$

The shortest vector has length $O\left(N^{\frac{(h-1)h}{2n}} B^{\frac{n-1}{2}}\right)$, bounded by $O\left(N^{\beta h}\right)$ to find small roots

$$B < O\left(N^{\frac{n-1}{n}\left(\frac{2\beta h}{n} - \frac{d(h-1)h}{n^2}\right)}\right) \overset{n=\frac{d}{\beta}h}{=} O\left(N^{\frac{n-1}{n}\left(2-\frac{h-1}{h}\right)\frac{\beta^2}{d}}\right)$$

$$\lim_{h,n\to\infty} \frac{n-1}{n}\left(2 - \frac{h-1}{h}\right)\frac{\beta^2}{d} = \frac{\beta^2}{d}$$

# Multivariate

Using the polynomials $g_{i,j,k,\ldots} = N^{h-i} f(x,y,\ldots)^i x^j y^k \ldots$ and $f(x)^h x^i y^j \ldots$ to construct a lattice and get polynomials with identical small roots over integers

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Determinant is hard to compute, bound is of the form $XY \cdots < O(N^x)$ where $x < X, y < Y, \ldots$ so they can't be too big

# Summary

LLL finds a short vector in a lattice

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Coppersmith algorithm can find small roots of univariate and bivariate polynomials mod a potentially unknown factor of $N$

# Usage

- Finding small/short solutions
- Recovering information with noise
- Miscellaneous

# Mertens conjecture and roots of $\zeta(t)$

$$|M(n)| = \left|\sum_{k=1}^{n} \mu(k)\right| < \sqrt{n}?$$

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Bound $c_\rho$ assuming Mertens and with LLL on roots

$$\rho < 2516 \implies \limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.06 \quad \liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.009$$

# RSA

$N = pq$ for primes $p, q$ and $e, d$ such that $ed = 1 \pmod{\lambda(N)}$.
Note that usually $ed = 1 \pmod{\phi(N)}$
Encryption: $c = m^e \pmod{N}$
Decryption: $m = c^d \pmod{N}$

# Franklin-Reiter Related Message Attack

$m_2 = f(m_1)$, $f$ a known polynomial and $c_1, c_2$ are ciphertexts of $m_1, m_2$
$x^e - c_1 \pmod{N}$ and $f(x)^e - c_2 \pmod{N}$ has $m_1$ as a root

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$$\gcd_{\frac{\mathbb{Z}}{N\mathbb{Z}}[x]}(x^e - c_1, f(x)^e - c_2) = x - m_1$$

# Coppersmith's Short Pad Attack

$m_2 = m_1 + r_1$ for some pad $r_1$, and $c_1, c_2$ are ciphertexts of $m_1, m_2$

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# Coppersmith's Short Pad Attack

$m_2 = m_1 + r_1$ for some pad $r_1$, and $c_1, c_2$ are ciphertexts of $m_1, m_2$

$$\mathrm{res}_x(f(x), g(x)) = 0 \iff f \text{ and } g \text{ shares a root}$$

$$f(y) = \mathrm{res}_x\left(x^e - c_1, (x+y)^e - c_2\right)$$

Find a small root of $f(y) \pmod N$ with coppersmith algorithm

# Known approximation of factor

If $p_0 \approx p$, find 'small roots' of $p + x \pmod{N}$ with coppersmith algorithm

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$$N = pq \quad p \approx r_p t, q \approx r_q t$$

$$t \approx \sqrt{\frac{N}{r_p r_q}} \implies N = (r_p t + x)(r_q t + y)$$

# Approximately similar prime factors

Assume we have modulus $N_i = p_i q_i$ with $p_i$ close to each other, construct a lattice with columns having 2 non-zero elements, $N_i, -N_j$ and the $i$th row lacking $\pm N_i$
Example:

$$\begin{pmatrix} N_2 & N_3 & 0 \\ -N_1 & 0 & N_3 \\ -N_1 & -N_2 & 0 \end{pmatrix}$$

Since $q_i N_j - q_j N_i = q_i q_j (p_i - p_j)$ is small, LLL is likely to find such a vector and we can take GCD

# Wiener attack

If $d$ is small, we can compute $d$ by simple algebraic means:

$$ed - 1 = k\phi(N) \implies \frac{e}{\phi(N)} - \frac{k}{d} = \frac{1}{d\phi(N)}$$

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$$\frac{e}{N} \approx \frac{k}{d}$$

Note that for $d < N^{\frac{1}{4}}$, $\frac{k}{d}$ is in the convergents of $\frac{e}{N}$'s continued fractions

# Boneh-Durfee attack

$$ed = 1 + x(p-1)(q-1) = 1 + x(N-y) \equiv 0 \pmod{e}$$

$$d < O\left(N^{\frac{7-2\sqrt{7}}{6} \approx 0.284}\right)$$

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Removing certain ‘bad vectors’:

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$$d < O\left(N^{\frac{1}{2}}\right)?$$

# Weak NTRU keys

$f, g \in \frac{\mathbb{Z}[x]}{x^N - 1}$, coefficients of $f, g$ are $-1, 0, 1$. $f_p f = 1 \pmod{p}$ and $h = p f_p g \pmod{q}$

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$$L = \begin{pmatrix} \lambda I_N & 0 \\ H & q I_n \end{pmatrix}$$

where $H$ is circulant matrix with first column being coefficients of $f_p g \pmod{q}$

$L \begin{pmatrix} f' \\ kq \end{pmatrix} = \begin{pmatrix} \lambda f' \\ g' \end{pmatrix}$ is hopefully short for some $k$. $pg' = f'h \pmod{q}$ breaks NTRU

# Coppersmith in the wild

Primes of the form $p = a + 2^t x + y$ with $a$ known and $t$ bruteforcable, $x, y$ unknown errors appeared in Taiwan's national Citizen Digital Certificate database

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Coppersmith method for bivariate polynomial and unknown modulus worked, but the theoretical bounds are not satisfied

# ROCA attack

Primes of the form $p = kM + (e^a \pmod{M})$ with $M$ being some primorial and $e = 65537$ was used, keys using these can be factored with coppersmith, hence the name the Return Of Coppersmith Attack

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$$N = (kM + e^a \bmod M)(lM + e^b \bmod M) \equiv e^{a+b} \pmod{M}$$

By bruteforcing $a$ in a certain way, we can construct the polynomial $xM + (65537^a \pmod{M})$ and find small roots

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# LLL algorithm and usage in cryptography

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libgen/scihub

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