1 Chapter 1

2 Chapter 2

2.1 Definitions

Spaces:

$\mathbb{R}^n$	Euclidean space
$D^n = \{x \in \mathbb{R}^n \| \|x\| \leq 1\}$	n-disk
$S^{n-1} = \{x \in \mathbb{D}^n \| \|x\| = 1\}$	$n-1$-sphere
$E^n = D^n - S^{n-1}$	n-cell
$I^n = \{x \in \mathbb{R}^n \| 0 \leq x_i \leq 1\}$	n-cube
$\partial I^n = \{x \in \mathbb{I}^n \| \exists i, x_i = 0, 1\}$	boundary of I^n
$\Delta^n = \Delta[n] = \{x \in \mathbb{R}^{n+1} \| x_i \geq 0, \sum_i x_i = 1\}$	n-simplex
$\partial \Delta^n = \{x \in \Delta^n \| \exists i, x_i = 0\}$	Boundary of n-simplex

Path: $u : [a, b] \to X$ from $x = u(a)$ to $y = u(b)$ (usually reparametrized to $[0, 1] \to X$)
Inverse path: $u^- : t \to u(1-t)$ from y to x
Product path: $u * v : t \to \begin{cases} u(2t) & t \leq \frac{1}{2} \\ v(2t-1) & \frac{1}{2} \leq t \end{cases}$
Constant path: $k_x : t \to x$
$\pi_0 : \mathrm{TOP} \to \mathrm{SET}$

- $\pi_0(X)$: Set of path connected components of X
- $\pi_0(f)([x]) = [f(x)]$

$W : \mathrm{TOP} \to \mathrm{CAT}$

- $W(X)$: Paths $u : [0, a] \to X$ and composition is defined on $[0, a+b]$ for associativity
- $W(f)(x) = f(x), W(f)(u) = f \circ u$

2.2 Homotopy notions

Homotopy: $H_t : X \times [0, 1] \to Y$ from $f = H_0 : X \to Y$ to $g = H_1 : X \to Y$; $H : f \simeq g$ (composition/inverse immediate)
Homotopy $H_t : X \to Y$ relative to $A \subset X$ if $H_t : A \to Y$ is independent of t
Homotopy between f and a constant map is a null homotopy
Null homotopy of $\mathrm{id}_X : X \to X$ is a contraction
Path category $W(X, Y)$

- Objects: $f : X \to Y$
- Morphisms: Homotopy $H_t : [0, a] \times X \to Y$ between f and g

hTOP is TOP quotiented by the homotopy relation.

hTOP	TOP
Isomorphic	Homotopy equivalent/Same homotopy type
Isomorphic to $\{*\}$	Contractible
Isomorphism	h-equivalence
Constant map	Null homotopic

Hom functors in hTOP of $f: X \to Y$:

$$f_*: [Z, X] \to [Z, Y], g \to fg \quad f^*: [Y, Z] \to [X, Z], h \to hf$$

Remark. *Generally lower index for covariant and upper index for contravariant*

TOP^0: Category of pointed spaces
hTOP^0: Quotient of TOP^0 by homotopy
Forgetful functor $\mathrm{TOP}^0 \to \mathrm{TOP}$ has a left adjoint, $X \to (X + \{*\}, *)$

Remark. *The smash product $A \wedge B = \frac{A \times B}{A \vee B}$ is always compatible with homotopies and is a tensor product in some appropriate subcategory, i.e. compactly generated spaces*

TOP(2): Pairs of topological spaces $A \subset X$
Note that the product we use here is not the categorical product, instead it is defined as

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$$

so that $(I^m, \partial I^m) \times (I^n, \partial I^n) = (I^{m+n}, \partial I^{m+n})$
TOP(3): Pairs of topological spaces $A \subset B \subset X$
TOP_B: Slice category, objects are morphisms $x: X \to B$ and morphisms are $f: X \to Y$ such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x"'] & & Y \arrow[dl, "y"] \\ & B & \end{tikzcd}$$

- A morphism from $\mathrm{id}_B : B \to B$ to $p: E \to B$ is a section of p
- If $p \cong \mathrm{id}_B$ in hTOP_B, then it is shrinkable

TOP^K: Coslice category, objects are morphisms $a: K \to A$ and morphisms are $f: A \to B$ such that

$$\begin{tikzcd} & K \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

- A morphism from $i: K \to X$ to $\mathrm{id}_K : K \to K$ is a retraction of i and i is an embedding
- $i: K \subset X$, then K is a retract of X
- If $i \cong \mathrm{id}_K$ in hTOP^K, then it is a deformation retract

Note that $\mathrm{TOP}_{\{*\}} \cong \mathrm{TOP}$ and $\mathrm{TOP}^{\{*\}} \cong \mathrm{TOP}^0$
$H_t : A \to B$ is a homotopy in the (co)slice category if each $H_t, t \in [0, 1]$ is a morphism in the (co)slice category, hence we get the quotient categories $\mathrm{hTOP}^K, \mathrm{hTOP}_B$.

2.3 Internal hom objects

Let Y^X or $F(X, Y)$ be the set of continuous maps from X to Y with the compact open topology. Suppose that X is locally compact, then Y^X is the exponential object, i.e.

$$\begin{tikzcd} X \times Y \arrow[d, dotted, "f^{\wedge} \times \mathrm{id}_Y"'] \arrow[dr, "f"] & \\ Z^Y \times Y \arrow[r, "e_{Y,Z}"'] & Z \end{tikzcd}$$

f induces $f^{\wedge}$ and $f^{\wedge}$ induces f, alternatively

$$\mathrm{Hom}\left(- \times Y, Z\right) \cong \mathrm{Hom}\left(-, Z^Y\right)$$

which also tells us the functors $-^Y$ is a right adjoint to $- \times Y$.
Unfortunately in categories with zero objects, i.e. TOP^0, then exponential objects generally dont exist unless the category is trivial as if Y^X exists, we have

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}\left(0 \times X, Y\right) \cong \mathrm{Hom}\left(0, Y^X\right) \cong \{*\}$$

However, we may have some form of tensor-hom adjuncation.
In the category TOP^0, we define $F^0(X, Y)$ as the subspace of pointed maps of $F(X, Y)$ and the constant map is the basepoint. Any pointed map $X \times Y \to Z$ induces a pointed map $X \to F^0(X, Y)$ if it sends $X \times y \cup x \times Y$ to z, hence it corresponds to maps from $X \wedge Y \to Z$. The adjuncation in this case reduces to

$$F^0\left(X \wedge Y, Z\right) \cong F^0\left(X, F^0\left(Y, Z\right)\right)$$

when X, Y are locally compact. This gives us our tensor-hom adjuncation.
If we quotient by homotopy and assume X is locally compact and $e^0_{X,Y}$ is continuous, then we get

$$[X \wedge Y, Z]^0 \cong \left[X, F^0(X, Y)\right]^0$$

2.4 Fundamental groupoid

$\Pi : \mathrm{TOP} \to \mathrm{GRPd}$

- $\Pi(X)$: Quotient of $W(X)$ by homotopy

Somewhat cleaner way to state van-Kampen theorem:

Theorem (Seifert–Van Kampen [May, Thm 2.7]). *Suppose that $\mathcal{U}$ is a covering of X such that if $U_1, U_2 \in \mathcal{U}$, then $U_1 \cap U_2 \in \mathcal{U}$. This turns $\mathcal{U}$ into a category where morphisms are inclusions, then*

$$\Pi(X) \cong \mathrm{colim}_{U \in \mathcal{U}}\, \Pi(U)$$

Choosing a base point, we get the functor $\pi_1 : \mathrm{TOP}^0 \to \mathrm{GRP}$.

Remark. *Proposition 2.7.3 of tom Dieck that the fundamental group of a monoid in* TOP^0 *is commutative and agrees with the monoid operation comes from a more general theorem, the Eckmann–Hilton argument*

Theorem (Eckmann-Hilton argument)**.** *If* $\cdot, *$ *are unital binary operations on* X *with units* $1.$ *and* 1_* *such that*

$$(a \cdot b) * (c \cdot d) = (a * b) \cdot (c * d)$$

Then $\cdot, *$ *coincide, are associative and commutative*

2.5 Enriching TOP

We give a groupoid structure, $\Pi(X, Y)$ to each hom set $\mathrm{Hom}_{\mathrm{TOP}}(X, Y)$ with homotopy as morphisms. This provides us with a 2-category, i.e.

$$\begin{tikzcd}
X \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=60, "h"'{name=H}] & Y \arrow[r, bend left=60, "u"{name=U}] \arrow[r, "v"{name=V, description}] \arrow[r, bend right=60, "w"'{name=W}] & Z
\arrow[Rightarrow, from=F, to=G, "\alpha"]
\arrow[Rightarrow, from=G, to=H, "\beta"]
\arrow[Rightarrow, from=U, to=V, "\gamma"]
\arrow[Rightarrow, from=V, to=W, "\delta"]
\end{tikzcd}$$

such that all the compositions makes sense, i.e.

$$(\delta\gamma)\,(\beta\alpha) = (\delta\beta)\,(\gamma\alpha)$$

We can enrich similar categories like TOP^0

3 Chapter 3

3.1 Definitions

Suppose $p : E \to B$ is surjective and $U \subset B$ is open

- **Trivialization** of p over U is a homeomorphism $p^{-1}(U) \to U \times F$
- p is **locally trivial** if a open covering $\mathcal{U}$ exists where a trivialization of p over $U \in \mathcal{U}$ exists for all U
- $\mathcal{U}$ is a **bundle chart**
- F is the **typical fibre**
- p is **trivial over** U if a bundle chart over U exists

- **Bundles/Fibre bundles** are locally trivial maps

Covering space/Covering of B is a locally trivial trivial map $p : E \to B$ with discrete fibres

- If $\phi_U : p^{-1}(U) \to U \times F$ is a trivialization, then $\phi_U^{-1}(U \times \{*\})$ are the **sheets** over U
- If $|F| = n$, then p is a n-fold covering
- A **trivial covering** is the covering $p : B \times F \to B$
- U is **admissible** or **evenly covered** if a trivialization exists
- E is the **total space** and B is the **base space**

3.2 Coverings with group actions

A **left G-principal covering** is a covering $p : E \to B$ and a properly discontinuous group action G on E such that $p(gx) = p(x)$ and the action on fibres are transitive
$\alpha \in \mathrm{Aut}(p)$ if $\alpha : p \to p$ is a morphism in TOP_B. These are **deck transformations**
The map $x \to gx$ gives a map $G \to \mathrm{Aut}(p)$

Theorem (Galois correspondence)**.** *Let $p : E \to B$ be a covering, then*

- *If E is connected,* $\mathrm{Aut}(p)$ *is a properly discontinuous action on E*
- *If B is locally path connected, H subgroup of* $\mathrm{Aut}(p)$*, then $E/H \to B$ is a covering*

A **right G-principal covering** is a covering $p : E \to B$ and a properly discontinuous group action G on E such that $p(xg) = p(x)$ and the action on fibres are transitive.
Let F be a set with a left G action, then the space $E \times_G F$ constructed by quotienting $E \times F$ by $(xg, f) = (x, gf)$ is an **associated covering**.

Remark. *Seems like for this part we need to assume that that G is a free action of the fibres as well and F is given the discrete topology*

Theorem. *The map $p_F : E \times_G F \to B, (x, f) \to p(f)$ is a covering with typical fibre F*

Proof. Suppose that $\mathcal{F}$ is the typical fibre of p.
First we show the typical fibre of p_F is F. It's immediate that the typical fibre is given by $\frac{\mathcal{F}\times F}{\sim}$ immediately showing discreteness. Next, notice that $\{(\mathfrak{f}, f) \,|\, f \in F\}$ are the representatives of $\frac{\mathcal{F}\times F}{\sim}$ for some arbitrary $\mathfrak{f}$ as supose $\mathfrak{f}' = \mathfrak{f}g$, then $(\mathfrak{f}', f) = (\mathfrak{f}, gf)$ and $(\mathfrak{f}, f) = (\mathfrak{f}, f')$ implies that either $\mathfrak{f}$ has a nontrivial stabalizer or $f = f'$, hence we need to assume the action is free on $\mathcal{F}$.
Next we show that this is indeed a covering. Suppose U has a trivialization, i.e. $p^{-1}(U) \cong U \times \mathcal{F}$. Then $\frac{p^{-1}(U)\times F}{\sim} \cong \frac{U\times\mathcal{F}\times F}{\sim} \cong U \times F$. Hence p_F is a covering with typical fibre F. $\square$

This gives us the functor

$$A(p) : G\text{-}\mathrm{SET} \to \mathrm{COV}_B$$

from the category of sets with a left G action to the category of covering spaces over B (a subcategory of TOP_B).
If $A(p)$ is an equivalence of categories, then p is the universal cover

3.3 Lifting

$F : X \to E$ is a **lifting** of $f : X \to B$ along $p : E \to B$ if $pF = f$, i.e. a morphism in TOP_B
If X is connected and p is a covering, liftings that agree somewhere are unique.
A map $p : E \to B$ has **homotopy lifting property** (HLP) for a space X if for each homotopy h_t and initial condition H_0, we can extend H_0 to the homotopy H_t such that the diagram commutes:

$$\begin{tikzcd} X \arrow[rr, dotted, "H_t"] \arrow[dr, "h_t"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

H is a lifting of h with initial conditions a. p is a **fibration** if it has HLP for all spaces

Theorem. *Coverings $p : E \to B$ are fibrations*

Proof. First show that projection maps $U \times F \to U$ are fibrations, then glue these projection maps and use uniqueness of liftings □

As liftings along coverings are unique, the diagram below is a pullback:

$$\begin{tikzcd} E^I \arrow[r, "p^I"] \arrow[d, "e_E^0"'] & B^I \arrow[d, "e_B^0"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

Let $p : E \to B$ be a map with HLP for I and $F_b = p^{-1}(b)$.
For every map $[v] \in \Pi(B)$, we obtain a well defined map $v_\sharp : \pi_0(F_b) \to \pi_0(F_c)$. Suppose $V : I \to E$ is a lifting of v with $V(0) = x$, then $v_\sharp[x] = [V(1)]$.
With this we obtain the **transport functor** $T_p : \Pi(B) \to \mathrm{SET}$

- $b \to \pi_0(B)$
- $[v] \to v_\sharp$

Let $p(x) = b$ and let $\partial_x : \pi_1(B, b) \to \pi_0(F_b, x), [v] \to v_\sharp(x)$ and $i : F_b \subset E$, then we have the exact sequence

$$\begin{tikzcd}
\pi_1\left(F_b, x\right) \arrow[r, "i_*"] & \pi_1\left(E, x\right) \arrow[r, "p_*"] & \pi_1(B, b) \arrow[dll, "\partial_x"] \\
\pi_0\left(F_b, x\right) \arrow[r, "i_*"] & \pi_0\left(E, x\right) \arrow[r, "p_*"] & \pi_0(B, b)
\end{tikzcd}$$

as well as the isomorphisms of sets $\partial_x : \frac{\pi_1(B,b)}{p_* \pi_1(E,x)} \cong \pi_0\left(F_b, x\right), i_* : \frac{\pi_0(F_b,x)}{\pi_1(B,b)} \cong \pi_0(E,x)$
For a covering $p : E \to B$ with B path connected, the exact sequence simplifies to

$$\begin{tikzcd}
1 \arrow[r] & \pi_1\left(E, x\right) \arrow[r, "p_*"] & \pi_1(B, b) \arrow[dll, "\partial_x"] \\
\pi_0\left(F_b, x\right) \arrow[r, "i_*"] & \pi_0\left(E, x\right) \arrow[r] & \{*\}
\end{tikzcd}$$

Furthermore suppose that $p : E \to B$ is a right G-principal covering with E path connceted, then we get the exact sequence

$$1 \longrightarrow \pi_1\left(E, x\right) \xrightarrow{p_*} \pi_1(B, b) \xrightarrow{\delta_x} G \longrightarrow 1$$

and the image of p_* is normal.

3.4 Coverings

Outline:

1. Construct the inverse X of $T : \mathrm{TRA}_B \to \mathrm{COV}_B$ that exists and is an equivalence of categories for sufficiently nice B
2. Construct the functor $\epsilon_B : \mathrm{TRA}_B \to \pi_b\text{-}\mathrm{SET}$ and the inverse η_b
3. Hence $A(p) : G\text{-}\mathrm{SET} \to \mathrm{COV}_B$ is an equivalence of categories iff the total space of p is simply connected

Let $\mathrm{TRA}_B = [\Pi(B), \mathrm{SET}]$, the transport functor in the previous section yields the functor $T : \mathrm{COV}_B \to \mathrm{TRA}_B$.
If B is path connected and T is an equivalence of categories, then B is a **transport space**.
A set $U \in B$ is **transport simple** if any paths in U between identical points are homotopic in B.
B is **semi-locally simply connected** if it has an open covering with transport simple sets.
B is **transport local** if it is path connected, locally path connected and semi-locally simply connected.

Theorem. *If B is then B is a transport space*

Proof. We need to construct the inverse of T, $X : \mathrm{TRA}_B \to \mathrm{COV}_B$. Let $\Phi : \Pi(B) \to \mathrm{SET}$ be some functor, we will construct a covering $p : X(\Phi) \to B$. As a set $X(\Phi) = \coprod_{b \in B} \Phi(b)$. To get a reasonable topology on it, we consider a covering $\mathcal{U}$ of B by transport simple path connected open sets. For every $b \in U \in \mathcal{U}$, we define $\phi_{U,b} : U \times \Phi(b) \to p^{-1}(U)$ and by gluing these maps together, we obtain a topology on $X(\Phi)$ and a covering p.
Verification of functoriality and inverse are somewhat direct from definition. $\square$

With this, consider the hom functor $\mathrm{Hom}_{\Pi(B)}(b, -) \in \mathrm{TRA}_B$ and let $p^b : E^b \to B$ be its associated covering. Then E^b is simply connected right $\mathrm{Hom}_{\Pi(B)}(b, b)$-principal covering.
Suppose that B is path connected, then $\Pi = \Pi(B)$ is a connected groupoid. Let $\Pi(x, y) = \mathrm{Hom}_{\Pi(B)}(x, y)$ and $\pi_b = \Pi(b, b)$
For a functor $F : \Pi \to \mathrm{SET}$, we have the left π_B-set $F(b)$ giving us the functor $\epsilon_b : \mathrm{TRA}_B \to \pi_b\text{-}\mathrm{SET}$.
For a left π_B-set A, we define the functor $\Pi(b, -) \times_{\pi_B} A : \Pi \to \mathrm{SET}$ where $A \times_G B$ is the set $A \times B$ quotiented by $(ag, b) = (a, gb)$. This gives us the functor $\eta_b : \pi_b\text{-}\mathrm{SET} \to \mathrm{TRA}_B$, the inverse of ϵ_b.
Finally we have the following categories and functors:

$$G\text{-}\mathrm{SET} \xrightarrow{A(p)} \mathrm{COV}_B \underset{X}{\overset{T}{\rightleftarrows}} \mathrm{TRA}_B \underset{\eta_b}{\overset{\epsilon_b}{\rightleftarrows}} \pi_b\text{-}\mathrm{SET}$$

where X exists if B is transport-local and ϵ_B, η_B require B to be path connected to exist.
Finally we have

Theorem. *The following are equivalent:*

- *B is a transport space, i.e. T is an equivalence of categories*
- *B has a universal right G-principal covering $p : E \to B$, i.e. $A(p)$ is an equivalence of categories*

Note that the exact sequences above imply that E is simply connected.
Define the **orbit category** $\mathrm{Or}(G)$ consisting of homogenous G-sets ($\frac{G}{H}$ for any subgroup H) and G-maps. This category is a strict subcategory of $G\text{-}\mathrm{SET}$, consisting only the transitive sets.
For a covering $p : E \to B$, we obtain the injective map $p_* : \pi_1(E, x) \to \pi_1(B, p(x))$ and the image is called the **characteristic subgroup** of p wrt x.
Let $p : E \to B$ be a simply connected covering, then the subcategory $A(p)(\mathrm{Or}(\pi_b))$ of COV_B is equivalent to the subcategory consisting of connected coverings. This tells us that the connected coverings of a transport space is determined by subgroups of the fundamental group.

Theorem. *Let B be a transport space and $p : E \to B$ a covering.*

- *The action of* $\mathrm{Aut}(p)$ *on E makes it a left-*$\mathrm{Aut}(p)$ *principal covering*

- *A simply connected covering is a universal covering*
- *Universal coverings are unique up to isomorphism*
- *The automorphism group of a universal cover is isomorphic to* $\pi_1(B,b)$
- E^b *is simply connected*
- *We have a Galois correspondence between isomorphism classes of connected coverings and subgroups of* $\pi_1(B,b)$

If B is not a transport space but is path connected and locally path connected, then we have a similar result where coverings by path connected total spaces are isomorphic iff the characteristic subgroups are conjugate in $\pi_1(B,b)$.
Suppose we have coverings $p : E \to B$ and $f : Z \to B$, then a covering $\Phi : Z \to E$ such that

$$\begin{tikzcd} Z \arrow[rr, dashed, "\Phi"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

exists iff $f_*\pi_1(Z,z) \subset p_*\pi_1(E,x)$ for some $f(z) = p(x)$.

If X is a topological group with identity x and $p : E \to X$ is a covering with E path connected and locally path connected, then for each $e \in p^{-1}(b)$, there exists a unique group structure on E such that e is the identity and p a homomorphism.

4 Chapter 4

4.1 Mapping cylinders

For a map $f : X \to Y$ **mapping cylinder** $Z(f)$ is constructed by the pushout

$$\begin{tikzcd} X + X \arrow[r, "\mathrm{id}+f"] \arrow[d, "\langle i_0, i_1\rangle"'] & X + Y \arrow[d, dashed, "\langle j, J\rangle"] \arrow[ddr, bend left, "\langle f, \mathrm{id}\rangle"] & \\ X \times I \arrow[r, dashed, "a"'] \arrow[drr, bend right, "f"'] & Z(f) \arrow[dr, dashed, "q"] & \\ & & Y \end{tikzcd}$$

We also have $Jq \cong \mathrm{id}$ as $X \times I \cong X$ and $f = qj$, j a closed immersion and q a homotopy equivalence. We see that Z has nice functorial properties. Suppose we have the homotopy commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\
X' \arrow[r, dashed, "f'"] \arrow[d, "\alpha'"'] & Y' \arrow[d, "\beta'"] \\
X'' \arrow[r, dashed, "f''"] & Y''
\end{tikzcd}$$

and let the homotopy equivalences be $\Phi : f'\alpha \cong \beta f, \Phi' : f''\alpha' \cong \beta' f'$
This induces the following homotopy commutative diagram

$$\begin{tikzcd}
X+Y \arrow[r] \arrow[d, "\alpha+\beta"'] & Z(f) \arrow[d, "Z(\alpha,\beta,\Phi)"] \\
X'+Y' \arrow[r] \arrow[d, "\alpha'+\beta'"'] & Z(f') \arrow[d, "Z(\alpha',\beta',\Phi'')"] \\
X''+Y'' \arrow[r] & Z(f'')
\end{tikzcd}$$

where each small square commutes in TOP and the whole diagram commutes in hTOP.
Given maps $f : A \to B$ and $g : A \to C$, we can construct the double mapping cylinder by either of the two pushouts:

$$\begin{tikzcd}
A+A \arrow[r, "f+g"] \arrow[d, "\langle i_0,i_1\rangle"'] & B+C \arrow[d, dashed, "\langle j_0,j_1\rangle"] \\
A \times I \arrow[r, dashed] & Z(f,g)
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "j^B"] \arrow[d, "j^C"'] & Z(f) \arrow[d, dashed] \\
Z(g) \arrow[r, dashed] & Z(f,g)
\end{tikzcd}$$

The functorality of the double mapping cylinder can be seem from the following commutative homotopy diagram:

$$
\begin{tikzcd}[column sep=small]
 & & & & & A+B \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow"] & & \\
 & A \arrow[rrr] \arrow[dr] \arrow[ddd] & & & Z(f) \arrow[dr] \arrow[ddd, dashed] & & A'+B' \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow"] & \\
 & & A' \arrow[rrr] \arrow[dr] \arrow[ddd] & & & Z(f') \arrow[dr] \arrow[ddd, dashed] & & A''+B'' \arrow[dl] \\
 & & & A'' \arrow[rrr] \arrow[ddd] & & & Z(f'') \arrow[ddd, dashed] & \\
 & Z(g) \arrow[rrr, dashed] \arrow[dr] & & & Z(f,g) \arrow[dr] & & & \\
A+C \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Rightarrow"] & & Z(g') \arrow[rrr, dashed] \arrow[dr] & & & Z(f',g') \arrow[dr] & & \\
 & A'+C' \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Rightarrow"] & & Z(g'') \arrow[rrr, dashed] & & & Z(f'',g') & \\
 & & A''+C'' \arrow[ur] & & & & &
\end{tikzcd}
$$

Suppose we have the homotopy commutative diagram

$$
\begin{tikzcd}
X_0 \arrow[r, "f_+"] \arrow[d, "f_-"'] & X_+ \arrow[d, "j_+"] \arrow[ddr, dashed, bend left] & \\
X_- \arrow[r, "j_-"'] \arrow[drr, dashed, bend right] & X & \\
 & & Z(f_-, f_+) \arrow[ul, dashed, "\phi"']
\end{tikzcd}
$$

The square is called a **homotopy pushout** or **homotopy cocartesian** if ϕ is a homotopy equivalence (i.e. a pushout in the category hTOP)
Let $f_\pm, j_\pm$ be inclusions and $X = X_+ \cup X_-$, then the diagram is a pushout in TOP.
Let $N(X_-, X_+) = X_- \times 0 \cup X_0 \times I \times X_+ \times 1$ be a subspace of $X \times I$ and let $p_N : N(X_-, X_+) \to X$ be the projection map.
The covering $X_\pm$ is **numerable** if p_N has a section. With this, we have the following condition to determine if the diagram above is a pushout in hTOP:

Theorem 4.1. *$Z(f_-, f_+) \cong X$ if the covering $X_\pm$ is numerable*

Given the projection maps $X \overset{f}{\leftarrow} X \times Y \overset{g}{\to} Y$, the double mapping cylinder $Z(f,g) = X \star Y$ is known as the **join** of X and Y

4.2 Suspensions and loops

Here we work in pointed categories
The **suspension** functor $\Sigma : \mathrm{TOP}^0 \to \mathrm{TOP}^0$ is given by

- $\Sigma X = S^1 \wedge X = \frac{X \times I}{X \times \partial I \cup \{x\} \times I}$
- $S\Sigma_* [X, Y]^0 = [\Sigma X, \Sigma Y]^0$

Note that Σ_* is a homomorphism if $X = \Sigma A$ and any pointed homotopy $H_t : X \to Y$ corresponds uniquely to a pointed map $\overline{K} : \Sigma X \to Y$
The set $[\Sigma X, Y]^0$ has a natural group structure by composition of homotopies. Furthermore as $[\Sigma^n X, Y]^0$ has n natural composition laws by composin the homotopies at the the ith coordinate and these satisfies the assumptions of Eckmann-Hilton argument, they are equivalent. This also tells us that the higher homotopy groups, $\pi_n(X) = [S^n, X]^0$, are commutative groups.
We can dualize everything above:
The **loop** space of X is $\Omega X = F^0 (S^1, X)$. This consists of loops in X with basepoint x. This is naturally a topological group with the product of loops.
There is a natural group structure on $\mathrm{Hom}_{\mathrm{TOP}^0} (X, \Omega Y)$ given by $[f] +_m [g] = [f][g]$.
As S^1 is locally compact, we have the tensor-hom adjuncation

$$[\Sigma X, Y]^0 \cong [X, \Omega Y]^0$$

and this commutes with the group operation.
In the set $[\Sigma X, \Omega Y]^0$, by the Eckmann-Hilton argument, the group operations on these two sets coincide and are commutative.

4.3 Group objects

Perhaps a nicer way of looking at group/any objects is via Yoneda's lemma, see [Wat] for more details:

Theorem 4.2 (Yoneda). *For any category C, we have a full and faithful functor, the Yoneda embedding $y_C : C \to [C^{\mathrm{op}}, \mathrm{SET}]$, given by $y_C(c) \to \mathrm{Hom}_C (-, c)$*

An object $g \in C$ is a **group object** if the functor $y_C(g)$ factors through GRP, i.e. the following diagram commutes where GRP $\to$ SET is the forgetful functor

$$\begin{tikzcd}
 & \mathrm{GRP} \arrow[dr] & \\
C^{\mathrm{op}} \arrow[ur, dashed, "G"] \arrow[rr, "y_C(g)"'] & & \mathrm{SET}
\end{tikzcd}$$

To give an explicit construction, recall that for a group G, we need to have a unit element, an inverse operation and a group operation, given by

$$
\begin{aligned}
e_G &: 1 \to G \\
\mathrm{inv}_G &: G \to G^{\mathrm{op}} \\
\cdot_G &: G \times G \to G
\end{aligned}
$$

such that the following diagrams commute to ensure associativity, unit and inverse holds:

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{id}_G \times \cdot_G"] \arrow[d, "\cdot_G \times \mathrm{id}_G"'] & G \times G \arrow[d, "\cdot_G"] \\
G \times G \arrow[r, "\cdot_G"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "{(e,\mathrm{id}_G)}"] \arrow[d, "{(\mathrm{id}_G,e)}"'] \arrow[dr, equal] & G \times G \arrow[d, "\cdot_G"] \\
G \times G \arrow[r, "\cdot_G"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "{(\mathrm{inv}_G,\mathrm{id}_G)}"] \arrow[d, "{(\mathrm{id}_G,\mathrm{inv}_G)}"'] \arrow[dr, "e"] & G \times G \arrow[d, "\cdot_G"] \\
G \times G \arrow[r, "\cdot_G"'] & G
\end{tikzcd}
$$

where $e : G \to G$ is the composite morphism $G \to 1 \overset{e_G}{\to} G$
The maps can immediately be constructed by Yoneda's lemma, take for instance the product map. If g is a group object with $G : C^{\mathrm{op}} \to \mathrm{GRP}$ as the functor to group and $f : c \to d$ is a morphism in C, then the following diagram commutes:

$$
\begin{tikzcd}
G(c) \times G(c) \arrow[r, "\cdot_{G(c)}"] & G(c) \\
G(d) \times G(d) \arrow[r, "\cdot_{G(d)}"'] \arrow[u, "G(f)\times G(f)"] & G(d) \arrow[u, "G(f)"']
\end{tikzcd}
$$

telling us we have a natural transformation $\cdot_G : G \times G \to G$. This gives us the morphism $\cdot_g : g \times g \to g$ by Yoneda's lemma and as the Yoneda embedding is full and faithful, the commutativity of the diagrams above defining a group is immediate.
Similarly one defines cogroups as group objects in the opposite category. With this view, it is immediate that if $c \in C$ is a cogroup, then $\mathrm{Hom}_C(c, -)$ is a functor to GRP.
As examples in hTOP^0, ΣX is a cogroup as $[\Sigma X, Y]^0$ is a group and ΩY is a group as $[X, \Omega Y]^0$ is a group.

4.4 Fibre sequence

Again here we work in pointed spaces. A map $f : X \to Y$ induces a map $f^* : [Y, B]^0 \to [X, B]^0$. The kernels of f is any element that gets sent to the basepoint of Y and the kernel of f^* is any element that gets sent to a nullhomotopic element. This allows us to define exact sequences of topological spaces.
A sequence of spaces $U \overset{f}{\to} V \overset{g}{\to} W$ is **h-coexact** if for all spaces B, the sequence

$$[U, B]^0 \overset{f^*}{\leftarrow} [V, B]^0 \overset{g^*}{\leftarrow} [W, B]^0$$

is exact.

The **cylinder** $XI = \frac{X \times I}{* \times I}$ describes homotopies in TOP^0 (morphisms $XI \to Y$ are homotopies).
The **cone** $CX = \frac{X \times I}{X \times 0 \cup * \times I} = X \wedge I$ describes homotopies starting from constant maps in TOP^0.
Define $C(f)$ as the pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "f_1"] \\
CX \arrow[r, "j"'] & C(f)
\end{tikzcd}$$

and by its universal property, we have the h-coexact sequence $X \xrightarrow{f} Y \xrightarrow{f_1} C(f)$ which can be iterated to create a long h-coexact sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} C(f) \xrightarrow{f_2} C\left(f_1\right) \xrightarrow{f_3} \ldots$$

Furthermore we have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "f"'] & CX \arrow[r, "p"] \arrow[d, "j"] & Y/i_1X \arrow[d] & = \Sigma X \\
Y \arrow[r, "f_1"] \arrow[d, "i_1"'] & C(f) \arrow[r, "p(f)"] \arrow[d, "f_2"] & C(f)/f_1Y \arrow[d] & = \Sigma X \\
CY \arrow[r, "j_1"'] & C\left(f_1\right) \arrow[r, "q(f)"'] & C\left(f_1\right)/j_1Y & = \Sigma X
\end{tikzcd}$$

Applying this to itself, we obtain

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & CX \arrow[d] & \\
Y \arrow[r, "f_1"] \arrow[d] & C(f) \arrow[r] \arrow[d, "f_2"'] & \Sigma X \arrow[dd, "\Sigma f \circ \iota"] \\
CY \arrow[r] & C\left(f_1\right) \arrow[ur, "q(f)"'] \arrow[dr, "p(f_1)"] \arrow[d, "f_3"'] & \\
& C\left(f_2\right) \arrow[r, "q(f_1)"'] & \Sigma Y
\end{tikzcd}$$

where $\iota : (x, t) \to (x, 1-t)$ to ensure commtuativity.
As $q(f), q\left(f_1\right)$ are homotopy equivalences and a sequence remains h-coexact if we replace elements with h-equivalent ones, we obtain the h-coexact sequence

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y$$

And we can apply this sequence to itself iteratively and noting that Σ and C commutes to obtain the **Puppe-sequence** or the **cofibre sequence** of f:

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y \to \Sigma C(f) \to \Sigma C(f) \to \Sigma^2 X \to \Sigma^2 Y \ldots$$

as $\Sigma f : \Sigma X \to \Sigma Y$

Applying the functor $[-, B]^0$, we see that from the 4th place onwards these are groups and from the 7th place onwards these are abelian groups.

For any map $f : X \to Y$, let $\mu : C(f) \to \Sigma X \vee C(f)$ be defined as

$$\mu(x,t) = \begin{cases} ((x,2t),*) & 2t \leq 1 \\ (*,(x,2t-1)) & 2t \geq 1 \end{cases}$$

and $\mu(y) = y$. This map is a **h-coaction** of the h-cogroup ΣX on $C(f)$ as we have

$$[\Sigma X, B]^0 \times [C(f), B]^0 \cong [\Sigma X \vee C(f), B]^0 \to [C(f), B]$$

where the last map is induced by μ.

With the map $Y \xrightarrow{f_1} C(f) \xrightarrow{p(f)} \Sigma X$ and any maps $\alpha_1, \alpha_2 : \Sigma X \to B$, this group action satisfies $(\alpha_1)\left(p(f)^*\alpha_2\right) = p(f)^*\left(\alpha_1\alpha_2\right)$ and f_1^* is an injective map on orbits of this action.

We can dualize everything as always.

A sequence of spaces $U \xrightarrow{f} V \xrightarrow{g} W$ is **h-exact** if for all spaces B, the sequence

$$[B,U]^0 \xrightarrow{f^*} [B,V]^0 \xrightarrow{g^*} [B,W]^0$$

is exact.

To dualize the cone, we use the exponential object adjuncation $[X \wedge I, Y]^0 \cong [X, F^0(Y,I)]^0$ and define $FY = F^0(Y,I)$. We then define $F(f)$ as the pullback

$$\begin{tikzcd} F(f) \arrow[r, dashed, "q"] \arrow[d, dashed, "f^1"'] & FY \arrow[d, "e^1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and by its universal property, we have the h-exact sequence $F(f) \xrightarrow{f_1} X \xrightarrow{f} Y$ which can be iterated to create a long h-exact sequence

$$\ldots \xrightarrow{f^4} F\left(f^2\right) \xrightarrow{f^3} F\left(f^1\right) \xrightarrow{f^2} F(f) \xrightarrow{f^1} X \xrightarrow{f} Y$$

We can dualize the diagrams above to obtain

$$
\begin{tikzcd}
Y & FY \arrow[l, "e^1"'] & \Omega Y \arrow[l, "i"'] \\
X \arrow[u, "f"] & F(f) \arrow[l, "f^1"'] \arrow[u, "q"'] & \Omega Y \arrow[l, "i(f)"'] \arrow[u] \\
FX \arrow[u, "e^1"] & F\left(f^1\right) \arrow[l, "q^1"] \arrow[u, "f^2"'] & \Omega Y \arrow[l, "j(f)"] \arrow[u]
\end{tikzcd}
$$

where $j(f)$ is a h-equivalence and using this on itself, we obtain

$$
\begin{tikzcd}
Y & FY \arrow[l] & \\
X \arrow[u, "f"] & F(f) \arrow[l, "f^1"'] \arrow[u] & \Omega Y \arrow[l, "i(f)"'] \arrow[dl, "j(f)"] \\
FX \arrow[u] & F\left(f^1\right) \arrow[l] \arrow[u, "f^2"] & \\
 & F\left(f^2\right) \arrow[u, "f^3"] & \Omega X \arrow[l, "j\left(f^1\right)"] \arrow[ul, "i\left(f^1\right)"'] \arrow[uu, "\iota\circ\Omega f"']
\end{tikzcd}
$$

and finally we obtain the dual long h-exact sequence

$$\Omega X \to \Omega Y \to F(f) \to X \to Y$$

and repeating this on the map $\Omega f : \Omega X \to \Omega Y$, we obtain the long h-exact sequence

$$\ldots \Omega^2 X \to \Omega^2 Y \to \Omega F(f) \to \Omega X \to \Omega Y \to F(f) \to X \to Y$$

known as the **fibre seqence** of f. Similarly applying the functor $[B, -]^0$, we see that from the 4th place onwards these are groups and from the 7th place onwards these are abelian groups.

Finally to dualize the group action, again let $f : X \to Y$ be any map. We have the h-action $m : \Omega Y \times F(f) \to F(f)$ defined as

$$m\left([f(t), (x, g(t))]\right) = \begin{cases} (x, f(2t)) & 2t \leq 1 \\ (x, g(2t-1)) & 2t \geq 1 \end{cases}$$

This map induces the map

$$[B, \Omega Y]^0 \times [B, F(f)]^0 \cong [B, \Omega Y \times F(f)]^0 \to [B, F(f)]^0$$

Furthermore with the maps $\Omega Y \overset{i(f)}{\to} F(f) \overset{f^1}{\to} X$ and any maps $\alpha_1, \alpha_2 : B \to \Omega Y$, this group action satisfies $\left(i(f)_* \alpha_1\right)(\alpha_2) = i(f)_* \left(\alpha_1 \alpha_2\right)$ and f^1_* is an injective map on the orbits of the action.

5 Chapter 5

References

[May] May, J. P. (1999). A concise course in algebraic topology. University of Chicago press.

[Wat] Waterhouse, W. C. (2012). Introduction to affine group schemes (Vol. 66). Springer Science & Business Media.