ESS 490/590 - Data Science for Earth and Planetary Systems - Spring 2021

Using automatic differentiation for seismic inversion

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Introduction

The forward problem: We know the seismic source function $f(t)$ and the properties of the rock medium where the seismic wave propagates, ρ and V_S. We want to model numerically the waveforms of the seismograms that will be observed at the receivers.

For that, we solve the seismic wave equation (here in 1D for ease of implementation and computational speed):

$\rho \frac{\partial v}{\partial t} = \frac{\partial s}{\partial x} + f(x)$ and $\frac{\partial s}{\partial t} = \mu \frac{\partial v}{\partial x}$

where v is the velocity, s is the stress, x is the direction of propagation, t is the time, and $\mu = \rho V_S^2$ is the shear modulus.

The inverse problem: We know the seimsic source function $f(t)$ and the seismograms at some receivers $d(x_{ir}, t)$ for some x_{ir} with $ir = 1, \cdots, nr$. We want to guess what are the values of the rock properties ρ and V_S.

How to solve the inverse problem: We define a loss function that is the difference between the actual observed seismograms and the synthetic seismograms obtained with some hypothetical value for ρ and V_S:

$$J = \frac{1}{2} \sum_{ir=1}^{nr} \sum_{it=1}^{nt} (v(x_{ir}, t) - d(x_{ir}, t))^2$$

At the beginning, we choose initial values ρ_0 and V_{S0} and we compute the corresponding values of the velocity at the receivers $v(x_{ir}, t)$ using forward modeling. We compute the corresponding value of the loss function, and the gradient of the loss function with respect to the rock properties $\frac{\partial J}{\partial \rho}$ and $\frac{\partial J}{\partial \mu}$. We use the gradient descent method to update the values of ρ and V_S and we continue until the loss $J(\rho, V_S)$ is small enough.

Use of automatic differentation: To implement the gradient descent method, we mainly need to compute the gradient of the loss function. To do that, we can use automatic differentiation and its PyTorch implementation.

Implementation of the forward problem

We will start by choosing a model for the source and the rock properties and solve the sesimic wave equation to compute synthetic seismograms. Then we will use these synthetics to solve the inverse problem and see if we can retrieve the rock properties that we have used initially.

Last start by importing the necessary Python modules.

In [1]:
```
import matplotlib.pyplot as plt
import numpy as np
import torch
from math import pi
from scipy.signal import butter, lfilter
```

Let us define the properties of the model.

In [2]:
```
# Size of the model
N = 200                                                         # Number of cells
dx = 100.0                                                      # Cell size
dt = 0.01                                                       # Time step
T = 400                                                         # Number of time steps
# Rock properties
rho = 2500.0                                                    # Rock density (uniform
# Source
f = np.sin(2 * pi * np.arange(start=0, step=dt, stop=1 + dt))  # Seismic source function
x0 = 81                                                         # Location of the source
# Receivers
x = [91, 101, 111, 121]                                         # Locations of the receivers
```

We are going to compute synthetics seismograms. Here is the function to do forward modeling. We use finite differences to solve the seismic wave equation, again for ease of implementation and computational speed. Note that we do not use absorbing layers at the boundaries of the model and we will have a reflection at the boundaries. We will need to keep only the beginnings of the synthetic sesimograms, where we see the first wave arrival but no sesimic reflection yet.

This function use the model properties and returns the synthetic seismograms at the receivers.

In [3]:
```
def forward(N, dx, T, dt, rho, mu, f, x0, x):
    # Initialization
    obs = np.zeros((T, len(x)))
    v = np.zeros(N + 3)
    s = np.zeros(N + 4)
    # Loop on time
    for i in range(0, T):
        # Compute stress
        ds = 1.0e10 * mu[2 : (N + 2)] * ((9.0 / 8.0) * (v[2 : (N + 2)] - v[1 : (N + 1)]) - \
                                         (1.0 / 24.0) * (v[3 : (N + 3)] - v[0 : N])) / dx
        s[2 : (N + 2)] = s[2 : (N + 2)] + ds * dt
        # Boundary condition (reflection)
        s[1] = - s[2]
        s[0] = - s[3]
        s[N + 2] = - s[N + 1]
        s[N + 3] = - s[N]
        # Compute velocity
        dv = (1.0 / rho) * ((9.0 / 8.0) * (s[2 : (N + 3)] - s[1 : (N + 2)]) - \
                            (1.0 / 24.0) * (s[3 : (N + 4)] - s[0 : (N + 1)])) / dx
        v[1 : (N + 2)] = v[1 : (N + 2)] + dv * dt
        # Add force
        if i < len(f):
            v[x0] = v[x0] + f[i]
        # Receivers
        for j in range(0, len(x)):
            obs[i, j] = v[x[j]]
    return obs
```

We choose a velocity model with one seimsic discontinuity between two different S-wave velocities.

In [4]:
```
vs1 = 2000.0
vs2 = 1800.0
mu = np.concatenate([np.repeat(rho * vs1**2, N - 93), np.repeat(rho * vs2**2 , 97)]) / 1.0e10
```

We run the forward simulation for this velocity model.

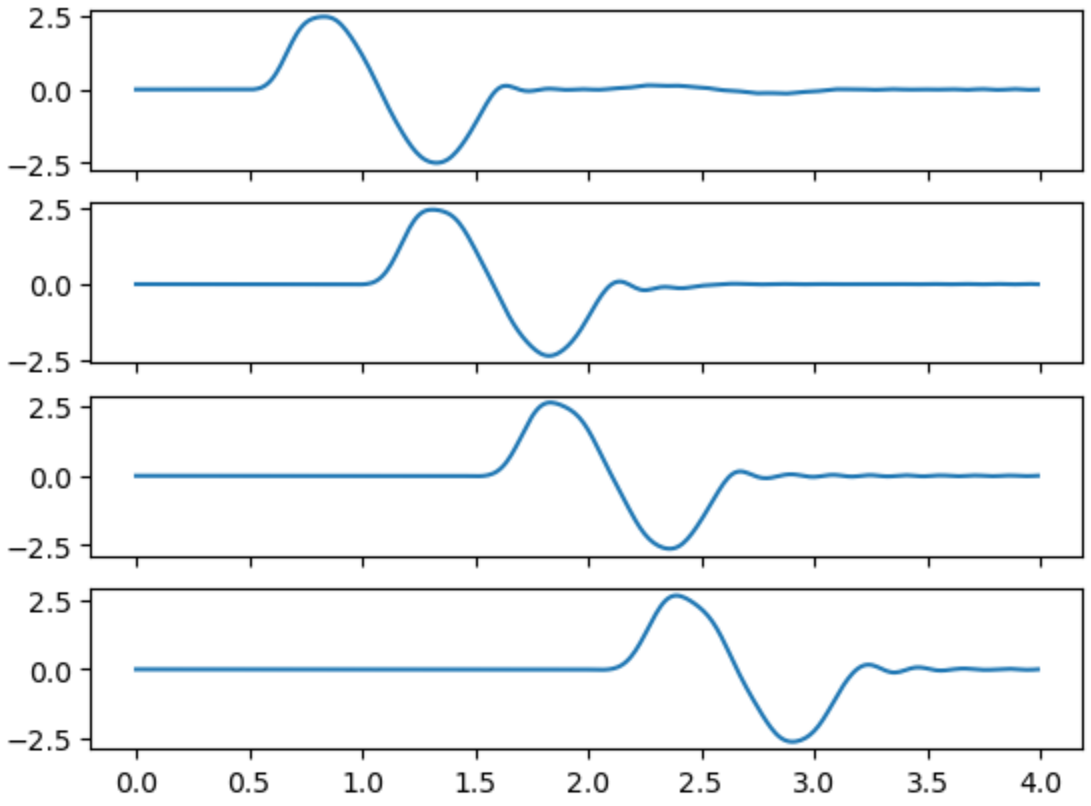
In [5]:
```
d = forward(N, dx, T, dt, rho, mu, f, x0, x)
```

We filter and plot the synthetics at the 4 receivers.

In [6]:
```
b, a = butter(4, 0.1, btype='low')
```

In [7]:
```
ax1 = plt.subplot(411)
plt.plot(dt * np.arange(0, T), lfilter(b, a, d[:, 0]))
plt.setp(ax1.get_xticklabels(), visible=False)
ax2 = plt.subplot(412)
plt.plot(dt * np.arange(0, T), lfilter(b, a, d[:, 1]))
plt.setp(ax2.get_xticklabels(), visible=False)
ax3 = plt.subplot(413)
plt.plot(dt * np.arange(0, T), lfilter(b, a, d[:, 2]))
plt.setp(ax3.get_xticklabels(), visible=False)
ax4 = plt.subplot(414)
plt.plot(dt * np.arange(0, T), lfilter(b, a, d[:, 3]))
```

Out[7]:
```
[<matplotlib.lines.Line2D at 0x1681c1f10>]
```



Solving the inverse problem using automatic differentiation

Now that we have our synthetics, we will see if we can use them to retrieve the initial values of the S-wave velocities that we have chosen. Here, we suppose that the value of the density ρ is known.

At each time step of the gradient descent method, we will need to:

- solve the forward problem with the current value of μ,
- compute the loss between the current and the target seismograms,
- compute the gradient of the loss with respect to μ,
- update the value of μ using the gradient.

We repeat the steps until the loss is small enough.

The forward modeling is done as we have done to create the synthetic seismograms. The only difference is that we use PyTorch tensors instead of Numpy arrays, in order to be able to compute the gradient using automatic differentation.

In [8]:
```
def forward_AD(N, dx, T, dt, rho, mu, f, x0, x):
    # Initialization
    obs = torch.zeros((T, len(x)))
    v = torch.zeros(N + 3)
    s = torch.zeros(N + 4)
    # Loop on time
    for i in range(0, T):
        # Compute stress
        ds = 1.0e10 * mu[2 : (N + 2)] * ((9.0 / 8.0) * (v[2 : (N + 2)] - v[1 : (N + 1)]) - \
                                        (1.0 / 24.0) * (v[3 : (N + 3)] - v[0 : N])) / dx
        s[2 : (N + 2)] = s[2 : (N + 2)] + ds * dt
        # Boundary condition (reflection)
        s[1] = - s[2]
        s[0] = - s[3]
        s[N + 2] = - s[N + 1]
        s[N + 3] = - s[N]
        # Compute velocity
        dv = (1.0 / rho) * ((9.0 / 8.0) * (s[2 : (N + 3)] - s[1 : (N + 2)]) - \
                           (1.0 / 24.0) * (s[3 : (N + 4)] - s[0 : (N + 1)])) / dx
        v[1 : (N + 2)] = v[1 : (N + 2)] + dv * dt
        # Add force
        if i < len(f):
            v[x0] = v[x0] + f[i]
        # Receivers
        for j in range(0, len(x)):
            obs[i, j] = v[x[j]]
    return obs
```

The loss is also computed with PyTorch tensors instead of Numpy arrays.

In [9]:
```
def loss_AD(obs, d):
    F = 0
    for j in range(0, obs.size()[1]):
        F = F + torch.sum(torch.square(obs[:, j] - torch.from_numpy(d[:, j]))) / 2
    return F
```

We now implement one step of the gradient descent method. At each step, we compute the current value of the loss, then the gradient of the loss with respect to the value of the shear modulus μ, and we update the value of μ.

In [10]:
```
def step(N, dx, T, dt, rho, mu, f, x0, x, d, alpha):
    # Compute the loss for the current value of mu
    obs = forward_AD(N, dx, T, dt, rho, mu, f, x0, x)
    F = loss_AD(obs, d)
    # Compute the gradient of the loss
    F.backward()
    # Specifically, we want the gradient with respect to mu
    dmu = mu.grad
    # Update the values of mu
    mu = mu - alpha * dmu
    mu.retain_grad()
    # Return the new value of mu
    return (mu, F)
```

We can now write the optimization function. At each step, we save the current value of mu and the value of the loss. We must choose the value of the learning rate α with which we multiply the gradient to get the new value of μ.

We also need to choose an initial value for μ. Here, we choose a uniform value over all the grid cells, equal to $1.0e^{10} Pa$, which corresponds to $V_S = 2000 m / s$. This is the value that was chosen for the left part of the domain to compute the synthetics. We will see if we can retrieve the drop in velocity using the inversion method.

In [11]:
```
def invert_mu(N, dx, T, dt, rho, f, x0, x, d, alpha, n_epochs):
    save_mu = np.zeros((N + 3, n_epochs))
    save_F = np.zeros(n_epochs)
    f = torch.from_numpy(f)
    mu = torch.from_numpy(np.repeat(1.0, N + 3))
    mu.requires_grad = True
    for epoch in range(0, n_epochs):
        (mu, F) = step(N, dx, T, dt, rho, mu, f, x0, x, d, alpha)
        save_mu[:, epoch] = mu.detach().numpy()
        save_F[epoch] = F
    return (mu, save_mu, save_F)
```

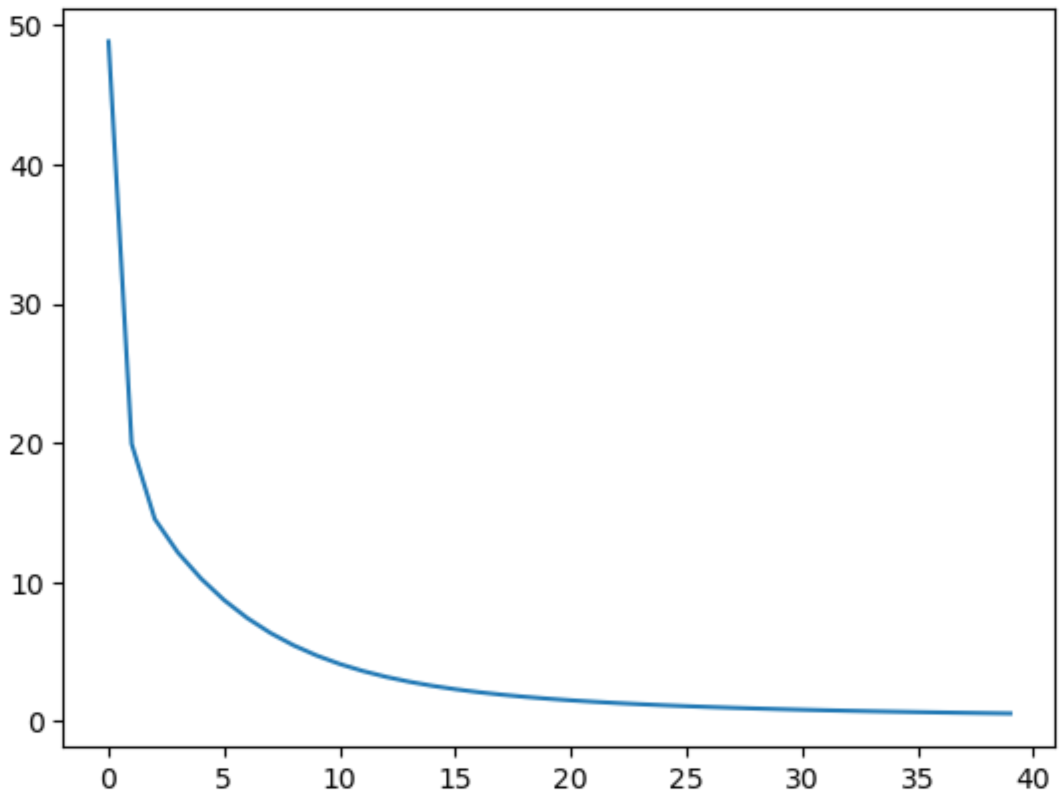
We can now run the optimization algorithm.

In [12]:
```
(mu, save_mu, save_F) = invert_mu(N, dx, T, dt, rho, f, x0, x, d, 0.001, 40)
```

Let us first plot the evolution of the loss over time.

In [13]:
```
plt.plot(np.arange(0, 40), save_F)
```

Out[13]:
```
[<matplotlib.lines.Line2D at 0x168283350>]
```



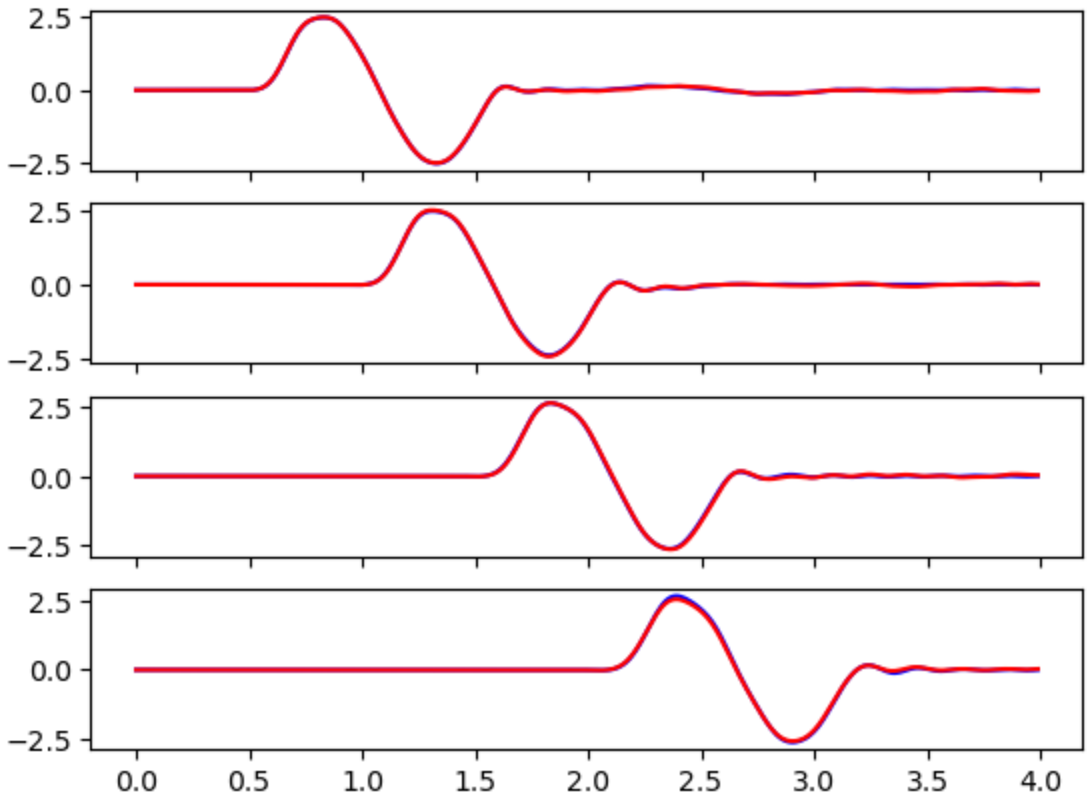
After 40 iterations, we see that the loss does not decrease much. We can also plot the seismograms computed with the last value of μ and compare it the true value of the synthetics.

In [14]:
```
obs = forward_AD(N, dx, T, dt, rho, mu, f, x0, x)
```

In [15]:
```
ax1 = plt.subplot(411)
plt.plot(dt * np.arange(0, T), lfilter(b, a, d[:, 0]), 'b')
plt.plot(dt * np.arange(0, T), lfilter(b, a, obs.detach().numpy()[:, 0]), 'r')
plt.setp(ax1.get_xticklabels(), visible=False)
ax2 = plt.subplot(412)
plt.plot(dt * np.arange(0, T), lfilter(b, a, d[:, 1]), 'b')
plt.plot(dt * np.arange(0, T), lfilter(b, a, obs.detach().numpy()[:, 1]), 'r')
plt.setp(ax2.get_xticklabels(), visible=False)
ax3 = plt.subplot(413)
plt.plot(dt * np.arange(0, T), lfilter(b, a, d[:, 2]), 'b')
plt.plot(dt * np.arange(0, T), lfilter(b, a, obs.detach().numpy()[:, 2]), 'r')
plt.setp(ax3.get_xticklabels(), visible=False)
ax4 = plt.subplot(414)
plt.plot(dt * np.arange(0, T), lfilter(b, a, d[:, 3]), 'b')
plt.plot(dt * np.arange(0, T), lfilter(b, a, obs.detach().numpy()[:, 3]), 'r')
```

Out[15]:

```
[<matplotlib.lines.Line2D at 0x1695ead20>]
```

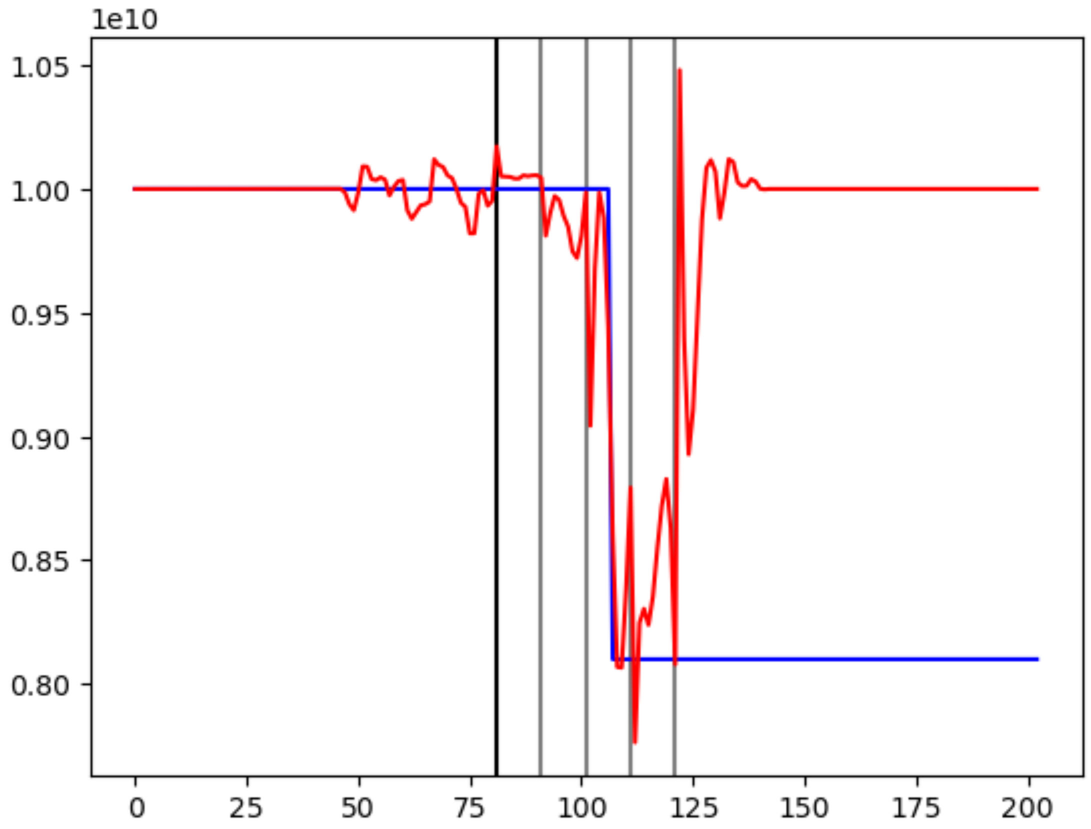


We can see that there is a good agreement between the theoretical seismograms and the obtained after the inversion. Let us now look at the value of μ.

In [16]:

```
plt.axvline(x0, color='black')
for j in range(0, len(x)):
    plt.axvline(x[j], color='grey')
plt.plot(np.arange(0, N + 3), np.concatenate([np.repeat(rho * vs1 * vs1, N - 93), np.repeat(rho * vs2 * vs
plt.plot(np.arange(0, N + 3), 1.0e10 * mu.detach().numpy(), 'r')
```

```
Out[16]: [<matplotlib.lines.Line2D at 0x169687dd0>]
```



The blue line represents the true value of the shear modulus, while the red line represents the value obtained from the inversion. The black vertical line represents the location of the source while the 4 grey lines represent the locations of the receivers. We note that in the area covered by the receivers we can retrieve the drop in the value of the shear modulus. However, in the right part of the domain that is not covered by the receivers, we cannot retrieve the true S-wave velocity.

For this small example and for the sake of the demonstration, we implemented the gradient descent method ourselves. In practice, you would probably use one of the optimizers already implemented in PyTorch. Additional work would also be needed for the computation of the forward model, the choice of the loss, and the initial value of the shear modulus.

For a general approach on the use of automatic differentiation for seismic inversion, please see the paper:

Zhu, W., Xu, K., Darve, E., and Beroza, G.C. (2021) A general approach to seismic inversion with automatic differentiation.

Computers & Geosciences, 151, https://doi.org/10.1016/j.cageo.2021.104751.

In this tutorial, we applied the method to seismic inversion of a velocity model but the same approach could be used for many inverse problems in geophysics.

```
In [ ]:
```