0.1 Time lags

0.1.1 Method

Some bla bla about cross correlaion and stacking

0.1.2 Cross-correlation of tremor recordings

Definitions

We define the Fourier transform $\hat{f}$ of the function f by:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{1}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega \tag{2}$$

We define the convolution product by:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau \tag{3}$$

We have:

$$(f \hat{*} g)(\omega) = \sqrt{2\pi} \hat{f}(\omega) \hat{g}(\omega) \tag{4}$$

and:

$$(\hat{fg})(\omega) = \frac{1}{\sqrt{2\pi}} \hat{f}(\omega) * \hat{g}(\omega) \tag{5}$$

We define the cross correlation by:

$$(f \otimes g)(t) = \int_{-\infty}^{\infty} f^*(\tau) g(t + \tau) d\tau \tag{6}$$

where f^* is the complex conjugate of f. We have:

$$(f \hat{\otimes} g)(\omega) = \sqrt{2\pi} \hat{f}^*(\omega) \hat{g}(\omega) \tag{7}$$

and:

$$(\hat{fg})(\omega) = \frac{1}{\sqrt{2\pi}} \hat{f}^*(\omega) \otimes \hat{g}(\omega) \tag{8}$$

Finally, we have:

$$(f(t) * g(t)) \otimes (f(t) * g(t)) = (f(t) \otimes f(t)) * (g(t) \otimes g(t)) \tag{9}$$

If we have a point source at ξ with a source function $f_k(t)$ for $k = 1, 2, 3$, then we can express the displacement $u(x, t)$ with the Green's function:

$$u_i(x, t) = \int_{-\infty}^{\infty} G_{ik}(x, \xi, t - \tau) f_k(\xi, \tau) d\tau \tag{10}$$

In the case of a moment tensor, we have:

$$u_i(x, t) = \int_{-\infty}^{\infty} \frac{\partial G_{ip}}{\partial \xi_q}(x, \xi, t - \tau) M_{pq}(\xi, \tau) d\tau \tag{11}$$

In the Fourier domain, we have:

$$\hat{u_i}(x, \omega) = \sqrt{2\pi} \hat{G_{ik}}(x, \xi, \omega) \hat{f_k}(\xi, \omega) \tag{12}$$

or:

$$\hat{u_i}(x, \omega) = \sqrt{2\pi} \frac{\partial \hat{G}_{ip}}{\partial \xi_q}(x, \xi, \omega) \hat{M_{pq}}(\xi, \omega) \tag{13}$$

When we compute the cross correlation between two components of the displacements, we have:

$$(u_i \otimes u_j)(x,t) = \int_{-\infty}^{\infty} u_i^*(x,\tau) u_j(x,t+\tau) d\tau \tag{14}$$

In the Fourier domain, we have:

$$\begin{aligned}(u_i \hat{\otimes} u_j)(x,\omega) &= \sqrt{2\pi}\hat{u_i}^*(x,\omega)\hat{u_j}(x,\omega) \\ &= (2\pi)^{\frac{3}{2}}[\hat{G_{ik}}^*(x,\xi,\omega)\hat{f_k}^*(\xi,\omega)][\hat{G_{jl}}(x,\xi,\omega)\hat{f_l}(\xi,\omega)] \\ &= (2\pi)^{\frac{3}{2}}[\frac{\partial \hat{G}_{ip}}{\partial \xi_q}^*(x,\xi,\omega)\hat{M_{pq}}^*(\xi,\omega)][\frac{\partial \hat{G}_{jr}}{\partial \xi_s}(x,\xi,\omega)\hat{M_{rs}}(\xi,\omega)]\end{aligned} \tag{15}$$

Change of coordinates

Point source Equation (15) can be written as:

$$\begin{pmatrix} (u_1 \hat{\otimes} u_1) & (u_1 \hat{\otimes} u_2) & (u_1 \hat{\otimes} u_3) \\ (u_2 \hat{\otimes} u_1) & (u_2 \hat{\otimes} u_2) & (u_2 \hat{\otimes} u_3) \\ (u_3 \hat{\otimes} u_1) & (u_3 \hat{\otimes} u_2) & (u_3 \hat{\otimes} u_3) \end{pmatrix} = (2\pi)^{\frac{3}{2}} \begin{pmatrix} \hat{G_{11}}^* & \hat{G_{12}}^* & \hat{G_{13}}^* \\ \hat{G_{21}}^* & \hat{G_{22}}^* & \hat{G_{23}}^* \\ \hat{G_{31}}^* & \hat{G_{32}}^* & \hat{G_{33}}^* \end{pmatrix} \begin{pmatrix} \hat{f_1}^* \\ \hat{f_2}^* \\ \hat{f_3}^* \end{pmatrix} \begin{pmatrix} \hat{f_1} & \hat{f_2} & \hat{f_3} \end{pmatrix} \begin{pmatrix} \hat{G_{11}} & \hat{G_{21}} & \hat{G_{31}} \\ \hat{G_{12}} & \hat{G_{22}} & \hat{G_{32}} \\ \hat{G_{13}} & \hat{G_{23}} & \hat{G_{33}} \end{pmatrix} \tag{16}$$

that is:

$$\hat{U} = (2\pi)^{\frac{3}{2}} \hat{G}^* \hat{f}^* \hat{f}^T \hat{G}^T \tag{17}$$

We define a new coordinate system with the unit vectors $n^{(1)}$, $n^{(2)}$ and $n^{(3)}$, and the matrix N by:

$$N = \begin{pmatrix} n_1^{(1)} & n_1^{(2)} & n_1^{(3)} \\ n_2^{(1)} & n_2^{(2)} & n_2^{(3)} \\ n_3^{(1)} & n_3^{(2)} & n_3^{(3)} \end{pmatrix} \tag{18}$$

In the new coordinate system, the Green's function is equal to $G' = N^T G N$, thus we have:

$$N^T \hat{U} N = (2\pi)^{\frac{3}{2}} \hat{G'}^* N^T \hat{f}^* \hat{f}^T N \hat{G'}^T \tag{19}$$

with:

$$N^T \hat{U} N = \begin{pmatrix} (u.n^{(1)} \hat{\otimes} u.n^{(1)}) & (u.n^{(1)} \hat{\otimes} u.n^{(2)}) & (u.n^{(1)} \hat{\otimes} u.n^{(3)}) \\ (u.n^{(2)} \hat{\otimes} u.n^{(1)}) & (u.n^{(2)} \hat{\otimes} u.n^{(2)}) & (u.n^{(2)} \hat{\otimes} u.n^{(3)}) \\ (u.n^{(3)} \hat{\otimes} u.n^{(1)}) & (u.n^{(3)} \hat{\otimes} u.n^{(2)}) & (u.n^{(3)} \hat{\otimes} u.n^{(3)}) \end{pmatrix} \tag{20}$$

If we choose N_1 such that:

$$N_1^T \hat{f} = \begin{pmatrix} \hat{F} \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

we get:

$$N_1^T \hat{U} N_1 = (2\pi)^{\frac{3}{2}} \hat{F}^* \hat{F} \begin{pmatrix} \hat{G'_{11}}^* \hat{G'_{11}} & \hat{G'_{11}}^* \hat{G'_{21}} & \hat{G'_{11}}^* \hat{G'_{31}} \\ \hat{G'_{21}}^* \hat{G'_{11}} & \hat{G'_{21}}^* \hat{G'_{21}} & \hat{G'_{21}}^* \hat{G'_{31}} \\ \hat{G'_{31}}^* \hat{G'_{11}} & \hat{G'_{31}}^* \hat{G'_{21}} & \hat{G'_{31}}^* \hat{G'_{31}} \end{pmatrix} \tag{22}$$

If we choose N_2 such that:

$$N_2^T \hat{f} = \begin{pmatrix} 0 \\ \hat{F} \\ 0 \end{pmatrix} \tag{23}$$

we get:

$$N_2^T \hat{U} N_2 = (2\pi)^{\frac{3}{2}} \hat{F}^* \hat{F} \begin{pmatrix} \hat{G'_{12}}^* \hat{G'_{12}} & \hat{G'_{12}}^* \hat{G'_{22}} & \hat{G'_{12}}^* \hat{G'_{32}} \\ \hat{G'_{22}}^* \hat{G'_{12}} & \hat{G'_{22}}^* \hat{G'_{22}} & \hat{G'_{22}}^* \hat{G'_{32}} \\ \hat{G'_{32}}^* \hat{G'_{12}} & \hat{G'_{32}}^* \hat{G'_{22}} & \hat{G'_{32}}^* \hat{G'_{32}} \end{pmatrix} \tag{24}$$

If we choose N_3 such that:

$$N_3^T \hat{f} = \begin{pmatrix} 0 \\ 0 \\ \hat{F} \end{pmatrix} \tag{25}$$

we get:

$$N_3^T \hat{U} N_3 = (2\pi)^{\frac{3}{2}} \hat{F}^* \hat{F} \begin{pmatrix} \hat{G'_{13}}^* \hat{G'_{13}} & \hat{G'_{13}}^* \hat{G'_{23}} & \hat{G'_{13}}^* \hat{G'_{33}} \\ \hat{G'_{23}}^* \hat{G'_{13}} & \hat{G'_{23}}^* \hat{G'_{23}} & \hat{G'_{23}}^* \hat{G'_{33}} \\ \hat{G'_{33}}^* \hat{G'_{13}} & \hat{G'_{33}}^* \hat{G'_{23}} & \hat{G'_{33}}^* \hat{G'_{33}} \end{pmatrix} \tag{26}$$

If we define strike ϕ, dip δ and rake λ, we can define the following vectors:

$$e_1 = \begin{pmatrix} \sin\phi \\ \cos\phi \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} \cos\phi \\ -\sin\phi \\ 0 \end{pmatrix},\ e_3 = \cos\delta e_2 - \sin\delta e_z = \begin{pmatrix} \cos\phi\cos\delta \\ -\sin\phi\cos\delta \\ -\sin\delta \end{pmatrix} \text{ and } e_4 = \sin\delta e_2 + \cos\delta e_z = \begin{pmatrix} \cos\phi\sin\delta \\ -\sin\phi\sin\delta \\ \cos\delta \end{pmatrix} \tag{27}$$

and the new coordinate system (u, v, w) with:

$$u = \cos\lambda e_1 - \sin\lambda e_3,\ v = -\sin\lambda e_1 - \cos\lambda e_3 \text{ and } w = e_4 \tag{28}$$

Thus we have:

$$u = \begin{pmatrix} \sin\phi\cos\lambda - \cos\phi\cos\delta\sin\lambda \\ \cos\phi\cos\lambda + \sin\phi\cos\delta\sin\lambda \\ \sin\delta\sin\lambda \end{pmatrix},\ v = \begin{pmatrix} -\sin\phi\sin\lambda - \cos\phi\cos\delta\cos\lambda \\ -\cos\phi\sin\lambda + \sin\phi\cos\delta\sin\lambda \\ -\sin\delta\cos\lambda \end{pmatrix} \text{ and } w = \begin{pmatrix} \cos\phi\sin\delta \\ -\sin\phi\sin\delta \\ \cos\delta \end{pmatrix} \tag{29}$$

We can choose:

$$N_1 = \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\lambda - \cos\phi\cos\delta\sin\lambda & -\sin\phi\sin\lambda - \cos\phi\cos\delta\cos\lambda & \cos\phi\sin\delta \\ \cos\phi\cos\lambda + \sin\phi\cos\delta\sin\lambda & -\cos\phi\sin\lambda + \sin\phi\cos\delta\sin\lambda & -\sin\phi\sin\delta \\ \sin\delta\sin\lambda & -\sin\delta\cos\lambda & \cos\delta \end{pmatrix} \tag{30}$$

to get Equation (19). To get Equations (21) and (23), we just have to permute the columns of N_1 to get N_2 and N_3. If we go back into the time domain, we have:

$$N_k^T \begin{pmatrix} u_1 \otimes u_1 & u_1 \otimes u_2 & u_1 \otimes u_3 \\ u_2 \otimes u_1 & u_2 \otimes u_2 & u_2 \otimes u_3 \\ u_3 \otimes u_1 & u_3 \otimes u_2 & u_3 \otimes u_3 \end{pmatrix} N_k = \begin{pmatrix} (F * G'_{1k}) \otimes (F * G'_{1k}) & (F * G'_{1k}) \otimes (F * G'_{2k}) & (F * G'_{1k}) \otimes (F * G'_{3k}) \\ (F * G'_{2k}) \otimes (F * G'_{1k}) & (F * G'_{2k}) \otimes (F * G'_{2k}) & (F * G'_{2k}) \otimes (F * G'_{3k}) \\ (F * G'_{3k}) \otimes (F * G'_{1k}) & (F * G'_{3k}) \otimes (F * G'_{2k}) & (F * G'_{3k}) \otimes (F * G'_{3k}) \end{pmatrix} \tag{31}$$

which can also be written as:

$$N_k^T \begin{pmatrix} u_1 \otimes u_1 & u_1 \otimes u_2 & u_1 \otimes u_3 \\ u_2 \otimes u_1 & u_2 \otimes u_2 & u_2 \otimes u_3 \\ u_3 \otimes u_1 & u_3 \otimes u_2 & u_3 \otimes u_3 \end{pmatrix} N_k = \begin{pmatrix} (F \otimes F) * (G'_{1k} \otimes G'_{1k}) & (F \otimes F) * (G'_{1k} \otimes G'_{2k}) & (F^S \otimes F^S) * (G'_{1k} \otimes G'_{3k}) \\ (F \otimes F) * (G'_{2k} \otimes G'_{1k}) & (F \otimes F) * (G'_{2k} \otimes G'_{2k}) & (F^S \otimes F^S) * (G'_{2k} \otimes G'_{3k}) \\ (F \otimes F) * (G'_{3k} \otimes G'_{1k}) & (F \otimes F) * (G'_{3k} \otimes G'_{2k}) & (F^S \otimes F^S) * (G'_{3k} \otimes G'_{3k}) \end{pmatrix} \tag{32}$$

Moment tensor We have:

$$M_{pq} = \int\int_\Sigma m_{pq} d\Sigma = \int\int_\Sigma \mu(\nu_p[u_q] + \nu_q[u_p]) d\Sigma = \mu A(\nu_p[u_q] + \nu_q[u_p]) \tag{33}$$

where ν is the normal to the fault surface and$[u]$ is the displacement discontinuity on the fault.
We define a new coordinates system with the unit vectors $n^{(1)}$, $n^{(2)}$ and $n^{(3)}$, and the matrix N by:

$$N = \begin{pmatrix} n_1^{(1)} & n_1^{(2)} & n_1^{(3)} \\ n_2^{(1)} & n_2^{(2)} & n_2^{(3)} \\ n_3^{(1)} & n_3^{(2)} & n_3^{(3)} \end{pmatrix} \tag{34}$$

In the new coordinates system, the Green's function is equal to $G' = N^T G N$, the moment tensor to $M' = N^T M N$, and the displacement to $u' = N^T u$. We choose N such that:

$$M' = \mu D A \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{35}$$

Therefore, we have:

$$u'_i \hat{\otimes} u'_j = (2\pi)^{\frac{3}{2}} \mu D^* A [\frac{\partial \hat{G}'_{i1}}{\partial \xi'_1}^* - \frac{\partial \hat{G}'_{i2}}{\partial \xi'_2}^*] \mu D A [\frac{\partial \hat{G}'_{j1}}{\partial \xi'_1} - \frac{\partial \hat{G}'_{j2}}{\partial \xi'_1}] \tag{36}$$

If we come back in the time domain, we have:

$$u'_i \otimes u'_j = \mu^2 A^2 (D * [\frac{\partial G'_{i1}}{\partial \xi'_1} - \frac{\partial G'_{i2}}{\partial \xi'_2}]) \otimes (D * [\frac{\partial G'_{j1}}{\partial \xi'_1} - \frac{\partial G'_{j2}}{\partial \xi'_2}]) \tag{37}$$

which can also be written as:

$$u'_i \otimes u'_j = \mu^2 A^2 (D \otimes D) * ([\frac{\partial G'_{i1}}{\partial \xi'_1} - \frac{\partial G'_{i2}}{\partial \xi'_2}] \otimes [\frac{\partial G'_{j1}}{\partial \xi'_1} - \frac{\partial G'_{j2}}{\partial \xi'_2}]) \tag{38}$$

If we choose N such that:

$$M' = \mu D A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{39}$$

we would get:

$$u'_i \otimes u'_j = \mu^2 A^2 (D \otimes D) * ([\frac{\partial G'_{i1}}{\partial \xi'_1} - \frac{\partial G'_{i3}}{\partial \xi'_3}] \otimes [\frac{\partial G'_{j1}}{\partial \xi'_1} - \frac{\partial G'_{j3}}{\partial \xi'_3}]) \tag{40}$$

If we choose N such that:

$$M' = \mu D A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{41}$$

we would get:

$$u'_i \otimes u'_j = \mu^2 A^2 (D \otimes D) * ([\frac{\partial G'_{i2}}{\partial \xi'_2} - \frac{\partial G'_{i3}}{\partial \xi'_3}] \otimes [\frac{\partial G'_{j2}}{\partial \xi'_2} - \frac{\partial G'_{j3}}{\partial \xi'_3}]) \tag{42}$$

How to compute the autocorrelation of the source term $F \otimes F$ or $D \otimes D$?

White noise At time t_i, we assume that the amplitude of the source function is A_i (random variable with expectancy $m = 0$ and standard deviation σ). We define a_i by:

$$P(A_i < p_0) = \int_{-\infty}^{p_0} a_i(p) dp \tag{43}$$

We have:

$$\int_{-\infty}^{\infty} a_i(p) dp = m \tag{44}$$

and:

$$\int_{-\infty}^{\infty} a_i^2(p) dp = m^2 + \sigma^2 \tag{45}$$

We suppose that the source function is a white noise, that is:

$$\int_{-\infty}^{\infty} (a_i(p) - m)(a_j(p) - m) dp = 0 \tag{46}$$

Thus, we have:

$$\int_{-\infty}^{\infty} a_i(p) a_j(p) dp = m^2 \tag{47}$$

We define the source time function $F(t_i) = A_i$ and we compute the autocorrelation. We have:

$$(F \otimes F)(t) = \int_{-\infty}^{\infty} F^*(\tau) F(t + \tau) d\tau \tag{48}$$

The expectancy of the term $F^*(\tau)F(t+\tau)$ is $m^2 + \sigma^2$ if $t = 0$ and m^2 if $t \neq 0$.

Stacking

0.1.3 Computation of amplitudes for P-, SV- and SH-waves

Changes of coordinates

We denote (x, y) the coordinates of the receiver array, (x_0, y_0) the coordinates of the tremor source, and d the depth of the tremor source.

In the $(\vec{e}_X, \vec{e}_Y, \vec{e}_Z)$ coordinate system, we have:

$$\vec{e}_R = \begin{pmatrix} \cos\beta \\ \sin\beta \\ 0 \end{pmatrix}, \vec{e}_T = \begin{pmatrix} -\sin\beta \\ \cos\beta \\ 0 \end{pmatrix} \text{ and } \vec{e}_Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{49}$$

with $\beta = \text{atan2}(y - y_0, x - x_0)$

In the $(\vec{e}_R, \vec{e}_T, \vec{e}_Z)$ coordinate system, we have:

$$\vec{e}_X = \begin{pmatrix} \cos\beta \\ -\sin\beta \\ 0 \end{pmatrix}, \vec{e}_Y = \begin{pmatrix} \sin\beta \\ \cos\beta \\ 0 \end{pmatrix} \text{ and } \vec{e}_Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{50}$$

For the direct wave, we have:

$$\vec{e}_P = \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}, \vec{e}_{SV} = \begin{pmatrix} \cos\alpha \\ 0 \\ -\sin\alpha \end{pmatrix} \text{ and } \vec{e}_{SH} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{51}$$

with $\alpha = \frac{\sqrt{(x-x_0)^2 + (y-y_0)^2}}{d}$

For the reflected wave, we have:

$$\vec{e}_P = \begin{pmatrix} \sin\alpha \\ 0 \\ -\cos\alpha \end{pmatrix}, \vec{e}_{SV} = \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix} \text{ and } \vec{e}_{SH} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \tag{52}$$

where α is computed with the RayTracing code.

In the $(\vec{e}_P, \vec{e}_{SV}, \vec{e}_{SH})$ coordinate system, we have for the direct wave:

$$\vec{e}_R = \begin{pmatrix} \sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}, \vec{e}_T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \vec{e}_Z = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \\ 0 \end{pmatrix} \tag{53}$$

For the reflected wave, we have:

$$\vec{e}_R = \begin{pmatrix} \sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}, \vec{e}_T = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \text{ and } \vec{e}_Z = \begin{pmatrix} -\cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} \tag{54}$$

We compute the seismic moment M in the $(\vec{e}_X, \vec{e}_Y, \vec{e}_Z)$ coordinate system. We have:

$$M_{ij} = u_i \nu_j + u_j \nu_i \tag{55}$$

with:

$$\vec{u} = \begin{pmatrix} -\cos\delta\cos\phi \\ \cos\delta\sin\phi \\ \sin\delta \end{pmatrix} \text{ and } \vec{\nu} = \begin{pmatrix} \sin\delta\cos\phi \\ -\sin\delta\sin\phi \\ \cos\delta \end{pmatrix} \tag{56}$$

where ϕ is the strike of the subducting plate, and δ is the dip of the subducting plate.
We then compute the value of M in the $(\vec{e}_P, \vec{e}_{SV}, \vec{e}_{SH})$ coordinate system. We have:

$$\begin{pmatrix} M_{RR} & M_{RT} & M_{RZ} \\ M_{TR} & M_{TT} & M_{TZ} \\ M_{ZR} & M_{ZT} & M_{ZZ} \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} M_{XX} & M_{XY} & M_{XZ} \\ M_{YX} & M_{YY} & M_{YZ} \\ M_{ZX} & M_{ZY} & M_{ZZ} \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{57}$$

For the direct wave, we have:

$$\begin{pmatrix} M_{PP} & M_{PSV} & M_{PSH} \\ M_{SVP} & M_{SVSV} & M_{SVSH} \\ M_{SHP} & M_{SHSV} & M_{SHSH} \end{pmatrix} = \begin{pmatrix} \sin\alpha & 0 & \cos\alpha \\ \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} M_{RR} & M_{RT} & M_{RZ} \\ M_{TR} & M_{TT} & M_{TZ} \\ M_{ZR} & M_{ZT} & M_{ZZ} \end{pmatrix} \begin{pmatrix} \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \\ \cos\alpha & -\sin\alpha & 0 \end{pmatrix} \tag{58}$$

For the reflected wave, we have:

$$\begin{pmatrix} M_{PP} & M_{PSV} & M_{PSH} \\ M_{SVP} & M_{SVSV} & M_{SVSH} \\ M_{SHP} & M_{SHSV} & M_{SHSH} \end{pmatrix} = \begin{pmatrix} \sin\alpha & 0 & -\cos\alpha \\ \cos\alpha & 0 & \sin\alpha \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} M_{RR} & M_{RT} & M_{RZ} \\ M_{TR} & M_{TT} & M_{TZ} \\ M_{ZR} & M_{ZT} & M_{ZZ} \end{pmatrix} \begin{pmatrix} \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & -1 \\ -\cos\alpha & \sin\alpha & 0 \end{pmatrix} \tag{59}$$

In the $(\vec{e}_P, \vec{e}_{SV}, \vec{e}_{SH})$ coordinate system, we have:

$$\vec{\Gamma} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{60}$$

from Equation 9.13.1 of Pujol (2003), we have:

$$\begin{pmatrix} A_P \\ A_{SV} \\ A_{SH} \end{pmatrix} = \begin{pmatrix} M_{PP} \\ M_{SVP} \\ M_{SHP} \end{pmatrix} \tag{61}$$

Getting the reflection, conversion and transmission coefficients

We compute the reflection, conversion and transmission coefficients at the interface between two homogeneous media, following Aki and Richards (2002, ch. 5.2).

SH-wave

We have $u_x = 0$, $u_z = 0$ and $\frac{\partial}{\partial y} = 0$. Thus the wave equations become:

$$\begin{aligned} \rho\frac{\partial^2 u_y}{\partial t^2} &= \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} \\ \sigma_{xy} &= \mu\frac{\partial u_y}{\partial x} \\ \sigma_{yz} &= \mu\frac{\partial u_y}{\partial z} \\ \sigma_{xx} &= \sigma_{yy} = \sigma_{zz} = \sigma_{xz} = 0 \end{aligned} \tag{62}$$

The incident SH-wave is of the form:

$$u_y(t) = \acute{S}_1 exp(i\omega(t - p_{\beta_1}x - q_{\beta_1}z)) \tag{63}$$

From Snell's law, we have $\frac{sinj_1}{\beta_1} = \frac{sinj_2}{\beta_2}$
At $z = 0$, we have $u_y(z^+) = u_y(z^-)$, and $\sigma_{yz}(z^+) = \sigma_{yz}(z^-)$ thus:

$$\begin{aligned} &\acute{S}_1 + \grave{S}_1 = \acute{S}_2 \\ &-\mu_1 i\omega q_{\beta_1}\acute{S}_1 + \mu_1 i\omega q_{\beta_1}\grave{S}_1 = -\mu_2 i\omega q_{\beta_2}\acute{S}_2 \end{aligned} \tag{64}$$

Therefore, using $q_{\beta_1} = \frac{cosj_1}{\beta_1}$ and $q_{\beta_2} = \frac{cosj_2}{\beta_2}$, we find:

$$
\begin{aligned}
\grave{S}_1 &= \acute{S}_1 \frac{\rho_1\beta_1 cosj_1 - \rho_2\beta_2 cosj_2}{\rho_1\beta_1 cosj_1 + \rho_2\beta_2 cosj_2} \\
\acute{S}_2 &= \acute{S}_1 \frac{2\rho_1\beta_1 cosj_1}{\rho_1\beta_1 cosj_1 + \rho_2\beta_2 cosj_2}
\end{aligned} \tag{65}
$$

0.1.4 P-wave

We have $u_y = 0$ and $\frac{\partial}{\partial y} = 0$. Thus the wave equations become:

$$
\begin{aligned}
\rho\frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} \\
\rho\frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} \\
\sigma_{xx} &= (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z} \\
\sigma_{yy} &= \lambda(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}) \\
\sigma_{zz} &= \lambda\frac{\partial u_x}{\partial x} + (\lambda + 2\mu)\frac{\partial u_z}{\partial z} \\
\sigma_{xz} &= \mu(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}) \\
\sigma_{xy} &= \sigma_{yz} = 0
\end{aligned} \tag{66}
$$

The incident P-wave is of the form:

$$
\begin{aligned}
u_x(t) &= \acute{P}_1 sini_1 exp(i\omega(t - p_{\alpha_1}x - q_{\alpha_1}z)) \\
u_z(t) &= \acute{P}_1 cosi_1 exp(i\omega(t - p_{\alpha_1}x - q_{\alpha_1}z))
\end{aligned} \tag{67}
$$

From Snell's law, we have $\frac{sini_1}{\alpha_1} = \frac{sini_2}{\alpha_2} = \frac{sinj_1}{\beta_1} = \frac{sinj_2}{\beta_2}$
At $z = 0$, we have $u_x(z^+) = u_x(z^-)$, $u_z(z^+) = u_z(z^-)$, $\sigma_{xz}(z^+) = \sigma_{xz}(z^-)$ and $\sigma_{zz}(z^+) = \sigma_{zz}(z^-)$ thus:

$$
\begin{aligned}
\acute{P}_1 sini_1 + \grave{P}_1 sini_1 + \grave{S}_1 cosj_1 &= \acute{P}_2 sini_2 + \acute{S}_2 cosj_2 \\
-\acute{P}_1 cosi_1 + \grave{P}_1 cosi_1 - \grave{S}_1 sinj_1 &= -\acute{P}_2 cosi_2 + \acute{S}_2 sinj_2 \\
\mu_1(-q\alpha_1\acute{P}_1 sini_1 + q_{\alpha_1}\grave{P}_1 sini_1 + q_{\beta_1}\grave{S}_1 cosj_1) + \mu_1(-p_{\alpha_1}\acute{P}_1 cosi_1 + p_{\alpha_1}\grave{P}_1 cosi_1 - p_{\beta_1}\grave{S}_1 sinj_1) &= \\
\mu_2(-q_{\alpha_2}\acute{P}_2 sini_2 - q_{\beta_2}\acute{S}_2 cosj_2) + \mu_2(-p_{\alpha_2}\acute{P}_2 cosi_2 + p_{\beta_2}\acute{S}_2 sinj_2) & \\
\lambda_1(p_{\alpha_1}\acute{P}_1 sini_1 + p_{\alpha_1}\grave{P}_1 sini_1 + p_{\beta_1}\grave{S}_1 cosj_1) + (\lambda_1 + 2\mu_1)(q_{\alpha_1}\acute{P}_1 cosi_1 + q_{\alpha_1}\grave{P}_1 cosi_1 - q_{\beta_1}\grave{S}_1 sinj_1) &= \\
\lambda_2(p_{\alpha_2}\acute{P}_2 sini_2 + p_{\beta_2}\acute{S}_2 cosj_2) + (\lambda_2 + 2\mu_2)(q_{\alpha_2}\acute{P}_2 cosi_2 - q_{\beta_2}\acute{S}_2 sinj_2) &
\end{aligned} \tag{68}
$$

that is:

$$
\begin{aligned}
-\acute{P}_2 sini_2 - \acute{S}_2 cosj_2 + \grave{P}_1 sini_1 + \grave{S}_1 cosj_1 &= -\acute{P}_1 sini_1 \\
\acute{P}_2 cosi_2 - \acute{S}_2 sinj_2 + \grave{P}_1 cosi_1 - \grave{S}_1 sinj_1 &= \acute{P}_1 cosi_1 \\
\mu_2 q_{\alpha_2}\acute{P}_2 sini_2 + \mu_2 p_{\alpha_2}\acute{P}_2 cosi_2 + \mu_2 q_{\beta_2}\acute{S}_2 cosj_2 - \mu_2 p_{\beta_2}\acute{S}_2 sinj_2 & \\
+\mu_1 q\alpha_1\grave{P}_1 sini_1 + \mu_1 p_{\alpha_1}\grave{P}_1 cosi_1 + \mu_1 q_{\beta_1}\grave{S}_1 cosj_1 - \mu_1 p_{\beta_1}\grave{S}_1 sinj_1 &= \\
\mu_1 q_{\alpha_1}\acute{P}_1 sini_1 + \mu_1 p_{\alpha_1}\acute{P}_1 cosi_1 & \\
-\lambda_2 p_{\alpha_2}\acute{P}_2 sini_2 - (\lambda_2 + 2\mu_2)q_{\alpha_2}\acute{P}_2 cosi_2 - \lambda_2 p_{\beta_2}\acute{S}_2 cosj_2 + (\lambda_2 + 2\mu_2)q_{\beta_2}\acute{S}_2 sinj_2 & \\
+\lambda_1 p_{\alpha_1}\grave{P}_1 sini_1 + (\lambda_1 + 2\mu_1)q_{\alpha_1}\grave{P}_1 cosi_1 + \lambda_1 p_{\beta_1}\grave{S}_1 cosj_1 - (\lambda_1 + 2\mu_1)q_{\beta_1}\grave{S}_1 sinj_1 &= \\
-\lambda_1 p_{\alpha_1}\acute{P}_1 sini_1 - (\lambda_1 + 2\mu_1)q_{\alpha_1}\acute{P}_1 cosi_1 &
\end{aligned} \tag{69}
$$

that is:

$$
\begin{aligned}
-\acute{P}_2 sini_2 - \acute{S}_2 cosj_2 + \grave{P}_1 sini_1 + \grave{S}_1 cosj_1 = -\acute{P}_1 sini_1 \\
\acute{P}_2 cosi_2 - \acute{S}_2 sinj_2 + \grave{P}_1 cosi_1 - \grave{S}_1 sinj_1 = \acute{P}_1 cosi_1 \\
(\mu_2 q_{\alpha_2} sini_2 + \mu_2 p_{\alpha_2} cosi_2)\acute{P}_2 + (\mu_2 q_{\beta_2} cosj_2 - \mu_2 p_{\beta_2} sinj_2)\acute{S}_2 \\
+(\mu_1 q_{\alpha_1} sini_1 + \mu_1 p_{\alpha_1} cosi_1)\grave{P}_1 + (\mu_1 q_{\beta_1} cosj_1 - \mu_1 p_{\beta_1} sinj_1)\grave{S}_1 = \\
(\mu_1 q_{\alpha_1} sini_1 + \mu_1 p_{\alpha_1} cosi_1)\acute{P}_1 \\
-(\lambda_2 p_{\alpha_2} sini_2 + (\lambda_2 + 2\mu_2) q_{\alpha_2} cosi_2)\acute{P}_2 - (\lambda_2 p_{\beta_2} cosj_2 - (\lambda_2 + 2\mu_2) q_{\beta_2} sinj_2)\acute{S}_2 \\
+(\lambda_1 p_{\alpha_1} sini_1 + (\lambda_1 + 2\mu_1) q_{\alpha_1} cosi_1)\grave{P}_1 + (\lambda_1 p_{\beta_1} cosj_1 - (\lambda_1 + 2\mu_1) q_{\beta_1} sinj_1)\grave{S}_1 = \\
-(\lambda_1 p_{\alpha_1} sini_1 + (\lambda_1 + 2\mu_1) q_{\alpha_1} cosi_1)\acute{P}_1
\end{aligned}
\tag{70}
$$

that is:

$$
\begin{aligned}
-\alpha_2 p_{\alpha_2}\acute{P}_2 - cosj_2\acute{S}_2 + \alpha_1 p_{\alpha_1}\grave{P}_1 + cosj_1\grave{S}_1 = -\alpha_1 p_{\alpha_1}\acute{P}_1 \\
cosi_2\acute{P}_2 - \beta_2 p_{\beta_2}\acute{S}_2 + cosi_1\grave{P}_1 - \beta_1 p_{\beta_1}\grave{S}_1 = cosi_1\acute{P}_1 \\
(2\rho_2\beta_2^2 p_{\alpha_2} cosi_2)\acute{P}_2 + \rho_2\beta_2(1 - 2\beta_2^2 p_{\beta_2}^2)\acute{S}_2 \\
+(2\rho_1\beta_1^2 p_{\alpha_1} cosi_1)\grave{P}_1 + \rho_1\beta_1(1 - 2\beta_1^2 p_{\beta_1}^2)\grave{S}_1 = \\
(2\rho_1\beta_1^2 p_{\alpha_1} cosi_1)\acute{P}_1 \\
-\rho_2\alpha_2(1 - 2\beta_2^2 p_{\alpha_2}^2)\acute{P}_2 + 2\rho_2\beta_2^2 p_{\beta_2} cosj_2\acute{S}_2 \\
+\rho_1\alpha_1(1 - 2\beta_1^2 p_{\alpha_1}^2)\grave{P}_1 - 2\rho_1\beta_1^2 p_{\beta_1} cosj_1\grave{S}_1 = \\
-\rho_1\alpha_1(1 - 2\beta_1^2 p_{\alpha_1}^2)\acute{P}_1
\end{aligned}
\tag{71}
$$

that is:

$$
M \begin{pmatrix} \acute{P}_2 \\ \acute{S}_2 \\ \grave{P}_1 \\ \grave{S}_1 \end{pmatrix} = N \begin{pmatrix} 0 \\ 0 \\ \acute{P}_1 \\ 0 \end{pmatrix} \tag{72}
$$

with:

$$
M = \begin{pmatrix}
-\alpha_2 p_{\alpha_2} & -cosj_2 & \alpha_1 p_{\alpha_1} & cosj_1 \\
cosi_2 & -\beta_2 p_{\beta_2} & cosi_1 & -\beta_1 p_{\beta_1} \\
2\rho_2\beta_2^2 p_{\alpha_2} cosi_2 & \rho_2\beta_2(1 - 2\beta_2^2 p_{\beta_2}^2) & 2\rho_1\beta_1^2 p_{\alpha_1} cosi_1 & \rho_1\beta_1(1 - 2\beta_1^2 p_{\beta_1}^2) \\
-\rho_2\alpha_2(1 - 2\beta_2^2 p_{\alpha_2}^2) & 2\rho_2\beta_2^2 p_{\beta_2} cosj_2 & \rho_1\alpha_1(1 - 2\beta_1^2 p_{\alpha_1}^2) & -2\rho_1\beta_1^2 p_{\beta_1} cosj_1
\end{pmatrix} \tag{73}
$$

and:

$$
N = \begin{pmatrix}
0 & 0 & -\alpha_1 p_{\alpha_1} & 0 \\
0 & 0 & cosi_1 & 0 \\
0 & 0 & 2\rho_1\beta_1^2 p_{\alpha_1} cosi_1 & 0 \\
0 & 0 & -\rho_1\alpha_1(1 - 2\beta_1^2 p_{\alpha_1}^2) & 0
\end{pmatrix} \tag{74}
$$

SV-wave

We have $u_y = 0$ and $\frac{\partial}{\partial y} = 0$. Thus the wave equations become:

$$\begin{aligned}
\rho\frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} \\
\rho\frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} \\
\sigma_{xx} &= (\lambda+2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z} \\
\sigma_{yy} &= \lambda(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}) \\
\sigma_{zz} &= \lambda\frac{\partial u_x}{\partial x} + (\lambda+2\mu)\frac{\partial u_z}{\partial z} \\
\sigma_{xz} &= \mu(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}) \\
\sigma_{xy} &= \sigma_{yz} = 0
\end{aligned} \tag{75}$$

The incident S-wave is of the form:

$$\begin{aligned}
u_x(t) &= \acute{S}_1 cosj_1 exp(i\omega(t - p_{\beta_1}x - q_{\beta_1}z)) \\
u_z(t) &= -\acute{S}_1 sinj_1 exp(i\omega(t - p_{\beta_1}x - q_{\beta_1}z))
\end{aligned} \tag{76}$$

From Snell's law, we have $\frac{sini_1}{\alpha_1} = \frac{sini_2}{\alpha_2} = \frac{sinj_1}{\beta_1} = \frac{sinj_2}{\beta_2}$
At $z=0$, we have $u_x(z^+) = u_x(z^-)$, $u_z(z^+) = u_z(z^-)$, $\sigma_{xz}(z^+) = \sigma_{xz}(z^-)$ and $\sigma_{zz}(z^+) = \sigma_{zz}(z^-)$ thus:

$$\begin{aligned}
\acute{S}_1 cosj_1 + \grave{P}_1 sini_1 + \grave{S}_1 cosj_1 &= \acute{P}_2 sini_2 + \acute{S}_2 cosj_2 \\
\acute{S}_1 sinj_1 + \grave{P}_1 cosi_1 - \grave{S}_1 sinj_1 &= -\acute{P}_2 cosi_2 + \acute{S}_2 sinj_2 \\
\mu_1(-q_{\beta_1}\acute{S}_1 cosj_1 + q_{\alpha_1}\grave{P}_1 sini_1 + q_{\beta_1}\grave{S}_1 cosj_1) + \mu_1(p_{\beta_1}\acute{S}_1 sinj_1 + p_{\alpha_1}\grave{P}_1 cosi_1 - p_{\beta_1}\grave{S}_1 sinj_1) &= \\
\mu_2(-q_{\alpha_2}\acute{P}_2 sini_2 - q_{\beta_2}\acute{S}_2 cosj_2) + \mu_2(-p_{\alpha_2}\acute{P}_2 cosi_2 + p_{\beta_2}\acute{S}_2 sinj_2) & \\
\lambda_1(p_{\beta_1}\acute{S}_1 cosj_1 + p_{\alpha_1}\grave{P}_1 sini_1 + p_{\beta_1}\grave{S}_1 cosj_1) + (\lambda_1+2\mu_1)(-q_{\beta_1}\acute{S}_1 sinj_1 + q_{\alpha_1}\grave{P}_1 cosi_1 - q_{\beta_1}\grave{S}_1 sinj_1) &= \\
\lambda_2(p_{\alpha_2}\acute{P}_2 sini_2 + p_{\beta_2}\acute{S}_2 cosj_2) + (\lambda_2+2\mu_2)(q_{\alpha_2}\acute{P}_2 cosi_2 - q_{\beta_2}\acute{S}_2 sinj_2) &
\end{aligned} \tag{77}$$

that is:

$$\begin{aligned}
-\acute{P}_2 sini_2 - \acute{S}_2 cosj_2 + \grave{P}_1 sini_1 + \grave{S}_1 cosj_1 &= -\acute{S}_1 cosj_1 \\
\acute{P}_2 cosi_2 - \acute{S}_2 sinj_2 + \grave{P}_1 cosi_1 - \grave{S}_1 sinj_1 &= -\acute{S}_1 sinj_1 \\
\mu_2 q_{\alpha_2}\acute{P}_2 sini_2 + \mu_2 p_{\alpha_2}\acute{P}_2 cosi_2 + \mu_2 q_{\beta_2}\acute{S}_2 cosj_2 - \mu_2 p_{\beta_2}\acute{S}_2 sinj_2 & \\
+\mu_1 q\alpha_1\grave{P}_1 sini_1 + \mu_1 p_{\alpha_1}\grave{P}_1 cosi_1 + \mu_1 q_{\beta_1}\grave{S}_1 cosj_1 - \mu_1 p_{\beta_1}\grave{S}_1 sinj_1 &= \\
\mu_1 q_{\beta_1}\acute{S}_1 cosj_1 - \mu_1 p_{\beta_1}\acute{S}_1 sinj_1 & \\
-\lambda_2 p_{\alpha_2}\acute{P}_2 sini_2 - (\lambda_2+2\mu_2)q_{\alpha_2}\acute{P}_2 cosi_2 - \lambda_2 p_{\beta_2}\acute{S}_2 cosj_2 + (\lambda_2+2\mu_2)q_{\beta_2}\acute{S}_2 sinj_2 & \\
+\lambda_1 p_{\alpha_1}\grave{P}_1 sini_1 + (\lambda_1+2\mu_1)q_{\alpha_1}\grave{P}_1 cosi_1 + \lambda_1 p_{\beta_1}\grave{S}_1 cosj_1 - (\lambda_1+2\mu_1)q_{\beta_1}\grave{S}_1 sinj_1 &= \\
-\lambda_1 p_{\beta_1}\acute{S}_1 cosj_1 + (\lambda_1+2\mu_1)q_{\beta_1}\acute{S}_1 sinj_1 &
\end{aligned} \tag{78}$$

that is:

$$\begin{aligned}
-\acute{P}_2 sini_2 - \acute{S}_2 cosj_2 + \grave{P}_1 sini_1 + \grave{S}_1 cosj_1 &= -\acute{S}_1 cosj_1 \\
\acute{P}_2 cosi_2 - \acute{S}_2 sinj_2 + \grave{P}_1 cosi_1 - \grave{S}_1 sinj_1 &= -\acute{S}_1 sinj_1 \\
(\mu_2 q_{\alpha_2} sini_2 + \mu_2 p_{\alpha_2} cosi_2)\acute{P}_2 + (\mu_2 q_{\beta_2} cosj_2 - \mu_2 p_{\beta_2} sinj_2)\acute{S}_2 & \\
+(\mu_1 q_{\alpha_1} sini_1 + \mu_1 p_{\alpha_1} cosi_1)\grave{P}_1 + (\mu_1 q_{\beta_1} cosj_1 - \mu_1 p_{\beta_1} sinj_1)\grave{S}_1 &= \\
(\mu_1 q_{\beta_1} cosj_1 - \mu_1 p_{\beta_1} sinj_1)\acute{S}_1 & \\
-(\lambda_2 p_{\alpha_2} sini_2 + (\lambda_2+2\mu_2)q_{\alpha_2} cosi_2)\acute{P}_2 - (\lambda_2 p_{\beta_2} cosj_2 - (\lambda_2+2\mu_2)q_{\beta_2} sinj_2)\acute{S}_2 & \\
+(\lambda_1 p_{\alpha_1} sini_1 + (\lambda_1+2\mu_1)q_{\alpha_1} cosi_1)\grave{P}_1 + (\lambda_1 p_{\beta_1} cosj_1 - (\lambda_1+2\mu_1)q_{\beta_1} sinj_1)\grave{S}_1 &= \\
(-\lambda_1 p_{\beta_1} cosj_1 + (\lambda_1+2\mu_1)q_{\beta_1} sinj_1)\acute{S}_1 &
\end{aligned} \tag{79}$$

that is:

$$
\begin{aligned}
-\alpha_2 p_{\alpha_2}\acute{P}_2 - cosj_2\acute{S}_2 + \alpha_1 p_{\alpha_1}\grave{P}_1 + cosj_1\grave{S}_1 &= -cosj_1\acute{S}_1 \\
cosi_2\acute{P}_2 - \beta_2 p_{\beta_2}\acute{S}_2 + cosi_1\grave{P}_1 - \beta_1 p_{\beta_1}\grave{S}_1 &= -\beta_1 p_{\beta_1}\acute{S}_1 \\
(2\rho_2\beta_2^2 p_{\alpha_2} cosi_2)\acute{P}_2 + \rho_2\beta_2(1-2\beta_2^2 p_{\beta_2}^2)\acute{S}_2 & \\
+(2\rho_1\beta_1^2 p_{\alpha_1} cosi_1)\grave{P}_1 + \rho_1\beta_1(1-2\beta_1^2 p_{\beta_1}^2)\grave{S}_1 &= \\
\rho_1\beta_1(1-2\beta_1^2 p_{\beta_1}^2)\acute{S}_1 & \\
-\rho_2\alpha_2(1-2\beta_2^2 p_{\alpha_2}^2)\acute{P}_2 + 2\rho_2\beta_2^2 p_{\beta_2} cosj_2\acute{S}_2 & \\
+\rho_1\alpha_1(1-2\beta_1^2 p_{\alpha_1}^2)\grave{P}_1 - 2\rho_1\beta_1^2 p_{\beta_1} cosj_1\grave{S}_1 &= \\
2\rho_1\beta_1^2 p_{\beta_1} cosj_1\acute{S}_1 &
\end{aligned}
\tag{80}
$$

that is:

$$
M \begin{pmatrix} \acute{P}_2 \\ \acute{S}_2 \\ \grave{P}_1 \\ \grave{S}_1 \end{pmatrix} = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ \acute{S}_1 \end{pmatrix} \tag{81}
$$

with:

$$
M = \begin{pmatrix}
-\alpha_2 p_{\alpha_2} & -cosj_2 & \alpha_1 p_{\alpha_1} & cosj_1 \\
cosi_2 & -\beta_2 p_{\beta_2} & cosi_1 & -\beta_1 p_{\beta_1} \\
2\rho_2\beta_2^2 p_{\alpha_2} cosi_2 & \rho_2\beta_2(1-2\beta_2^2 p_{\beta_2}^2) & 2\rho_1\beta_1^2 p_{\alpha_1} cosi_1 & \rho_1\beta_1(1-2\beta_1^2 p_{\beta_1}^2) \\
-\rho_2\alpha_2(1-2\beta_2^2 p_{\alpha_2}^2) & 2\rho_2\beta_2^2 p_{\beta_2} cosj_2 & \rho_1\alpha_1(1-2\beta_1^2 p_{\alpha_1}^2) & -2\rho_1\beta_1^2 p_{\beta_1} cosj_1
\end{pmatrix} \tag{82}
$$

and:

$$
N = \begin{pmatrix}
0 & 0 & 0 & -cosj_1 \\
0 & 0 & 0 & -\beta_1 p_{\beta_1} \\
0 & 0 & 0 & \rho_1\beta_1(1-2\beta_1^2 p_{\beta_1}^2) \\
0 & 0 & 0 & 2\rho_1\beta_1^2 p_{\beta_1} cosj_1
\end{pmatrix} \tag{83}
$$

0.1.5 Ray tracing

We solve the eikonal and the transport equations following Čeverný (2001, ch. 3.1).

Eikonal equation

Following Čeverný (2001, ch. 2.4), the eikonal equation is:

$$
\nabla T . \nabla T = \frac{1}{V^2} \tag{84}
$$

with $V = \alpha$ or $V = \beta$. Using the Hamiltonian, it can also be written as:

$$
\mathcal{H}(x_i, p_i) = \frac{1}{2}(p_i^2 - \frac{1}{V^2}) = 0 \tag{85}
$$

where $p_i = \frac{\partial T}{\partial x_i}$.
We define the auxiliary variable σ by:

$$
\frac{dx_i}{d\sigma} = \frac{\partial \mathcal{H}}{\partial p_i} \text{ and } \frac{dp_i}{d\sigma} = -\frac{\partial \mathcal{H}}{\partial x_i} \tag{86}
$$

We get:

$$
\frac{dT}{d\sigma} = \frac{\partial T}{\partial x_i}\frac{\partial x_i}{\partial \sigma} = p_i \frac{\partial \mathcal{H}}{\partial p_i} = \frac{1}{V^2} \tag{87}
$$

thus we have:

$$
T = T_0 + \frac{1}{V^2}\sigma \text{ and } \sigma = V^2(T - T_0) \tag{88}
$$

Constant velocity We have:

$$\frac{dp_i}{d\sigma} = -\frac{\partial \mathcal{H}}{\partial x_i} = \frac{1}{2}\frac{\partial}{\partial x_i}(\frac{1}{V^2}) = -\frac{1}{V^3}\frac{\partial V}{\partial x_i} = 0 \tag{89}$$

thus:

$$\begin{aligned} p_1 &= p_{10} \\ p_2 &= p_{20} \\ p_3 &= p_{30} \end{aligned} \tag{90}$$

We have:

$$x_i = x_{i0} + \frac{\partial \mathcal{H}}{\partial p_i}\sigma = x_{i0} + p_i\sigma \tag{91}$$

thus:

$$\begin{aligned} x_1 &= x_{10} + p_{10}V^2(T - T_0) \\ x_2 &= x_{20} + p_{20}V^2(T - T_0) \\ x_3 &= x_{30} + p_{30}V^2(T - T_0) \end{aligned} \tag{92}$$

Constant gradient of velocity We write the velocity as $V = az + b$.
We have:

$$\frac{dp_1}{d\sigma} = 0, \; \frac{dp_2}{d\sigma} = 0 \text{ and } \frac{dp_3}{d\sigma} = -\frac{1}{V^3}\frac{\partial V}{\partial z} = -\frac{a}{(az+b)^3} \tag{93}$$

thus:

$$\begin{aligned} p_1 &= p_{10} \\ p_2 &= p_{20} \\ p_3 &= p_{30} - \frac{a}{(az+b)^3}\sigma = p_{30} - \frac{a}{az+b}(T - T_0) \end{aligned} \tag{94}$$

We have:

$$\begin{aligned} x_1 &= x_{10} + p_{10}\sigma \\ x_2 &= x_{20} + p_{20}\sigma \\ x_3 &= x_{30} + p_{30}\sigma - \frac{1}{2}\frac{a}{(az+b)^3}\sigma^2 \end{aligned} \tag{95}$$

thus:

$$\begin{aligned} x_1 &= x_{10} + p_{10}(az+b)^2(T - T_0) \\ x_2 &= x_{20} + p_{20}(az+b)^2(T - T_0) \\ x_3 &= x_{30} + p_{30}(az+b)^2(T - T_0) - \frac{1}{2}a(az+b)(T - T_0)^2 \end{aligned} \tag{96}$$

Transport equation

Following Červený (2001, ch. 2.4), the transport equation is:

$$2\nabla T.\nabla(\sqrt{\rho V^2}A) + \sqrt{\rho V^2}A\nabla^2 T = 0 \tag{97}$$

with $V = \alpha$ or $V = \beta$ and A is the amplitude of the P-wave or one of the two components of the S-wave.

Constant velocity We have $(\nabla T)_i = p_i = p_{i0}$ thus $\nabla^2 T = 0$ and the wave equation becomes:

$$2\nabla T.\nabla(\sqrt{\rho V^2}A) = 0 \tag{98}$$

As ρ and V are constant, we get:

$$\nabla T.\nabla A = p_i \frac{\partial A}{\partial x_i} = 0 \tag{99}$$

However, we have:

$$\frac{\partial A}{\partial \sigma} = \frac{\partial A}{\partial x_i}\frac{\partial x_i}{\partial \sigma} = \frac{\partial A}{\partial x_i}\frac{\partial \mathcal{H}}{\partial p_i} = \frac{\partial A}{\partial x_i} p_i \tag{100}$$

Thus:

$$\frac{\partial A}{\partial \sigma} = 0 \text{ that is } A = A_0 \tag{101}$$

Constant gradient of velocity We have:

$$\nabla^2 T = \frac{a^2}{(az+b)^2}(T - T_0) \tag{102}$$

If we assume constant density, we get the transport equation:

$$2A\nabla T.\nabla V + 2V\nabla T.\nabla A + A\frac{a^2}{az+b}(T - T_0) = 0 \tag{103}$$