Algorithmic Operation Research

Theodora Panagea - 1115201400135
Anna-Aikaterini Kavvada - 1115201500050

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1

Let $C \subseteq \mathbb{R}^n$ be a convex set with $x_1, ...x_k \in C$ and let θ_1 , . . . ,$\theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0$ and $\theta_1 + ... + \theta_k = 1$. Show that $\theta_1 x_1 + ... + \theta_k x_k \in C$.

Solution:

We will start by proving the question for $i = 3$ which is trivial. So, suppose $i = 3$ and we also have $x_1, x_2, x_3 \in C$ and $\theta_1 + \theta_2 + \theta_3 = 1$ with $\theta_1, \theta_2, \theta_3 \geq 0$. We will show that

$$\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \in C \quad (1)$$

There is at least one of the θ_i that does not equal 1. Now, assume, without loss of generality that $\theta_i \neq 1$, where

$$\theta_1 + \theta_2 + \theta_3 = 1 \iff$$

$$\theta_2 + \theta_3 = 1 - \theta_1 \iff \quad ,(1 - \theta_1 > 0)$$

$$\frac{\theta_2 + \theta_3}{1 - \theta_1} = 1 \Rightarrow$$

$$1 = m_1 + m_2 = \frac{\theta_2 + \theta_3}{1 - \theta_1} \Rightarrow$$

$$m_1 = \frac{\theta_2}{1 - \theta_1} \quad (2) \quad and \quad m_2 = \frac{\theta_3}{1 - \theta_1} \quad (3) \Rightarrow$$

$$\theta_2 = m_1(1 - \theta_1) \quad and \quad \theta_3 = m_2(1 - \theta_2)$$

Thus, from $(1), (2) \quad and \quad (3)$, we have

$$\theta_1 x_1 + (1 - \theta_1)(m_1 x_1 + m_2 x_3) \quad (4)$$

Since C is convex and $x_2, x_3 \in C$, we conclude that $(m_1 x_1 + m_2 x_3) \in C$ as "extensions". Thus, since both points are in the convex C, then the function is in C as well.

Now that we have proven that the function belongs to set C for $i = 3$, we can prove with the same steps that it is also true for the general example $i = k$ where

$$\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 + \quad ... \quad + \theta_k \in C \quad (5)$$

. Following the method above, we find that

$$\theta_1 + \theta_2 + ... + \theta_k = 1 \iff$$

$$\theta_2 + \theta_3 + \quad ... \quad + \theta_k = 1 - \theta_1 \iff \quad , (1 - \theta_1 > 0)$$

$$\frac{\theta_2 + \theta_3 + \quad ... \quad + \theta_k}{1 - \theta_1} = 1 \Rightarrow$$

$$m_1 = \frac{\theta_2}{1 - \theta_1} \quad (6) \quad ... \quad m_{k-1} = \frac{\theta_k}{1 - \theta_1} \quad (7)$$

. As in the proof for $i = 3$ it is easy to see that from the $(5), (6) \quad and \quad (7)$ we have

$$\theta_1 x_1 + (1 - \theta_1)(m_1 x_1 + m_2 x_3 + \quad ... \quad + m_{k-1} x_k) \quad (8)$$

. Thus, we show that since C is convex and all points $x_1, x_2, ..., x_k \in C$ then $(m_1 x_1 + m_2 x_3 + \quad ... \quad + m_{k-1} x_k) \in C$. So the function given is proved to be convex and belong to set C.

2

Show that a set is convex if and only if its intersection with any line is convex.

Solution:

We have a convex set S. Suppose we have a convex set A.

$\Rightarrow$

Let's take two points $p_1, p_2 \in S \cup A$, and point p is on the line segment of $p_1 p_2$.

- $p \in S$ because S is convex
- $p \in A$ because A is convex

So $p \in S \cap A$

Since the two points p_1 and p_2 are arbitrary, we can assume that $S \cap A$ as well. (even in the case where $S \cap A$ is null it is convex)

$\Leftarrow$

Let $L \cap C$ which is convex for all lines in L. Assume that C is not convex, thus $\exists x, y \in C$, with $\lambda \in [0, 1]$, where

$$\lambda x + (1 - \lambda) y \notin C$$

. But, the line $ax + (1 - a)y, a \in \mathbb{R}$ intersects with C, thus, the intersection is convex.

The line segment between x, y belongs to the intersection $\lambda x + (1 - \lambda)y$, thus belonging to C as well, which negates our previous assumption, proving that C is indeed convex.

3

Show that a set is affine if and only if its intersection with any line is affine.

Solution:

As proved above in question (3), the intersection of convex set S with any line is convex.

Let's take two points x_1 and x_2, where $x_1, x_2 \in S$. From the proof above, we know that the line between these two points is convex. Thus, the convex combination of x_1 and x_2 belongs to the intersection and expectedly to S. To conclude, since the line belongs to convex set S, the set is affine.

4

A set C is midpoint convex, if whenever two points $a, b \in C$, the average or midpoint $(a+b)/2$ is in C. Obviously, a convex set is midpoint convex. Prove that if C is closed and midpoint convex, then C is convex.

Solution:

Let $x, y \in C$ and assume that z is a point on the line segment between x and y. We construct a sequence $\mathrm{z}_i :=(\mathrm{x}_i + y_i)/2$, where $\mathrm{x}_0 = x, y_0 = y, and$

$$\mathrm{x}_{i+1} = \begin{cases} z_i & z_i \leq z \\ x_i & z_i > z \end{cases} \quad y_{i+1} = \begin{cases} z_i & z_i \leq z \\ y_i & z_i > z \end{cases} \qquad (0)$$

By this construction, z is always on the line segment between x_i and y_i, so we have $||z_i - z|| \leq ||x - y||2^{-i}$, which means z_i converges to z. By midpoint convexity of C, we have $z_i \in C$ for all i, and because C is closed we have $z \in C$. This is true for all z on the line segment between x and y, so C is convex.

5

Show that the convex hull of a set S is the intersection of all convex sets that contain S.(The same method can be used to show that the conic, or affine, or linear hull of a set S is the intersection of all conic sets, or affine sets, or subspaces that contain S.

Solution:

Let A be the set of all convex combinations of points in S. Since:

- $S \subseteq Conv(S)$
- $\forall x \in A$ is a convex combination of points in $Conv(S)$.

Hence, we have, as proven in question 2, that $A \subseteq Conv(S)$.
To prove the reverse inclusion, we show that A is a convex subspace containing S. Since $Conv(S)$ is the smallest possible convex subspace that contains S, it will show that $A \supseteq Conv(S)$.
We have that $S \subseteq A$, and we have to show that A is a convex subspace.
Assume two points $x_1, x_2 \in A$.
By definition there is:

- $y_1, ..., y_p, y_{p+1}, ..., y_q$
- $a_1, ..., a_p, a_{p+1}, ..., a_q$, such that
- $x_1 = \sum_{i=1}^{p} a_i y_i, \quad x_2 = \sum_{i=p+1}^{q} a_i y_i \quad$ and
- $\sum_{i=1}^{p} a_i = \sum_{i=p+1}^{q} a_i = 1$.

For any $\beta \in \mathbb{R}$, we have
$\overline{x} = \beta x_1 + (1-\beta)x_2 = \sum_{i=1}^{p} \beta a_i y_i + \sum_{i=p+1}^{q} (1-\beta) a_i y_i$,
with $\sum_{i=1}^{p} \beta a_i + \sum_{i=p+1}^{q} (1-\beta) a_i = 1$
. Thus, we conclude that $\overline{x}$ is a convex combination of points in S, and we have $\overline{x} \in A$, showing that A is a convex subset.
Now we proved both sides, showing that the convex hull of a set S is the intersection of all convex sets that contain S.

6

What is the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n : a^T x = b_1\}$ *and* $\{x \in \mathbb{R}^n : a^T x = b_2\}$*?*

Solution:

Let x_1 be a point in the first hyperplane, and consider the line L, that goes through x_1, in the direction of a. The equation of L is $x_1 + at, t \in \mathbb{R}$. Now we have to find the intersection of L with the second hyperplane:

$$a^T(x_1 + at) = b2 \quad \Leftrightarrow \quad t = \frac{b_2 - a^T x_1}{a^T a} \quad \Leftrightarrow \quad t = \frac{b_2 - b_1}{a^T a}$$

.

Therefore, let's call the intersection point x_2, which is equal to:

$$x_2 = x_1 + a\frac{b_2 - b_1}{a^T a}$$

.

Thus, we only have to find the distance between x_1 and x_2:

$$||x_1 - x_2|| = \frac{|b_2 - b_1|}{||a||^2}$$

.

7

Let a and b be distinct points in $\mathbb{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b is a halfspace.

Solution:

The euclidean norm is nonnegative, therefore:

$$
\begin{gathered}
||x-a||_2 \leq ||x-b||_2 \quad \Leftrightarrow \quad ||x-a||_2^2 \leq ||x-b||_2^2 \quad \Leftrightarrow \\
\Leftrightarrow \quad (x-a)^T(x-a) \leq (x-b)^T(x-b) \quad \Leftrightarrow \\
\Leftrightarrow \quad x^Tx - 2a^Tx + a^Ta \leq x^Tx - 2b^Tx + b^Tb \quad \Leftrightarrow \\
\Leftrightarrow \quad 2(b-a)^Tx \leq b^Tb - a^Ta
\end{gathered}
$$

Therefore, the set is indeed halfspace, if we substitute $c = 2(b-a)$ and $d = b^Tb - a^Ta$.