Algorithmic Operation Research
Homework 4

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Exercise 1

Consider the problem

$$\begin{aligned} &\textit{minimize} \qquad 2x_1 + 3|x_2 - 10| \\ &\textit{subject to} \quad |x_1 + 2| + |x_2| \leqslant 5 \end{aligned}$$

and reformulate it a linear programming problem.
Solution:
Our **goal** is to replace the absolute values with a variable y_i. The absolute values can be written as:

$$|x_2 - 10| = \max\{-x_2 + 10, x_2 - 10\} = y_1$$
$$|x_1 + 2| = \max\{-x_1 + 2, -x_1 - 2\} = y_2$$
$$|x_2| = \max\{x_2, -x_2\} = y_3$$

Furthermore, we also have to add some constraints for the absolute values that we replaced:

$$y_1 \geqslant x_2 - 10$$
$$y_1 \geqslant -x_2 + 10$$

In that case, the minimum possible value of y_1 is $|x_2 - 10|$. Similarly, for y_2, y_3:

$$
\begin{aligned}
y_2 &\geqslant x_1 + 2 \\
y_2 &\geqslant -x_1 - 2 \\
y_3 &\geqslant x_2 \\
y_3 &\geqslant -x_2
\end{aligned}
$$

To conclude, the problem reformulates as follows:

$$
\begin{array}{ll}
\text{minimize} & 2x_1 + 3y_1 \\
\text{subject to} & y_2 + y_3 \leqslant 5 \\
& y_1 \geqslant x_2 - 10 \\
& y_1 \geqslant -x_2 + 10 \\
& y_2 \geqslant x_1 + 2 \\
& y_2 \geqslant -x_1 - 2 \\
& y_3 \geqslant x_2 \\
& y_3 \geqslant -x_2
\end{array}
$$

Exercise 2

(Road Lighting) Consider a road divided in n segments that is illuminated by m lamps. Let p_j be the power of the j-th lamp. The illumination I_i of the i-th segment is assumed to be $\sum_{j=1}^{m} a_{ij}p_j$ where a_{ij} are known coefficients. Let I_i^* be the desired illumination of road i. We are interested in choosing the lamp powers p_j so that the illuminations I_i are close to the desired I_i^*. Provide a reasonable linear programming formulation of this problem.

Solution:

Our **goal** is to minimize the difference between the desired (optimal) value and I_i^* and the current illumination I_i the segment i of the road has.
Thus, we ask in a mathematical form, for:

$$\text{minimize} \sum_{i=1}^{n} |I_i^* - I_i|$$

where $I_i = \sum_{j=1}^{m} a_{ij}p_j$. So the cost function can be written as:

$$\text{minimize} \sum_{i=1}^{n} |I_i^* - \sum_{j=1}^{m} a_{ij}p_j|$$

From the above function, knowing that a_{ij} being coefficients, we conclude that p_j is our variable of the equation. That leads us to the following constraint:

$$p_j \geqslant 0$$

In conclusion, we end up with the following linear programming formulation:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} |I_i^* - I_i| \\ \text{subject to} \quad & p_j \geqslant 0 \end{aligned}$$

Exercise 3

Consider a school district with with I neighborhoods, J schools and G grades at each school. Each school j has capacity C_jg for grade g. In each neighborhood i, the student population of grade i is S_{ig}. Finally, the distance of school j from neighborhood i is d_{ij}. Formulate a linear programming problem whose objective is to assign all students to schools, while minimizing the total distance traveled by all students. (You may ignore the fact that numbers of students must be integer).

Solution:

Suppose x_{ijg} students from neighborhood i going to school j and being in g grade.

Goal: minimize the distance that the students have to travel, thus we want the min sum of the distance each student has to travel.

$$\text{minimize} \qquad \sum_{i \in I} \sum_{j \in J} \sum_{g \in G} x_{ijg} d_{ij} \tag{1}$$

Restrictions:

- All students from a neighborhood i being on grade g must get in a school.
$$S_{ig} = \sum_{j \in J} x_{ijg} \qquad \forall g \in G \text{ and } i \in I \tag{2}$$
- Each school can take only C_{jg} student for every grade.
$$\sum_{i \in I} x_{ijg} \leqslant C_{jg} \qquad \forall j \in J \text{ and } g \in G \tag{3}$$
- We can't have negative numbers.
$$x \geqslant 0 \tag{4}$$

Thus, we conclude to the following linear programming formulation:

$$\begin{aligned} \text{minimize} \ & \sum_{i \in I} \sum_{j \in J} \sum_{g \in G} x_{ijg} d_{ij} \\ \text{subject to} \ & \sum_{j \in J} x_{ijg} = S_{ig} \qquad \forall g \in G \text{ and } i \in I \\ & \sum_{i \in I} x_{ijg} \leqslant C_{jg} \qquad \forall j \in J \text{ and } g \in G \\ & x \geqslant 0 \end{aligned}$$

Exercise 4

Consider a set P described by linear inequality constraints

$$P = \{x \in \mathbb{R}^n : a_i' x \leqslant b_i, i = 1, \dots, m\}$$

A ball with center y and radius r is defined as all the set of all points within distance r from y. We are interested in finding a ball with the largest possible radius, which is entirely contained within the set P. Provide a linear programming formulation of this problem.

Solution: The optimal solution is to examine a point on the boundary of P. So, we want to maximize the shortest distance (i.e. the line segment) between y and the boundary point.

If we take the product of y and the vector a_i', then the distance and therefore, the problem would be:

$$\begin{array}{rll} \text{minimize} & \max_i\{-|a_i' y - b_i|\} & \\ \text{subject to} & a_i' y \leqslant b_i, & i = 1, \dots, m \end{array}$$