From Linear to Conic Duality

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Introduction

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Introduction

Linear Programming

- Specific computational techniques
- A number of general results, including those of (LP) Duality Theorem

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Introduction

Linear Duality $\rightarrow$ one of the most important techniques of Linear Programming

But...

Nonlinear cases $\rightarrow$ not applicable!

Solution: Extension of (LP) techniques to (CP).

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Introduction

Generic (LP) Problem:

$$\min_x \{c^T x | Ax \geq B\}$$

Generic (CP) Problem:

$$\min_x \{c^T x | Ax \geq_{\kappa} b\},$$

where **K** is a regular cone (convex, pointed, closed and with a nonempty interior).

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What is Conic Duality?

What is Conic Duality?

(LP): The need to get to a lower bound on the optimal value

c^*, where

$$c^* = \min_x \{c^T x | Ax \geq b\},$$

with constraint

$$\langle \lambda, A \rangle \equiv \lambda^T Ax \geq \lambda^T b \quad (Cons(\lambda))$$

and weight vectors

$$\lambda \geq 0$$

What is Conic Duality?

(LP) Dual:

$$\max\{b^T\lambda | \lambda \geq 0, A^T\lambda = c\},$$

After all, it is nothing but finding the best lower bound one can get in this fashion.

What is Conic Duality?

(CP): Following the same scheme of LP, with the following primal

$$\min\{c^T x | Ax \geq_\kappa b\}$$

(?)What are the "admissible" weight vectors

$$\langle \lambda, Ax \rangle \geq \langle \lambda, b \rangle$$

is a consequence of the vector inequality $Ax \geq_\kappa b$?

What is Conic Duality?

(?)What are the "admissible" weight vectors

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is a consequence of the vector inequality $Ax \geq_{\kappa} b$?

Answer: The same as to say what are the weight vectors λ such that

$$\forall \alpha \geq_{\kappa} 0 : \quad \langle \lambda, \alpha \rangle \geq 0$$

What is Conic Duality?

Observation:

When:

- x is a feasible solution to (CP)
- λ is an admissible weight vector, i.e., $\lambda \in K^*$ where K^* is dual cone of K

Then:

$$(A^*\lambda)^T x \equiv \langle \lambda, Ax \rangle \geq \langle \lambda, b \rangle$$

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What is Conic Duality?

For $(A^*\lambda)^T x \equiv \langle \lambda, Ax \rangle \geq \langle \lambda, b \rangle$, we need:

- $A^*\lambda = c$
- $c^T x = (A^*\lambda)^T x = \langle \lambda, Ax \rangle \geq \langle b, \lambda \rangle$

$\Rightarrow \langle b, \lambda \rangle$: Lower bound on the optimal value of (CP).

What is Conic Duality?

(CP) Dual: *The best bound is the optimal value in the problem:*

$$\max\{\langle b, \lambda\rangle | A^*\lambda = c, \lambda \geq_{\kappa^*} 0\}$$

which is the (CP) dual.

Conic Duality Theorem

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Conic Duality Theorem

To establish Conic Duality (CD), we need:

- Proof of symmetry
- Weak Duality
- Regular Duality
- Strong Duality

Conic Duality Theorem

- Proof of Symmetry

$$\text{(P)}\ \min\{c^T x | Ax \geq_\kappa b\}$$
$$\text{(D)}\ \max\{\langle b, \lambda\rangle | A^*\lambda = c, \lambda \geq_{\kappa^*} 0\}$$

The (D) can be reformed as:

$$\text{(D')}\ \min\{-\langle b, \lambda\rangle | A^*\lambda = c, \lambda \geq_{\kappa^*} 0\}$$

It's trivial that the dual of the (D') $\Rightarrow$ (P)

Conic Duality Theorem

- Weak Duality

Theorem

If (D) feasible, and if (P) subfeasible, then the subvalue of (P) is upper-bounded by the value of (D).
If (P) and (D) feasible, then the value of (P) is upper-bounded by the value of (D), and both are finite.

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Conic Duality Theorem

- Regular Duality

Theorem

The (D) is feasible and has finite value β if and only if the (P) is subfeasible and has finite subvalue γ. Also, $\beta = \gamma$.

The proof consists of applications of the Farkas lemma.

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Conic Duality Theorem

The Farkas Lemma

Lemma

Let $K \subseteq V$ be a closed convex cone, and $b \in W$. The system $Ax = b, x \in K$ is subfeasible if and only if every $y \in W$ with $A^T y \in K^*$ also satisfies $\langle y, b \rangle \geq 0$

Conic Duality Theorem

- Strong Duality

Theorem

If (P) is feasible, has finite value γ and has interior point $\tilde{x}$, then (D) is feasible and has finite value $\beta = \gamma$.

The proof uses the Slater's constraint qualification.

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Geometry of Primal and Dual Problems

Geometry of Primal and Dual Problems

(P) $\min\{c^T x | Ax \geq_{\kappa} b\}$

(D) $\max\{\langle b, \lambda\rangle | A^* \lambda = c, \lambda \geq_{\kappa^*} 0\}$

At first, they look completely different...

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Geometry of Primal and Dual Problems

But...

The difference is just in data representation $\rightarrow$ Geometrically similar!

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Geometry of Primal and Dual Problems

In (D):

- $max\langle b, \lambda \rangle$
  - $L_* = \{\lambda : A^*\lambda = c\}$: affine plane
  - K_*: cone

In (P):

- $y = Ax - b$: image of "true design" variables x
- $x \in R^n$
- $y \in L = \{y = Ax - b : x \in R^n\}$: affine
- $iff\ x \in R^n \rightarrow$ feasible $\Rightarrow y \in K$

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Geometry of Primal and Dual Problems

The objective:

$$c^T x = \langle d, Ax - b\rangle + \text{const} \equiv c \in \mathrm{Im}A^*$$

where:

- (P) equals: $\min_y \{\langle d, y\rangle : y \in L, y \geq_\kappa 0\}$
- $L = \mathrm{Im}A - b$
- $A^* d = c, \quad d$: any vector

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Geometry of Primal and Dual Problems

Thus...

The primal problem, geometrically, is the problem of minimizing a linear form over the intersection of the affine plane L with the cone K.

The dual problem, similarly, is to maximize another linear form over the intersection of the affine plane L_* with the dual cone K_*.

Geometry of Primal and Dual Problems

If conditions for (P) are not satisfied, the problem is:

- unbounded below, or
- infeasible

$\Rightarrow$ We reject (P) from the very beginning.

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Geometry of Primal and Dual Problems

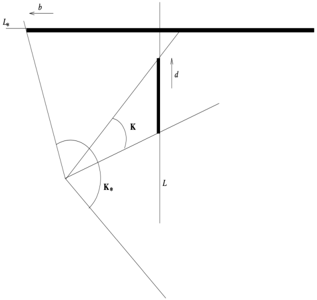

Figure: Primal-dual pair of conic problems
[bold: primal (vertical segment) and dual (horizontal ray) feasible sets]

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Geometry of Primal and Dual Problems

- K_* dual of K
- $L \perp L_*$

NOTE: The duality is completely symmetric (Weak Conic Duality Theorem)

$$\Rightarrow \quad (K_*)_* = K$$
$$\Rightarrow \quad (L^\perp)^\perp = L$$

From Linear to Conic Duality

Algorithmic Operational Research

So...

Our Github Repository

So...

(?!)Where can all these maths, actually be applied?!

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Applications of Conic Duality

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Applications of Conic Duality

- Self-Dual Cones
- Second Order Cone Programming (SOCP)
- Semidefinite Programming (SDP)
- Distributionally Robust Learning (DRL)

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Applications of Conic Duality

More specifically, DRL is necessary for Machine Learning systems: When applied in the real world, performance may be significantly degraded, because of different distribution between test and training data.
DRL considers the worst-case distribution, minimizing the reweighted training loss and creating classifiers.

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Is Something Wrong?

Is Something Wrong?

Well...

CP duality Theorem $\rightarrow$ weaker than LP duality Theorem

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Is Something Wrong?

LP:

- Feasibility (even non-strict) and boundedness of either primal or dual $\Rightarrow$ solvability of (P) and (D) equality between their optimal values

CP:

- If one of (P), (D) is essentially strictly feasible and bounded, then the other is solvable and $Opt(P) = Opt(D)$
- If both are essentially strictly feasible, then both are solvable, with $Opt(P) = Opt(D)$

Consequences

Consequences

- Sufficient condition for infeasibility
- The linear inequality of scalar (S) is a consequence of a feasible system of linear inequalities $Ax \geq b$ *iff* (S) can be obtained from linear vector (V) and the trivial $1 \geq 0$
- Robustness: are the properties for the (P) (feasibility, solvability, etc), stable with respect to perturbations of the data (inexact data, adding noise during data processing)

Bibliography

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Thank you, next

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