1 Introduction

The goal is to create a GalSim ChromaticObject with realistic color gradients using high resolution multi-band images (most likely from HST) as input. This is done by modeling the preconvolution chromatic surface brightness profile $f(\vec{x}, \lambda)$ as a sum of two or more separable chromatic surface brightness profiles, each with a particular asserted SED. I.e.,

$$f(\vec{x}, \lambda) = \sum_j S_j(\lambda) a_j(\vec{x}), \tag{1}$$

where $S_j(\lambda)$ is the jth SED asserted as part of the decomposition, and $a_j(\vec{x})$ is the spatial component of the jth separable chromatic profile.

The required set of inputs is:

- Two or more HST images of the same galaxy in different filters, $I_i(\vec{x})$, where i labels the different filters.
- The 2D noise covariance function for each image, $\xi_i(\Delta\vec{x})$. If the noise is uncorrelated, then this could be simply the noise variance ($\xi_i(\Delta\vec{x} = 0) = \sigma_i^2$, $\xi_i(\Delta\vec{x} \neq 0) = 0$).
- The chromatic HST PSF, $\Pi(\vec{x}, \lambda)$.
- The HST throughput for each image filter, $T_i(\lambda)$.
- Two or more SEDs, $S_j(\lambda)$, to use in the decomposition in Equation 1.

The model for the observed images is:

$$\begin{aligned} I_i(\vec{x}) &= \int T_i(\lambda) \left[\Pi(\vec{x}, \lambda) \ast f(\vec{x}, \lambda)\right] \mathrm{d}\lambda + \eta_i(\vec{x}) && (2) \\ &= \int T_i(\lambda) \sum_j S_j(\lambda) \left[\Pi(\vec{x}, \lambda) \ast a_j(\vec{x})\right] \mathrm{d}\lambda + \eta_i(\vec{x}) && (3) \end{aligned}$$

where $\eta_i(\vec{x})$ indicates (potentially spatially correlated) Gaussian noise in image i and the $\ast$ symbol indicates (spatial) convolution. The noise $\eta_i(\vec{x})$ is related to the noise covariance function via $\langle \eta_i(\vec{x}_l)\eta_i(\vec{x}_m)\rangle = \xi_i(\vec{x}_l - \vec{x}_m)$, where angle brackets indicate averaging over realizations of the noise. Note that we also assume that ξ_i is even: $\xi_i(\Delta\vec{x}) = \xi_i(-\Delta\vec{x})$.

The convolution is easier to work with in Fourier space where it becomes a mode-by-mode product. Indicating the Fourier transform of the real-space quantity $g(\vec{x})$ with $\tilde{g}(\vec{k})$, the model in Fourier space is

$$\begin{aligned} \tilde{I}_i(\vec{k}) &= \int T_i(\lambda) \sum_j S_j(\lambda) \tilde{\Pi}(\vec{k}, \lambda) \tilde{a}_j(\vec{k}) \, \mathrm{d}\lambda + \tilde{\eta}(\vec{k}) && (4) \\ &= \sum_j \left[\int T_i(\lambda) S_j(\lambda) \tilde{\Pi}(\vec{k}, \lambda) \, \mathrm{d}\lambda\right] \tilde{a}_j(\vec{k}) + \tilde{\eta}(\vec{k}) && (5) \\ &= \sum_j \tilde{\Pi}^{\mathrm{eff}}_{ij}(\vec{k}) \tilde{a}_j(\vec{k}) + \tilde{\eta}(\vec{k}), && (6) \end{aligned}$$

where the effective PSF for the ith filter and jth SED is

$$\tilde{\Pi}^{\text{eff}}_{ij}(\vec{k}) = \int T_i(\lambda) S_j(\lambda) \tilde{\Pi}(\vec{k}, \lambda)\, \mathrm{d}\lambda. \tag{7}$$

The crux of the problem is to solve for the (complex-valued) $\tilde{a}_j(\vec{k})$ and to propagate the statistics of the noise.

2 Solving for $\tilde{a}_j(\vec{k})$

There are several possible ways to estimate the required $\tilde{a}_j(\vec{k})$.

2.1 No noise

We'll look first at the case where one ignores noise entirely ($\eta(\vec{x}) = \tilde{\eta}(\vec{k}) = 0$).

If the number of asserted SEDs N_j is larger than the number of input images N_i, then the system of equations represented by Equation 6 is underdetermined. While in principle this could still be solvable using some prior constraints, for now, we'll just ignore this possibility.

If $N_j = N_i$, then we can exactly solve Equation 6 using matrix inversion for each Fourier mode $\vec{k}$:

$$\tilde{a}_j(\vec{k}) = \sum_i [(\tilde{\Pi}^{\text{eff}}(\vec{k}))^{-1}]_{ij} \tilde{I}_i(\vec{k}). \tag{8}$$

If there are more images than spectra ($N_i > N_j$), then the system of equations is overdetermined and no longer has an exact solution. Instead, there is a (usually) unique "least-squares" solution for each $\vec{k}$ obtained by minimizing

$$\sum_i \left| \tilde{I}_i(\vec{k}) - \sum_j \tilde{\Pi}^{\text{eff}}_{ij}(\vec{k}) \tilde{a}_j(\vec{k}) \right|^2. \tag{9}$$

2.2 Uncorrelated noise

Stationary, uncorrelated Gaussian pixel noise in the ith image is completely described by its variance σ_i^2. In Fourier space, the variance of each mode is independent of $\vec{k}$ and proportional to σ_i^2, where the constant of proportionality depends on the particular Fourier conventions employed. We can therefore write down a likelihood for $\tilde{a}_j(\vec{k})$ as

$$\chi^2(\vec{k}) = -2 \log \mathcal{L}(\vec{k}) = \sum_i \frac{1}{\sigma_i^2} \left| \tilde{I}_i(\vec{k}) - \sum_j \tilde{\Pi}^{\text{eff}}_{ij}(\vec{k}) \tilde{a}_j(\vec{k}) \right|^2. \tag{10}$$

This is a *weighted* least-squares problem. Note that the weights only matter for determining the Fourier coefficients $\tilde{a}_j(\vec{k})$ if the number of input images is greater than the number of asserted spectra. When these quantities are equal, it is always possible to find $\tilde{a}_j(\vec{k})$ such that $\tilde{I}_i(\vec{k}) = \sum_j \tilde{\Pi}^{\text{eff}}_{ij}(\vec{k}) \tilde{a}_j(\vec{k})$. The weights do still matter in this case if we're interested in propagating the noise, however.

2.3 Correlated noise

If the noise covariance function for a particular image $\xi_i(\vec{\Delta x})$ is non-zero away from the origin, then the variance of different Fourier modes is not constant, but proportional to the noise power spectrum, which is the Fourier transform of the noise covariance function:

$$P_i(\vec{k}) = \int \xi_i(\Delta\vec{x})e^{-2\pi i\vec{k}\cdot\Delta\vec{x}}. \tag{11}$$

Due to assumed translation invariance of the noise (though note that we do not assume isotropy), the covariance of different Fourier modes vanishes. More explicitly, if the noise in image i and mode $\vec{k_l}$ is $\tilde{\eta}_i(\vec{k_l})$, then

$$\langle\tilde{\eta}_i^*(\vec{k_l})\tilde{\eta}_i(\vec{k_m})\rangle = \delta(\vec{k_l} - \vec{k_m})P_i(\vec{k_l}). \tag{12}$$

Once the noise power spectrum has been computed, therefore, the only change to the likelihood in Equation 10 is that the mode variance now depends on the particular Fourier mode in question:

$$-2\log\mathcal{L}(\vec{k}) = \sum_i \frac{1}{P_i(\vec{k})}\left|\tilde{I}_i(\vec{k}) - \sum_j \tilde{\Pi}_{ij}^{\mathrm{eff}}(\vec{k})\tilde{a}_j(\vec{k})\right|^2. \tag{13}$$

This is essentially the same weighted least squares problem as in the uncorrelated noise case – the only difference being that the weights now depend on $\vec{k}$. Written in matrix notation, the solution for a particular $\vec{k}$ mode is:

$$\tilde{a} = \left(\tilde{\Pi}^{\mathrm{eff},\dagger}W\tilde{\Pi}^{\mathrm{eff}}\right)^{-1}\tilde{\Pi}^{\mathrm{eff},\dagger}W\tilde{I} \tag{14}$$

where

$$W_{ij} = \delta_{ij}/\sqrt{P_i}. \tag{15}$$

The covariance matrix for the elements of $\tilde{a}$ is given by

$$\Sigma = \left(\tilde{\Pi}^{\mathrm{eff},\dagger}W\tilde{\Pi}^{\mathrm{eff}}\right)^{-1}. \tag{16}$$

3 Propagating the noise covariance

At this point, we have the $\tilde{a}_j(\vec{k})$ necessary to represent a chromatic surface brightness profile in Fourier space as

$$\tilde{f}(\vec{k},\lambda) = \sum_j S_j(\lambda)\tilde{a}_j(\vec{k}). \tag{17}$$

The uncertainty in each $\tilde{a}_j(\vec{k})$ is uncorrelated from one $\vec{k}$ to the next (since we defined independent likelihoods for each $\vec{k}$), but does, in general, possess correlations between the different SED components j and j' for a given $\vec{k}$. Using $\Sigma_{jj'}(\vec{k})$ to represent these covariances, we have

$$\langle\delta\tilde{a}_j^*(\vec{k}_l)\delta\tilde{a}_{j'}(\vec{k}_m)\rangle = \delta(\vec{k}_l - \vec{k}_m)\Sigma_{jj'}(\vec{k}_l). \tag{18}$$

Fortunately, $\Sigma_{jj'}(\vec{k})$ is analytically computable for weighted least squares problems.

This $\Sigma_{jj'}(\vec{k})$ can then be transformed alongside the $\tilde{a}_j(\vec{k})$ to effect shears, rotations, dilations, etc., the same way that noise is transformed in existing GalSim routines.

Finally, the model for creating a (Fourier-domain) output image $\tilde{I}^{\rm out}(\vec{k})$, convolving by an output chromatic PSF $\tilde{\Pi}^{\rm out}(\vec{k},\lambda)$ and drawing through an output filter with transmission $T^{\rm out}(\lambda)$, is

$$\tilde{I}^{\rm out}(\vec{k}) = \int T^{\rm out}(\lambda)\tilde{\Pi}^{\rm out}(\vec{k},\lambda)\tilde{f}(\vec{k},\lambda)\,{\rm d}\lambda \tag{19}$$

$$= \int T^{\rm out}(\lambda)\tilde{\Pi}^{\rm out}(\vec{k},\lambda)\sum_j S_j(\lambda)\tilde{a}_j(\vec{k})\,{\rm d}\lambda \tag{20}$$

$$= \sum_j \tilde{\Pi}_j^{\rm out,eff}(\vec{k})\tilde{a}_j(\vec{k}) \tag{21}$$

where the effective PSF for the jth component of the output image is

$$\tilde{\Pi}_j^{\rm out,eff}(\vec{k}) = \int \tilde{\Pi}^{\rm out}(\vec{k},\lambda)T^{\rm out}(\lambda)S_j(\lambda)\,{\rm d}\lambda. \tag{22}$$

Propagating the (potentially transformed) covariance spectrum $\Sigma_{jj'}(\vec{k})$ into the final output noise power spectrum $P^{\rm out}(\vec{k})$ then follows the normal propagation of errors formula:

$$P^{\rm out}(\vec{k}) = \sum_{jj'} \tilde{\Pi}_j^{\rm out,eff,*}(\vec{k})\Sigma_{jj'}(\vec{k})\tilde{\Pi}_{j'}^{\rm out,eff}(\vec{k}). \tag{23}$$

This power spectrum can then be used in the existing correlated noise whitening and symmetrizing GalSim routines.

4 Implementation notes

GalSim uses the methodology developed by Bernstein & Gruen (2013) to interpolate discretely sampled surface brightness profiles in real and Fourier space. Here we investigate the impact of this interpolation on the Wiener-Khinchin theorem, which relates the real-space autocorrelation function to the Fourier-space power spectrum.

The discrete version of the Wiener-Khinchin theorem is derived as follows. We are interested in the (co)-variance of the Discrete Fourier Transform amplitudes of some discrete real space samples with discrete auto-covariance function $\xi\left[r\right]$ (we follow the convention of Rowe++14 in using square brackets to indicate the arguments of a discretly sampled function, and reserve parentheses to indicate arguments of continuous objects). Note that since we're using DFTs, ξ is implicitly periodic: $\xi\left[r\right] = \xi\left[r+N\right]$.

The 1D DFT of an N-point sampled function is:

$$\tilde{b}_k = \sum_{j=-N/2}^{N/2-1} b_j e^{-2\pi ijk/N}. \tag{24}$$

We are interested in the quantity $\left\langle \tilde{b}_k^* \tilde{b}_{k'} \right\rangle$ where the angle brackets indicate averaging over noise realizations.

$$
\begin{aligned}
\left\langle \tilde{b}_k^* \tilde{b}_{k'} \right\rangle &= \left\langle \sum_{j=-N/2}^{N/2-1} b_j e^{+2\pi i jk/N} \sum_{j'=-N/2}^{N/2-1} b_{j'} e^{-2\pi i j'k'/N} \right\rangle && \text{Sub in FT definition} \\
&= \sum_{j=-N/2}^{N/2-1} \sum_{j'=-N/2}^{N/2-1} \left\langle b_j b_{j'} \right\rangle e^{2\pi i (jk - j'k')/N} && \text{Interchange expectation and summation} \\
&= \sum_{j=-N/2}^{N/2-1} \sum_{j'=-N/2}^{N/2-1} \xi\left[j - j'\right] e^{2\pi i (jk - j'k')/N} && \xi\left[r\right] \text{ definition} \\
&= \sum_{j=-N/2}^{N/2-1} \sum_{r=-N/2-j}^{N/2-1-j} \xi\left[r\right] e^{2\pi i (jk - (r+j)k')/N} && \text{change variables } r = j' - j \\
&= \sum_{j=-N/2}^{N/2-1} \sum_{r=-N/2}^{N/2-1} \xi\left[r\right] e^{-2\pi i r k'/N} e^{2\pi i j (k-k')/N} && \text{simplify using periodicty of } \xi \text{ and exp} \\
&= P[k'] \sum_{j=-N/2}^{N/2-1} e^{2\pi i j (k-k')/N} && \text{definition of } P\left[k\right] \\
&= N P[k'] \sum_{N=0}^{\infty} \delta_{k'}^{k+N} && e^{2\pi i j(k-k')/N} \text{ evenly samples unit circle unless } k - k' \in N\mathbb{Z}... \\
&= N P[k'] \delta_{k'}^{k}. && \text{...but we really only care about } k \in \left[-N/2, N/2-1\right]
\end{aligned}
$$

Now we try the same thing using the Bernstein & Gruen continuously interpolated Fourier transform. Recall their result:

$$
\begin{aligned}
\tilde{F}(u) &= \int F(x) e^{-2\pi i u x} \, \mathrm{d}x && (25) \\
&\approx \tilde{K}_x(u) \sum_{k=-N/2}^{N/2-1} \tilde{b}_k K_u(u - k/N) && (26)
\end{aligned}
$$

where K_x is an asserted real-space interpolant, $\tilde{K}_x$ is its Fourier transform, and K_u is an asserted Fourier-space interpolant. Note that the final equality is exact if and only if

$$
K_u(v) = e^{i\pi v} \frac{\operatorname{sinc} N v}{\operatorname{sinc} v}. \tag{27}
$$

We are interested in the (co)-variance of Fourier amplitudes:

$$
\begin{aligned}
\left\langle \tilde{F}^*(u)\tilde{F}(u')\right\rangle &= \left\langle \tilde{K}_x^*(u)\sum_{k=-N/2}^{N/2-1}\tilde{b}_k^* K_u^*(u-k/N)\tilde{K}_x(u')\sum_{k'=-N/2}^{N/2-1}\tilde{b}_{k'}K_u(u'-k'/N)\right\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{Sub in GalSim approximate FT definition} \\
&= \tilde{K}_x^*(u)\tilde{K}_x(u')\sum_{k=-N/2}^{N/2-1}\sum_{k'=-N/2}^{N/2-1}K_u^*(u-k/N)K_u(u'-k'/N)\left\langle \tilde{b}_k^*\tilde{b}_{k'}\right\rangle \qquad \text{Rearrange} \\
&= \tilde{K}_x^*(u)\tilde{K}_x(u')\sum_{k=-N/2}^{N/2-1}\sum_{k'=-N/2}^{N/2-1}K_u^*(u-k/N)K_u(u'-k'/N)NP[k']\delta_{k'}^{k} \qquad \text{definition of } P[k] \\
&= N\tilde{K}_x^*(u)\tilde{K}_x(u')\sum_{k=-N/2}^{N/2-1}K_u^*(u-k/N)K_u(u'-k/N)P[k] \qquad \text{sum over Kronecker } \delta
\end{aligned}
$$

This is as far as I've been able to push (ran out of δ functions!). Assuming there are no further simplifications, one implication is that Fourier amplitudes of an interpolated noise image are not uncorrelated. Equivalently, the continuous auto-covariance function of interpolated discretely stationary noise is not itself stationary. This makes some intuitive sense, I think. For example, the variance of interpolated points seems like it should always be less than the variance of the input samples, implying that the continuously regarded noise is not stationary.