

Chapter 2

Axiomatic Probability Theory

2.1 Axioms of probability

In the previous chapter, we have introduced sample spaces and event spaces. Now we would like to be able to express that certain events are more (or less) likely than others. Therefore, we are going to measure the probability of events in a mathematically precise sense.

Definition 2.1 A finite measure is a function $\mu : \mathcal{S} \rightarrow \mathbb{R} : S \mapsto \mu(S)$ that maps elements from a set of sets \mathcal{S} (formally a *σ -algebra*) to real numbers. Such a measure has the following properties:

1. $\mu(S) \in \mathbb{R}$ for $S \in \mathcal{S}$,
2. $\mu\left(\bigcup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} \mu(S_i)$ if $S_i \cap S_j = \emptyset$ for $i \neq j$.

Notice that we are restricting ourselves to finite measures here, i.e. the value of the measure can never be infinite. This restriction makes sense as probabilities are finite as well. Property ?? is known as *countable additivity*.

If we let $S = \bigcup_{i=1}^n S_i$ for some positive natural number n and disjoint S_i and $S_j = \emptyset$ for $j > n$ we get $\mu(S) = \mu\left(\bigcup_{i=1}^{\infty} S_i\right)$. By countable additivity, this expands to

$$(2.1) \quad \mu(S) = \mu\left(\bigcup_{i=1}^n S_i \cup \bigcup_{j=n+1}^{\infty} \emptyset\right) = \sum_{i=1}^n \mu(S_i) + \sum_{j=n+1}^{\infty} \mu(\emptyset).$$

Since the S_i are disjoint, we must have $\mu(S) = \sum_{i=1}^n \mu(S_i)$ and it follows that $\mu(\emptyset) = 0$. That is the empty set has measure 0 holds true for all measures. Furthermore, we also see from the above derivation that countable additivity implies finite additivity, i.e. $\mu(S) = \sum_{i=1}^n \mu(S_i)$ for finite positive n (again, this only holds if the S_i are disjoint).

Examples of measures are not hard to find. In fact, we have already seen a measure, namely the function $|\cdot|$ that counts the elements of a set (check yourself that it really is a measure). Another measure is the Dirac-measure that is related to the characteristic function of a set. While the characteristic function tells you whether any object belongs to a given set, the Dirac-measure tells you whether any set contains a given object. Let us call the object in question a . Then its Dirac measure $\delta_a(S) = 1$ iff $a \in S$ and 0 otherwise (check again that the Dirac-measure indeed is a measure).

Apart from these examples, there is one measure, however, that is going to be the star of the rest of this script, namely the **probability measure**.

Definition 2.2 A probability measure $\mathbb{P} : \mathcal{A} \rightarrow \mathbb{R} : A \mapsto \mathbb{P}(A)$ on an event space \mathcal{A} associated with a sample space Ω has the following properties:

1. $\mathbb{P}(A) \geq 0$ for all $A \in \mathcal{A}$,
2. $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$,
3. $\mathbb{P}(\Omega) = 1$.

Notice that we only added Property 3 to the general definition of a measure. Hence, a **probability** (the value that the probability measure assigns to an event) will always lie in the real interval $[0, 1]$. The above three axioms for a probability measure are often referred to as *axioms of probability* or *Kolmogorov axioms* after their inventor [Andrey Kolmogorov](#).

We have already discussed uniform probabilities in the previous chapter. We can now formally explain what we meant by that. The uniform probability measure is the measure \mathbb{P} such that $\mathbb{P}(\{\omega\}) = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$. At this point, the distinction between sample and event spaces becomes important. We cannot measure the elements of a sample space, only the elements of an event space! Recall our convention that we will always assume that $\mathcal{A} = \mathcal{P}(\Omega)$ which obviously contains a singleton for each element in Ω . Using this assumption, the uniform probability measure is indeed well-defined. Whenever we talk about *uniform probability*, we either mean the uniform probability measure or, more often, the real value $\frac{1}{|\Omega|}$ to which this measure uniformly evaluates.

In order to create a tight relationship between a sample space, an event space and a probability measure, we introduce the concept of a **probability space**. Probability spaces are also known as (probabilistic) **experiments**.

Definition 2.3 A probability space is a triple $(\Omega, \mathcal{A}, \mathbb{P})$, consisting of a sample space Ω , an event space \mathcal{A} and a probability measure \mathbb{P} .

If we roll a die, for example, we have the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$ and, by convention, the event space $\mathcal{A} = \mathcal{P}(\Omega)$. If we add the uniform probability measure, we have constructed a *probabilistic experiment*. We can use it to answer a couple of questions. For example, we might wonder about the probability of obtaining an even number. By Property 2 of our definition, this probability is given by

$$(2.2) \quad \mathbb{P}(\{2, 4, 6\}) = \mathbb{P}(\{2\} \cup \{4\} \cup \{6\}) = \mathbb{P}(\{2\}) + \mathbb{P}(\{4\}) + \mathbb{P}(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

Notice that this calculation is rather cumbersome. After all, we might just have evaluated $\mathbb{P}(\{2, 4, 6\})$ directly. However, the above calculation points to an interesting fact. In order to fully specify a probability measure, it suffices to specify the measure on the singleton sets of the event space. By countable additivity, this assignment already specifies the measure on the entire event space, as we can construct any event as a countable union of singletons.

It is important to point out that we just chose the uniform probability measure as the one that seems “natural” for a die roll. However, nobody is forcing us to do so. In fact, Definition ?? allows us to impose arbitrary probability measures.

Exercise 2.1 Let us consider a rigged die. Take $(\Omega, \mathcal{A}, \mathbb{P})$ with Ω and \mathcal{A} as in the uniform die-roll example before, but now use the probability measure

$$\mathbb{P} = \{(\{1\}, 0), (\{2\}, \frac{1}{12}), (\{3\}, \frac{1}{6}), (\{4\}, \frac{1}{6}), (\{5\}, \frac{1}{3}), (\{6\}, \frac{1}{4})\}.$$

1. Verify that \mathbb{P} is indeed a probability measure.
2. Compute the probability of obtaining a number strictly smaller than 5 in this experiment.

2.2 Probability of Unions of Events

We have seen how to compute probabilities of events if they can be formed as unions of *disjoint* events. The natural question to ask is what to do if we want to compute the probability of the *union of non-disjoint events*. In order to reason about this problem, we first take a step back and think

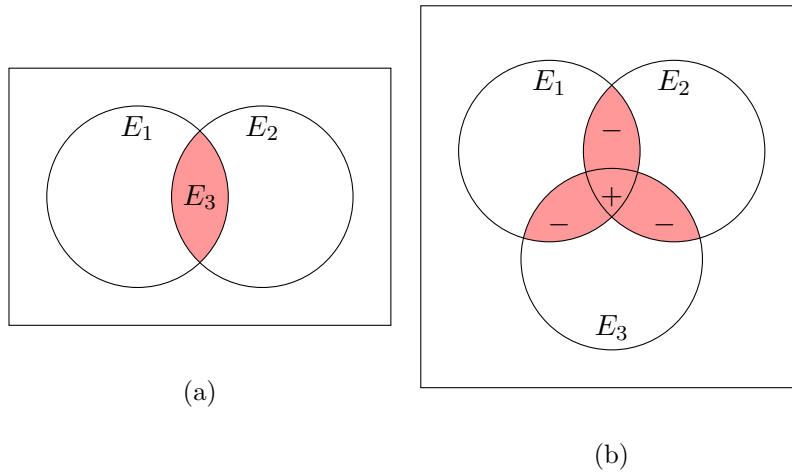


Figure 2.1: ???: Two overlapping events E_1 and E_2 . Their intersection (the coloured region) gets counted twice if we add up their probabilities.

???: Venn diagram with 3 events. First we deduct $E_1 \cap E_2, E_1 \cap E_3, E_2 \cap E_3$ in order to prevent double counting and then we add in $E_1 \cap E_2 \cap E_3$. Deductions and additions are indicated by pluses and minuses.

about the outcomes of our probability space. We know that each relevant event contains at least one outcome (since $\mathbb{P}(\emptyset) = 0$, we can safely ignore it). Now let us assume that we take the union of events E_1 and E_2 with $E_1 \cap E_2 = E_3 \neq \emptyset$. This means that the outcomes in E_3 are contained in both E_1 and E_2 . This situation is illustrated in fig. ???. Now if we apply the rule that we have for disjoint events, we get

$$\begin{aligned}
 (2.3) \quad \mathbb{P}(E_1 \cup E_2) &= \mathbb{P}(E_1) + \mathbb{P}(E_2) = \mathbb{P}(((E_1 \setminus E_3) \cup E_3) \cup ((E_2 \setminus E_3) \cup E_3)) \\
 &= \mathbb{P}(E_1 \setminus E_3) + \mathbb{P}(E_3) + \mathbb{P}(E_2 \setminus E_3) + \mathbb{P}(E_3)
 \end{aligned}$$

Exercise 2.2 Try to justify each of the above equalities using your knowledge about sets, event spaces and the probability measure. In doing so, you will realize that one of the above equalities does not hold. Which one?

What happens here is that we count E_3 , the outcomes that are in both events of interest, twice. Thus, what we will need to is to subtract E_3 one time. This leads us to the following formulation:

$$(2.4) \quad \mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2)$$

Notice that this is fully general in that it is true even if E_1 and E_2 were disjoint. In that case, their intersection would be empty. We can generalize this principle to the (countable) union of an arbitrary number of events.

This will give us a principled way of calculating the probability of any union of events. This calculation technique is known as the **Inclusion-Exclusion principle**.

Theorem 2.1 (Inclusion-Exclusion principle) *The probability of any (countable) union of events E_1, \dots, E_n can be computed as*

$$\mathbb{P}\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n (-1)^{i+1} \left(\sum_{j_1 < \dots < j_i} \mathbb{P}(E_{j_1} \cap \dots \cap E_{j_i}) \right)$$

We are now going to proceed with a combinatorial proof of the Inclusion-Exclusion principle. It is very elegant but invokes the [binomial theorem](#) which we do not expect you to know. For completeness sake we will prove the binomial theorem at the end of this chapter. For now, just trust us that it exists and is correct.

We are going to focus on a particular outcome ω that is contained in m events E_1, \dots, E_m . Notice that we can safely neglect all events which do not contain ω , since ω is not going to contribute to their probability.

For all the E_i , $1 \leq i \leq m$ in which ω is contained, it is certainly true that ω is also contained in their intersections. Now the Inclusion-Exclusion-principle adds up the probabilities of intersections of a given size. Notice that any intersection of more than m events will not contain ω as we intersect with at least one event that does not contain ω . Thus, we only need to consider intersections of our ω -containing m sets.

The first intersection is trivial, as it just consists of one event. How many ways are there to pick one out of m events? The answer is $\binom{m}{1}$. This is the number of times that ω contributes to the overall probability. At this point we have an overestimate of that probability (compare this to eq. ??). Next we subtract the probabilities of the mutual intersections. By the same reasoning as before, the contribution of ω is deducted $\binom{m}{2}$ times which gives us an underestimate since $\binom{m}{1} \leq \binom{m}{2}$ for $m \geq 3$.

What we want is that in total we only count ω 's contribution to the overall probability once. That is we want that

$$(2.5) \quad 1 = \sum_{i=1}^m (-1)^{i-1} \binom{m}{i}$$

And now we are right on our way towards exploiting the binomial theorem. Let us first state it.

$$(2.6) \quad (p + q)^m = \sum_{i=0}^m \binom{m}{i} p^i q^{m-i}$$

It may not be entirely obvious how to use the theorem in our situation, but hold on. First observe that the sum in eq. ?? actually starts counting

at 0 whereas in eq. ?? it starts counting from 1. However, we already know that $\binom{m}{0} = 1$. Thus we can replace the the left hand side in eq. ?? with this binomial coefficient and then subtract 1 from the right hand side. This gives us ??. This is the equality that we are going to proof in what follows. Since we derived it from eq. ??, it implies that if ?? holds, so does ??.

$$(2.7) \quad \sum_{i=0}^m (-1)^i \binom{m}{i} = 0$$

Now we notice that 0 in turn can be expressed as $(1 - 1)$. Furthermore, any power of 0 is still 0. Thus we rewrite again to get

$$(2.8) \quad \sum_{i=0}^m \binom{m}{i} (-1)^i = (1 - 1)^m$$

This now looks a lot like the binomial theorem, doesn't it? Just one component is missing. We have our q from eq. ?? which is -1 in this case. We still need to account for p , though. Luckily, setting $p = 1$ and taking powers does not change anything. Thus we can safely write

$$(2.9) \quad \sum_{i=0}^m \binom{m}{i} (-1)^i 1^{m-i} = (1 - 1)^m$$

This equality is now true by the binomial theorem (eq. ??) which it is an instantiation of. We have thus succeeded in proofing the Inclusion-Exclusion principle. If the proof is not entirely clear to you, please go over it again.

At this point we have done our fair share of math and found out how to calculate the probability of a union of events. Now we should ask ourselves, what the probability of a union of events even tells us. Observe that an event occurs whenever we draw an outcome from our sample space that is contained in that event. By taking the union of events E_1, \dots, E_n we form a new event E that (possibly) contains more outcomes than each of the original events. Thus, the probability of the E will be higher than (or the same as) the probability of each of E_1, \dots, E_n . What we are measuring then, is the probability that *any* of the events E_1, \dots, E_n . Crucially, we do not care anymore which one of them occurs.

What we are missing is a way to express the probability that a given number of events occur *together*. This concept is so important that we have a dedicated name for it, that of **joint probability**.

Definition 2.4 *The joint probability of a (countable) set of events $\{E_1, \dots, E_n\}$ is defined as*

$$\mathbb{P}(E_1, \dots, E_n) = \mathbb{P}(E_1 \cap \dots \cap E_n)$$

Wow, that was simple! We don't even need to prove another rule for calculating the joint probability. After all, we already know how to take the intersection of sets. Annoyingly, one problem remains: our definition of event spaces does not guarantee that they contain the intersections of their members. Or does it? Well, let us see whether we can “paraphrase” what an intersection is.

$$(2.10) \quad E_1 \cap E_2 = \Omega \setminus ((\Omega \setminus E_1) \cup (\Omega \setminus E_2))$$

[REMARK PHILIP: Should we maybe put this into chapter one when we introduce event spaces?]

All the operations on the right hand side are defined for events spaces. We have thus solved our problem. To convince yourself that this is correct, you may want to consult fig. ?? . We also do not claim that this is the only valid “paraphrase”. Feel free to find others, if you like!

At this point we are capable to do most probabilistic computations that we will encounter in this course. From here on, it is all about making our lives easier. For example, how would you solve the following problem.

Exercise 2.3 *You are observing a panel of 200 light bulbs and you know that at least one of them will light up once you press a button. What is the probability that any except the 87th bulb will light up? Note: this is a conceptual exercise. For the very keen ones, you can obtain the probability for each bulb to be turned on by typing the following into the Python interpreter:*

```
import numpy

probabilities = numpy.random.rand(1,200)
probabilities/probabilities.sum()
```

The point of the above exercise is that it will be awfully cumbersome to compute the probability of the union of the singletons E_i where $1 \leq i \leq 200$ and $i \neq 87$. On the other hand we can easily look up $\mathbb{P}(E_{87})$. The question is whether we can exploit this simpler calculation to help us answer the original question. Here we will again make use of the properties of event spaces. For any event E in our event space we also have $\Omega \setminus E$ in the same space. Furthermore, E and $\Omega \setminus E$ are disjoint which by our probability axioms means that we can simply add up their probabilities if we want to calculate the probability of their union. But what's the union of E and $\Omega \setminus E$? It's exactly Ω . From our axioms we know that $\mathbb{P}(\Omega) = 1$. By simple algebraic manipulations we find that

$$(2.11) \quad \mathbb{P}(\Omega \setminus E) = 1 - \mathbb{P}(E)$$

Thus if we want to find the probability that any but the 87th bulb will light up, we simply compute the probability that it will light up and subtract that from 1. This is a rather general strategy that comes in handy whenever the probability of an event is hard to compute. Maybe the probability of the complement of that event will be easier to compute.

Exercise 2.4 *Show that in general*

$$\mathbb{P}(E_1 \setminus (E_1 \cap E_2)) = \mathbb{P}(E_1) - \mathbb{P}(E_1 \cap E_2)$$

2.3 Conditional probability and independence

After we have seen how to measure the probability of events, we are now going to introduce another tremendously important concept, that of **conditional probability** measures.

Definition 2.5 *The probability of an event E_i conditioned on another event E_j is defined as*

$$\mathbb{P}(E_i|E_j) = \frac{\mathbb{P}(E_i \cap E_j)}{\mathbb{P}(E_j)}$$

The function $\mathbb{P}(\cdot|E_j)$ is often referred to as conditional probability measure.

Before we get into the math of conditional probabilities, let us try to understand what this concept event signifies. When we are computing the conditional probability of an event E_i , we scale it to the probability of the conditioning event E_j . Now if $E_j \neq \Omega$, $\mathbb{P}(E_j)$ might be smaller than 1. Thus, this rescaling assumes *that E_j has already occurred*. In other words, we are excluding all outcomes that are not in E_j from further consideration (event though they may be in E_i). The interpretation of conditional probabilities that is that they are the probabilities of events assuming that another event has already occurred.

Another interpretation is that when working with a conditional probability measure, we are in fact working in a new probability space, where $\Omega_{new} = E_j$, i.e. our sample space is the conditioning event. Notice that this also means that our probability measure will change. In particular it will become the measure from def. ??.

Now here comes the cool part: although we have introduced a new concept, all the properties of probability measures that we know by now will seamlessly carry over to conditional probabilities if we can prove that the conditional probability measure is a probability measure according to our axioms.

Exercise 2.5 Use the axioms from def. ?? to prove that $\mathbb{P}(\cdot|E_j)$ is a probability measure.

We will make use of conditional probabilities quite a lot in this course. We will later see a way in which they help us to decompose joint probability distributions. For now, we are going to focus on the fact that they are also related to the idea of **independence** between events.

Definition 2.6 Two events E_1, E_2 are said to be independent if

$$\mathbb{P}(E_1, E_2) = \mathbb{P}(E_1) \times \mathbb{P}(E_2)$$

Independence between events is denoted as $E_1 \perp E_2$.

This definition relates to conditional probabilities in the following way: assume that $E_1 \perp E_2$. Then we get

$$(2.12) \quad \mathbb{P}(E_1|E_2) = \frac{\mathbb{P}(E_1 \cap E_2)}{\mathbb{P}(E_2)} = \frac{\mathbb{P}(E_1) \times \mathbb{P}(E_2)}{\mathbb{P}(E_2)} = \mathbb{P}(E_1)$$

Notice however, that $E_1 \perp E_2 \Rightarrow \mathbb{P}(E_1|E_2) = \mathbb{P}(E_1)$ but not the other way around.

Exercise 2.6 Find a probability space with at least three events such that $\mathbb{P}(E_1|E_2) = \mathbb{P}(E_1)$ but $\mathbb{P}(E_1, E_2) \neq \mathbb{P}(E_1) \times \mathbb{P}(E_2)$.

Independence will prove to be a useful concept in later chapters. More precisely, we will often just *assume* that two events (or random variable – see the next chapter) are independent. Although this assumption may not be true, it will allow us to formulate much simpler probabilistic models.

2.4 A Remark on the interpretation of probabilities*

This concludes our introduction of axiomatic probability theory. We now know that a probability is a real number in $[0, 1]$. For all that we are going to do in this course (and in most follow-up courses) this is fully sufficient. However, some of you may wonder what a “natural” interpretation of probabilities would be. There are two dominating views on that. One postulates that we were to take A LOT (read: almost infinitely many) samples from a sample space, the probability of an event is its frequency amongst these samples divided by the total number of samples taken. For those of you who know limits, this can be formalized as $\mathbb{P}(E) = \lim_{n \rightarrow \infty} \frac{\#E}{n}$. This view is known as the frequentist view.

The second view postulates that probabilities are an expression for degrees of belief. Basically, if you assign $\mathbb{P}(E)$ to an event E , then $\mathbb{P}(E)$ is the strength your personal belief that E will occur. This latter view is known as the Bayesian view.

Which conception of probability you choose is a philosophical matter and does not really impact the math. That is why we will not care about this issue in this course. However, it is useful to at least be aware of these two views (if only to appear knowledgeable in a conversation you may have with your philosopher friends).

2.5 The binomial theorem*

The binomial theorem from eq. ?? is actually not that hard to prove. We will do so by induction. As a base case we choose $m = 0$. Then the equality is easy to see.

$$(2.13) \quad (p + q)^0 = 1 = \binom{0}{0} p^0 q^0$$

Next, we assume that the theorem holds for $m = n$. What we want to show is that it also holds for $m = n + 1$. We achieve this by algebraic manipulation.

(2.14)

$$(p+q)^{n+1} = (p+q)^n \times (p+q)$$

$$(2.15) \quad = (p+q)^n p + (p+q)^n q$$

$$(2.16) \quad = p \sum_{i=0}^n \binom{n}{i} p^i q^{n-i} + q \sum_{i=0}^n \binom{n}{i} p^i q^{n-i}$$

$$(2.17) \quad = \sum_{i=0}^n \binom{n}{i} p^{i+1} q^{n-i} + \sum_{i=0}^n \binom{n}{i} p^i q^{n+1-i}$$

$$(2.18) \quad = \sum_{j=1}^{n+1} \binom{n}{j-1} p^j q^{n+1-j} + \sum_{i=0}^n \binom{n}{i} p^i q^{n+1-i}$$

$$(2.19) \quad = \binom{n}{n} p^{n+1} q^{(n+1)-(n+1)} + \sum_{k=1}^n \binom{n}{k-1} p^k q^{n+1-k} \\ + \binom{n}{0} p^0 q^{n+1} + \sum_{k=1}^n \binom{n}{k} p^k q^{n+1-k}$$

$$(2.20) \quad = q^{n+1} + p^{n+1} + \sum_{k=1}^n \left(\binom{n}{k} + \binom{n}{k-1} \right) p^i q^{n+1-k}$$

$$(2.21) \quad = q^{n+1} + p^{n+1} + \sum_{k=1}^n \left(\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \right) p^i q^{n+1-k}$$

$$(2.22) \quad = q^{n+1} + p^{n+1} + \sum_{k=1}^n \left(\frac{n!(n+1-k)}{k!(n+1-k)!} + \frac{n!k}{k!(n-k+1)!} \right) p^k q^{n+1-k}$$

$$(2.23) \quad = q^{n+1} + p^{n+1} + \sum_{k=1}^n \left(\frac{n!(n+1)}{k!(n+1-k)!} \right) p^i q^{n+1-k}$$

$$(2.24) \quad = q^{n+1} + p^{n+1} + \sum_{k=1}^n \binom{n+1}{k} p^k q^{n+1-k}$$

$$(2.25) \quad = \sum_{i=0}^{n+1} \binom{n+1}{i} p^i q^{n+1-i}$$

Let us clarify some parts of the proof. In line ??, we switch the variable i in the first summand to $j = i + 1$. The reason why we do this is because we want to achieve congruence with the exponents of the second summand. In the following line we uniformly name the variables k . Since k has to run over a common range, we chop off the ends of both sums that stick out. In the first sum of line ?? that is the summand that corresponds to $j = n + 1$ and in the second sum it is the summand that corresponds to $i = 0$. We pull out both of them in line ?? and then collapse the sums in line ??. The

following lines are basically just an exercise in manipulation fractions. The jump from the second-to-last to the last line is allowed because

$$q^{n+1} = \binom{n+1}{0} p^0 q^{n+1-0}$$

and

$$p^{n+1} = \binom{n+1}{n+1} p^{n+1} q^{(n+1)-(n+1)}$$

which are exactly the quantities that we need to add to make our sum reach from 0 to $n+1$. This completes the proof.

Further reading

A very quick and dirty introduction to measure theory is provided by Maya Gupta and can be found [here](#). If you are looking for something more extensive that also motivates event spaces and the like you may want to take a look at [this script](#) by Ross Leadbatter and Stamatis Cambanis (which has also been published as a book).