GENERAL ASSEMBLY

INTRO TO DATA SCIENCE

REGRESSION & REGULARIZATION

I. INTRO TO REGRESSION
II. LINEAR AND POLYNOMIAL
III. REGULARIZATION

EXERCISES:
III. IMPLEMENTING MULTIPLE REGRESSION AND POLYNOMIAL REGRESSION IN PYTHON

I. LINEAR REGRESSION

	continuous	*categorical*
supervised	???	???
unsupervised	???	???

	continuous	*categorical*
supervised	regression	classification
unsupervised	dimension reduction	clustering

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ε = **residual** (the prediction error)

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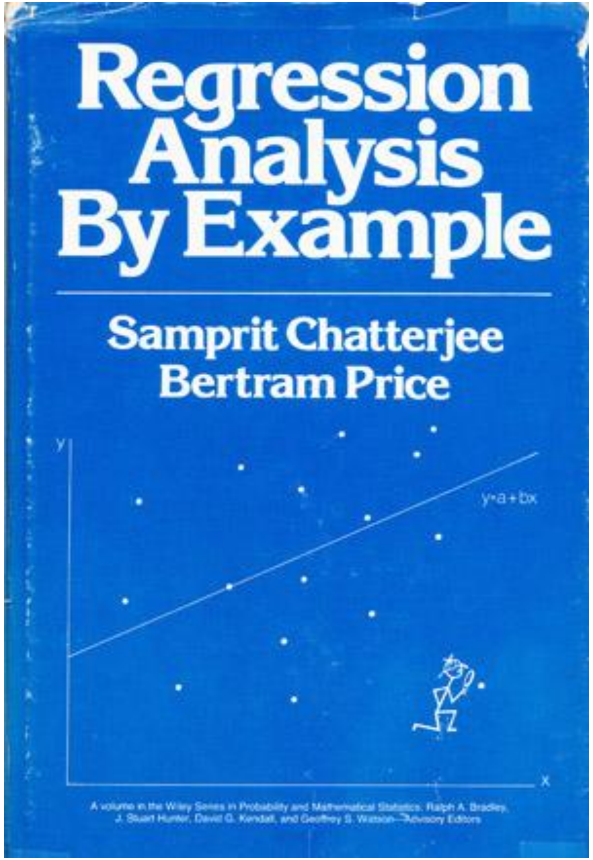
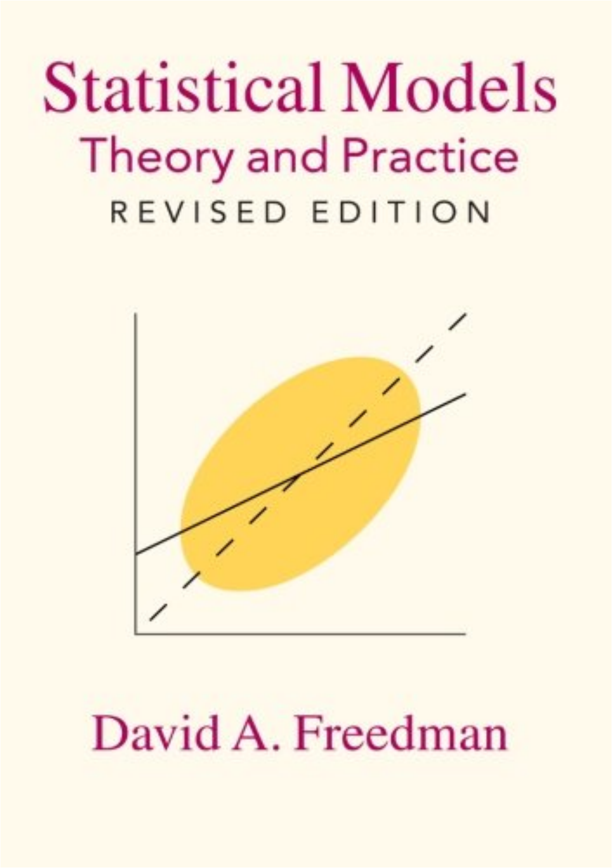
Linear regression involves several technical assumptions and is often presented with lots of mathematical formality.

The math is not very important for our purposes, but you should check it out if you get serious about solving regression problems.


Regression Analysis By Example
Samprit Chatterjee
Bertram Price
y
y=a+bx
x
A volume in the Wiley Series in Probability and Mathematical Statistics. Ralph A. Bradley, J. Stuart Hunter, David G. Kendall, and Geoffrey S. Watson—Advisory Editors


Statistical Models
Theory and Practice
REVISED EDITION
David A. Freedman

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But again, if you get serious about regression, you should learn how this works!

II: POLYNOMIAL REGRESSION

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"Although polynomial regression fits a *nonlinear* model to the data, as a statistical estimation problem it is *linear*, in the sense that the regression function E(y|x) is linear in the unknown parameters that are estimated from the data. For this reason, polynomial regression is considered to be a special case of multiple linear regression." -- Wikipedia

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A: This model violates one of the assumptions of linear regression!

This model displays **multicollinearity**, which means the predictor variables are highly correlated with each other.

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \ldots + \beta_n x^n + \varepsilon$$

```
> x <- seq(1, 10, 0.1)
> cor(x^9, x^10)
[1] 0.9987608
```

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NOTE

This results in a *singularity*. We will see an example of this in just a minute!

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OPTIONAL NOTE

These polynomial functions form an *orthogonal basis* of the function space.

So far, we've seen how polynomial regression allows us to fit complex nonlinear relationships, and even to avoid multicollinearity (by using basis functions).

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Q: Can a regression model be too complex?

III: REGULARIZATION

Recall our earlier discussion of **overfitting**.

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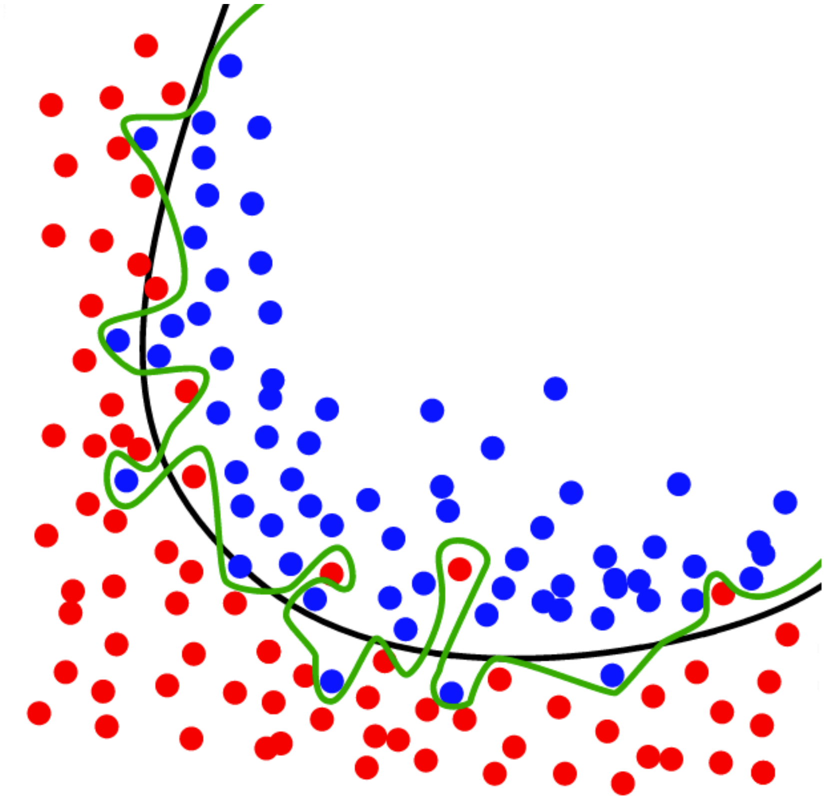
In other words, an overfit model matches the **noise** in the dataset instead of the **signal**.



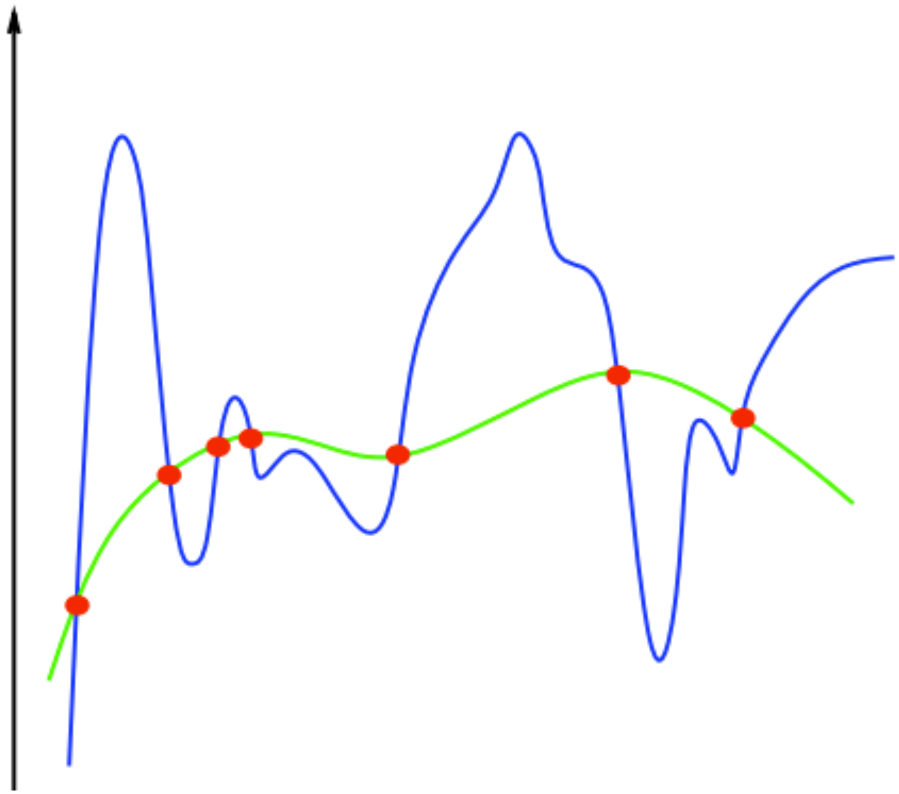
source: http://upload.wikimedia.org/wikipedia/commons/1/19/Overfitting.svg

The same thing can happen in regression.

It’s possible to design a regression model that matches the noise in the data instead of the signal.

This happens when our model becomes *too complex* for the data to support.

OVERFITTING EXAMPLE (REGRESSION)



source: http://www.mit.edu/~9.520/spring12/slides/class02/class02.pdf

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Ex 1: $\Sigma\,|\beta_i|$
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A: One method is to define complexity as a function of the size of the coefficients.

Ex 1: $\Sigma\,|\beta_i|$ this is called the **L1-norm**

Ex 2: $\Sigma\,{\beta_i}^2$ this is called the **L2-norm**

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Regularization refers to the method of preventing **overfitting** by explicitly controlling model **complexity**.

These regularization problems can also be expressed as:

L1 regularization: $min(\|y - \boldsymbol{x} \bullet \boldsymbol{\beta}\|^2 + \lambda\|\boldsymbol{\beta}\|)$

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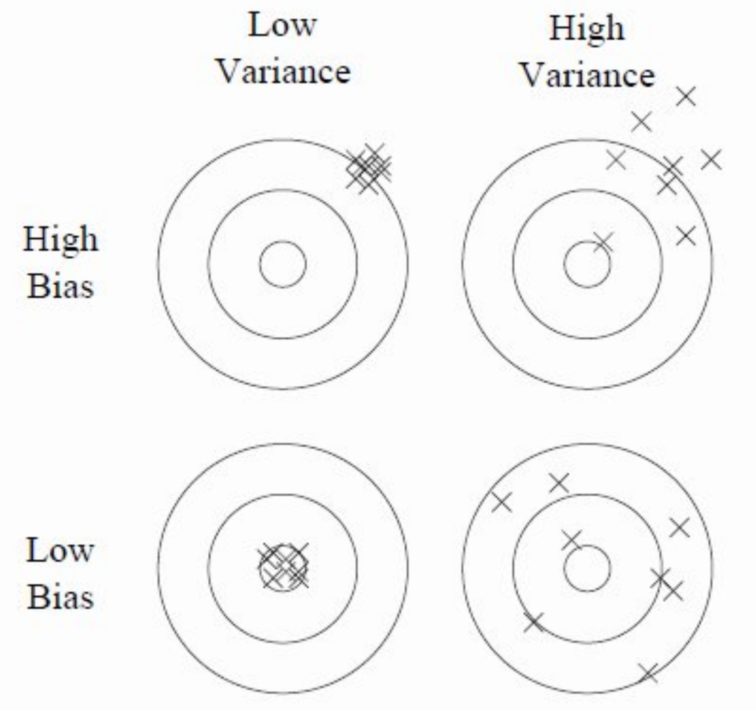

Figure 1: Bias and variance in dart-throwing.

source: http://homes.cs.washington.edu/~pedrod/papers/cacm12.pdf

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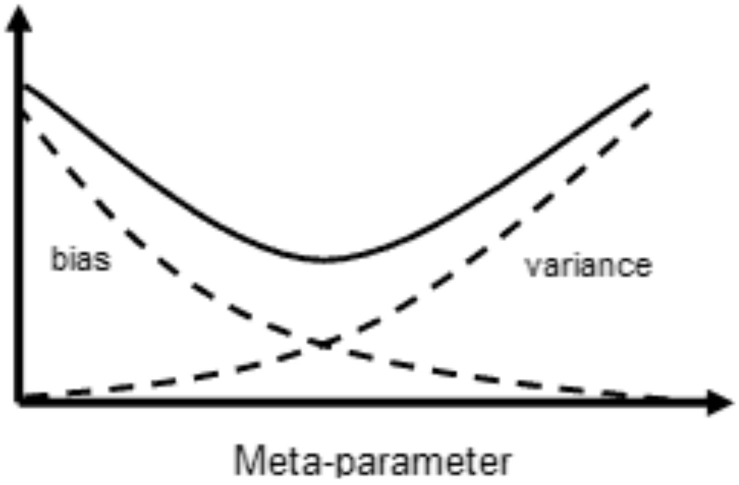
It turns out (after some math) that the generalization error in our model can be decomposed into a bias component and variance component.

This is another example of the **bias-variance tradeoff**.



source: http://www.isu.edu/chem/images/kalivasmeta.gif

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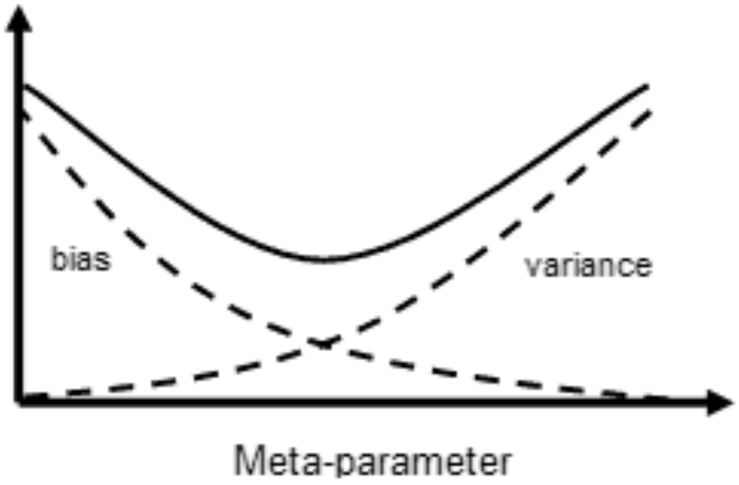

NOTE

The "meta-parameter" here is the lambda we saw above.

A more typical term is "hyperparameter".

source: http://www.isu.edu/chem/images/kalivasmeta.gif

This tradeoff is regulated by a **hyperparameter** λ, which we've already seen:

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L2 regularization: $y = \Sigma\, \beta_i x_i + \varepsilon \quad st. \quad \Sigma\, \beta_i^2 < \lambda$

So regularization represents a method to trade away some variance for a little bias in our model, thus achieving a better overall fit.

EX: POLYNOMIAL REGRESSION & REGULARIZATION