

INTRO to DATA SCIENCE

REGRESSION & REGULARIZATION

I. INTRO TO REGRESSION

II. LINEAR AND POLYNOMIAL

III. REGULARIZATION

EXERCISES:

**III. IMPLEMENTING MULTIPLE REGRESSION AND POLYNOMIAL
REGRESSION IN PYTHON**

I. LINEAR REGRESSION

	<i>continuous</i>	<i>categorical</i>
<i>supervised</i>	???	???
<i>unsupervised</i>	???	???

	<i>continuous</i>	<i>categorical</i>
<i>supervised</i>	regression	classification
<i>unsupervised</i>	dimension reduction	clustering

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ε = **residual** (the prediction error)

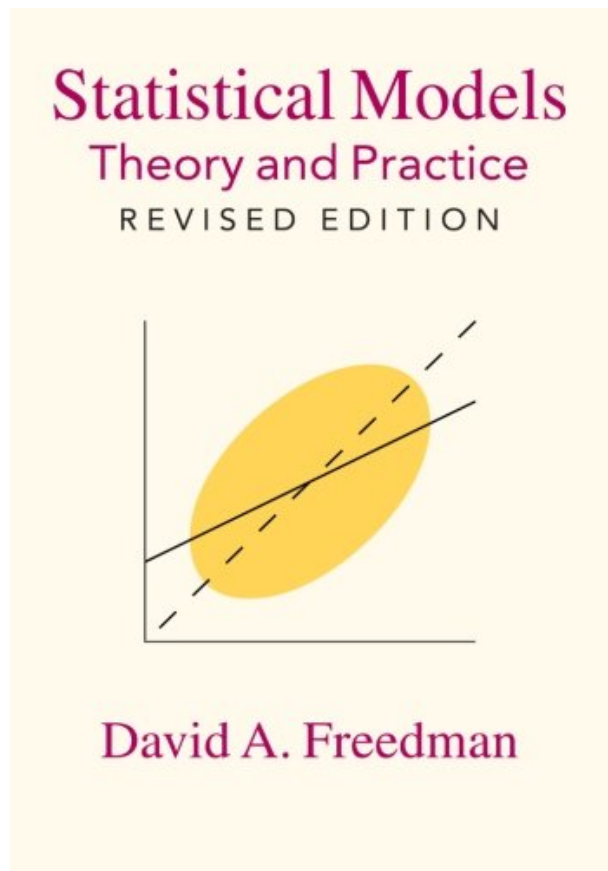
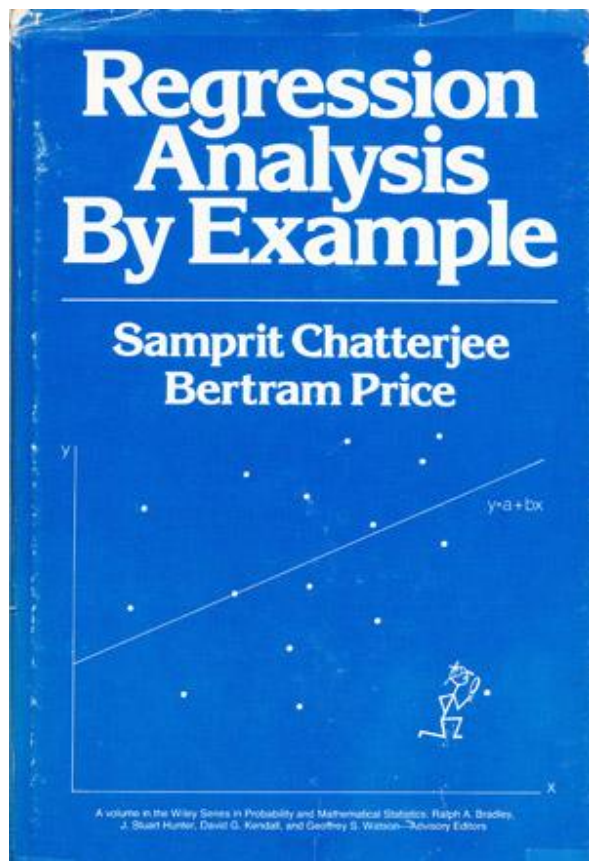
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Linear regression involves several technical assumptions and is often presented with lots of mathematical formality.

The math is not very important for our purposes, but you should check it out if you get serious about solving regression problems.



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But again, if you get serious about regression, you should learn how this works!

INTRO TO DATA SCIENCE

II: POLYNOMIAL REGRESSION

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“Although polynomial regression fits a *nonlinear* model to the data, as a statistical estimation problem it is *linear*, in the sense that the regression function $E(y|x)$ is linear in the unknown parameters that are estimated from the data. For this reason, polynomial regression is considered to be a special case of multiple linear regression.” -- Wikipedia

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A: This model violates one of the assumptions of linear regression!



This model displays **multicollinearity**, which means the predictor variables are highly correlated with each other.

$$y = \alpha + \beta_1 x + \beta_2 x^2 + \dots + \beta_n x^n + \varepsilon$$

```
> x <- seq(1, 10, 0.1)
> cor(x^9, x^10)
[1] 0.9987608
```

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NOTE

This results in a *singularity*. We will see an example of this in just a minute!

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OPTIONAL NOTE

These polynomial functions form an *orthogonal basis* of the function space.

So far, we've seen how polynomial regression allows us to fit complex nonlinear relationships, and even to avoid multicollinearity (by using basis functions).

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Q: Can a regression model be too complex?

III: REGULARIZATION

Recall our earlier discussion of **overfitting**.

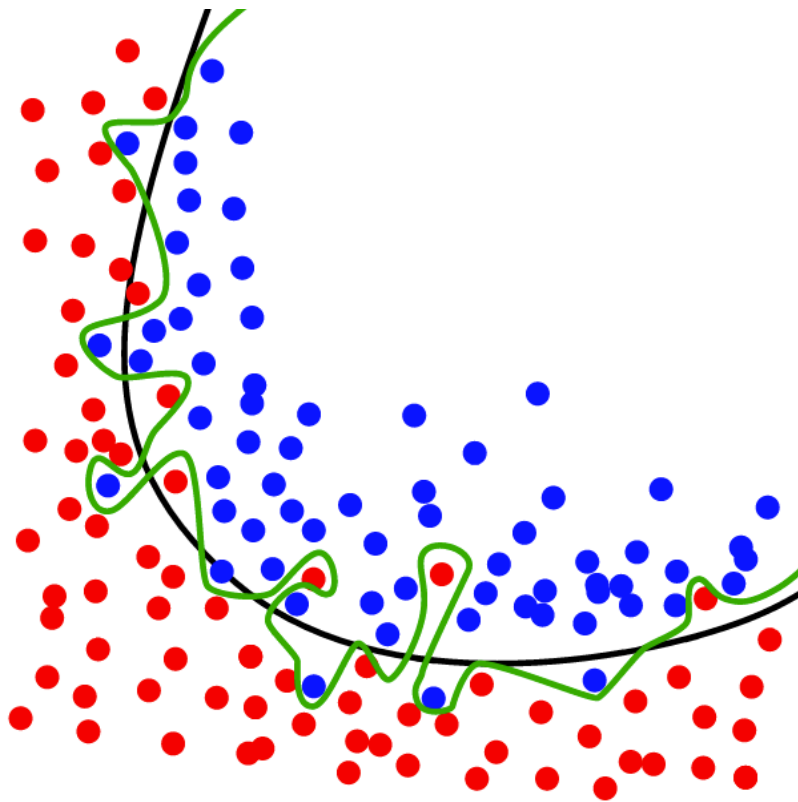
Recall our earlier discussion of **overfitting**.

When we talked about this in the context of classification, we said that it was a result of matching the training set too closely.

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In other words, an overfit model matches the **noise** in the dataset instead of the **signal**.



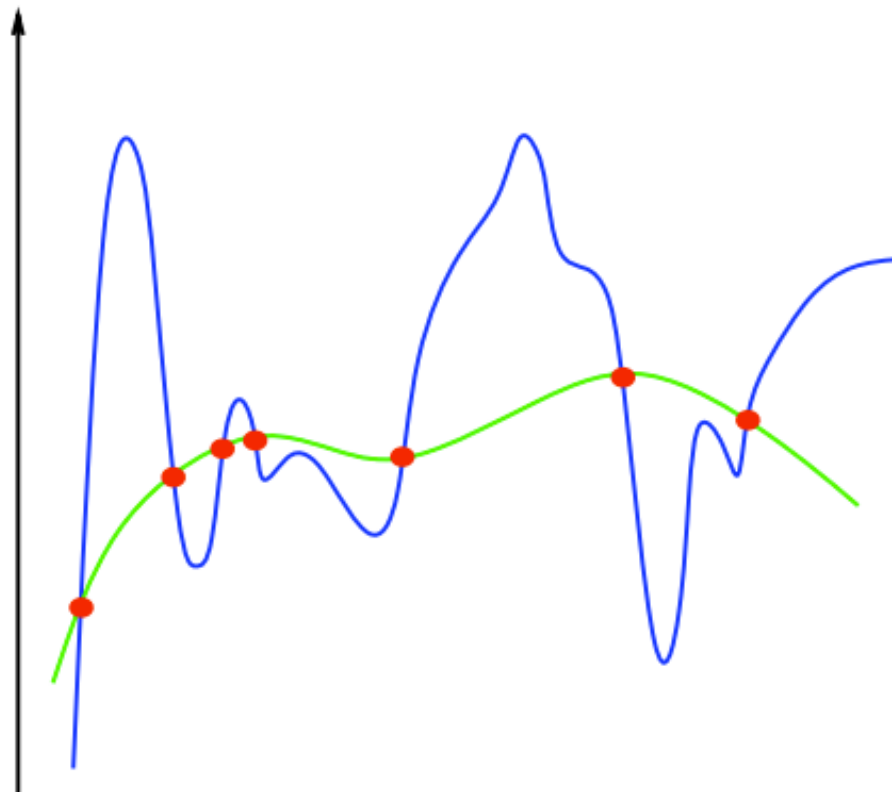
The same thing can happen in regression.

It's possible to design a regression model that matches the noise in the data instead of the signal.

This happens when our model becomes *too complex* for the data to support.

OVERFITTING EXAMPLE (REGRESSION)

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A: One method is to define complexity as a function of the size of the coefficients.

Ex 1: $\sum |\beta_i|$ this is called the **L1-norm**

Ex 2: $\sum \beta_i^2$ this is called the **L2-norm**

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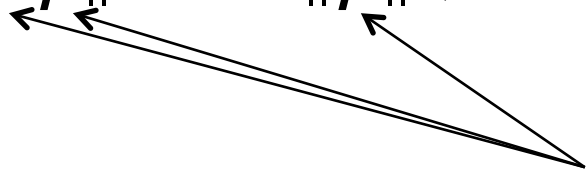
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Regularization refers to the method of preventing **overfitting** by explicitly controlling model **complexity**.

These regularization problems can also be expressed as:

L1 regularization: $\min(\|y - x \bullet \beta\|^2 + \lambda \|\beta\|)$

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Q: Can anyone see what it is?



vectors

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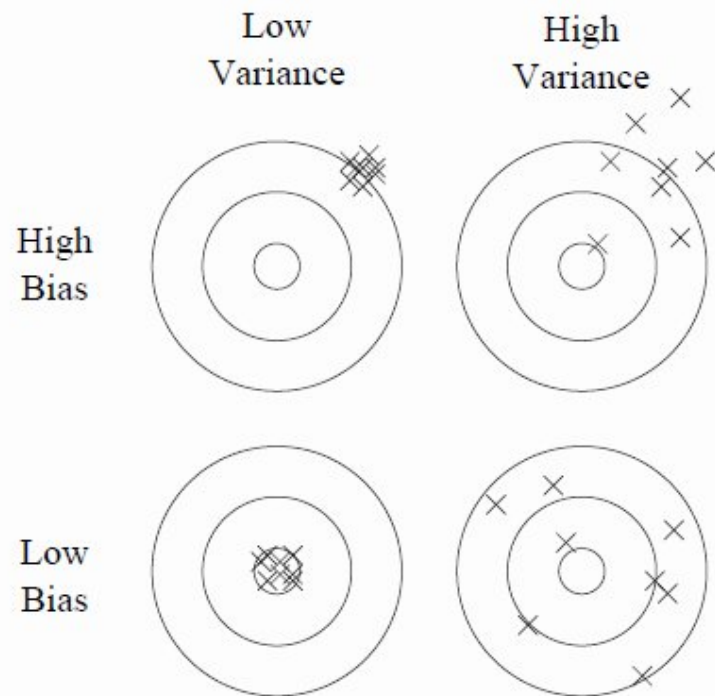


Figure 1: Bias and variance in dart-throwing.

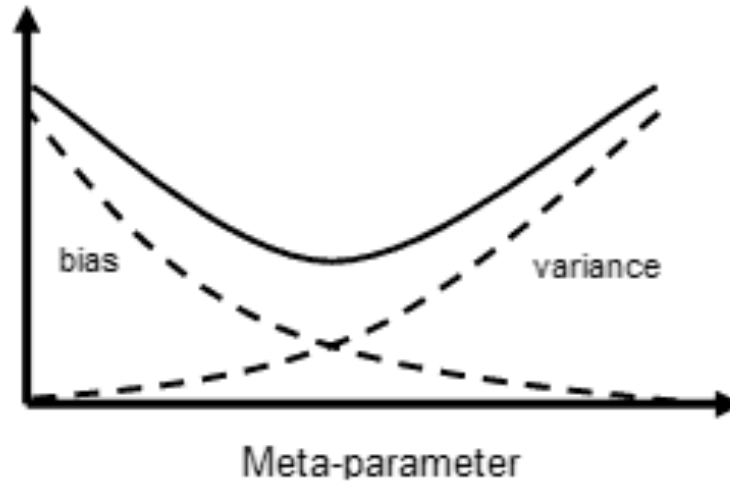
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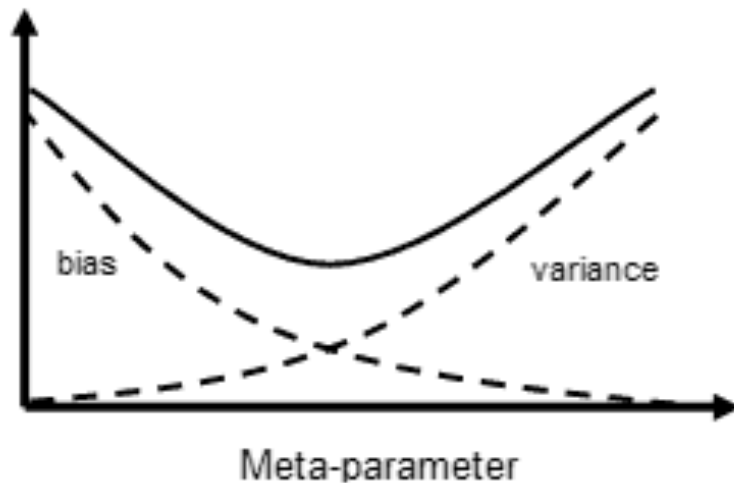
Variance refers to predictions that are *generally* inaccurate.

It turns out (after some math) that the generalization error in our model can be decomposed into a bias component and variance component.

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NOTE

The “meta-parameter” here is the λ we saw above.

A more typical term is “hyperparameter”.

This tradeoff is regulated by a **hyperparameter** λ , which we've already seen:

L1 regularization: $y = \sum \beta_i x_i + \varepsilon \quad st. \quad \sum |\beta_i| < \lambda$

L2 regularization: $y = \sum \beta_i x_i + \varepsilon \quad st. \quad \sum \beta_i^2 < \lambda$

So regularization represents a method to trade away some variance for a little bias in our model, thus achieving a better overall fit.

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**EX: POLYNOMIAL
REGRESSION &
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