CSCI3230 / ESTR3108 2021-22 First Term Assignment 1

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University Guideline on Academic Honesty:
`http://www.cuhk.edu.hk/policy/academichonesty/`

Faculty of Engineering Guidelines to Academic Honesty:
`http://www.erg.cuhk.edu.hk/erg-intra/upload/documents/ENGG_Discipline.pdf`

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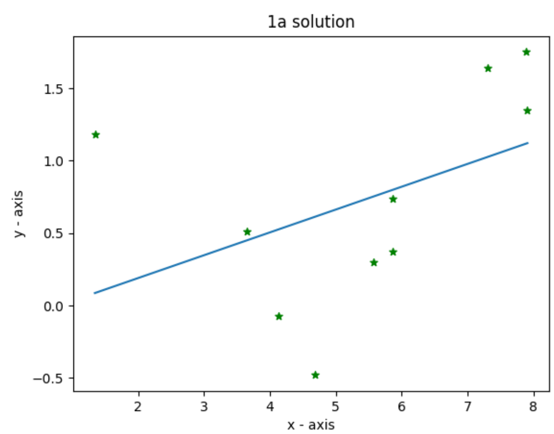
1a)

By $\Theta = (X^T X)^{-1} X^T Y$

$$\Theta = \begin{bmatrix} -0.12471382 \\ 0.15748272 \end{bmatrix}$$

which $\theta_0 = -0.12471382, \theta_1 = 0.15748272$

Figure is shown at right hand side.



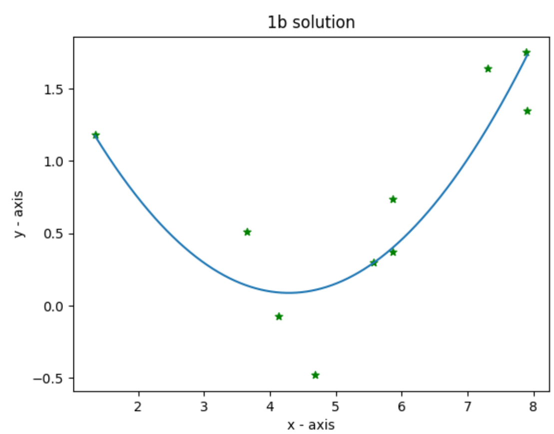
1b)

By $\Theta = (X^T X)^{-1} X^T Y$

$$\Theta = \begin{bmatrix} 2.38902063 \\ -1.07370721 \\ 0.1252789 \end{bmatrix}$$

which $\theta_0 = 2.38902063, \theta_1 = -1.07370721,$
$\theta_2 = 0.1252789$

Figure is shown at right hand side.



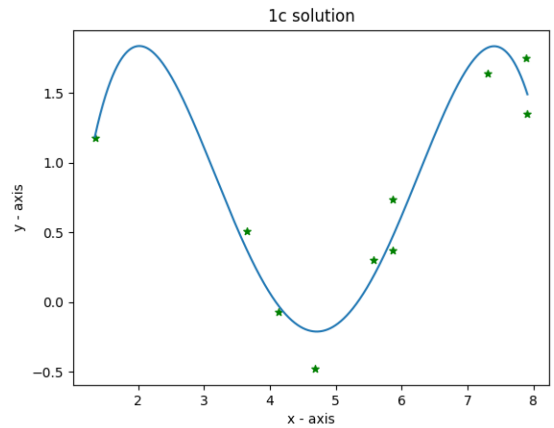
1c)

By $\Theta = (X^T X)^{-1} X^T Y$

$$\Theta = \begin{bmatrix} -6.80776406 \\ 10.91680274 \\ -4.60513794 \\ 0.73170903 \\ -0.03884038 \end{bmatrix}$$

which $\theta_0 = -6.80776406, \theta_1 = 10.91680274,$
$\theta_2 = -4.60513794, \theta_3 = 0.73170903,$
$\theta_4 = -0.03884038$

Figure is shown at right hand side.



1d)

By

$$EPE(X) = E((y - \hat{f}(X))^2)$$

With summarising values,
In 1a) , $EPE(X) = 9.415644$
In 1b) , $EPE(X) = 5.013401$
In 1c) , $EPE(X) = 19.504463$

Therefore,
1a) is underfitting,
1b) is relatively a good one,
1c) is overfitting.

2a)

By differentiating towards θ_0,

$$\begin{aligned}
\frac{\partial J(\Theta, \theta_0)}{\partial \theta_0} &= \sum_{i=1}^{m} 2(X^{(i)} - \bar{X})\Theta + \theta_0 + \bar{X}\Theta - Y^{(i)} \\
&= \sum_{i=1}^{m} 2X^{(i)}\Theta - \bar{X}\Theta + \theta_0 - Y^{(i)} \\
&= 2m\bar{X}\Theta - m\bar{X}\Theta + m\theta_0 - m\bar{Y} \\
&= m\bar{X}\Theta + m\theta_0 - m\bar{Y}
\end{aligned}$$

By letting $\dfrac{\partial J(\Theta, \theta_0)}{\partial \theta_0} = 0$,

$$\begin{aligned}
0 &= m\bar{X}\Theta + m\theta_0 - m\bar{Y} \\
0 &= \bar{X}\Theta + \theta_0 - \bar{Y} \\
\theta_0 &= \bar{Y} - \bar{X}\Theta
\end{aligned}$$

Therefore, when $\theta_0 = \bar{Y} - \bar{X}\Theta$, we will get the minimum value for $J(\Theta, \theta_0)$.

2b)

Considering $Y^{(i)} = Y_c^{(i)} + \bar{Y}$ and plugging in θ_0,

$$\begin{aligned} J(\Theta, \theta_0) = J_c(\Theta) &= \sum_{i=1}^{m} (X_c^{(i)}\Theta + \bar{Y} - \bar{X}\Theta + \bar{X}\Theta - Y_c^{(i)} - \bar{Y})^2 + \lambda \|\Theta\|_2^2 \\ &= \sum_{i=1}^{m} (X_c^{(i)}\Theta - Y_c^{(i)})^2 + \lambda \|\Theta\|_2^2 \end{aligned}$$

2c)

$$J_c(\Theta) = \sum_{i=1}^{m} (X_c^{(i)}\Theta - Y_c^{(i)})^2 + \lambda \|\Theta\|_2^2$$

By considering the minimum value of this formula can be obtained by the sum of each minimum part of summation, so I will remove the summation sign here and then take derivatives,

$$\begin{aligned} J_c'(\Theta) &= \frac{\partial}{\partial \Theta}((X_c\Theta - Y_c)^T (X_c\Theta - Y_c) + \lambda \Theta^T \Theta) \\ &= \frac{\partial}{\partial \Theta}(\Theta^T X_c^T X_c \Theta - \Theta^T X_c^T Y_c - Y_c^T X_c \Theta + Y_c^T Y_c + \lambda \Theta^T \Theta) \\ &= 2X_c^T X_c \Theta - X_c^T Y_c - X_c^T Y_c + 2\lambda\Theta \\ &= 2X_c^T X_c \Theta - 2X_c^T Y_c + 2\lambda\Theta \end{aligned}$$

By taking $J_c' = 0$,

$$\begin{aligned} 0 &= 2X_c^T X_c \Theta - 2X_c^T Y_c + 2\lambda\Theta \\ 2X_c^T Y_c &= 2X_c^T X_c \Theta + 2\lambda\Theta \\ X_c^T Y_c &= (X_c^T X_c + \lambda I)\Theta \\ \Theta &= (X_c^T X_c + \lambda I)^{-1} X_c^T Y_c \end{aligned}$$

Therefore, the analytic solution $\hat{\Theta} = (X_c^T X_c + \lambda I)^{-1} X_c^T Y_c$.

3a)

Logistic model

$$P(\hat{y}=1 \mid x_1, x_2) = \frac{1}{1+e^{-(\theta_0+\theta_1 x_1+\theta_2 x_2)}}$$

Cross-entropy function

$$-y^{(i)} \ln(P(\hat{y}=1 \mid x_1, x_2)) - (1-y^{(i)}) \ln(1-P(\hat{y}=1 \mid x_1, x_2))$$

Plug-in logistic model,

$$-y^{(i)} \ln(\frac{1}{1+e^{-(\theta_0+\theta_1 x_1+\theta_2 x_2)}}) - (1-y^{(i)}) \ln(1-\frac{1}{1+e^{-(\theta_0+\theta_1 x_1+\theta_2 x_2)}})$$

3b) New regression model $\hat{\Theta}$

$$\begin{aligned}
\hat{\Theta} &= \Theta - \alpha \sum_{i=1}^{m} \left(\frac{1}{e^{-X^{(i)T}\Theta}} - y^{(i)} \right) X^{(i)} \\
&= \begin{bmatrix} -1 \\ 1.5 \\ 0.5 \end{bmatrix} - 0.1 \Bigg(\begin{bmatrix} 0.47726569 \\ 0.16513393 \\ 0.37226724 \end{bmatrix} + \begin{bmatrix} 0.4192142 \\ 0.1270219 \\ 0.18403503 \end{bmatrix} + \begin{bmatrix} 0.47539489 \\ 0.17019137 \\ 0.34656288 \end{bmatrix} \\
&+ \begin{bmatrix} 1.39724159 \\ 0.84113944 \\ 1.20581949 \end{bmatrix} + \begin{bmatrix} 1.75330048 \\ 1.38510738 \\ 1.32023526 \end{bmatrix} + \begin{bmatrix} -1.01899756 \\ 1.53210518 \\ 0.51181202 \end{bmatrix} \Bigg) \\
&= \begin{bmatrix} -1.01899756 \\ 1.53210518 \\ 0.51181202 \end{bmatrix}
\end{aligned}$$

Therefore, the updated function is

$$P(\hat{y}=1 \mid x_1, x_2) = \frac{1}{1+e^{-(-1.01899756+1.53210518 x_1+0.51181202 x_2)}}$$

3c)

Index 1 (y = 0): $P(\hat{y}=1 \mid x_1, x_2) = 0.656064168$, FP
Index 2 (y = 0): $P(\hat{y}=1 \mid x_1, x_2) = 0.571187019$, FP
Index 3 (y = 0): $P(\hat{y}=1 \mid x_1, x_2) = 0.493790599$, TN
Index 4 (y = 1): $P(\hat{y}=1 \mid x_1, x_2) = 0.653157925$, TP
Index 5 (y = 1): $P(\hat{y}=1 \mid x_1, x_2) = 0.644341574$, TP
Index 6 (y = 1): $P(\hat{y}=1 \mid x_1, x_2) = 0.621692311$, TP
Therefore, we have 3 TP, 1 TN, 2 FP, 0 FN

Accuracy $= \frac{3+1}{3+1+2+0} = \frac{2}{3}$
Precision $= \frac{3}{3+2} = \frac{3}{5}$
Recall $= \frac{3}{3+0} = 1$

4a)
I will find the partial derivatives of

$$P(\hat{y}_i = 1 \mid X) = \frac{e^{X^T\Theta_i}}{\sum_{k=1}^{K} e^{X^T\Theta_k}}$$

by quotient rule.

Case 1: $i = j$

$$\begin{aligned}
\frac{\partial P(\hat{y}_i = 1 \mid X)}{\partial X^T\Theta_j} = \frac{\partial P(\hat{y}_i = 1 \mid X)}{\partial X^T\Theta_i} &= \frac{e^{X^T\Theta_i}\sum_{k=1}^{K} e^{X^T\Theta_k} - (e^{X^T\Theta_i})^2}{(\sum_{k=1}^{K} e^{X^T\Theta_k})^2} \\
&= \frac{e^{X^T\Theta_i}}{\sum_{k=1}^{K} e^{X^T\Theta_k}} - \left(\frac{e^{X^T\Theta_i}}{\sum_{k=1}^{K} e^{X^T\Theta_k}}\right)^2 \\
&= P(\hat{y}_i = 1 \mid X) - (P(\hat{y}_i = 1 \mid X))^2 \\
&= P(\hat{y}_i = 1 \mid X)(1 - P(\hat{y}_j = 1 \mid X))
\end{aligned}$$

Case 2: $i \neq j$

$$\begin{aligned}
\frac{\partial P(\hat{y}_i = 1 \mid X)}{\partial X^T\Theta_j} &= \frac{0 - (e^{X^T\Theta_i})(e^{X^T\Theta_j})}{(\sum_{k=1}^{K} e^{X^T\Theta_k})^2} \\
&= -\left(\frac{(e^{X^T\Theta_i})(e^{X^T\Theta_j})}{(\sum_{k=1}^{K} e^{X^T\Theta_k})(\sum_{k=1}^{K} e^{X^T\Theta_k})}\right) \\
&= -P(\hat{y}_i = 1 \mid X)P(\hat{y}_j = 1 \mid X)
\end{aligned}$$

Therefore,

$$\frac{\partial P(\hat{y}_i = 1 \mid X)}{\partial X^T\Theta_j} = \begin{cases} P(\hat{y}_i = 1 \mid X)(1 - P(\hat{y}_j = 1 \mid X)) & \text{if } x = y \\ -P(\hat{y}_i = 1 \mid X)P(\hat{y}_j = 1 \mid X) & \text{if } x \neq y \end{cases}$$

4b)
Considering

$$\begin{aligned}
\frac{\partial X^T\Theta_j}{\partial \Theta_j} &= (X^T)^T \\
&= X
\end{aligned}$$

Therefore, by using a) and above result,

$$\begin{aligned}
\frac{\partial P(\hat{y}_i = 1 \mid X)}{\partial \Theta_j} &= \frac{\partial P(\hat{y}_i = 1 \mid X)}{\partial X^T\Theta_j}\frac{\partial X^T\Theta_j}{\partial \Theta_j} \\
&= \begin{cases} P(\hat{y}_i = 1 \mid X)(1 - P(\hat{y}_j = 1 \mid X))X & \text{if } x = y \\ -P(\hat{y}_i = 1 \mid X)P(\hat{y}_j = 1 \mid X)X & \text{if } x \neq y \end{cases}
\end{aligned}$$

4c)
Considering

$$
\begin{aligned}
\frac{\partial E(\Theta)}{\partial P(\hat{y}_i = 1 \mid X)} &= \frac{\partial - \sum_{i=1}^{K} y_i \ln P(\hat{y}_i = 1 \mid X)}{\partial P(\hat{y}_i = 1 \mid X)} \\
&= - \sum_{i=1}^{K} \left(\frac{\partial y_i \ln P(\hat{y}_i = 1 \mid X)}{\partial P(\hat{y}_i = 1 \mid X)} \right) \\
&= - \sum_{i=1}^{K} \left(0 + y_i \frac{1}{P(\hat{y}_i = 1 \mid X)} \right) \\
&= - \sum_{i=1}^{K} \left(\frac{y_i}{P(\hat{y}_i = 1 \mid X)} \right) \\
&= \frac{-1}{P(\hat{y}_i = 1 \mid X)} && \text{Since only one of the } y_i \text{ will be 1}
\end{aligned}
$$

Therefore, combining above and b),

Case 1: $i = j$

$$
\begin{aligned}
\frac{\partial E(\Theta)}{\partial \Theta_j} &= \frac{\partial E(\Theta)}{\partial P(\hat{y}_i = 1 \mid X)} \frac{\partial P(\hat{y}_i = 1 \mid X)}{\partial X^T \Theta_j} \frac{\partial X^T \Theta_j}{\partial \Theta_j} \\
&= \frac{-1}{P(\hat{y}_i = 1 \mid X)} P(\hat{y}_i = 1 \mid X)(1 - P(\hat{y}_j = 1 \mid X))X \\
&= (P(\hat{y}_j = 1 \mid X) - 1)X \\
&= XP(\hat{y}_j = 1 \mid X) - X \\
&= XP(\hat{y}_j = 1 \mid X) - Xy_j && \text{By case condition, } i = j \text{ means } j \text{ is the true class, so } y_j \text{ must be 1}
\end{aligned}
$$

Case 2: $i \neq j$

$$
\begin{aligned}
\frac{\partial E(\Theta)}{\partial \Theta_j} &= \frac{\partial E(\Theta)}{\partial P(\hat{y}_i = 1 \mid X)} \frac{\partial P(\hat{y}_i = 1 \mid X)}{\partial X^T \Theta_j} \frac{\partial X^T \Theta_j}{\partial \Theta_j} \\
&= \frac{-1}{P(\hat{y}_i = 1 \mid X)} (-P(\hat{y}_i = 1 \mid X)(P(\hat{y}_j = 1 \mid X))X) \\
&= P(\hat{y}_i = 1 \mid X)X \\
&= XP(\hat{y}_j = 1 \mid X) - Xy_j && \text{By case condition, } i \neq j \text{ means } j \text{ is } \textbf{NOT} \text{ the true class, so } y_j \text{ must be 0}
\end{aligned}
$$

Combining 2 cases,

$$
\frac{\partial E(\Theta)}{\partial \Theta_j} = XP(\hat{y}_j = 1 \mid X) - Xy_j
$$