Numerical ODE Solvers

Theorem 1 (Picard's Theorem[2]) *Suppose that $f(\cdot,\cdot)$ is a continuous function of its arguments in a region U of the (x, y) plane which contains the rectangle*

$$\mathsf{R} = \{(x, y) : x_0 \le x \le X_M, \quad |y - y_0| \le Y_M\},$$

where $X_M > x_0$ and $Y_M > 0$ are constants. Suppose also, that there exists a positive constant L such that

$$\boxed{|f(x, y) - f(x, z)| \le L|y - z|}$$

holds whenever (x, y) and (x, z) lie in the rectangle R. *Finally, letting*

$$M = \max\{|f(x, y)| : (x, y) \in \mathsf{R}\},$$

suppose that $M(X_M - x_0) \le Y_M$. Then there exists a unique continuously differentiable function $x \mapsto y(x)$, defined on the closed interval $[x_0, X_M]$

- Lipschitz's condition

Euler's method

A simple derivation of Euler's method proceeds by first integrating the differential equation (1) between two consecutive mesh points x_n and x_{n+1} to deduce that

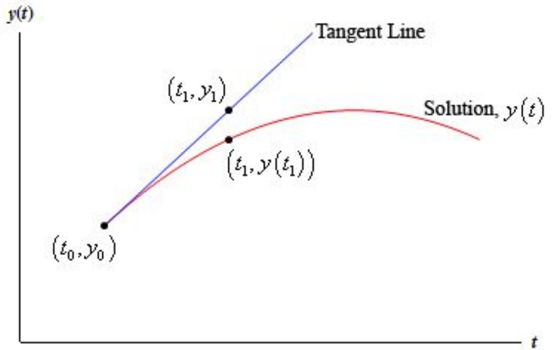
$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))\,\mathrm{d}x\,, \qquad n = 0, \ldots, N-1\,, \tag{14}$$

and then applying the numerical integration rule

$$\int_{x_n}^{x_{n+1}} g(x)\,\mathrm{d}x \approx h g(x_n)\,,$$

called the **rectangle rule**, with $g(x) = f(x, y(x))$, to get

$$y(x_{n+1}) \approx y(x_n) + h f(x_n, y(x_n))\,, \qquad n = 0, \ldots N-1\,, \qquad y(x_0) = y_0\,.$$



Euler’s method - generalization

This then motivates the definition of Euler’s method. The idea can be generalised by replacing the rectangle rule in the derivation of Euler’s method with a one-parameter family of integration rules of the form

$$\int_{x_n}^{x_{n+1}} g(x)\,\mathrm{d}x \approx h\left[(1-\theta)g(x_n) + \theta g(x_{n+1})\right] \ ,$$

with $\theta \in [0,1]$ a parameter. On applying this in (14) with $g(x) = f(x, y(x))$ we find that

$$\begin{aligned} y(x_{n+1}) &\approx y(x_n) + h\left[(1-\theta)f(x_n, y(x_n)) + \theta f(x_{n+1}, y(x_{n+1}))\right] \ , \quad n = 0, \ldots, N-1 \ , \\ y(x_0) &= y_0 \ . \end{aligned}$$

This then motivates the introduction of the following one-parameter family of methods: given that y_0 is supplied by (2), define

$$y_{n+1} = y_n + h\left[(1-\theta)f(x_n, y_n) + \theta f(x_{n+1}, y_{n+1})\right] \ , \quad n = 0, \ldots, N-1 \ .$$

Euler’s method - analytical example

Given the initial value problem $y' = x - y^2$, $y(0) = 0$, *on the interval of* $x \in [0, 0.4]$, *we compute an approximate solution using the* θ*-method, for* $\theta = 0$, $\theta = 1/2$ *and* $\theta = 1$, *using the step size* $h = 0.1$. *The results are shown in Table 1. In the case of the two implicit methods, corresponding to* $\theta = 1/2$ *and* $\theta = 1$, *the nonlinear equations have been solved by a fixed-point iteration.*

k	x_k	y_k for $\theta = 0$	y_k for $\theta = 1/2$	y_k for $\theta = 1$
0	0	0	0	0
1	0.1	0	0.00500	0.00999
2	0.2	0.01000	0.01998	0.02990
3	0.3	0.02999	0.04486	0.05955
4	0.4	0.05990	0.07944	0.09857

Table 1: The values of the numerical solution at the mesh points

Euler's method - analytical example 2

For comparison, we also compute the value of the analytical solution $y(x)$ *at the mesh points* $x_n = 0.1 * n$, $n = 0, \ldots, 4$. *Since the solution is not available in closed form, we use a Picard iteration to calculate an accurate approximation to the analytical solution on the interval* $[0, 0.4]$ *and call this the "exact solution". Thus, we consider*

$$y_0(x) \equiv 0 \,, \qquad y_k(x) = \int_0^x \left(\xi - y_{k-1}^2(\xi)\right) \, d\xi \,, \quad k = 1, 2, \ldots \,.$$

Hence,

$$\begin{aligned}
y_0(x) &\equiv 0 \,, \\
y_1(x) &= \frac{1}{2}x^2 \,, \\
y_2(x) &= \frac{1}{2}x^2 - \frac{1}{20}x^5 \,, \\
y_3(x) &= \frac{1}{2}x^2 - \frac{1}{20}x^5 + \frac{1}{160}x^8 - \frac{1}{4400}x^{11} \,.
\end{aligned}$$

Euler's method - analytical example 3

Using MAPLE V, we obtain the solution in terms of Bessel functions:

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dsolve({diff(y(x),x) + y(x)*y(x) = x, y(0)=0}, y(x));
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$$\mathrm{y}(x) = -\frac{\sqrt{x}\left(\dfrac{\sqrt{3}\,\mathrm{BesselK}(\frac{-2}{3}, \frac{2}{3}\,x^{3/2})}{\pi} - \mathrm{BesselI}(\frac{-2}{3}, \frac{2}{3}\,x^{3/2})\right)}{\dfrac{\sqrt{3}\,\mathrm{BesselK}(\frac{1}{3}, \frac{2}{3}\,x^{3/2})}{\pi} + \mathrm{BesselI}(\frac{1}{3}, \frac{2}{3}\,x^{3/2})}$$

k	x_k	$y(x_k)$
0	0	0
1	0.1	0.00500
2	0.2	0.01998
3	0.3	0.04488
4	0.4	0.07949

Table 2: Values of the "exact solution" at the mesh points

Euler's method - error analysis

$$|e_n| \;\le\; |e_0| \exp\left(L\frac{x_n - x_0}{1-\theta L h}\right) + \frac{h}{L}\left\{\left|\frac{1}{2}-\theta\right| M_2 + \frac{1}{3} h M_3\right\}\left[\exp\left(L\frac{x_n - x_0}{1-\theta L h}\right) - 1\right]$$

Euler’s method - General explicit one-step method

A general explicit one-step method may be written in the form:

$$y_{n+1} = y_n + h\Phi(x_n, y_n; h)\,, \quad n = 0, \ldots, N-1\,, \quad y_0 = y(x_0)\,[= \text{specified by } (2)]\,,$$

where $\Phi(\cdot, \cdot; \cdot)$ is a continuous function of its variables. For example, in the case of Euler’s method, $\Phi(x_n, y_n; h) = f(x_n, y_n)$, while for the improved Euler method

$$\Phi(x_n, y_n; h) = \frac{1}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right]\,.$$

Consider the general one-step method (22) where, in addition to being a continuous function of its arguments, Φ is assumed to satisfy a Lipschitz condition with respect to its second argument; namely, there exists a positive constant L_Φ such that, for $0 \le h \le h_0$ and for the same region R *as in Picard’s theorem,*

$$|\Phi(x, y; h) - \Phi(x, z; h)| \le L_\Phi |y - z|, \qquad \text{for } (x, y),\ (x, z) \text{ in } \mathrm{R}\,.$$

Then, assuming that $|y_n - y_0| \le Y_M$, it follows that

$$|e_n| \le \mathrm{e}^{L_\Phi(x_n - x_0)}|e_0| + \left[\frac{\mathrm{e}^{L_\Phi(x_n - x_0)} - 1}{L_\Phi}\right] T\,, \quad n = 0, \ldots, N\,,$$

where $T = \max_{0 \le n \le N-1} |T_n|\,.$

Euler’s method - no more theory

Forward method

$$y'(t) \approx \frac{y(t+h) - y(t)}{h},$$

$$\Downarrow$$

$$y_{n+1} = y_n + hf(t_n, y_n).$$

Backward method

$$y'(t) \approx \frac{y(t) - y(t-h)}{h},$$

$$\Downarrow$$

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}).$$

Euler’s method - practical example

$$y' + 2y = 2 - \mathbf{e}^{-4t} \qquad y(0) = 1$$

Use Euler’s Method with a step size of $h = 0.1$ to find approximate values of the solution at $t = 0.1, 0.2, 0.3,$ 0.4, and 0.5. Compare them to the exact values of the solution at these points.

Euler’s method - practical example (2)

This is a fairly simple linear differential equation so we’ll leave it to you to check that the solution is

$$y(t) = 1 + \frac{1}{2}\mathbf{e}^{-4t} - \frac{1}{2}\mathbf{e}^{-2t}$$

n order to use Euler’s Method we first need to rewrite the differential equation into the right form

$$y' = 2 - \mathbf{e}^{-4t} - 2y$$

From this we can see that $f(t, y) = 2 - \mathbf{e}^{-4t} - 2y$. Also note that $t_0 = 0$ and $y_0 = 1$. We can now start doing some computations.

$$f_0 = f(0, 1) = 2 - \mathbf{e}^{-4(0)} - 2(1) = -1$$
$$y_1 = y_0 + h\, f_0 = 1 + (0.1)(-1) = 0.9$$

Euler's method - practical example (3)

So, the approximation to the solution at $t1 = 0.1$ is $y1=0.9$

$$f_1 = f(0.1, 0.9) = 2 - \mathbf{e}^{-4(0.1)} - 2(0.9) = -0.470320046$$
$$y_2 = y_1 + h\, f_1 = 0.9 + (0.1)(-0.470320046) = 0.852967995$$

Therefore, the approximation to the solution at

$$t_2 = 0.2 \text{ is } y_2 = 0.852967995.$$

I'll leave it to you to check the remainder of these computations.

$$f_2 = -0.155264954 \qquad y_3 = 0.837441500$$
$$f_3 = 0.023922788 \qquad y_4 = 0.839833779$$
$$f_4 = 0.1184359245 \qquad y_5 = 0.851677371$$

Euler’s method - practical example (4)

Time, t_n	Approximation	Exact	Error
$t_0 = 0$	$y_0 = 1$	$y(0) = 1$	0 %
$t_1 = 0.1$	$y_1 = 0.9$	$y(0.1) = 0.925794646$	2.79 %
$t_2 = 0.2$	$y_2 = 0.852967995$	$y(0.2) = 0.889504459$	4.11 %
$t_3 = 0.3$	$y_3 = 0.837441500$	$y(0.3) = 0.876191288$	4.42 %
$t_4 = 0.4$	$y_4 = 0.839833779$	$y(0.4) = 0.876283777$	4.16 %
$t_5 = 0.5$	$y_5 = 0.851677371$	$y(0.5) = 0.883727921$	3.63 %

Euler's method - practical example 2 (1)

Repeat the previous example only this time give the approximations at

$t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. Use $h = 0.1$, $h = 0.05$, $h = 0.01$, $h = 0.005$, and $h = 0.001$ f

Approximations

Time	Exact	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	0.9414902	0.9313244	0.9364698	0.9404994	0.9409957	0.9413914
$t = 2$	0.9910099	0.9913681	0.9911126	0.9910193	0.9910139	0.9910106
$t = 3$	0.9987637	0.9990501	0.9988982	0.9987890	0.9987763	0.9987662
$t = 4$	0.9998323	0.9998976	0.9998657	0.9998390	0.9998357	0.9998330
$t = 5$	0.9999773	0.9999890	0.9999837	0.9999786	0.9999780	0.9999774

Euler's method - practical example 2 (2)

Percentage Errors

Time	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	1.08 %	0.53 %	0.105 %	0.053 %	0.0105 %
$t = 2$	0.036 %	0.010 %	0.00094 %	0.00041 %	0.0000703 %
$t = 3$	0.029 %	0.013 %	0.0025 %	0.0013 %	0.00025 %
$t = 4$	0.0065 %	0.0033 %	0.00067 %	0.00034 %	0.000067 %
$t = 5$	0.0012 %	0.00064 %	0.00013 %	0.000068 %	0.000014 %

We can see from these tables that decreasing h does in fact improve the accuracy of the approximation as we expected.

Euler’s method - practical example 3 (2)

Use Euler’s Method to find the approximation to the solution at

$$y' - y = -\frac{1}{2}\mathbf{e}^{\frac{t}{2}}\sin(5t) + 5\mathbf{e}^{\frac{t}{2}}\cos(5t) \qquad y(0) = 0$$

$t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. Use $h = 0.1$, $h = 0.05$, $h = 0.01$, $h = 0.005$, and $h = 0.001$ for the approximations.

We’ll leave it to you to check the details of the solution process. The solution to this linear first order differential equation is.

$$y(t) = \mathbf{e}^{\frac{t}{2}}\sin(5t)$$

Euler’s method - practical example 3 (2)

Approximations

Time	Exact	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	-1.58100	-0.97167	-1.26512	-1.51580	-1.54826	-1.57443
$t = 2$	-1.47880	0.65270	-0.34327	-1.23907	-1.35810	-1.45453
$t = 3$	2.91439	7.30209	5.34682	3.44488	3.18259	2.96851
$t = 4$	6.74580	15.56128	11.84839	7.89808	7.33093	6.86429
$t = 5$	-1.61237	21.95465	12.24018	1.56056	0.0018864	-1.28498

Euler's method - practical example 3 (3)

Percentage Errors

Time	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
t = 1	38.54 %	19.98 %	4.12 %	2.07 %	0.42 %
t = 2	144.14 %	76.79 %	16.21 %	8.16 %	1.64 %
t = 3	150.55 %	83.46 %	18.20 %	9.20 %	1.86 %
t = 4	130.68 %	75.64 %	17.08 %	8.67 %	1.76 %
t = 5	1461.63 %	859.14 %	196.79 %	100.12 %	20.30 %

Euler’s method - practical example 3 (4)

So, with this example Euler’s Method does not do nearly as well as it did on the first IVP. Some of the observations we made in Example 2 are still true however. Decreasing the size of **h** decreases the error as we saw with the last example and would expect to happen. Also, as we saw in the last example, decreasing **h** by a factor of **10** also decreases the error by about a factor of **10**.

However, unlike the last example increasing **t** sees an increasing error. This behavior is fairly common in the approximations. We shouldn’t expect the error to decrease as **t** increases as we saw in the last example. Each successive approximation is found using a previous approximation. Therefore, at each step we introduce error and so approximations should, in general, get worse as **t** increases.

Below is a graph of the solution (the line)

as well as the approximations

(the dots) for **h = 0.05**