1996 年数学(一)真题解析

一、填空题

(1)**【答案】** $\ln 2$.

【解】 由$\lim\limits_{x\to\infty}\left(\dfrac{x+2a}{x-a}\right)^{x}=\lim\limits_{x\to\infty}\left[\left(1+\dfrac{3a}{x-a}\right)^{\frac{x-a}{3a}}\right]^{x\cdot\frac{3a}{x-a}}=\mathrm{e}^{3a}=8$,

得 $3a=3\ln 2$,即得 $a=\ln 2$.

(2)**【答案】** $2x+2y-3z=0$,

【解】 设所求的平面方程为 $\pi:Ax+By+Cz+D=0$,

因为该平面经过原点,所以 $D=0$,

又因为该平面经过点$(6,-3,2)$,所以 $6A-3B+2C=0$,

又因为该平面与平面 $4x-y+2z=8$ 垂直,则 $4A-B+2C=0$,

解得 $B=A,C=-\dfrac{3}{2}A$,故所求平面为 $\pi:Ax+Ay-\dfrac{3}{2}Az=0$,即 $\pi:2x+2y-3z=0$.

(3)**【答案】** $y=\mathrm{e}^x(C_1\cos x+C_2\sin x)+\mathrm{e}^x$($C_1,C_2$ 为任意常数).

【解】 特征方程为 $\lambda^2-2\lambda+2=0$,特征根为 $\lambda_{1,2}=1\pm\mathrm{i}$,

$y''-2y'+2y=0$ 的通解为 $y=\mathrm{e}^x(C_1\cos x+C_2\sin x)$;

显然 $y=\mathrm{e}^x$ 为方程 $y''-2y'+2y=\mathrm{e}^x$ 的一个特解,

故 $y''-2y'+2y=\mathrm{e}^x$ 的通解为 $y=\mathrm{e}^x(C_1\cos x+C_2\sin x)+\mathrm{e}^x$($C_1,C_2$ 为任意常数).

(4)**【答案】** $\dfrac{1}{2}$.

【解】 $\dfrac{\partial u}{\partial x}=\dfrac{1}{x+\sqrt{y^2+z^2}}$,

$$\frac{\partial u}{\partial y}=\frac{1}{x+\sqrt{y^2+z^2}}\cdot\frac{y}{\sqrt{y^2+z^2}},$$

$$\frac{\partial u}{\partial z}=\frac{1}{x+\sqrt{y^2+z^2}}\cdot\frac{z}{\sqrt{y^2+z^2}},$$

$$\left.\frac{\partial u}{\partial x}\right|_{(1,0,1)}=\frac{1}{2},\quad \left.\frac{\partial u}{\partial y}\right|_{(1,0,1)}=0,\quad \left.\frac{\partial u}{\partial z}\right|_{(1,0,1)}=\frac{1}{2},\quad \overrightarrow{AB}=\{2,-2,1\},$$

$$\cos\alpha=\frac{2}{3},\quad \cos\beta=-\frac{2}{3},\quad \cos\gamma=\frac{1}{3},$$

则所求的方向导数为$\left.\dfrac{\partial u}{\partial x}\right|_{(1,0,1)}\cos\alpha+\left.\dfrac{\partial u}{\partial y}\right|_{(1,0,1)}\cos\beta+\left.\dfrac{\partial u}{\partial z}\right|_{(1,0,1)}\cos\gamma=\dfrac{1}{2}$.

(5)**【答案】** 2.

【解】 因为 $|\boldsymbol{B}|=\begin{vmatrix}1&0&2\\0&2&0\\-1&0&3\end{vmatrix}=10\neq0$,所以矩阵 $\boldsymbol{B}$ 可逆,

由矩阵秩的性质得 $r(\boldsymbol{AB})=r(\boldsymbol{A})=2$.

二、选择题

(1)**【答案】** (D).

【解】 $P(x,y)=\dfrac{x+ay}{(x+y)^2}$, $Q(x,y)=\dfrac{y}{(x+y)^2}$,

$$\frac{\partial P}{\partial y}=\frac{a(x+y)^2-2(x+y)(x+ay)}{(x+y)^4}=\frac{a(x+y)-2(x+ay)}{(x+y)^3},\quad \frac{\partial Q}{\partial x}=\frac{-2y}{(x+y)^3},$$

由$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$得$a(x+y)-2(x+ay)=-2y$,得$a=2$,应选(D).

(2)【答案】 (B).

【解】 因为$\lim\limits_{x\to 0}\frac{f''(x)}{|x|}=1>0$,所以由极限保号性,存在$\delta>0$,当$0<|x|<\delta$时,$\frac{f''(x)}{|x|}>0$,即$f''(x)>0$,从而$f'(x)$在$(-\delta,\delta)$内单调递增.

再由$f'(0)=0$得$\begin{cases}f'(x)<0,x\in(-\delta,0),\\ f'(x)>0,x\in(0,\delta)\end{cases}$得$f(0)$为$f(x)$的极小值,应选(B).

(3)【答案】 (A).

【解】 因为正项级数$\sum\limits_{n=1}^{\infty}a_n$收敛,所以$\sum\limits_{n=1}^{\infty}a_{2n}$收敛,

由$\left|(-1)^n\left(n\tan\frac{\lambda}{n}\right)a_{2n}\right|\sim\lambda a_{2n}$得级数$\sum\limits_{n=1}^{\infty}\left|(-1)^n\left(n\tan\frac{\lambda}{n}\right)a_{2n}\right|$收敛,

故$\sum\limits_{n=1}^{\infty}(-1)^n\left(n\tan\frac{\lambda}{n}\right)a_{2n}$绝对收敛,应选(A).

(4)【答案】 (C).

【解】 $F(x)=\int_0^x(x^2-t^2)f(t)\mathrm{d}t=x^2\int_0^x f(t)\mathrm{d}t-\int_0^x t^2f(t)\mathrm{d}t$, $F'(x)=2x\int_0^x f(t)\mathrm{d}t$,

由$\lim\limits_{x\to 0}\frac{F'(x)}{x^3}=2\lim\limits_{x\to 0}\frac{\int_0^x f(t)\mathrm{d}t}{x^2}=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x}=f'(0)\neq 0$得$k=3$.

(5)【答案】 (D).

【解】 将行列式按第一行展开,得

$$\begin{vmatrix}a_1&0&0&b_1\\0&a_2&b_2&0\\0&b_3&a_3&0\\b_4&0&0&a_4\end{vmatrix}=a_1A_{11}+b_1A_{14}=a_1M_{11}-b_1M_{14}$$

$$=a_1\begin{vmatrix}a_2&b_2&0\\b_3&a_3&0\\0&0&a_4\end{vmatrix}-b_1\begin{vmatrix}0&a_2&b_2\\0&b_3&a_3\\b_4&0&0\end{vmatrix}$$

$$=a_1a_4(a_2a_3-b_2b_3)-b_1b_4(a_2a_3-b_2b_3)$$

$$=(a_1a_4-b_1b_4)(a_2a_3-b_2b_3),$$

应选(D).

三、

(1)【解】 弧长 $l=2\int_0^{\pi}\sqrt{r^2(\theta)+r'^2(\theta)}\,\mathrm{d}\theta=2\int_0^{\pi}\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,\mathrm{d}\theta$

$$=2\sqrt{2}a\int_0^{\pi}\sqrt{1+\cos\theta}\,\mathrm{d}\theta=4a\int_0^{\pi}\cos\frac{\theta}{2}\mathrm{d}\theta$$

$$=8a\int_0^{\pi}\cos\frac{\theta}{2}\mathrm{d}\left(\frac{\theta}{2}\right)=8a\int_0^{\frac{\pi}{2}}\cos t\,\mathrm{d}t=8a.$$

(2)【解】 令$y=f(x)=\sqrt{6+x}$,

因为 $f'(x)=\dfrac{1}{2\sqrt{6+x}}>0$,所以 $\{x_n\}$ 单调.

由 $x_1=10>x_2=4$ 得数列 $\{x_n\}$ 单调递减,

再由 $x_n>0$ 得数列 $\{x_n\}$ 单调递减且有下界,故数列 $\{x_n\}$ 收敛.

令 $\lim\limits_{n\to\infty}x_n=A$,由 $x_{n+1}=\sqrt{6+x_n}$ 得 $A=\sqrt{6+A}$,解得 $A=3$.

四、

(1)**【解】** 令 $S_1:z=1(x^2+y^2\leqslant 1)$,取下侧,则

$$\iint\limits_S(2x+z)\mathrm{d}y\mathrm{d}z+z\mathrm{d}x\mathrm{d}y=\oiint\limits_{S+S_1}(2x+z)\mathrm{d}y\mathrm{d}z+z\mathrm{d}x\mathrm{d}y-\iint\limits_{S_1}(2x+z)\mathrm{d}y\mathrm{d}z+z\mathrm{d}x\mathrm{d}y,$$

由高斯公式得

$$\oiint\limits_{S+S_1}(2x+z)\mathrm{d}y\mathrm{d}z+z\mathrm{d}x\mathrm{d}y=-3\iiint\limits_{\Omega}\mathrm{d}v=-3\int_0^1\mathrm{d}z\iint\limits_{x^2+y^2\leqslant z}\mathrm{d}x\mathrm{d}y=-3\pi\int_0^1 z\mathrm{d}z=-\frac{3\pi}{2};$$

$$\iint\limits_{S_1}(2x+z)\mathrm{d}y\mathrm{d}z+z\mathrm{d}x\mathrm{d}y=-\iint\limits_{x^2+y^2\leqslant 1}\mathrm{d}x\mathrm{d}y=-\pi,$$

故 $\displaystyle\iint\limits_S(2x+z)\mathrm{d}y\mathrm{d}z+z\mathrm{d}x\mathrm{d}y=-\frac{\pi}{2}$.

(2)**【解】** $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}=\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}$, $\quad\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}=-2\dfrac{\partial z}{\partial u}+a\dfrac{\partial z}{\partial v}$,

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2},$$

$$\frac{\partial^2 z}{\partial x\partial y}=-2\frac{\partial^2 z}{\partial u^2}+a\frac{\partial^2 z}{\partial u\partial v}-2\frac{\partial^2 z}{\partial v\partial u}+a\frac{\partial^2 z}{\partial v^2}=-2\frac{\partial^2 z}{\partial u^2}+(a-2)\frac{\partial^2 z}{\partial v\partial u}+a\frac{\partial^2 z}{\partial v^2},$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2}&=-2\left(\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial y}\right)+a\left(\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial y}\right)\\&=-2\left(-2\frac{\partial^2 z}{\partial u^2}+a\frac{\partial^2 z}{\partial u\partial v}\right)+a\left(-2\frac{\partial^2 z}{\partial v\partial u}+a\frac{\partial^2 z}{\partial v^2}\right)\\&=4\frac{\partial^2 z}{\partial u^2}-4a\frac{\partial^2 z}{\partial u\partial v}+a^2\frac{\partial^2 z}{\partial v^2},\end{aligned}$$

代入整理得

$$(5a+10)\frac{\partial^2 z}{\partial v\partial u}+(-a^2+a+6)\frac{\partial^2 z}{\partial v^2}=0,$$

于是 $\begin{cases}5a+10\neq 0,\\-a^2+a+6=0,\end{cases}$ 解得 $a=3$.

五、【解】 令 $S(x)=\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{n^2-1}(-1<x<1)$,

则 $S(x)=\dfrac{1}{2}\left(\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{n-1}-\sum_{n=2}^{\infty}\frac{x^n}{n+1}\right)$,

$S(0)=0$;

当 $x\neq 0$ 时,

$$\begin{aligned}S(x)&=\frac{x}{2}\sum_{n=2}^{\infty}\frac{x^{n-1}}{n-1}-\frac{1}{2x}\sum_{n=2}^{\infty}\frac{x^{n+1}}{n+1}=\frac{x}{2}\sum_{n=1}^{\infty}\frac{x^n}{n}-\frac{1}{2x}\sum_{n=3}^{\infty}\frac{x^n}{n}\\&=-\frac{x}{2}\ln(1-x)-\frac{1}{2x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n}-x-\frac{x^2}{2}\right)\end{aligned}$$

$$= \left(\frac{1}{2x} - \frac{x}{2}\right)\ln(1-x) + \frac{1}{2} + \frac{x}{4},$$

故 $\sum\limits_{n=2}^{\infty} \dfrac{1}{(n^2-1)2^n} = S\left(\dfrac{1}{2}\right) = \dfrac{5}{8} - \dfrac{3}{4}\ln 2.$

六、【解】 曲线 $y = f(x)$ 在点 $(x, f(x))$ 的切线为

$$Y - f(x) = f'(x)(X - x),$$

令 $X = 0$ 得 $Y = f(x) - xf'(x)$,

由题意得 $f(x) - xf'(x) = \dfrac{1}{x}\int_0^x f(t)\mathrm{d}t$,整理得 $xf(x) - x^2 f'(x) = \int_0^x f(t)\mathrm{d}t$,

两边求导得 $f'(x) + xf''(x) = 0$,即 $[xf'(x)]' = 0$,

解得 $xf'(x) = C_1$,或 $f'(x) = \dfrac{C_1}{x}$,故 $f(x) = C_1 \ln x + C_2$ (C_1, C_2 为任意常数).

七、【证明】 (1) 由泰勒公式得

$$f(0) = f(c) + f'(c)(0-c) + \frac{f''(\xi_1)}{2!}(0-c)^2,\quad 0 < \xi_1 < c,$$

$$f(1) = f(c) + f'(c)(1-c) + \frac{f''(\xi_2)}{2!}(1-c)^2,\quad c < \xi_2 < 1,$$

两式相减得

$$f'(c) = f(1) - f(0) + \frac{c^2}{2}f''(\xi_1) - \frac{(1-c)^2}{2}f''(\xi_2).$$

(2) $|f'(c)| \leqslant |f(1)| + |f(0)| + \dfrac{c^2}{2}|f''(\xi_1)| + \dfrac{(1-c)^2}{2}|f''(\xi_2)|$

$$\leqslant 2a + \frac{b}{2}[c^2 + (1-c)^2],$$

由 $c^2 \leqslant c, (1-c)^2 \leqslant 1-c$ 得 $c^2 + (1-c)^2 \leqslant 1$,故 $|f'(c)| \leqslant 2a + \dfrac{b}{2}$.

八、【证明】 (1) 令 $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi} = k$,

$$\boldsymbol{A}^2 = (\boldsymbol{E} - \boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}})(\boldsymbol{E} - \boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}) = \boldsymbol{E} + (k-2)\boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}},$$

则 $\boldsymbol{A}^2 = \boldsymbol{A}$ 的充分必要条件是 $k = 1$,即 $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi} = 1$.

(2) **方法一** 当 $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi} = 1$ 时,由 $\boldsymbol{A}^2 = \boldsymbol{A}$ 得 $\boldsymbol{A}(\boldsymbol{E} - \boldsymbol{A}) = \boldsymbol{O}$,从而 $r(\boldsymbol{A}) + r(\boldsymbol{E} - \boldsymbol{A}) \leqslant n$;

再由 $r(\boldsymbol{A}) + r(\boldsymbol{E} - \boldsymbol{A}) \geqslant r(\boldsymbol{E}) = n$ 得 $r(\boldsymbol{A}) + r(\boldsymbol{E} - \boldsymbol{A}) = n$,

因为 $\boldsymbol{\xi}$ 为非零向量,所以 $\boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}} \neq \boldsymbol{O}$,从而 $\boldsymbol{E} - \boldsymbol{A} = \boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}} \neq \boldsymbol{O}$,即 $r(\boldsymbol{E} - \boldsymbol{A}) \geqslant 1$,

故 $r(\boldsymbol{A}) < n$,即 $\boldsymbol{A}$ 是不可逆矩阵.

方法二 令 $\boldsymbol{B} = \boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}$,矩阵 $\boldsymbol{B}$ 的特征值为 $\lambda_1 = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi} = 1, \lambda_2 = \cdots = \lambda_n = 0$,

矩阵 $\boldsymbol{A}$ 的特征值为 $\lambda_1 = 0, \lambda_2 = \cdots = \lambda_n = 1$,则 $|\boldsymbol{A}| = |\boldsymbol{E} - \boldsymbol{B}| = 0$,

故 $r(\boldsymbol{A}) < n$,即 $\boldsymbol{A}$ 不可逆.

九、【解】 (1) 令 $\boldsymbol{A} = \begin{pmatrix} 5 & -1 & 3 \\ -1 & 5 & -3 \\ 3 & -3 & c \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,则 $f(x_1, x_2, x_3) = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{X}$,

因为二次型的秩为 2,所以 $|\boldsymbol{A}| = 0$,

由 $|\boldsymbol{A}| = \begin{vmatrix} 5 & -1 & 3 \\ -1 & 5 & -3 \\ 3 & -3 & c \end{vmatrix} = 24c - 72 = 0$,得 $c = 3$.

容易验证，此时 $\boldsymbol{A}$ 的秩是 2. $\boldsymbol{A}$ 的特征多项式为

$$|\lambda \boldsymbol{E}-\boldsymbol{A}|=\begin{vmatrix}\lambda-5 & 1 & -3\\ 1 & \lambda-5 & 3\\ -3 & 3 & \lambda-3\end{vmatrix}=\lambda(\lambda-4)(\lambda-9),$$

故所求特征值为 $\lambda_1=0,\lambda_2=4,\lambda_3=9$.

(2) 二次型 f 的标准形为 $$f=4y_2^2+9y_3^2,$$

由此可知 $f(x_1,x_2,x_3)=1$ 所表示的曲面是椭圆柱面.

十、填空题

(1)**【答案】** $\dfrac{3}{7}$.

【解】 设 $A_1=\{$抽取的为 A 厂产品$\}$，$A_2=\{$抽取的为 B 厂产品$\}$，$B=\{$抽取的为次品$\}$，

$$P(A_1)=0.6,\quad P(A_2)=0.4,\quad P(B\mid A_1)=0.01,\quad P(B\mid A_2)=0.02,$$

则 $$P(A_1\mid B)=\frac{P(A_1B)}{P(B)}=\frac{P(A_1)P(B\mid A_1)}{P(A_1)P(B\mid A_1)+P(A_2)P(B\mid A_2)}$$

$$=\frac{0.6\times0.01}{0.6\times0.01+0.4\times0.02}=\frac{3}{7}.$$

(2)**【答案】** $\dfrac{2}{\sqrt{2\pi}}$.

【解】 令 $U=\xi-\eta$，

因为 ξ,η 相互独立且都服从正态分布 $N\left(0,\left(\frac{1}{\sqrt{2}}\right)^2\right)$，所以 $U\sim N(0,1)$，

于是 $$E(|\xi-\eta|)=E(|U|)=\int_{-\infty}^{+\infty}|u|\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\mathrm{d}u=\frac{2}{\sqrt{2\pi}}\int_0^{+\infty}ue^{-\frac{u^2}{2}}\mathrm{d}u$$

$$=\frac{2}{\sqrt{2\pi}}\int_0^{+\infty}e^{-\frac{u^2}{2}}\mathrm{d}\left(\frac{u^2}{2}\right)=\frac{2}{\sqrt{2\pi}}\Gamma(1)=\frac{2}{\sqrt{2\pi}}.$$

十一、【解】 (1)$P\{X=1,Y=1\}=P\{\xi=1,\eta=1\}=P\{\xi=1\}P\{\eta=1\}=\dfrac{1}{9}$，

$P\{X=1,Y=2\}=0$，$\quad P\{X=1,Y=3\}=0$；

$P\{X=2,Y=1\}=P\{\xi=1,\eta=2\}+P\{\xi=2,\eta=1\}=\dfrac{2}{9}$，

$P\{X=2,Y=2\}=P\{\xi=2,\eta=2\}=P\{\xi=2\}P\{\eta=2\}=\dfrac{1}{9}$，

$P\{X=2,Y=3\}=0$；

$P\{X=3,Y=1\}=P\{\xi=3,\eta=1\}+P\{\xi=1,\eta=3\}=\dfrac{2}{9}$，

$P\{X=3,Y=2\}=P\{\xi=3,\eta=2\}+P\{\xi=2,\eta=3\}=\dfrac{2}{9}$，

$P\{X=3,Y=3\}=\dfrac{1}{9}$，

故(X,Y)的联合分布律为

X	Y		
	1	2	3
1	$\frac{1}{9}$	0	0
2	$\frac{2}{9}$	$\frac{1}{9}$	0
3	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

(2) 随机变量 X 的边缘分布律为

$$X \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{9} & \frac{3}{9} & \frac{5}{9} \end{pmatrix},$$

故 $E(X) = \frac{1}{9} + \frac{6}{9} + \frac{15}{9} = \frac{22}{9}$.