Lecture 14

Solving Linear Recurrence Relations

linear homogeneous relation of degree k with constant coefficients(常数系数为k的线性齐次关系)

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

Basic idea

$$Let's \quad a_n = r^n$$

◇ Bring $a_n = r^n$ back to the recurrence relation:

i.e., $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$,

$$r^{n-k}(r^k - c_1 r^{k-1} - \cdots - c_k) = 0$$

◇ The solutions to the *characteristic equation* can yield an explicit formula for the sequence.

$$(r^k - c_1 r^{k-1} - \cdots - c_k) = 0$$

K degree

$$a_n = \sum_{i=1}^{k} c_i a_{n-1}$$
$$characteristic\ equation:$$
$$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$$
$$a_n = \sum_{i=1} k\alpha_i r_i^n$$

The Case of Degenerate Roots

If the CE has only 1 rot r_0, then

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

Linear Nonhomogeneous Recurrence Relations

线性非齐次递推关系

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$
$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}\ called\ :$$

associated homogeneous recurrence relation

F(n) only depend on n !!

Theorem

If $a_n = p(n)$ is any particular solution to the linear nonhomogeneous relation with constant coefficients.

$a_n = p(n) + h(n)$

*$a_n = h(n)$ is any solution to **associated homogeneous recurrence relation**

let $p(n) = cn + d$

make p(n) into relation, and make it to 0, solve this equation to find c and d

like:

We try a degree-t polynomial as the particular solution $p(n)$.

Let $p(n) = cn + d$, then

$cn + d = 3(c(n-1) + d) + 2n$, which means
$(2c+2)n + (2d - 3c) = 0$.

We get $c = -1$ and $d = -3/2$. Thus,

$p(n) = -n - 3/2$

求n个数字的和为C，求有多少种情况

> 这类问题可以理解为数学题目的求解，也可以理解为多个人瓜分一定数量的东西。总的来说是有很多变量的和组成一个大的总体。

A: 对于所有变量的限制只有非负

使用组合数，$\binom{C+n-1}{n-1}$

B：对于变量有数值范围的情况

使用***generation functions***，找到x^C的系数

方程用开始时的变量范围确定——指数为变量的值、系数为方案数

Example

A:

How many solutions are there to the equation $x_1 + x_2 + x_3 = 17$, where x_1, x_2, x_3 are nonnegative integers?

This is **equivalent** to the problem of **r-combinations from a set with n elements** when **repetition** is allowed.

$$C(n+r-1, r-1) = C(19, 2)$$

B:

Find the number of solutions of $x_1 + x_2 + x_3 = 17$, where x_1, x_2, x_3 are nonnegative integers with$2 \le x_1 \le 5, 3 \le x_2 \le 6, 4 \le x_3 \le 7$.

Using ***generating functions***, the number is the **coefficient** of x_{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$$