Lec 17

Partial Ordering

1. **Definition:** A relation R on a set S is called a partial ordering, or partial order, if it is $reflexive$, $antisymmetric$, and $transitive$. A set S together with a partial ordering R is called a partially ordered set, or poset, denoted by (S, R). Members of S are called elements of the poset.
2. $Example:$
   S = $\{1, 2, 3, 4, 5, 6\}$, R denotes the "|"" relation
   Is R reflexive? Yes
   Is R antisymmetric? Yes
   Is R transitive? Yes
   R is a partial ordering

Comparability

1. **Definition:** The elements a and b of a poset $(S, \preccurlyeq)$ are comparable if either a $\preccurlyeq$ b or b $\preccurlyeq$ a. Otherwise, a and b are called incomparable.

Total Ordering

- **Definition:** If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preccurlyeq$ is called a total order or a linear order. A totally ordered set is also called a chain.

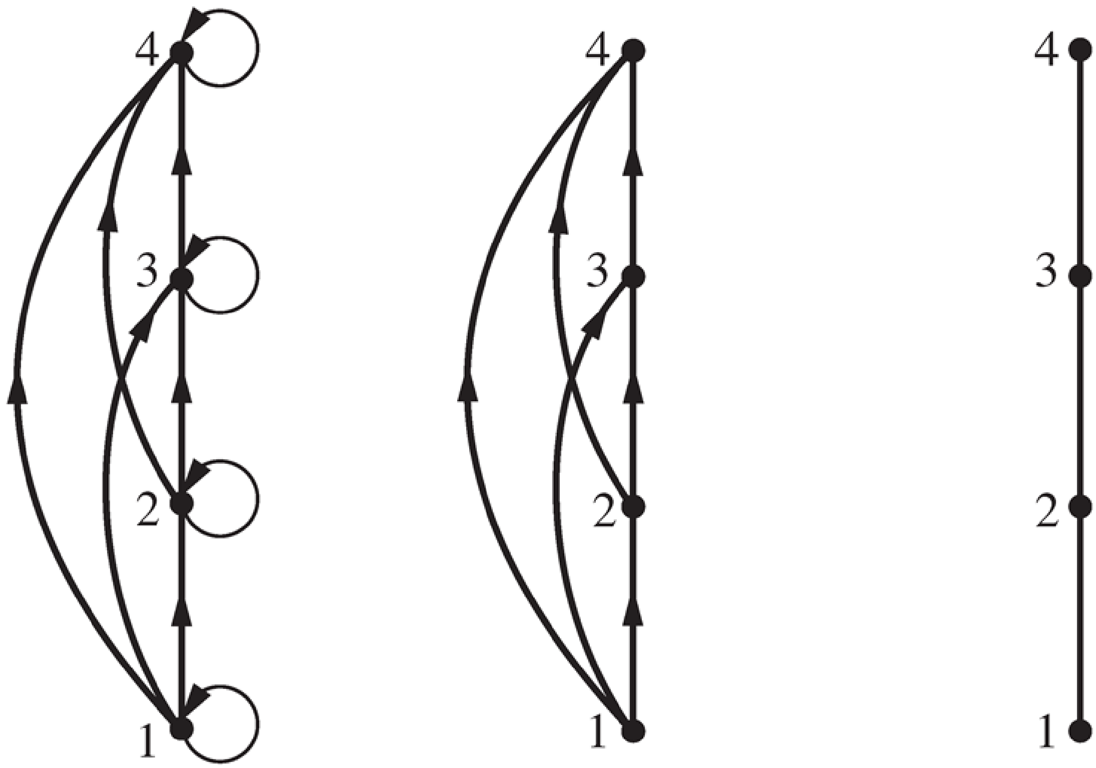
Lexicographic Ordering

Definition Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), i.e., $(a_1, a_2) \prec (b_1, b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ then $a_2 \prec_2 b_2$.

- **Example** Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.
    - *discreet* $\prec$ *discrete*
    - *discreet* $\prec$ *discreetness*

Hasse Diagram

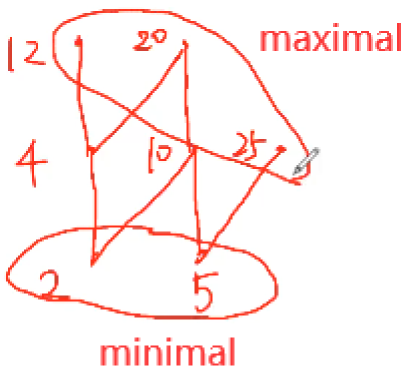
1. A Hasse diagram is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.



2. **Maximal and Minimal Elements**

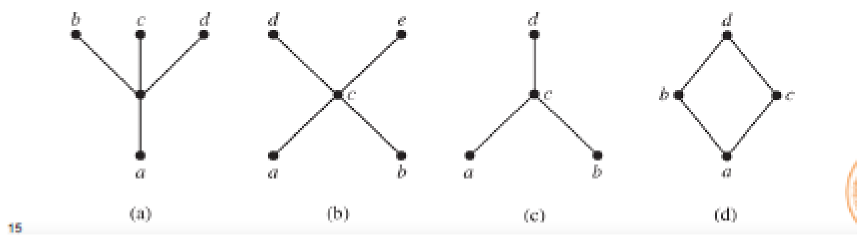
- **Definition** a is a *maximal* (resp. *minimal*) element in poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$ (resp. $b \prec a$).

  Example Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and minimal?



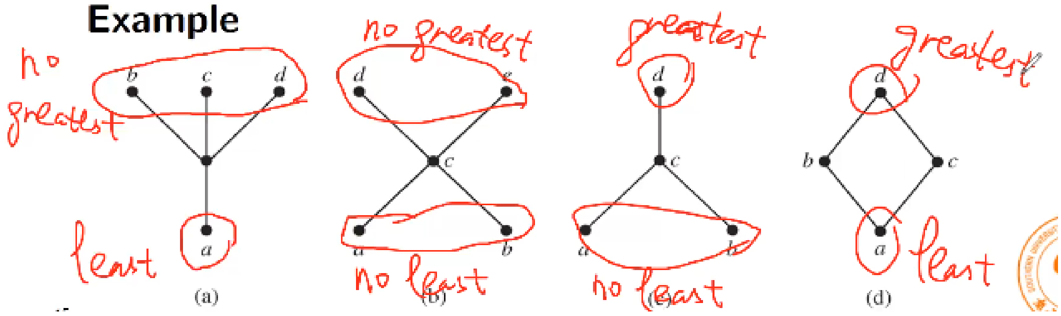
- **Definition** a is the *greatest* (resp. *least*) element of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ (resp. $a \preccurlyeq b$) for all $b \in S$.

  Example



if exist just have only one element!!!

Example



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Definition Let A be a subset of a poset $(S, \preccurlyeq)$.

- $u \in S$ is called an *upper bound* (resp. *lower bound*) of A if $a \preccurlyeq u$ (resp. $u \preccurlyeq a$) for all $a \in A$.
- $x \in S$ is called the *least upper bound* (resp. *greatest lower bound*) of A if x is an upper bound (resp. lower bound) that is less than any other upper bound (resp. lower bound) of A.

Example Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

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3.