Lec 16

Closures of Relations

1. The set S is called the reflexive closure of R if it:
   - contains R
   - is reflexive
   - is minimal (is contained in every reflexive relation Q that contains R (R $\subseteq$ Q), i.e. , S $\subseteq$ Q)
2. **Definition:** Let R be a relation on a set A. A relation S on A with property P is called the closure of R with respect to P if S is subset of every relation Q (S $\subseteq$ Q) with property P that contains R (R $\subseteq$ Q).
3. *Find Transitive Closure*
   - Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path
   - **Method**: Path in Directed Graphs
     - Let R be relation on a set A. There is a path of length n from a to b ***if and only*** if (a, b) $\in$ R^n .
       - proof : (by induction)

         i.h. there is a path of length n from a to b if and only if (x,b) $\in$ R^n

         i.s. there is a path of length n+1

         iff (a,x) $\in$ R; iff (a,b) $\in$ R^n
     -
4.

Connectivity Relation

1. **Lemma:** Let A be a set with n elements, and R a relation on A. If there is a path from a to b with a $\neq$ b, then there exists a path of length $\leq$ n $-$ 1.

   $$R^* = \bigcup_{k=1}^{n} R^k \Longrightarrow R^* = \bigcup_{k=1}^{n} R^k$$

   For upper is n is a = b;

   We also can use 0 1 Matrix

   $$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots M_R^{[n]}$$

   Procedure :

   - transClosure (M_R: zero-one n $\times$ n matrix)
     // computes R^* with zero-one matrices
     A := B := M_R;
     for i := 2 to n
     A := A $\odot$ M_R
     B := B $\vee$ A

return B
// B is the zero-one matrix for R^*

- O(n^4)
- improve algorithm
  - **procedure**

    Warshall (MR: zero-one n $\times$ n matrix)
    // computes R^* with zero-one matrices
    W := M_R;
    for k := 1 to n

    â€‹ for i := 1 to n

    â€‹   for j := 1 to n

    â€‹ $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
    return W
  - Time Complex: $O(n^3)$
  - w_{ij} = 1 means there is a path from i to j going only through nodes $\leq$ k.

$$W_{ij}^{[k]} = W_{ij}^{[k-1]} \vee (W_{ik}^{[k-1]} \wedge W_{kj}^{[k-1]})$$

2. **Lemma:** The transitive closure of a relation R equals the connectivity relation R^*.
3. **Theorem:** The transitive closure of a relation R equals the connectivity relation R^*.

   Proof:

   - R^* is transitive
   - $R^* \subseteq S$ whenever S is a transitive relation containing R
     - If (a, b) $\in R^*$ and (b, c) $\in R^*$, then there are paths from a to b and from b to c in R. Thus, there is a path from a to c in R. This means that (a, c) $\in R^*$.
     - Suppose that S is a transitive relation containing R

       Then S^n is also transitive and $S^n \subseteq S$.

       We have $S^* \subseteq S$. Thus, $R^* \subseteq S^* \subseteq S$
       - $S^* = \bigcup_{k=1}^{n} S^k$
         $R \subseteq S \rightarrow R^* \subseteq S^*$

Equivalence Relation

1. **Definition:** A relation R on a set A is called an ***equivalence relation*** if it is $reflexive$, $symmetric$, and $transitive$.
2. ## Equivalence Class

   1. **Definition:** Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a, denoted by $[a]_R$. When only one relation is considered, we use the notation [a].

$$[a]_R = b : (a, b) \in R$$

   $Example :$

- A = {0, 1, 2, 3, 4, 5, 6}
  R = {(a, b) : a $\equiv$ b mod 3}
  [0] = [3] = [6] = {0, 3, 6}
  [1] = [4] = {1, 4}
  [2] = [5] = {2, 5}
- "Strings a and b have the same length."

  [a] = the set of all strings of the same length as a
- "Integers a and b have the same absolute value."

  [a] = the set {a, $-$a}
- "Real numbers a and b have the same fractional part (i.e., a $-$ b $\in$ **Z**)."

  [a] = the set $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$

2. **Theorem :** Let R be an equivalence relation on a set A. The following statements are equivalent:

   (i) a R b
   (ii) [a] = [b]
   (iii) [a] $\cap$ [b] $\neq \emptyset$;

   (i) $\rightarrow$ (ii) $\rightarrow$ (iii) $\rightarrow$ (i)

   $(i) \rightarrow (ii)$

   a R b (a,b) $\in$ R $[a] \subseteq [b] \quad [b] \subseteq [a]$

   $x \in [a] \;\; (a, x) \in R \quad (b, a) \in R \Rightarrow ((b, x) \in R \quad x \in [b])$

   $(ii) \rightarrow (iii)$

   $[a] = [b] \quad [a] \cap [b] \quad [a] \neq \emptyset$
   $(a, a) \in R \quad a \in [a] \quad [a] \cap [b] \neq \emptyset$

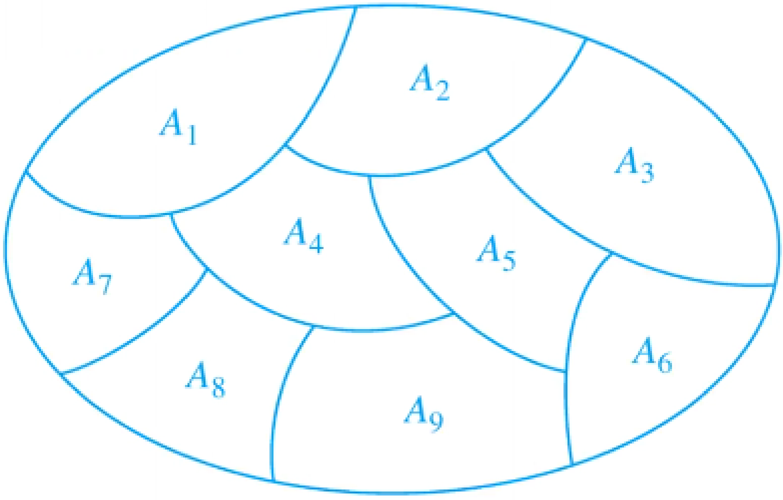
   $(iii) \rightarrow (i)$

   $Suppose \quad [a] \cap [b] \neq \emptyset \quad c \in [a] \cap [b]$
   $(a, c) \in R \quad and \quad (b, c) \in R$
   $from \;\; (b, c) \in R \;\; by \;\; symmetric \;\; we \;\; have \;\; (c, b) \in R$
   $by \;\; transitive \;\; we \;\; have \;\; (a, b) \in R$

Partition of a Set S

1. **Definition:** Let S be a set. A collection of nonempty subsets of S $A_1, A_2, A_3, \ldots, A_k$ is called a partition of S if:

$$A_i \cap A_j = \emptyset, \quad i \neq j \;\; and \;\; S = \bigcup_{i=1}^{k} A_i$$



2. $\color{blue}{Example:}$

   $A = \{0, 1, 2, 3, 4, 5, 6\}$
   $A_1 = \{0, 3, 6\}, A_2 = \{1, 4\}, A_3 = \{2, 5\}$

Equivalence Classes and Partitions

1. **Theorem :** Let R be an equivalence relation on a set A. Then union of all the equivalence classes of R is A:

$$A = \bigcup_{a \in A} [a]_R$$

2. **Theorem:** The equivalence classes form a partition of A.
3. **Theorem:** Let $A_1, A_2, A_3, \ldots, A_i, \ldots$ be a partition of S. Then there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

   $\color{red}{Equivalence\ \ relation \longrightarrow Equivalence\ \ classes \longrightarrow partition\ \ of\ \ A \longrightarrow Equivalence\ \ relation}$