The natural logarithm[*]

Ragnar Nymoen

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1 Logarithms and rates of change

We often make use of the approximation

$$\ln\left(1 + \frac{\Delta Y}{Y}\right) = \ln\left(\frac{Y + \Delta Y}{Y}\right) = \ln\left(Y + \Delta Y\right) - \ln Y$$
$$\approx \frac{\Delta Y}{Y}, \quad \frac{\Delta Y}{Y} \text{ is small.}$$

We will see how this approximation works, and how good it is.

When we consider marginal changes we have the precise relationship, i.e., from the derivative of a function

$$\frac{d \ln Y}{dY} = \frac{1}{Y}, \text{ and hence } d \ln Y = \frac{dY}{Y}.$$

To get to the approximation we need the following properties of the logarithmic function: $\ln X$:

$$\ln X = \begin{cases} = (X - 1), & X = 1 \\ < (X - 1), & X \neq 1 \end{cases}$$

The relationships holds because the function $f(X) = (X - 1)$ is a straight line with slope coefficient $\frac{d(X-1)}{dX} = 1$, while the function $g(X) = \ln X$ is concave, with a slope coefficient that decreases with increasing X:

$$\frac{d \ln X}{dX} = \frac{1}{X} \begin{cases} < 1, X > 1 \\ > 1, X < 1. \end{cases}$$

This result is illustrated in figure 1.

If we let $X = \frac{Y + \Delta Y}{Y} = 1 + \frac{\Delta Y}{Y}$, we get the relationship between logarithms and rates of change:

$$\ln\left(1 + \frac{\Delta Y}{Y}\right) = \ln\left(\frac{Y + \Delta Y}{Y}\right) = \ln\left(Y + \Delta Y\right) - \ln Y \leq \frac{\Delta Y}{Y}.$$

Changes in logarithms will therefore never give larger values than the exact rates of change. But how good is the approximation, and why does it become poorer when

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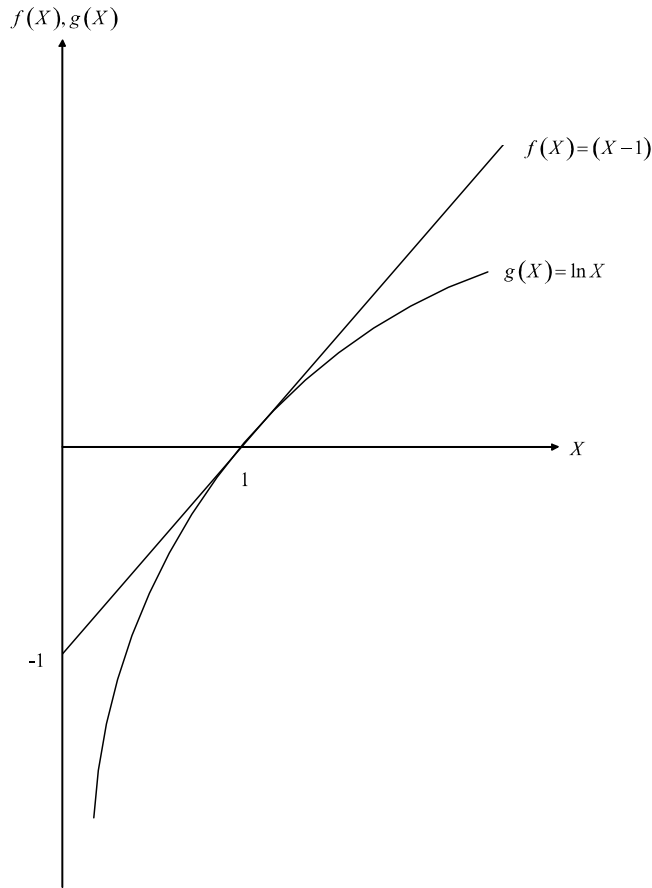[*]This note is a translation of Appendix 3.A in **?**.

Figure 1: The log-function as a an approximation.

the rate of changes becomes larger? We can investigate this by taking a second order Taylor expansion of $\ln X$ around the value $X_0 = 1$:

$$\ln X \approx \ln X_0 + \frac{1}{X_0}(X - X_0) - \left(\frac{1}{X_0^2}\right)\frac{(X - X_0)^2}{2}$$

$$= \ln 1 + 1(X - 1) - \frac{1}{2}(X - 1)^2$$

$$\ln X \approx (X - 1) - \frac{1}{2}(X - 1)^2. \tag{1}$$

If we substitute $X = \frac{Y + \Delta Y}{Y}$, as above, we can see how close the approximation is, depending on the size of the rate of change:

$$\ln\left(\frac{Y + \Delta Y}{Y}\right) = \ln(Y + \Delta Y) - \ln Y \approx \frac{\Delta Y}{Y} - \frac{1}{2}\left(\frac{\Delta Y}{Y}\right)^2.$$

Clearly the approximation is best for smallish rates of change, but it also works well for a 10 % change in Y, as the table shows:

$\frac{\Delta Y}{Y}$	0.01	0.05	0.1	0.2	0.5	1
$\ln\left(1 + \frac{\Delta Y}{Y}\right)$	0.00995	0.04879	0.09531	0.18232	0.405 47	0.693 15