Esercizio 3 (10 punti)
Siano X e Y variabili aleatorie indipendenti e supponiamo che X abbia distribuzione normale di valore atteso 1 e varianza 4 ed Y abbia distribuzione esponenziale di valore atteso 2.
(i) Calcolare $P(X - X^2 > 0, Y - Y^2 + 2 > 0)$;
(ii) posto $T = 2X - Y$, calcolare $E(T)$, $Var(T)$ e $Cov(T, Y)$.

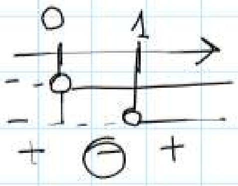
$X \sim \mathcal{N}(1, 4 = \sigma^2)$ $\qquad$ $Y \sim \mathcal{E}xp\left(\frac{1}{2}\right)$ $\qquad$ $E(Y) = 2 = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{2}$

(i) $\underline{P}(X - X^2 > 0,\ Y - Y^2 + 2 > 0) \underset{\text{X e Y indipendenti}}{=} \underline{P}(X - X^2 > 0) \cdot \underline{P}(Y - Y^2 + 2 > 0) = (0,1915) \cdot (0,6321) = 0,121$

$$\underline{P}(X - X^2 > 0) = \underline{P}(0 < X < 1) = \underline{P}\left(\frac{0-1}{2} < \underbrace{\frac{X-1}{2}}_{= Z \sim \mathcal{N}(0,1)} < \frac{1-1}{2}\right)$$

$X - X^2 > 0$
$X^2 - X < 0$
$X(X-1) < 0$

$\begin{cases} X > 0 \\ X > 1 \end{cases}$



$= \underline{P}(-0,5 < Z < 0)$

$= \phi(0) - \phi(-0.5)$

$= 0,5 - 1 + \phi(0.5)$

$= 0,5 - 1 + 0,6915 = 0,1915$

$$\underline{P}(Y - Y^2 + 2 > 0) = \underline{P}(-1 < Y < 2) = \underline{P}(0 \leq Y < 2) = F_Y(2) - F_Y(0)$$

Y è esponenziale
$\Rightarrow$ Y assume solo valori ≥ 0

$$= 1 - e^{-\frac{1}{2}\cdot 2} - \cancel{1} + \cancel{e^{-\frac{1}{2}\cdot 0}}$$

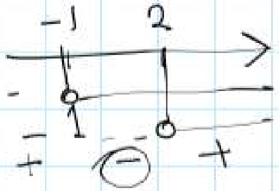
$$= 1 - e^{-1} = 0,6321$$

$Y - Y^2 + 2 > 0$

$Y^2 - Y - 2 < 0$

$(Y-2)(Y+1) < 0$

$$\begin{cases} Y > 2 \\ Y > -1 \end{cases}$$



(ii) $T = 2X - Y$ $\quad \mathbb{E}(T) = \mathbb{E}(2X - Y) = 2\mathbb{E}(X) - \mathbb{E}(Y) = 2\cdot 1 - 2 = 0$

(ii) $\quad T = 2X - Y \qquad \mathbb{E}(T) = \mathbb{E}(2X - Y) = 2\,\mathbb{E}(X) - \mathbb{E}(Y) = 2\cdot 1 - 2 = 0$

$$\mathrm{Var}(T) = \mathrm{Var}(2X - Y) = 4\,\mathrm{Var}\,X + \mathrm{Var}\,Y = 4\cdot 4 + \frac{1}{\left(\frac{1}{2}\right)^2} = 16 + 4 = 20$$

perché X e Y sono indipendenti

$$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}\,X + b^2\,\mathrm{Var}\,Y + 2ab\,\mathrm{Cov}(X, Y)$$

$$\mathrm{Cov}(T, Y) = \mathrm{Cov}(2X - Y, Y) = 2\,\cancel{\mathrm{Cov}(X, Y)} - \mathrm{Cov}(Y, Y) = -\mathrm{Var}(Y) = -4$$

$$\mathrm{Cov}(aX + bY, Z) = a\,\mathrm{Cov}(X, Z) + b\,\mathrm{Cov}(Y, Z)$$

Esercizio 3 Set 6

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Esercizio 4 Nell'estrazione con reinserimento di 2 biglie da un'urna contenente numeri da 1 a 5, sia X il numero di volte che esce un numero pari, e sia Y il numero massimo estratto.
(i) Determinare la densità di probabilità congiunta $p(x, y) = P(X = x, Y = y)$ e le densità marginali, verificando che la distribuzione di probabilità di X è di tipo binomiale.
(ii) Ricavare il valore atteso di $X + Y$ e la varianza di $X - Y$.
(iii) Calcolare il coefficiente di correlazione di (X, Y).

$X \in \{0, 1, 2\}$ $\qquad$ $Y \in \{1, 2, 3, 4, 5\}$

$p(0,1) = P(X=0, Y=1) = \frac{1}{25}$ $\qquad$ $p(1,1) = P(X=1, Y=1) = 0$

$p(0,2) = P(X=0, Y=2) = 0$ $\qquad$ $p(1,2) = P(X=1, Y=2) = \frac{2}{25}$

$p(0,3) = P(X=0, Y=3) = \frac{3}{25}$ $\qquad$ $p(1,3) = P(X=1, Y=3) = \frac{2}{25}$

$p(0,4) = P(X=0, Y=4) = 0$ $\qquad$ $p(1,4) = P(X=1, Y=4) = \frac{4}{25}$

$p(0,5) = P(X=0, Y=5) = \frac{5}{25}$ $\qquad$ $p(1,5) = P(X=1, Y=5) = \frac{4}{25}$

$p(2,1) = P(X=2, Y=1) = 0$

$p(2,2) = P(X=2, Y=2) = \frac{1}{25}$

$p(2,3) = P(X=2, Y=3) = 0$

$p(2,4) = P(X=2, Y=4) = \frac{3}{25}$

$p(2,5) = P(X=2, Y=5) = 0$

x \ y	1	2	3	4	5	$p_X(x)$
0	$1/25$	0	$3/25$	0	$5/25$	$\frac{9}{25}$
1	0	$\frac{2}{25}$	$\frac{2}{25}$	$\frac{4}{25}$	$\frac{4}{25}$	$\frac{12}{25}$
2	0	$\frac{1}{25}$	0	$\frac{3}{25}$	0	$\frac{4}{25}$
$p_Y(y)$	$1/25$	$3/25$	$5/25$	$7/25$	$9/25$	1

$P(X=0) = \binom{2}{0}\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^2 = \frac{9}{25}$

$P(X=1) = \binom{2}{1}\left(\frac{2}{5}\right)\left(\frac{3}{5}\right) = 2 \cdot \frac{6}{25}$

$P(X=2) = \binom{2}{2}\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^0 = \frac{4}{25}$

$X \sim Bin\left(2, \frac{2}{5}\right)$

(ii) $E(X+Y) = E(X) + E(Y) = \frac{4}{5} + \frac{103}{25} = \frac{123}{25}$

$E(X) = np = 2 \cdot \frac{2}{5} = \frac{4}{5}$

$E(Y) = 1 \cdot \frac{1}{25} + 2 \cdot \frac{3}{25} + 3 \cdot \frac{5}{25} + 4 \cdot \frac{7}{25} + 5 \cdot \frac{9}{25} = \frac{103}{25}$

$Var(X-Y) = Var(X) + Var(Y) - 2\,Cov(X,Y) = \frac{12}{25} + \frac{21284}{625} + 2 \cdot \frac{42}{125} = \frac{22004}{625}$

$Var(X) = np(1-p) = 2 \cdot \frac{2}{5} \cdot \frac{3}{5} = \frac{12}{25}$

$$\mathrm{Var}(X) = np(1-p) = 2 \cdot \frac{2}{5} \cdot \frac{3}{5} = \frac{12}{25}$$

$$\mathbb{E}(Y^2) = 1^2 \cdot \frac{1}{25} + 2^2 \cdot \frac{3}{25} + 3^2 \cdot \frac{5}{25} + 4^2 \cdot \frac{7}{25} + 5^2 \cdot \frac{9}{25} = \frac{427}{25}$$

$$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = \frac{427}{25} - \left(\frac{103}{25}\right)^2 = \frac{21284}{625}$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}(X \cdot Y) - \mathbb{E}(X)\mathbb{E}(Y) = \frac{74}{25} - \frac{4}{5} \cdot \frac{103}{25} = -\frac{42}{125}$$

$$\mathbb{E}(X \cdot Y) = 1 \cdot 2 \cdot \frac{2}{25} + 1 \cdot 3 \cdot \frac{2}{25} + 1 \cdot 4 \cdot \frac{4}{25} + 1 \cdot 5 \cdot \frac{4}{25} + 2 \cdot 2 \cdot \frac{1}{25} + 2 \cdot 4 \cdot \frac{3}{25}$$

$$= \frac{74}{25}$$

(iii) $$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}X \, \mathrm{Var}Y}} = \frac{-\frac{42}{125}}{\sqrt{\frac{12}{25} \cdot \frac{21284}{625}}} = \frac{-\frac{\cancel{42}^{21}}{\cancel{125}}}{\frac{\cancel{2}\sqrt{3} \cdot 2\sqrt{5321}}{\cancel{125}}} = \frac{-21}{2\sqrt{15963}} \cong -0.0831$$

Esercizio 5 Si considerino le variabili aleatorie $X_1, \ldots, X_{36}$ indipendenti ed identicamente distribuite, tali che

$$E(X_i) = 10\,\text{sec}, \quad Var(X_i) = 16\,\text{sec}^2.$$

Posto $Y = X_1 + \cdots + X_{36}$,
(i) determinare $E(Y)$ e $Var(Y)$;
(ii) calcolare un'approssimazione per $P(Y > 6\,\text{min})$ e $P(Y > 6\,\text{min} \mid Y < 7\,\text{min})$.

(i) $E(Y) = E(X_1 + \ldots + X_{36}) = E(X_1) + \ldots + E(X_{36}) = 36 \cdot 10 = 360\,s$

$Var(Y) = Var(X_1 + \ldots + X_{36}) = Var(X_1) + \ldots + Var(X_{36}) = 36 \cdot 16 = 576\,s^2$

Sia X una v.a. con densità

$$f(x) = \begin{cases} cx^2 + |x| & -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \text{altrimenti} \end{cases} = \begin{cases} 9x^2 - x & -\frac{1}{2} < x < 0 \\ 9x^2 + x & 0 < x < \frac{1}{2} \\ 0 & \text{altrimenti} \end{cases}$$

(i) Determinare c

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1 = \int_{-1/2}^{1/2} f(x)\,dx = \int_{-1/2}^{0} (cx^2 - x)\,dx + \int_{0}^{1/2} (cx^2 + x)\,dx$$

$$= \left[c\,\frac{x^3}{3} - \frac{x^2}{2} \right]_{-1/2}^{0} + \left[c \cdot \frac{x^3}{3} + \frac{x^2}{2} \right]_{0}^{1/2}$$

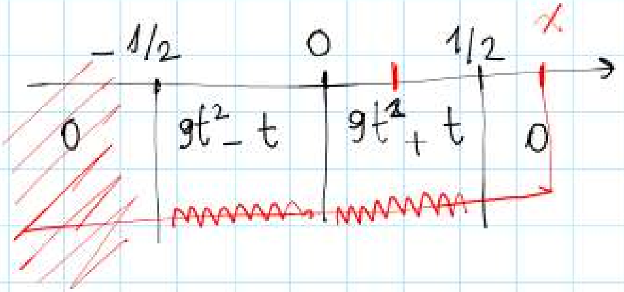
$$= + \frac{c \cdot \frac{1}{8}}{3} + \frac{\frac{1}{4}}{2} + \frac{c \cdot \frac{1}{8}}{3} + \frac{\frac{1}{4}}{2} =$$

$$= \frac{c}{24} + \frac{1}{8} + \frac{c}{24} + \frac{1}{8} = \frac{1}{12}c + \frac{1}{4} \Rightarrow 1 = \frac{1}{12}c + \frac{1}{4}$$

$$12 = c + 3$$

$$\Rightarrow c = 9$$

(ii) $F(x) = \int_{-\infty}^{x} f(t)\,dt$



Per $x < -\frac{1}{2}$, $F(x) = 0$

Per $-\frac{1}{2} \le x < 0$, $F(x) = \int_{-1/2}^{x} (9t^2 - t)\,dt = \left[\cancel{9}^{3}\frac{t^3}{\cancel{3}_1} - \frac{t^2}{2}\right]_{-\frac{1}{2}}^{x}$

$$= 3x^3 - \frac{x^2}{2} - \left(-3\cdot\frac{1}{8} - \frac{1/4}{2}\right)$$

$$= 3x^3 - \frac{x^2}{2} + \frac{1}{2}$$

Per $0 \le x < \frac{1}{2}$ $\quad F(x) = \int_{-1/2}^{0} (9t^2 - t)\,dt + \int_{0}^{x} (9t^2 + t)\,dt$

$$= \left[3t^3 - \frac{t^2}{2}\right]_{-\frac{1}{2}}^{0} + \left[3t^3 + \frac{t^2}{2}\right]_{0}^{x}$$

$$= +\frac{3}{8} + \frac{1}{8} + 3x^3 + \frac{x^2}{2} = 3x^3 + \frac{x^2}{2} + \frac{1}{2}$$

Per $x \geqslant \frac{1}{2}$ $\quad F(x) = \int_{-1/2}^{0} (9t^2 - t)\,dt + \int_{0}^{1/2} (9t^2 + t)\,dt + \cancel{\int_{1/2}^{x} 0\,dt}$

$$= \frac{1}{2} + \left[3t^3 + \frac{t^2}{2}\right]_0^{1/2} = \frac{1}{2} + \underbrace{\frac{3}{8} + \frac{1}{8}}_{\frac{1}{2}} = 1 \quad \text{ok}$$

(iii) $\quad E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_{-1/2}^{1/2} x f(x)\,dx = \int_{-1/2}^{0} x(9x^2 - x)\,dx + \int_{0}^{1/2} x(9x^2 + x)\,dx$

$9x^3 - x^2 \qquad 9x^3 + x^2$

$$= \left[9 \cdot \frac{x^4}{4} - \frac{x^3}{3}\right]_{-1/2}^{0} + \left[9 \cdot \frac{x^4}{4} + \frac{x^3}{3}\right]_0^{1/2}$$

$$= -\cancel{\frac{9}{64}} - \cancel{\frac{1}{24}} + \cancel{\frac{9}{64}} + \cancel{\frac{1}{24}} = 0$$

$\text{Var}(X) = E(X^2) = \int_{-1/2}^{0} x^2(9x^2 - x)\,dx + \int_{0}^{1/2} x^2(9x^2 + x)\,dx$

$9x^4 - x^3 \qquad 9x^4 + x^3$

$/64 \qquad /24 \qquad /64 \qquad /24$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) = \int_{-1/2}^{0} x^2(9x^2 - x)\,dx + \int_{0}^{1/2} x^2(9x^2 + x)\,dx$$

$$9x^4 - x^3 \qquad\qquad 9x^4 + x^3$$

$$= \left[9\cdot\frac{x^5}{5} - \frac{x^4}{4}\right]_{-1/2}^{0} + \left[9\cdot\frac{x^5}{5} + \frac{x^4}{4}\right]_{0}^{1/2}$$

$$= +\frac{23}{320} + \frac{23}{320} = \frac{23}{160}$$

Esercizio 1 prova intercorso 1/06/2020 Gruppo 1

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Esercizio 1 (10 punti)
Un gioco consiste nel lancio ripetuto di due dadi regolari. In ogni lancio dei due dadi si vince se l'esito del primo dado è strettamente minore di quello del secondo dado. Sia X la variabile aleatoria che rappresenta il lancio in cui si ottiene il primo successo.
(i) Determinare $P(X = k)$, $k = 1, 2, \ldots$; **(ii)** calcolare $P(X > 5 \mid X > 2)$;
(iii) determinare il valore di n tale che $P(X \leq n)$=95/144.

2 dadi $\quad x_1 < x_2$

1, 2	1, 3	...	1, 6	5 casi favorevoli
2, 3	2, 4	...	2, 6	4
3, 4	3, 5	3, 6		3
4, 5	4, 6			2
5, 6				1
				15 casi favorevoli

p = prob. vittoria = $\frac{15}{36}$ $\qquad X \sim \text{Geom}\left(\frac{15}{36}\right)$

(i) $\underline{P}(X=k) = \left(1 - \frac{15}{36}\right)^{k-1} \cdot \left(\frac{15}{36}\right) = \left(\frac{21}{36}\right)^{k-1} \cdot \left(\frac{15}{36}\right)$

(ii) $\underline{P}(X > 5 \mid X > 2) = \underline{P}(X > 3) = 1 - \underline{P}(X \leq 3) =$

$= 1 - \underline{P}(X=1) - \underline{P}(X=2) - \underline{P}(X=3)$

$= 1 - \frac{15}{36} - \frac{21}{36} \cdot \frac{15}{36} - \left(\frac{21}{36}\right)^2 \cdot \frac{15}{36} = \frac{343}{1728}$

(iii) n :

$$\underline{P}(X \leq n) = \frac{95}{144}$$

$$\Vert$$

$$1 - \underline{P}(X > n) = 1 - \left(\frac{21}{36}\right)^n$$

$$\Rightarrow 1 - \left(\frac{21}{36}\right)^n = \frac{95}{144}$$

$$\left(\frac{21}{36}\right)^n = \frac{49}{144}$$

$$\left(\frac{7}{12}\right)^n = \frac{7^2}{12^2} = \left(\frac{7}{12}\right)^2 \Rightarrow n = 2$$

$$n \log\left(\frac{21}{36}\right) = \log\left(\frac{21}{36}\right)^n = \log \frac{49}{144}$$

$$n = \frac{\log \frac{49}{144}}{\log \frac{21}{36}} = \frac{\log\left(\frac{7}{12}\right)^2}{\log \frac{7}{12}} = \frac{2 \cancel{\log \frac{7}{12}}}{\cancel{\log \frac{7}{12}}} = 2$$

$$\underline{P}(X \leq n) = \sum_{k=1}^{n} \underline{P}(X = k) = \sum_{k=1}^{n} \left(\frac{21}{36}\right)^{k-1} \cdot \frac{15}{36} = \frac{15}{36} \sum_{k=1}^{n} \left(\frac{21}{36}\right)^{k-1}$$

$k - 1 = h$

$$\sum_{h=0}^{N} x^h = \frac{1 - x^{N+1}}{1 - x}$$

$$= \frac{15}{36} \sum_{h=0}^{n-1} \left(\frac{21}{36}\right)^h = \cancel{\frac{15}{36}} \cdot \frac{1 - \left(\frac{21}{36}\right)^n}{\cancel{1 - \frac{21}{36}}}$$

$$= 1 - \left(\frac{21}{36}\right)^n$$