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$$E: \begin{pmatrix}-2\\1\\-6\\2\end{pmatrix} + \lambda\begin{pmatrix}-3\\-9\\7\\25\end{pmatrix} + y\begin{pmatrix}2\\6\\-8\\-20\end{pmatrix} = x$$

$$\Rightarrow \text{Basis für } E: \left\{\begin{pmatrix}-3\\-9\\7\\25\end{pmatrix}, \begin{pmatrix}2\\6\\-8\\-20\end{pmatrix}\right\}$$

$$\text{Löse} \begin{pmatrix}-3 & -9 & 7 & 25\\ 2 & 6 & -8 & -20\end{pmatrix}$$

$$\xrightarrow{I+2II} \begin{pmatrix}1 & 3 & -9 & -15\\ 2 & 6 & -8 & -20\end{pmatrix}$$

$$\xrightarrow{II-2I} \begin{pmatrix}1 & 3 & -9 & -15\\ 0 & 0 & 10 & 10\end{pmatrix}$$

$$\xrightarrow{II:10} \begin{pmatrix}1 & 3 & -9 & -15\\ 0 & 0 & 1 & 1\end{pmatrix}$$

$$\xrightarrow{I+9II} \begin{pmatrix}1 & 3 & 0 & -6\\ 0 & 0 & 1 & 1\end{pmatrix}$$

$$\xrightarrow{c\leftrightarrow b} \begin{array}{c} \begin{matrix}a & c & b & d\end{matrix} \\ \begin{pmatrix}1 & 0 & 3 & -6\\ 0 & 1 & 0 & 1\end{pmatrix}\end{array}$$

$$\Rightarrow \mathbb{L} = \left\{\begin{pmatrix}a\\c\\b\\d\end{pmatrix} : \lambda\begin{pmatrix}-3\\0\\1\\0\end{pmatrix} + y\begin{pmatrix}6\\-1\\0\\1\end{pmatrix}\right\}$$

$$\Rightarrow A = \begin{pmatrix}-3 & 1 & 0 & 0\\ 6 & 0 & -1 & 1\end{pmatrix}$$

$$\Rightarrow b = A\begin{pmatrix}-2\\1\\-6\\2\end{pmatrix} = \begin{pmatrix}6+1\\ -12+6+2\end{pmatrix} = \begin{pmatrix}7\\-4\end{pmatrix}$$

$$\Rightarrow E = \{x \mid Ax = b\}$$

$$A_H := \begin{pmatrix}-3 & 1 & 0 & 0\\ 6 & 0 & -1 & 1\\ 3 & -3 & -6 & 6\end{pmatrix} \quad b_H = \begin{pmatrix}7\\-4\\-3\end{pmatrix} \Rightarrow E \cap H = \{x \mid A_H x = b_H\}$$

$$\rightsquigarrow \begin{array}{c} \begin{matrix}b & d & c & a\end{matrix} \\ \left(\begin{array}{cccc|c}1 & 0 & 0 & -3 & 7\\ 0 & 1 & -1 & 6 & -4\\ -3 & 6 & -6 & 3 & -3\end{array}\right)\end{array}$$

$$\xrightarrow{III+3I} \left(\begin{array}{cccc|c}1 & 0 & 0 & -3 & 7\\ 0 & 1 & -1 & 6 & -4\\ 0 & 6 & -6 & -6 & 18\end{array}\right)$$

$$\xrightarrow{III-6II} \left(\begin{array}{cccc|c}1 & 0 & 0 & -3 & 7\\ 0 & 1 & -1 & 6 & -4\\ 0 & 0 & 0 & -42 & 42\end{array}\right)$$

$$\xrightarrow{III:42} \left(\begin{array}{cccc|c}1 & 0 & 0 & -3 & 7\\ 0 & 1 & -1 & 6 & -4\\ 0 & 0 & 0 & 1 & -1\end{array}\right)$$

$$\xrightarrow{I+3III} \left(\begin{array}{cccc|c}1 & 0 & 0 & 0 & 4\\ 0 & 1 & -1 & 0 & 2\\ 0 & 0 & 0 & 1 & -1\end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} \begin{matrix}b & d & a & c\end{matrix} \\ \left(\begin{array}{cccc|c}1 & 0 & 0 & 0 & 4\\ 0 & 1 & 0 & -1 & 2\\ 0 & 0 & 1 & 0 & -1\end{array}\right)\end{array}$$

$$\Rightarrow \begin{pmatrix}b\\d\\a\\c\end{pmatrix} = \lambda\begin{pmatrix}0\\1\\0\\1\end{pmatrix} + \begin{pmatrix}4\\2\\-1\\0\end{pmatrix} \Rightarrow E \cap H = \mathbb{L} = \left\{x \;\middle|\; x = \lambda\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \begin{pmatrix}-1\\4\\0\\2\end{pmatrix}, \lambda \in \mathbb{R}\right\}$$

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$\langle x, y \rangle := 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$

z.z. ① $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$

Gilt da $\cdot$ distributiv über $+$ ist in $\mathbb{R}$

② $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$

selbiges

③ $\langle x, y \rangle = \overline{\langle y, x \rangle}$

Gilt da $\operatorname{Im}(2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2) = 0$

und $\cdot$ in $\mathbb{R}$ kommutativ ist

④ $\langle x, x \rangle \geq 0$

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle = 2a^2 + ab + ba + b^2$$

$$= \overset{\geq 0}{a^2} + \overset{\geq 0}{(a+b)^2}$$

$$\geq 0$$

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z.Z. (1) $\|x\| \geq 0$

$|x_i| \geq 0 \ \forall x_i \Rightarrow \max\{|x_1|, \dots, |x_n|\} \geq 0$

(2) $\|x\| = 0 \Leftrightarrow x = 0$

$\max\{|x_1|, \dots, |x_n|\} = 0$

$\Rightarrow \exists x_i: \ |x_i| \geq |x_j| \ \forall j \ \wedge \ |x_i| = 0$

Da $|x_j| \geq 0 \ \forall j$ muss $|x_j| = 0 \ \forall j$

Dann $x_j = 0 \ \forall j$ und $x = 0$.

(3) $\|\lambda x\| = |\lambda| \cdot \|x\|$

$\max\{|\lambda x_1|, \dots, |\lambda x_n|\} = |\lambda| \max\{|x_1|, \dots, |x_n|\}$

$\Rightarrow \|\lambda x\| = |\lambda| \cdot \|x\|$

(4) $\|x+y\| \leq \|x\| + \|y\|$

Sei i sodass $|x_i| = \max\{|x_1|, \dots, |x_n|\}$

und y_j, $x_k + y_k$ analog.

Dann gilt

$\|x+y\|$

$= \max\{|x_1+y_1|, \dots, |x_n+y_n|\}$

$= |x_k + y_k|$

Dreiecksungl. in $\mathbb{R}$ $\leq |x_k| + |y_k|$

$x_i \geq \forall x_k$, $y_j \geq \forall y_k$ $\leq |x_i| + |y_j|$

per Def. $= \|x\| + \|y\|$

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Sei $A := \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$

Gilt $A^T A = I$, so ist (a,b,c,d) eine ONB

(äquiv. zu (a,b,c,d) sind paarweise orthogonal)

$$A^T A = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = I$$

∎

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Seien v_1, v_2, v_3 nicht orthogonal und w_i ihre orthogonalisierten Gegenstücke. Es gilt

$$w_1 = v_1$$

$$w_i = v_i - \sum_{j=1}^{i-1} \frac{\langle v_i, w_j \rangle}{\|w_j\|} w_j$$

$$\Rightarrow w_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 0 \end{pmatrix} - \frac{\langle v_2, w_1 \rangle}{\|w_1\|} w_1 = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 0 \end{pmatrix} - \frac{-4}{2} w_1 = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$w_3 = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 7 \end{pmatrix} - \frac{\langle v_3, w_1 \rangle}{\|w_1\|} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|} w_2$$

$$= \begin{pmatrix} 1 \\ 3 \\ 3 \\ 7 \end{pmatrix} - \frac{6}{2} w_1 - \frac{0}{\|w_2\|} w_2 = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 7 \end{pmatrix} - \begin{pmatrix} -3 \\ -3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 0 \\ 4 \end{pmatrix}$$