PS „Diskrete Mathematik“

Musterlösung zur Aufgabe 45

Aufgabe 45 Sei π eine Permutation (eine Bijektion) der Menge $\{1, \ldots, n\}$ mit einer (disjunkten) Zykeldarstellung $\pi = c_1 \ldots c_k$. Zeigen Sie, dass $\pi^{-1} = c_1^{-1} \ldots c_k^{-1}$ wobei für einen Zykel c der Zykel c^{-1} durch Umkehrung der Reihenfolge seiner Elemente entsteht. D.h., für $c = (a_1 \ldots a_m)$, ist $c^{-1} = (a_m \ldots a_1)$.

Lösung: Consider first cycles, i.e., let $c = (a_1 \ldots a_m)$ be a cycle, and consider the cycle denoted for now by $c^* = (a_m \ldots a_1)$. We have $c \circ c^*(x) = x$ for $x \notin \{a_1, \ldots, a_m\}$. Moreover, for $i \in \{2, \ldots, m\}$, $c \circ c^*(a_i) = c(a_{i-1}) = a_i$ and $c \circ c^*(a_1) = c(a_m) = a_1$. Hence $c \circ c^* = \mathrm{id}$. Similarly, we see that $c^* \circ c = \mathrm{id}$, showing that indeed $c^{-1} = c^*$.
Now, notice that permutations with disjoint set of movable elements commute: Let α and β be two such permutations. We have that

$$\alpha \circ \beta(x) = \begin{cases} \alpha(x) & \alpha(x) \neq x,\ \beta(x) = x \\ \beta(x) & \beta(x) \neq x,\ \alpha(x) = x \\ x & \alpha(x) = \beta(x) = x \end{cases} \quad = \quad \beta \circ \alpha(x).$$

Here, in the second line we used that if x is movable for β, then so is $\beta(x)$ due to injectivity of β. Moreover, if α and β have disjoint sets of movable elements, such have also α and β^{-1} as well as α^{-1} and β, since α and α^{-1} have the same set of movable elements. To see this, let x be a movable element of α^{-1}. This means $x \neq \alpha^{-1}(x)$ and therefore (since α is a permutation, so certainly injective) $\alpha(x) \neq \alpha(\alpha^{-1}(x)) = x$. This shows that the movable elements of α^{-1} are movable for α too. Since further $\alpha = (\alpha^{-1})^{-1}$ we get the opposite inclusion too.
We next notice that $(\alpha \circ \beta)^{-1} = \alpha^{-1} \circ \beta^{-1}$ as

$$\alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1} = \alpha \circ \alpha^{-1} \circ \beta \circ \beta^{-1} = \mathrm{id} \circ \mathrm{id} = \mathrm{id}$$

and

$$\alpha^{-1} \circ \beta^{-1} \circ \alpha \circ \beta = \alpha^{-1} \circ \alpha \circ \beta^{-1} \circ \beta = \mathrm{id} \circ \mathrm{id} = \mathrm{id}\,.$$

The statement now follows by induction on k, the number of disjoint cycles in π. We have just shown the property for $k = 2$ and previously also for $k = 1$. Assume the property holds for any permutation with k disjoint cycles, i.e., $(c_1 \circ \cdots \circ c_k)^{-1} = c_1^{-1} \circ \cdots \circ c_k^{-1}$ and consider a permutation with $k+1$ disjoint cycles, $\pi = c_1 \circ \cdots \circ c_k \circ c_{k+1}$. We have $\pi = \pi_k \circ c_{k+1}$ where π_k and c_{k+1} have disjoint set of movable elements. Therefore,

$$\pi^{-1} = \pi_k^{-1} \circ c_{k+1}^{-1} \stackrel{(IH)}{=} c_1^{-1} \circ \cdots \circ c_k^{-1} \circ c_{k+1}^{-1}.$$