Numerical Methods: Notes

Nis Sarup

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1 Preliminaries

1.1 Error, Accuracy, and Stability

1.1.2 Roundoff Error

- Two floats added gives rounding errors.
- The smallest float can be effectively reduced to zero if the difference is large.

1.1.3 Truncation Error

- Are created when you move from a continual value to a discrete.

1.1.4 Stability

- An early small error used in subsequent calculations and by that makes the result diverge more and more from the real answer makes a method unstable

2 Solution of Linear Algebraic Equations

2.0.1 Introduction

- If $N = M$, then there are as many equations as unknowns, and there is a good chance of solving for a unique solution set of x_j 's. Otherwise, if $N \neq M$, things are even more interesting.

2.0.2 Tasks of Computational Linear Algebra

- When $N = M$:
  - Solution of the matrix equation $A \cdot x = b$ for an unknown vector x
  - Solution of more than one matrix equation $A \cdot x_j = b_j$
  - Calculation of the matrix A^{-1} that is the matrix inverse of a square matrix A
  - Calculation of the determinant of a square matrix A
- If $M < N$, or if $M = N$ but the equations are degenerate:
  - Singular value decomposition of a matrix A
- If $M > N$:
  - Generally no solution vector x
  - The set of equations is said to be overdetermined
  - The best "compromise"solution is sought, the one that comes closest to satisfying all equations simultaneously.
  - If closeness is defined in the least-squares sense then the overdetermined linear problem reduces to a (usually) solvable linear problem
  - Linear least-squares problem:
    * Linear least-squares problem: $(A^T \cdot A) \cdot x = (A^T \cdot b)$
    * The above equations is called the normal equations of the linear least-squares problem
    * Direct solution of the normal equations not generally the best to find least-squares solution

2.1 Gauss-Jordan Elimination

- Gauss-Jordan elimination produces both the solution of the equations for one or more right-hand side vectors b, and also the matrix inverse A^{-1}
- Principal deficiencies
  - Requires all the right-hand sides to be stored and manipulated at the same time
  - When the inverse matrix is not desired, Gauss-Jordan is three times slower than the best alternative technique for solving a single linear set
- For inverting a matrix, Gauss-Jordan elimination is about as efficient as any other direct method
- Using inverse matrix with a new right-hand side to get additional solutions is higly susceptible to roundoff errors
- Gauss-Jordan elimination should not be your method of first choice for solving linear equations

2.1.1 Elimination on Column-Augmented Matrices

- $[\mathbf{A}] \cdot [\mathbf{x}_0 \sqcup \mathbf{x}_1 \sqcup \mathbf{x}_2 \sqcup \mathbf{Y}] = [\mathbf{b}_0 \sqcup \mathbf{b}_1 \sqcup \mathbf{b}_2 \sqcup \mathbf{1}]$
  - Where $\mathbf{A}$ and $\mathbf{Y}$ is square matrices, the $\mathbf{b}_i$'s and $\mathbf{x}_i$'s are column vectors and $\mathbf{1}$ is the identity matrix
  - Simultaneously solves the linear sets:
    * $\mathbf{A} \cdot \mathbf{x}_0 = \mathbf{b}_0$
    * $\mathbf{A} \cdot \mathbf{x}_1 = \mathbf{b}_1$
    * $\mathbf{A} \cdot \mathbf{x}_2 = \mathbf{b}_2$
    * $\mathbf{A} \cdot \mathbf{Y} = \mathbf{1}$
  - Interchanging any two rows of $\mathbf{A}$ and the corresponding rows of the $\mathbf{b}$'s and of $\mathbf{1}$ does not change the solution $\mathbf{x}$'s and $\mathbf{Y}$
  - Likewise if we replace any row in $\mathbf{A}$ by a linear combination of itself and any other row, as long as we do the same linear combination of the rows of the $\mathbf{b}$'s and $\mathbf{1}$
  - Interchanging any two columns of $\mathbf{A}$ gives the same solution set only if we simultaneously interchange corresponding rows of the $\mathbf{x}$'s and of $\mathbf{Y}$
- Gauss-Jordan elimination uses one or more of the above operations to reduce the matrix A to the identity matrix.
- When this is accomplished, the right-hand side becomes the solution set.