Mathematics for Computer Science Engineers

HYPOTHESIS and INFERENCE

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UNIT-3 HYPOTHESIS and INFERENCE

Distribution Free Tests

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Distribution Free Tests.

- The samples are not required to come from any specific distribution.

- While distribution free tests do require assumptions for their validity, these assumptions are somewhat less restrictive than the assumptions needed for the *t* test.

- Distribution-free tests are sometimes called nonparametric tests.
- We discuss two distribution-free tests in this section. The first, called the Wilcoxon signed-rank test, is a test for a population mean, analogous to the one-sample t test discussed

The second, called the Wilcoxon rank-sum test, or the Mann– Whitney test, is analogous to the two-sample t test discussed

The Wilcoxon Signed-Rank Test:

Example:

- The nickel content, in parts per thousand by weight, is measured for six welds.

- The results are $9.3, 0.9, 9.0, 21.7, 11.5, and\ 13.9$.

- Let μ represent the mean nickel content for this type of weld.

The Wilcoxon Signed-Rank Test:

- It is desired to test $H_0 : \mu \geq 12$ $versus$ $H_1 : \mu < 12.$
- The Student's t test is not appropriate, because there are two outliers, 0.9 and 21.7, which indicate that the population is not normal.
- The Wilcoxon signed-rank test can be used in this situation.

The Wilcoxon Signed-Rank Test:

- To compute the rank-sum statistic, we begin by subtracting 12 from each sample observation to obtain differences. The difference closest to 0, ignoring sign, is assigned a rank of 1.

- The difference next closest to 0, again ignoring sign, is assigned a rank of 2, and so on.

- Finally, the ranks corresponding to negative differences are given negative signs. The following table shows the results.

The Wilcoxon Signed-Rank Test:

X	X-12	Rank
11.5	-0.5	-1
13.9	1.9	2
9.3	-2.7	-3
9.0	-3.0	-4
21.7	9.7	5
0.9	-11.1	-6

- Let $H_0 : \mu \geq 12$, so a small value of S+ will provide evidence against H_0.
- We observe $S+ = 7$. The P-value is the probability of observing a value of $S+$ that is less than or equal to 7 when H_0 is true.
- For sample size $n = 6$, we find that the probability of observing a value of 4 or less is 0.1094.
- The probability of observing a value of 7 or less must be greater than this, so we conclude that $P > 0.1094$, and thus do not reject H_0.

Ties:

- Sometimes two or more of the quantities to be ranked have exactly the same value. Such quantities are said to be tied. The standard method for dealing with ties is to assign to each tied observation the average of the ranks they would have received if they had differed slightly.

- For example, the quantities 3, 4, 4, 5, 7 would receive the ranks 1, 2.5, 2.5, 4, 5

- The quantities 12, 15, 16, 16, 16, 20 would receive the ranks 1, 2, 4, 4, 4, 6.

Differences of Zero:

- If the mean under H_0 is μ_0, and one of the observations is equal to μ_0, then its difference is 0, which is neither positive nor negative.

- An observation that is equal to μ_0 cannot receive a signed rank. The appropriate procedure is to drop such observations from the sample altogether, and to consider the sample size to be reduced by the number of these observations.

The Wilcoxon Rank-Sum Test:

- The Wilcoxon rank-sum test, also called the Mann–Whitney test, can be used to test the difference in population means in certain cases where the populations are not normal.

- Two assumptions are necessary.

- First the populations must be continuous.

- Second, their probability density functions must be identical in shape and size; the only possible difference between them being their location.

The Wilcoxon Rank-Sum Test:

- Let $X_1, \ldots, X_m$ be a random sample from one population and let $Y_1 \ldots, Y_n$ be a random sample from the other.
- We adopt the notational convention that when the sample sizes are unequal, the smaller sample will be denoted $X_1, \ldots, X_m$.
- Thus the sample sizes are m and n, with $m \leq n$.
- Denote the population means by μ_X and μ_Y , respectively.

The Wilcoxon Rank-Sum Test:

- The test is performed by ordering the $m + n$ values obtained by combining the two samples, and assigning ranks $1, 2, \ldots, m + n$ to them.

- The test statistic, denoted by W, is the sum of the ranks corresponding to $X_1, \ldots, X_m$.

The Wilcoxon Rank-Sum Test:

- Since the populations are identical with the possible exception of location, it follows that if $\mu_X < \mu_Y$, the values in the X sample will tend to be smaller than those in the Y sample.

- So the rank sum W will tend to be smaller as well.

- By similar reasoning, if $\mu_X > \mu_Y$, W will tend to be larger.

The Wilcoxon Rank-Sum Test:

Example:

- Resistances, in m, are measured for five wires of one type and six wires of another type. The results are as follows:

$$X: \ 36\ 28\ 29\ 20\ 38$$
$$Y: \ 34\ 41\ 35\ 47\ 49\ 46$$

- Use the Wilcoxon rank-sum test to test $H_0: \mu_X \geq \mu_Y \ versus H_1: \mu_X < \mu_Y$.

The Wilcoxon Rank-Sum Test:
Solution:

We order the 11 values and assign the ranks.

	Rank	Value
20	1	X
28	2	X
29	3	X
34	4	Y
35	5	Y
36	6	X

	Rank	Value
38	7	X
41	8	Y
46	9	Y
47	10	Y
49	11	Y

The Wilcoxon Rank-Sum Test:

Solution:

$$W = 1 + 2 + 3 + 6 + 7 = 19.$$

- To determine the P-value, we consult Table A.6 (in Appendix A).

- We note that small values of W provide evidence against $H_0: \mu_X \geq \mu_Y$, so the P value Is the area in the left-hand tail of the null distribution. Entering the table with $m = 5$ and $n = 6$ we find that the area to the left of $W = 19$ is 0.0260. This is the P-value

Large-Sample Approximation

When the sample size n is large, the test statistic S+ is approximately normally distributed.

A rule of thumb is that the normal approximation is good if $n > 20$.

It can be shown by advanced methods that under H0, S+ has

mean= n(n + 1)/4

and variance= n(n + 1)(2n + 1)/24

The Wilcoxon signed-rank test is performed by computing the z-score of S+, and then using the normal table to find the P-value.

The z-score is

$$z = \frac{S_+ - n(n+1)/4}{\sqrt{n(n+1)(2n+1)/24}}$$

The article “Exact Evaluation of Batch-Ordering Inventory Policies in Two-Echelon Supply Chains with Periodic Review” (G. Chacon, Operations Research, 2001: 79–98) presents an evaluation of a reorder point policy, which is a rule for determining when to restock an inventory. Costs for 32 scenarios are estimated. Let μ represent the mean cost. Test $H0 : \mu \geq 70$ versus $H1 : \mu < 70$. The data is presented in Table

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Distribution Free Tests

X	X	X
79.26	22.39	10.08
80.79	118.39	7.28
82.07	118.46	6.87
82.14	20.32	6.23
57.19	16.69	4.57
55.86	16.50	4.09
42.08	15.95	140.09
41.78	15.16	140.77
100.01	14.22	
100.36	11.64	
30.46	11.48	
30.27	11.28	

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Distribution Free Tests

x	$x - 70$	Signed Rank	x	$x - 70$	Signed Rank	x	$x - 70$	Signed Rank
79.26	9.26	1	30.27	–39.73	–12	11.48	–58.52	–23
80.79	10.79	2	22.39	–47.61	–13	11.28	–58.72	–24
82.07	12.07	3	118.39	48.39	14	10.08	–59.92	–25
82.14	12.14	4	118.46	48.46	15	7.28	–62.72	–26
57.19	–12.81	–5	20.32	–49.68	–16	6.87	–63.13	–27
55.86	–14.14	–6	16.69	–53.31	–17	6.23	–63.77	–28
42.08	–27.92	–7	16.50	–53.50	–18	4.57	–65.43	–29
41.78	–28.22	–8	15.95	–54.05	–19	4.09	–65.91	–30
100.01	30.01	9	15.16	–54.84	–20	140.09	70.09	31
100.36	30.36	10	14.22	–55.78	–21	140.77	70.77	32
30.46	–39.54	–11	11.64	–58.36	–22			

Solution

The sample size is $n = 32$, so the mean is $n(n+1)/4 = 264$ and the variance is $n(n+1)(2n+1)/24 = 2860$. The sum of the positive ranks is $S_+ = 121$. We compute

$$z = \frac{121 - 264}{\sqrt{2860}} = -2.67$$

Since the null hypothesis is of the form $H_0: \mu \geq \mu_0$, small values of S_+ provide evidence against H_0. Thus the P-value is the area under the normal curve to the left of $z = -2.67$. This area, and thus the P-value, is 0.0038.

Large-Sample Approximation:

- When both sample sizes m and n are greater than 8, it can be shown by advanced methods that the null distribution of the test statistic W is approximately normal with mean $m(m+n+1)/2$ and variance $mn(m+n+1)/12$.

- $z - score\ is$

$$z = \frac{W - m(m+n+1)/2}{\sqrt{mn(m+n+1)/12}}$$

The article “Cost Analysis Between SABER and Design Bid Build Contracting Methods” (E. Henry and H. Brothers, Journal of Construction Engineering and Management, 2001:359–366) presents data on construction costs for 10 jobs bid by the traditional method (denoted X) and 19 jobs bid by an experimental system (denoted Y). The data, in units of dollars per square meter, and their ranks, are presented in . Test $H0 : \mu X \le \mu Y$ versus $H1 : \mu X > \mu Y$.

TABLE 6.2 Data for Example 6.20

Value	Rank	Sample	Value	Rank	Sample
57	1	X	613	16	X
95	2	Y	622	17	Y
101	3	Y	708	18	X
118	4	Y	726	19	Y
149	5	Y	843	20	Y
196	6	Y	908	21	Y
200	7	Y	926	22	X
233	8	Y	943	23	Y
243	9	Y	1048	24	Y
341	10	Y	1165	25	X
419	11	Y	1293	26	X
457	12	X	1593	27	X
584	13	X	1952	28	X
592	14	Y	2424	29	Y
594	15	Y			

Solution

The sum of the X ranks is $W = 1+12+13+16+18+22+25+26+27+28 = 188$. The sample sizes are $m = 10$ and $n = 19$. We use the normal approximation and compute

$$z = \frac{188 - 10(10+19+1)/2}{\sqrt{10(19)(10+19+1)/12}}$$

$$= 1.74$$

Large values of W provide evidence against the null hypothesis. Therefore the P-value is the area under the normal curve to the right of $z = 1.74$. From the z table we find that the P-value is 0.0409.

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