The Real Tau Conjecture is True on Average

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Introduction

- We study the number of **real zeros** of real univariate polynomials.
- A polynomial f is called t-**sparse** if it has at most t monomials.
- By Descartes' rule: a t-sparse polynomial f has at most $t-1$ positive real zeros.For a product $f_1 \cdots f_k$ of k such polynomials, there are at most $k(t-1)$ positive real zeros.
- Question: what about a **sum of** m **such products**?

$$F = \sum_{i=1}^{m} \prod_{j=1}^{k_i} f_{ij} \tag{1.1}$$

- Each f_{ij} is t-sparse. F has the structure $\Sigma\Pi\Sigma\Pi$ (a depth-4 arithmetic circuit) with parameters:

$$m, \quad k := \max_i k_i, \quad t.$$

The Real τ-Conjecture

The number of real zeros of a polynomial F of the form (1.1) is bounded by a polynomial in $m, \ k, \ t$

Significance:

- Proposed by Koiran (1996).
- If true, it implies a major result in **complexity theory:**

$$\mathrm{VP} \neq \mathrm{VNP} \text{ over } \mathbb{C}.$$

- The conjecture connects algebraic geometry (real zeros) with computational complexity.
- Even a weaker bound (polynomial in $m, k, t, 2^{\max k_i}$) would separate VP from VNP.

Setup

We fix a smooth function $f : \mathbb{R}^N \to \mathbb{R}$ such that

$$\{u \in \mathbb{R}^N : \nabla f(u) = 0\}$$

has measure zero. For regular values $a \in \mathbb{R}$, the fiber

$$H := f^{-1}(a)$$

is a smooth hypersurface in $\mathbb{R}^N$.

Pushforward Measure

Given a probability distribution on $\mathbb{R}^N$ with density ρ, we define

$$\rho_f(a) := \int_H \frac{\rho}{\|\nabla f\|} \, dH,$$

ρ_f is the pushforward measure w.r.t f of the measure on $\mathbb{R}^N$.

ρ_f is the pushforward measure with respect to f of the measure on $\mathbb{R}^N$ with density ρ.

Scaling Rule

For $\lambda \in \mathbb{R}^*$,

$$\rho_{\lambda f}(\lambda a) = \frac{1}{|\lambda|}\rho_f(a).$$

Conditional Density on a Level Set

Let $H := f^{-1}(a)$ for a regular value a such that $\rho_f(a) > 0$. Define the conditional density for $u \in H$ as

$$\rho_H(u) := \frac{1}{\rho_f(a)} \frac{\rho(u)}{\|\nabla f(u)\|}.$$

Conditional Expectation

Using the conditional density, we define

$$\mathbb{E}(Z \mid f = a) := \int_H Z\, \rho_H \, dH, \qquad H = f^{-1}(a),$$

for a nonnegative measurable function $Z : \mathbb{R}^N \to [0, \infty)$.

Equivalent Formula

$$\mathbb{E}(Z \mid f = a)\, \rho_f(a) = \int_H \frac{Z\rho}{\|\nabla f\|} \, dH,$$

Proposition 2.1

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a smooth function such that $\{u \in \mathbb{R}^N : \nabla f(u) = 0\}$ has measure zero. Let ρ be a probability density on $\mathbb{R}^N$ and $Z : \mathbb{R}^N \to [0, \infty)$ be measurable. Then:

$$\mathbb{E}(Z) = \int_{\mathbb{R}} \mathbb{E}(Z \mid f = a)\, \rho_f(a) \, da.$$

Lemma 2.2

$$dH = \frac{\|\nabla f\|}{|\partial_{u_1} f|}\, du_2 \cdots du_N.$$

Corollary

If H is parametrized by $(u_2, \ldots, u_N)$, then

$$\rho_f(a) = \int_{\mathbb{R}^{N-1}} \frac{\rho}{|\partial_{u_1} f|}\, du_2 \cdots du_N,$$

and

$$\mathbb{E}(Z \mid f = a)\, \rho_f(a) = \int_{\mathbb{R}^{N-1}} Z\, \frac{\rho}{|\partial_{u_1} f|}\, du_2 \cdots du_N.$$

Products of Gaussians

Setup

Let $y_1, \ldots, y_k$ be independent standard Gaussian random variables. Denote by ϖ_k the density of their product $y_1 \cdots y_k$.

Density Expression

$$\varpi_k(a) = \int_{(y_2,\ldots,y_k)\in\mathbb{R}^{k-1}} \varphi\left(\frac{a}{y_2\cdots y_k}\right)\varphi(y_2)\cdots\varphi(y_k)\frac{dy_2\cdots dy_k}{|y_2|\cdots|y_k|}$$

where $\varphi(y) = (2\pi)^{-\frac{1}{2}} e^{-y^2/2}$.

General Case

If $y_i \sim \mathcal{N}(0, \sigma_i^2)$ are independent, then

$$\rho_f(a) = \frac{1}{\sigma_1\cdots\sigma_k}\varpi_k\left(\frac{|a|}{\sigma_1\cdots\sigma_k}\right)$$

Properties of ϖ_k (Lemma 2.5)

1. ϖ_k is monotonically decreasing on $(0, \infty)$ and symmetric: $\varpi_k(-a) = \varpi_k(a)$.
2. For $0 < \delta \leq \frac{1}{2}$ and $a \in \mathbb{R}^*$, $\varpi_k(a) \leq |a|^{\delta - 1}$.
3. For $a \in \mathbb{R}^*$, $\varpi_k(a) \leq e\,|a|^{\frac{1}{k} - 1}$.

The Rice Formula

The Rice formula is a key tool in the theory of random fields. Provides an integral expression for the expected number of zeros of random functions. We apply it to the space $\mathbb{R}[X]_{\leq D}$ — polynomials of degree $\leq D$ in one variable. A random polynomial is parametrized as $F_u(X)$, where $u \in \mathbb{R}^N$ are random parameters.

Statement

Consider $F : \mathbb{R} \to \mathbb{R}$, $F(x) = F_u(x)$. The number of $\#\{x \in [0,1] : F(x) = 0\}$ becomes a random variable. Assuming $F(x)$ has density $\rho_F(x)$, the Rice Formula states, under some assumption:

$$\mathbb{E}[\#\{x \in [0,1] : F(x) = 0\}] = \int_0^1 \mathbb{E}(|F'(x)| \mid F(x) = 0)\, \rho_{F(x)}(0)\, dx$$

Examples

(1) Fix integers $0 = d_1 < d_2 < \cdots < d_t$. Then

$$F_u(x) = u_1 x^{d_1} + u_2 x^{d_2} + \cdots + u_t x^{d_t}$$

parametrizes sparse polynomials with support $\{d_1, \ldots, d_t\}$. Note that for all $x \in \mathbb{R}$, $F(x)$ is a nonconstant linear function of the argument u. In particular, $F(x)$ does not have singular values.

(2) Fix integers $0 = d_1 < d_2 < \cdots < d_t$ and $0 = e_1 < e_2 < \cdots < e_t$. The family

$$F_{u,v}(x) = (u_1 + u_2 x^{d_2} + \cdots + u_t x^{d_t})(v_1 + v_2 x^{e_2} + \cdots + v_t x^{e_t})$$

parametrizes products of two sparse polynomials with supports $\{d_1, \ldots, d_t\}$ and $\{e_1, \ldots, e_t\}$.Thus for all $x \in \mathbb{R}$, $F(x)$ is surjective and 0 is its only singular value.

The Rice Inequality

Theorem

Let $\mathbb{R}^N \times [x_0, x_1] \to \mathbb{R}, (u, x) \mapsto F_u(x)$ be a smooth function such that, for all $x \in [x_0, x_1]$, $\{u \in \mathbb{R}^N : \nabla F(x)(u) = 0\}$ has measure zero. Moreover, we assume that, for almost all $u \in \mathbb{R}^N$, the function $[x_0, x_1] \to \mathbb{R}$ has only finitely many zeros. Further, let a probability density ρ be given on $\mathbb{R}^N$. We assume there exists an integrable function $g : [x_0, x_1] \to [0, \infty)$ and $\varepsilon > 0$ such that for all $x \in [x_0, x_1]$ and almost all $a \in (-\varepsilon, \varepsilon)$ we have

$$\mathbb{E}(|F'_x(x)| \mid F(x) = a)\, \rho_{F(x)}(a) \leq g(x).$$

Then, for a random u with the density ρ, we can bound the expected number of zeros of the random function $x \mapsto F_u(x)$ in the interval $[x_0, x_1]$ as follows:

$$\mathbb{E}(\#\{x \in [x_0, x_1] : F(x) = 0\}) \leq \int_{x_0}^{x_1} g(x)\, dx.$$

Lemma

A C^1 function $f : [x_0, x_1] \to \mathbb{R}$ with only finitely many turning points satisfies

$$N(f) := \#\{x \in (x_0, x_1) : f(x) = 0\} \leq \lim_{\delta \to 0} \frac{1}{2\delta} \int_{x_0}^{x_1} \mathbf{1}_{\{|f(x)|<\delta\}} \, |f'(x)| \, dx.$$

In fact, for sufficiently small $\delta > 0$, the right-hand side equals $N(f) + \eta$, where $\eta = 0, \frac{1}{2}, 1$ according to as none, one or both of the numbers x_0, x_1 are zeros of f.

Proof:(To be shown separately)

Conditional Expectations of Random Linear Combinations

Setup

- $u_1, \ldots, u_m$ — independent real random variables with densities $\varphi_1, \ldots, \varphi_m$.
- Fix real weights $w_1, \ldots, w_m$ (not all zero) and define:

$$f := w_1 u_1 + \cdots + w_m u_m.$$

- We study bounds for $\mathbb{E}(|u_i| \mid f = a)\rho_f(a)$.

Lemma 4.1 (Bound on ρ_f)

If $\|\varphi_i\|_\infty \leq A$ for all i, then

$$\|\rho_f\|_\infty \leq \frac{A}{\max_i |w_i|}.$$

In particular, $\|\rho_f\|_\infty \leq A$ if $w_i = 1$ for some i.

Definition 4.2 (Convenient Density)

A probability density φ on $\mathbb{R}$ is called **convenient** if:

- φ is monotonically decreasing on $(0, \infty)$ and symmetric: $\varphi(-u) = \varphi(u)$.
- $\mathbb{E}_\varphi := \int_{\mathbb{R}} |u|\varphi(u)\, du < \infty$.

Lemma 4.3 (Properties of Convenient Densities)

If φ and ψ are convenient densities, then:

1. $|u|\varphi(u) \leq \frac{1}{2}$.
2. $\int_{\mathbb{R}} |u|\, \varphi(u)\psi(u)\, du \leq 1$.

Products of independent Gaussian random variables $\varpi_k$have convenient densities.

Some bounds

Proposition 4.4

Consider

$$f = w_1 u_1 + \cdots + w_m u_m, \quad (w_1, \ldots, w_m) \neq 0.$$

If the density φ_i of u_i is convenient, then for any $a \in \mathbb{R}$,

$$\mathbb{E}(|u_i| \mid f = a)\, \rho_f(a) \leq \frac{1}{|w_i|}$$

Proposition 4.5

Suppose u_i has a convenient density φ_i with $\mathbb{E}_{\varphi_i} \leq B$, for $i = 2, \ldots, m$. Further, assume the density φ_1 of u_1 satisfies

$$\forall u \quad \varphi_1(u) \leq C|u|^{\delta-1}$$

for some constants $C > 0$ and $0 < \delta \leq 1$. Then, for all $w_2, \ldots, w_m \in \mathbb{R}$, the random linear combination

$$f := u_1 + w_2 u_2 + \cdots + w_m u_m$$

satisfies for $i \geq 2$ and all $a \in \mathbb{R}$,

$$\mathbb{E}(|u_i| \mid f = a)\, \rho_f(a) \leq C(\delta^{-1} + B)\, |w_i|^{\delta-1}.$$

Random Linear Combinations of Functions

Setup

- Fix analytic functions $w_1, \ldots, w_m : [x_0, x_1] \to \mathbb{R}$.
- For $u \in \mathbb{R}^m$, define the linear combination:

$$F(x) = \sum_{i=1}^{m} w_i(x) u_i.$$

- Assume $w_1, \ldots, w_m$ do not have a common zero in $[x_0, x_1]$.

Lemma 5.1

The set of $u \in \mathbb{R}^m$ such that $\sum_{i=1}^{m} w_i(x) u_i$ has infinitely many zeros is of measure zero.

Bound on Expected Number of Real Zeros

Proposition 5.2

Suppose A, B are constants such that

$$\forall i, \quad \|\varphi_i\|_\infty \leq A, \quad \mathbb{E}_{\varphi_i} \leq B.$$

Then for $F(x) = u_1 + \sum_{i=2}^m w_i(x) u_i$, we have for all $x \in [x_0, x_1]$ and $a \in \mathbb{R}$:

$$\mathbb{E}(|F'(x)| \mid F(x) = a)\, \rho_F(a) \leq AB \sum_{i=2}^m |w_i'(x)|.$$

Therefore,

$$\mathbb{E}[\#\{x \in [x_0, x_1] : F(x) = 0\}] \leq AB \sum_{i=2}^m \int_{x_0}^{x_1} |w_i'(x)|\, dx.$$

Corollary 5.3

In the situation of Proposition 5.2, if all functions w_i are monotically increasing, then

$$\mathbb{E}\#\{x \in [x_0, x_1] : F(x) = 0\} \leq AB \sum_{i=2}^{m} (w_i(x_1) - w_i(x_0)).$$

In particular, for monomials $w_i(x) = x^{d_i}$ with $0 = d_1 < d_2 < \cdots < d_m$, the random fewnomial $F(x) = \sum_{i=1}^{m} u_i x^{d_i}$ satisfies

$$\mathbb{E}\#\{x \in [0, 1] : F(x) = 0\} \leq AB\,(m - 1).$$

Remark

Better bounds exist for particular distributions of the coefficients (e.g. Gaussian), yielding sharper rates such as $O(\sqrt{m \log m})$ for some random sparse polynomials.

Proposition 5.5

- For $i = 2, \ldots, m$, each u_i has a *convenient* density φ_i with $\mathbb{E}_{\varphi_i} \leq B$.
- The density φ_1 of u_1 satisfies

$$\varphi_1(u) \leq C\,|u|^{\delta-1} \quad \text{for all } u,$$

  for constants $C \geq 1$ and $0 < \delta \leq 1$.
- Define the random linear combination

$$F(x) := u_1 + \sum_{i=2}^{m} w_i(x) u_i.$$

For all $x \in [x_0, x_1]$ and all $a \in \mathbb{R}$ the following holds:

$$\mathbb{E}\big(|F'(x)| \mid F(x) = a\big)\, \rho_{F(x)}(a) \;\leq\; C\big(\delta^{-1} + B\big) \sum_{i=2}^{m} \frac{|w_i'(x)|}{\max\{|w_i(x)|,\ |w_i(x)|^{1-\delta}\}}$$

Definition 5.6 (Logarithmic Variation)

For a function $q : [x_0, x_1] \to (0, \infty)$, define

$$\mathrm{LV}(q) := \int_{x_0}^{x_1} \left| \frac{d}{dx} \ln q(x) \right|, dx = \int_{x_0}^{x_1} \frac{|q'(x)|}{q(x)} \, dx.$$

Lemma 5.7 (Properties of Logarithmic Variation)

1. If q is monotonically increasing, then $\mathrm{LV}(q) = \ln q(x_1) - \ln q(x_0)$.
2. $\mathrm{LV}(q_1 q_2) \leq \mathrm{LV}(q_1) + \mathrm{LV}(q_2)$.
3. $\mathrm{LV}(q^r) = |r| \, \mathrm{LV}(q)$ for $r \in \mathbb{R}$.

Bound via Logarithmic Variation

For a finite subset $S \subset \mathbb{N}$, define the polynomial

$$\alpha_S(x) := \sum_{s \in S} x^{2s}.$$

Let subsets $S_i \subset \mathbb{N}$ satisfy $0 \in S_i$, $|S_i| \leq t$, for $1 \leq i \leq \ell$ we choose $1 \leq k \leq \ell$ and define

$$q(x) := \left(\frac{\alpha_{S_1}(x) \cdots \alpha_{S_k}(x)}{\alpha_{S_{k+1}}(x) \cdots \alpha_{S_\ell}(x)} \right)^{1/2}.$$

Proposition 5.8

Let $d \in \mathbb{N}$ and $0 < \delta \leq 1$. The function $w : [0,1] \to [0,\infty)$, $x \mapsto q(x)x^d$, satisfies:

$$\mathrm{LV}(q) \leq \tfrac{1}{2} \ln t.$$

Moreover,

$$\int_0^1 \frac{|w'(x)|}{\max\{w(x), w(x)^{1-\delta}\}}\, dx \leq 2\,\mathrm{LV}(q) + kt + \frac{1}{\delta}$$

Sum of Product of Sparse Polynomials

Setting

We assume without loss of generality that

$$\forall i, j, \quad S_{ij} \subset \mathbb{N}, \quad 0 \in S_{ij}, \quad |S_{ij}| \leq t,$$

and define

$$F(x) := \sum_{i=1}^{m} f_{i1}(x) \cdots f_{ik_i}(x)\, x^{d_i},$$

where the degree pattern is $0 = d_1 \leq d_2 \leq \cdots \leq d_m$.

Probabilistic Model

For each i, j and $s \in S_{ij}$, fix a convenient probability density φ_{ijs} on $\mathbb{R}$ and assume constants A, B exist such that

$$\forall i, j, s : \quad \|\varphi_{ijs}\|_\infty \leq A, \quad \mathbb{E}_{\varphi_{ijs}} \leq B.$$

Random Polynomials

$$f_{ij}(x) = \sum_{s \in S_{ij}} u_{ijs} x^s,$$

with independent real coefficients u_{ijs} having convenient densities φ_{ijs}. The goal: to study the expected number of real zeros of the resulting random polynomial F.

Generating Functions

For each support S_{ij}, define:

$$\alpha_{ij}(x) := \sum_{s \in S_{ij}} x^{2s}, \qquad \beta_{ij}(x) := \sum_{s \in S_{ij}} x^s.$$

Note: $\mathbb{E}(f_{ij}(x)^2) = \alpha_{ij}(x)$ if $\mathbb{E}(u_{ijs}^2) = 1$, since $\mathbb{E}(u_{ijs}) = 0$.

Structure of the Polynomial Map

Lemma 6.1

Let

$$N := \sum_{i=1}^{m} \sum_{j=1}^{k_i} |S_{ij}|$$

denote the number of parameters. For $x \in \mathbb{R}$, consider the polynomial map

$$F(x) : \mathbb{R}^N \to \mathbb{R}, \quad u = (u_{ijs}) \mapsto F(x),$$

as defined in (6.1). Then, for all $x \in \mathbb{R}$:

(a) $F(x)$ is surjective and nonconstant.

(b) All nonzero $a \in \mathbb{R}$ are regular values of $F(x)$.

(c) 0 is a singular value of $F(x)$ only if $k_1 = \cdots = k_m = 1$.

We analyze the case $m = 1$, corresponding to one product

$$g(x) := f_1(x) \cdots f_k(x),$$

where each $f_j(x)$ is a random t-sparse polynomial

$$f_j(x) = \sum_{s \in S_j} u_{js} x^s,$$

with $0 \in S_j$ and $|S_j| \leq t$ for all j.

We also write

$$\beta_j(x) := \sum_{s \in S_j} x^s.$$

By Lemma 6.1, every nonzero $a \in \mathbb{R}$ is a regular value of $g(x)$, so the conditional density $\rho_{g(x)}(a)$ and conditional expectations with respect to $g(x) = a$ are well defined whenever $\rho_{g(x)}(a) > 0$.

Lemma 6.2

For all $x \in \mathbb{R}$ and all nonzero $a \in \mathbb{R}$, we have:

$$\mathbb{E}\left(\left|\frac{f_j'(x)}{f_j(x)}\right| \Big| g(x) = a\right) \rho_{g(x)}(a) \leq AB\frac{\beta_j'(x)}{|a|},$$

and

$$\mathbb{E}\Big(|g'(x)|\Big|g(x) = a\Big) \rho_{g(x)}(a) \leq AB\sum_{j=1}^{k} \beta_j'(x).$$

6.2. Polynomials with Nonzero Constant Coefficient

We consider the special case

$$d_1 = d_2 = \cdots = d_m = 0,$$

so each f_{ij} almost surely has a nonzero constant coefficient. This case is significantly simpler to analyze than the general one.

Theorem 6.3

Under the assumptions from the beginning of Section 6, the random polynomial

$$F = \sum_{i=1}^{m} f_{i1} \cdots f_{ik_i}$$

satisfies

$$\mathbb{E}\#\{x \in [0,1] : F(x) = 0\} \leq AB\,(k_1 + \cdots + k_m)\,(t-1).$$

6.3. Proof of main result — Setup

Gaussian coefficients and weight functions

Specialize Section 6 to the case where all coefficients u_{ijs} are standard Gaussian. For $1 \leq i \leq m$ define the auxiliary analytic weight functions

$$q_i(x) := \prod_{j=1}^{k_i} \left(\frac{\alpha_{ij}(x)}{\alpha_{1j}(x)} \right)^{1/2}, \qquad (\text{recall } \alpha_{ij}(x) = \sum_{s \in S_{ij}} x^{2s})$$

(note $q_i(x) > 0$ for all x and $q_1(x) = 1$).

Analytic weight and reduction

Define the analytic weight functions

$$w_i(x) := q_i(x) x^{d_i}, \qquad 1 \leq i \leq m \quad (\text{and } w_1(x) = 1).$$

Reduce counting zeros of the structured random polynomial F to studying the random linear combination

$$R(x) := \sum_{i=1}^{m} u_i q_i(x) x^{d_i} = u_1 + u_2 q_2(x) x^{d_2} + \cdots + u_m q_m(x) x^{d_m},$$

where the coefficients u_i follow the distribution ϖ_{k_i} (product of k_i Gaussians).

Proposition 6.4

For $x \in \mathbb{R}$ and nonzero $a \in \mathbb{R}$ we have

$$\mathbb{E}(|F'(x)| \mid F(x) = a)\, \rho_{F(x)}(a) \leq$$

$$\frac{1}{\sqrt{2\pi}} \sum_{i=1}^{m} \sum_{j=1}^{k_i} \beta'_{ij}(x) + \sum_{i=1}^{m} \left| \frac{q'_i(x)}{q_i(x)} \right| + \mathbb{E}(|R'(x)| \mid R(x) = a)\, \rho_{R(x)}(a).$$

Crucial part of the proof

For fixed $x \in \mathbb{R}$ and nonzero $a \in \mathbb{R}$ note that $f_{ij}(x)$ is a centered Gaussian random variable having the variance $\alpha_{ij}(x)$ (recall (6.2) and (6.3)). So we may write

$$f_{ij}(x) = \alpha_{ij}(x)^{\frac{1}{2}} v_{ij}$$

with independent standard Gaussian random variables v_{ij}. Hence, if we abbreviate $u_i := v_{i1} \cdots v_{ik_i}$ and put

$$p_i(x) := (\alpha_{i1}(x) \cdots \alpha_{ik_i}(x))^{-\frac{1}{2}},$$

then

$$g_i(x) = f_{i1}(x) \cdots f_{ik_i}(x) = \alpha_{i1}(x)^{\frac{1}{2}} \cdots \alpha_{ik_i}(x)^{\frac{1}{2}} v_{i1} \cdots v_{ik_i} = p_i(x)^{-1} u_i.$$

By its definition, the random variable u_i has the distribution ϖ_{k_i}. It is a convenient distribution. We also note that

$$q_i(x) = \frac{p_1(x)}{p_i(x)}$$

With these notations, we can write

$$F(x) = \sum_{i=1}^{m} g_i(x) x^{d_i} = \sum_{i=1}^{m} \frac{u_i}{p_i(x)} x^{d_i} = \frac{1}{p_1(x)} \sum_{i=1}^{m} u_i q_i(x) x^{d_i} = \frac{1}{p_1(x)} R(x).$$

Final Proof

The right-hand term in the statement of Proposition 6.4 can be bounded with Proposition 5.5. Indeed, due to Lemma 2.5 we know that $\varpi_{k_1}(a) \leq e\,|a|^{\frac{1}{2k_1}-1}$ for all a. Applying Proposition 5.5 with the parameters $B = 1$, $C = e$, and $\delta = (2k_1)^{-1}$ yields

$$\mathbb{E}(|R'(x)| \mid R(x) = a)\rho_{R(x)}(a) \leq e(2k_1+1)\sum_{i=2}^{m} \frac{|w_i'(x)|}{\max\{w_i(x), w_i(x)^{1-\frac{1}{2k_1}}\}}.$$

Applying Proposition 6.4 implies for $x \in \mathbb{R}$ and $a \in \mathbb{R}^*$, recalling that $w_i(x) := q_i(x)x^{d_i}$,

$$\mathbb{E}(|F'(x)| \mid F(x) = a)\rho_{F(x)}(a) \leq \frac{1}{\sqrt{2\pi}}\sum_{i=1}^{m}\sum_{j=1}^{k_i}\beta_{ij}'(x) + \sum_{i=1}^{m}\left|\frac{q_i'(x)}{q_i(x)}\right|$$

$$+e(2k_1+1)\sum_{i=2}^{m} \frac{|w_i'(x)|}{\max\{w_i(x), w_i(x)^{1-\frac{1}{2k_1}}\}} =: g(x).$$

$$\int_0^1 \frac{|w_i'(x)|}{\max\{w_i(x), w_i(x)^{1-\frac{1}{2k_1}}\}} \, dx \leq 2\mathrm{LV}(q_i) + k_i t + 2k_1.$$

The function $g(x)$ on the right-hand side of (6.13) is integrable:

$$\int_0^1 g(x)\, dx \leq \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{m} \sum_{j=1}^{k_i} (t-1) + \sum_{i=1}^{m} \mathrm{LV}(q_i) + e(2k_1+1) \sum_{i=2}^{m} (2\mathrm{LV}(q_i) + k_i t + 2k_1$$

By Proposition 5.8 we can bound $\mathrm{LV}(q_i) \leq \frac{1}{2} k_i \ln t$. Moreover, Theorem 3.2 can be applied (see Lemma 6.1) and states that

$$\mathbb{E}(\#\{x \in [0,1] : F(x) = 0\}) \leq \int_0^1 g(x)\, dx.$$

$$\mathbb{E}(\#\{x \in [0,1] : F(x) = 0\}) \leq \frac{1}{\sqrt{2\pi}} (k_1 + \cdots + k_m)(t-1)$$

$$+ (k_1 + \cdots + k_m) \ln t$$

$$+ e(2k_1+1)((k_2 + \cdots + k_m)(2\ln t + t) + (m-1)2k_1) = \mathcal{O}(k^2 m t).$$