On Real Roots of Dense and Sparse Random Polynomials

$$f(x) = \underbrace{a_0 + a_1 x^3 + a_2 x^{11} + \cdots + a_{20} x^{200}}_{15 \text{ terms.}}$$

# of real roots.

$f(x) = x^{100} + x^{10} - x$

<u>Descartes sign rule:</u>

If the # of sign changes is t in $f(x)$
the # of +ve real roots is
t or $t-2$ or $t-4, \ldots$

$f(-x)$ -ve

$\boxed{\text{Sparsity of } f \text{ is the no. of nonzero monomials}}$

For a K-sparse polynomials the total no. of real roots is $\le \boxed{2K-2}$.

$(x - x^5 + x^{10} \quad + \infty$

<u>Ex:</u>

$$f(x) = \underbrace{(x+1)}_{2} \underbrace{(x^3 - 3x + 4)}_{3} + \underbrace{(x^2+2)}_{2} \underbrace{(x^5 - 4x^3 + 3)}_{3}$$

The Real-τ Conjecture :- (Pascal Koiran)

$$F = \sum_{i=1}^{m} \underbrace{\left(\prod_{j=1}^{K_i} f_{ij} \right)}$$

$K := \max_i (K_i)$

$\rightarrow$ depth 4 $\Sigma\Pi\Sigma\Pi$ circuit

where f_{ij} are - t sparse

then the # of real roots of F is $O((mKt)^c)$

equivalently $O(\text{poly}(mKt))$

$O(m\, t^K) \rightarrow$

Arithmetic Circuits :

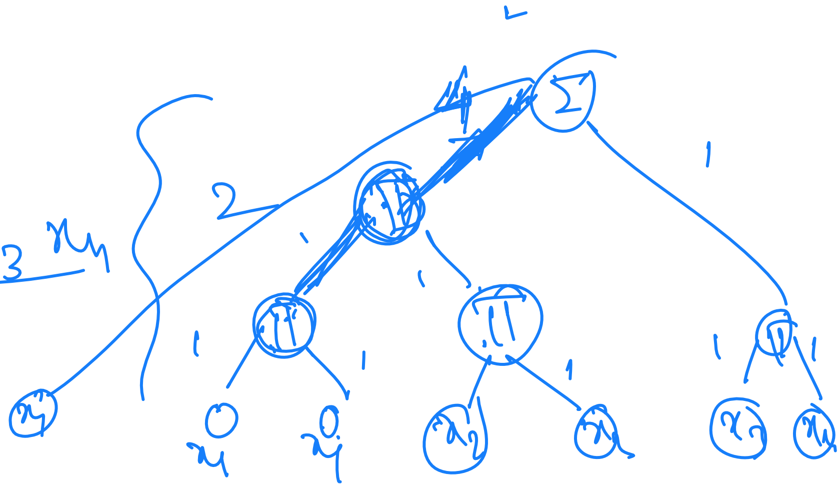
$f \in \mathbb{F}[x_0, \ldots, x_n]$

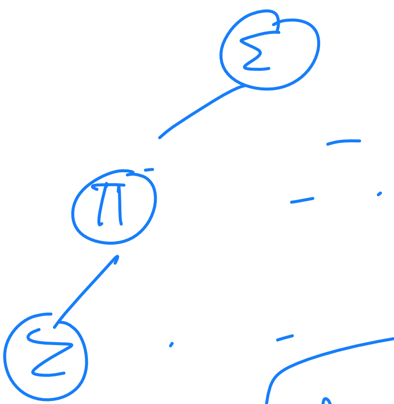
$$f = \sum_{I \in S} a_I \underline{x}^I$$

Nodes: Σ ; Π

$f(x) = 4x_1^2 x_2^2 + x_3 x_4$

$+2x_4$





$\Sigma\Pi\Sigma\Pi$ arithmetic circuits.

$\Sigma\Pi$ circuit

$$\boxed{f_{ij} = \sum_{\alpha_i = 1}^{t} a_{\alpha_i} x^{\alpha_i}} \rightarrow \text{depth-2}$$

Bürgisser showed

→ The Real-τ conjecture holds in average case.

→ $O(mk^2t)$.

$\cancel{VP^0 \neq VNP^0}$

P vs NP

$VP_{\mathbb{C}} \neq VNP_{\mathbb{C}}$

coupled with some upper bounds $\boxed{VP \neq VNP}$

Valiant's P $\quad \overline{VP} \stackrel{?}{=} VP$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$a_i \sim N(0,1)$$

$E_n :=$ Expected value of number of real zeros.

(1995)

(*) Bürgisser showed the expected no. of real roots for k-sparse poly. is $O(\sqrt{k}\log k)$

(*) 2020 3 MPI Jindal/Pandey/.

→ this is $O(\sqrt{k})$. AND it's tight.

Theorem 1:- Let $S \subseteq \mathbb{N}$ be the support of f.

$|S| = k$. $\quad S = \{e_1, \dots, e_k\}$

$$f = \sum_{i=1}^{k} \alpha_i x^{e_i} \qquad \alpha_i \neq 0$$

$$\boxed{E_S^K \le \frac{2}{\pi}\sqrt{K-1}}$$

Thm 2: $\exists$ a seq^n $S_n \subset \mathbb{N}$ $|S_n| = K+2$

s.t $\forall$ $K \geq 3$ $\boxed{E_{S_n} \geq \frac{\pi-\sqrt{3}}{16\pi}\sqrt{K} + \frac{1}{7}}$

Theorem (Kac '43) (Dense case)

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$a_i \sim N(0,1)$

$$\boxed{E_n = \frac{1}{\pi}\int_{-\infty}^{\infty}\sqrt{\frac{1}{(t^2-1)^2} - \frac{(n+1)^2 t^{2n}}{(t^{2n+2}-1)^2}}\,dt}$$

$$\boxed{E_n \sim O(\log n)}$$

Edelman-Kostlan's derivation of Kac's integral:-

$$\vec{a} = \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} ; \quad v(t) = \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix} \in \mathbb{R}^{n+1} \quad t \in \mathbb{R}.$$

$\vec{a} \leftrightarrow f$

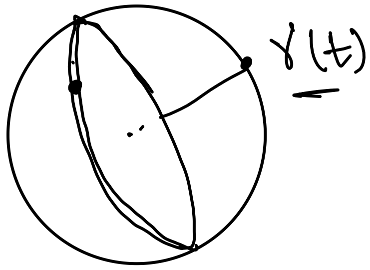
moment curve

t is a root iff $a \cdot v(t) = 0$

$\alpha := a/\|a\|$; $\gamma(t) = v(t)/\|v(t)\|$

Let S^n be the unit sphere on $\mathbb{R}^{n+1}$.

$\alpha, \gamma(t) \in S^n$.



Observation:-

If $a \in N(0, I)$

then $\alpha = a/\|a\| \sim$ Unif on S^n.

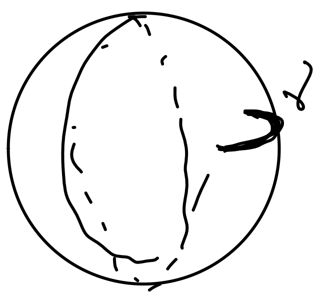
(*) density of a is independent of the orientation.

(*) **Def^n:** (Equator) $P_\perp := \{ x \in S^n \mid x \perp P \}$

Let γ be a curve on S^n,

$$\gamma_\perp := \{ P_\perp : P \in \gamma \}$$

Def^n: The multiplicity of a point $Q \in \gamma_\perp$ is the number of equators in $\gamma_\perp$ that contain Q [denoted by $m(Q)$]



Def^n: A curve on S^n is called "rectifiable" if the length is finite.

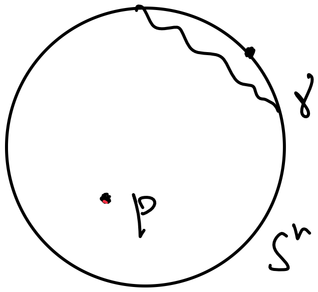
Lemma:- for a rectifiable curve γ.

$$\boxed{|\gamma_\perp| = \frac{|\gamma|}{\pi} A_n}$$

area swept by equators includes overlapping)

area of S^n.

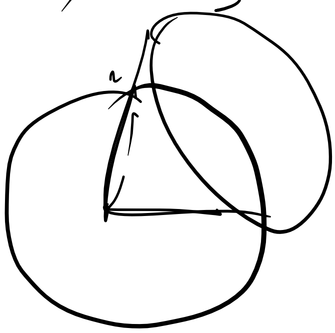
$E_n :=$ expected # of real roots.



let $p \in S^n$

$\boxed{m(p) = \#\text{ real roots of the poly. corresponding to } p.}$

$n=2$ / S^2



$\gamma(t) = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} \to$ squashed cone

$\boxed{\gamma(t) \text{ is rectifiable}}$

$$E_n = \int_{p \in S^n} m(p) \times \underbrace{\text{density of } p}_{= 1/A_n}$$

$$= 1/A_n \cdot \boxed{\int_{p \in S^n} m(p)}$$

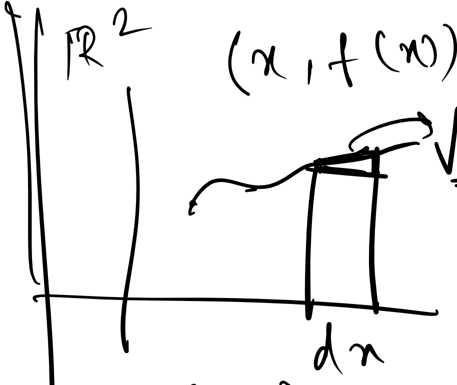
Area swept by $\gamma_\perp = |\gamma_\perp|$

$= \frac{|\gamma|}{\pi} \cdot A_n$

$$\boxed{E_n = \frac{|\gamma|}{\pi}}$$

$$E_n = \frac{1}{\pi} \int_{-\infty}^{\infty} \|\gamma'(t)\| \cdot dt$$

after doing messy algebra to revive Kac's integral.



$$\sqrt{1+f'(x)^2}\,dx = \|\gamma'(x)\|$$

$$\gamma = \begin{pmatrix} x \\ f(x) \end{pmatrix} \qquad \|\gamma'(x)\| = \sqrt{1+f'(x)^2}$$

Sparsity Conj

$f \in \mathbb{F}[x_0, \dots, x_n]$ with $\deg_{x_i}(f) < d$

if $f = g \cdot h$ $\quad g, h \in \mathbb{F}[x_0, \dots, x_n]$

then $\|f\|^{O(\text{poly}(d))} \geq \|g\|$

$s = \|f\| :=$ sparsity of f.

$$\#\text{ real roots} \leq O(\sqrt{K})$$

$$(\#\text{ real roots})^2 \leq K \to \text{sparsity}$$

$$K \geq (\text{avg \# real roots of } f)^2 \geq (\text{avg \# real roots of } g)^2$$

$$= (n - \text{avg \# imaginary roots})^2$$

upper bound w.r.t $\|g\|$