# Previous Vector Techniques: A Reminder

- Vector and Cartesian equations of a straight line
- Vector, dot product and Cartesian equations of a plane
- Angles between 2 lines, 2 planes, or a line and a plane
- Intersections of 2 lines or a line and a plane
- Shortest distances between any combination of points, lines and planes (parallel or skew)

## Scalar/dot product

- $a.b = a_1b_1 + a_2b_2 + a_3b_3$ where $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
- $\cos\theta = \frac{a.b}{|a||b|}$
- Perpendicular lines have $\theta = 90°$ so $a.b = 0$ where $a, b$ are the respective direction vectors

## Equation of a line

- $r = a + \lambda b$ where $a$ is any point on the line and $b$ is the direction vector (found by calculating the difference between any 2 points on the line)
- $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$ is an equivalent Cartesian equation

  Unlikely to happen at A level but if the direction vector is parallel to an axis e.g. $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ then we obviously can't divide by zero in the Cartesian equation. The solution is to give two separate equations e.g. $r = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ would become $\frac{x-2}{1} = \frac{y+3}{2}$ AND $z = 5$. The z-coordinate is fixed and essentially independent of $x$ and $y$

## Equation of a plane

- $(r - a).n = 0$ or $r.n = a.n$ where $a$ is a point on the plane and $n$ is a vector perpendicular to the plane
- $n_1x + n_2y + n_3z = a.n$ is the equivalent Cartesian equation, found by substituting $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ into the above vector equation
- $r = a + \lambda b + \mu c$ is also equivalent, where $a$ is a fixed point on the plane and $b, c$ are direction vectors on (parallel to) the plane, often found by calculating the difference between points on the plane
- To convert from $r = a + \lambda b + \mu c$ to Cartesian, set $x = a_1 + \lambda b_1 + \mu c_1$ and the equivalent for the $y-$coordinate, then eliminate $\mu$ simultaneously to get an expression for $\lambda$ in terms of $x$ and $y$, then repeat the trick to get $\mu$ in terms of $x$ and $y$, then substitute into $z = a_3 + \lambda b_3 + \mu c_3$
  HOWEVER, it is much easier once you have studied FP1 vectors to take the cross product of the two direction vectors to find a suitable normal vector.

## Angles between things

- 2 intersecting lines: use $\cos\theta = \frac{b_1.b_2}{|b_1||b_2|}$ with direction vectors $b_1$ , $b_2$
- 2 planes: equal to the acute angle between the two normal vectors, using the dot product
- Line and plane: 90 – (angle between line direction and plane normal)

## Intersections between things

- 2 lines (suppose $r_1 = a + \lambda b$ and $r_2 = c + \mu d$): Set all three components equal to each other. Solve the first two simultaneously to find $\lambda$ and $\mu$. If the solution works for the third equation, you have the intersection point. If not then there is no intersection.
- Line and plane: Substitute $x = a_1 + \lambda b_1$ , $y = a_2 + \lambda b_2$ and $z = a_3 + \lambda b_3$ into the Cartesian form of the plane equation, then solve the (linear) equation to find $\lambda$. Substitute $\lambda$ back into the line equation to find the point.

## Distances between things

- Plane and origin: If the plane is $r.n = a.n$ , divide through by $|n|$ to make the normal vector a unit normal (called $\hat{n}$). Then $r.\hat{n} = a.\hat{n}$ and the distance between the plane and the origin is $|a.\hat{n}|$
- Two parallel planes: Express both using the same normal vector i.e. $r.\hat{n} = a_1.\hat{n}$ and $.\hat{n} = a_2.\hat{n}$ Then the distance between the planes is $|a_1.\hat{n} - a_2.\hat{n}|$ i.e. the difference between their (plus or minus) distances to the origin.
- Point and plane – suppose $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $n_1x + n_2y + n_3z + d = 0$
  Shortest distance formula, from the formula booklet: $\left|\frac{n_1a_1+n_2b_2+n_3b_3+d}{|n|}\right|$
  To understand this, construct a parallel plane through your point and then find the distance between the two parallel planes. To do so, supposing the plane is $r.\hat{n} = a.\hat{n}$ (note use of unit normal vector) and the point has position vector $p$, make the new plane $r.\hat{n} = p.\hat{n}$ and then find the distance between those two planes
- Point and line: $P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ and $r = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
  We need to find $Q$ on the line such that $\overrightarrow{PQ}$ is perpendicular to the line.
  Take $Q$ as a generic point on the line i.e. $\overrightarrow{OQ} = \begin{pmatrix} a_1 + \lambda b_1 \\ a_2 + \lambda b_2 \\ a_3 + \lambda b_3 \end{pmatrix}$
  Then $\overrightarrow{PQ} = \begin{pmatrix} a_1 + \lambda b_1 - p_1 \\ a_2 + \lambda b_2 - p_2 \\ a_3 + \lambda b_3 - p_3 \end{pmatrix}$
  Set $\overrightarrow{PQ}.b = 0$, solve to find $\lambda$ and then evaluate $|\overrightarrow{PQ}|$
  (Or find $|\overrightarrow{PQ}| = \sqrt{\text{a quadratic in } \lambda}$ and minimise the quadratic)
- 2 parallel lines: take any point on one line and find the minimum distance to the other line
- 2 skew lines: We are looking for the points $P$ and $Q$ on the two respective lines for which $\overrightarrow{PQ}$ is perpendicular to both lines (since the shortest distance is always perpendicular). So take generic points $P$ and $Q$ in terms of $\lambda$ and $\mu$ and find $\overrightarrow{PQ}$. For each of the two direction vectors of the lines, set $\overrightarrow{PQ}.\text{direction vector} = 0$ giving two linear equations in $\lambda$ and $\mu$. Solve simultaneously, then substitute back into $\overrightarrow{PQ}$ and find $|\overrightarrow{PQ}|$. Note: you get a formula to achieve this more quickly in FP1