Proof by Induction - Recurrance Relations

The Method

This method of proof by induction is used for questions of the form:

$U_1 = x$ and $U_{n+1} = f(U_n)$
Prove by induction that $U_n = g(n)$

There are several steps to this proof:

1. Prove $U_n = g(n)$ true for $n = 1$
2. Assume $U_k = g(k)$ (This is the same as saying '*Assume $U_n = g(n)$ is true for $n = k$*')
3. Substitute in $k+1$ ($U_{k+1} = f(U_k)$)
4. Rearrange to be in the same form as $g(n)$, except all instances of n are replaced by $k+1$.
5. As this is true for $n = 1$, and true for $n = k+1$, this is true for all $n \in \mathbb{N}$.
6. You have proved the statement via induction.

An Example

Question

Given $u_1 = 8$ and $u_{n+1} = 4u_n - 9n$, prove $u_n = 4^n + 3n + 1$

Step 1

$u_1 = 4^1 + 3(1) + 1$
$u_1 = 4 + 3 + 1$
$u_1 = 8$
✓ True for $n = 1$

Step 2

Assume $u_n = 4^n + 3n + 1$ true for $n = k$.
$u_k = 4^k + 3k + 1$

Step 3

$u_{k+1} = 4(u_k) - 9k$
$u_{k+1} = 4(4^k + 3k + 1) - 9k$

Step 4

$u_{k+1} = 4^{k+1} + 12k + 4 - 9k$
$u_{k+1} = 4^{k+1} + 3k + 4$
$u_{k+1} = 4^{k+1} + 3(k+1) + 1$

> In the last part of this step, $3k + 4$ is rearranged into $3(k+1) + 1$. This does not appear to be a normal thing to do, but, if guided by the question, and what it is asking you to prove, it becomes far

> more understandable.

Step 5

✓ True for $n = k + 1$ if true for $n = k$.
$\therefore$ True for all $n \in \mathbb{N}$.

Step 6

$\therefore u_n = 4^n + 3n + 1$ must be true.