Implicit Differentiation

An explicit equation expresses the dependent variable in terms of the independent variable e.g. $y = f(x)$

An implicit equation is one which relates the variables not in an explicit fashion e.g. $x^2 + 2xy - y^2 = 0$

The chain rule allows us to differentiate implicit equations.
In a (notationally dubious) nutshell, the idea is that $\frac{d}{dx} = \frac{d}{dy} \times \frac{dy}{dx}$
i.e. instead of differentiating with respect to x (which we can't do because we don't have a function of x), we differentiate with respect to y and multiply the answer by $\frac{dy}{dx}$

Slightly more accurately:

$$\boxed{\frac{d}{dx} f(y) = \left(\frac{d}{dy} f(y)\right) \times \frac{dy}{dx}}$$

e.g. Differentiate with respect to x the equation $y^2 + \sin y - x^2 = e^{5x}$

We only need use implicit differentiation when we are differentiating a function of y. So we get:

$$2y\frac{dy}{dx} + \cos y\frac{dy}{dx} - 2x = 5e^{5x}$$

e.g. Find an expression for $\frac{dy}{dx}$ in terms of x and y if $e^{3y} - 3y^4 = \sin x$

Differentiating the equation gives:

$$3e^{3y}\frac{dy}{dx} - 12y^3\frac{dy}{dx} = \cos x$$

Then take out a factor of $\frac{dy}{dx}$ and divide through:

$$\frac{dy}{dx}(3e^{3y} - 12y^3) = \cos x$$
$$\frac{dy}{dx} = \frac{\cos x}{3e^{3y} - 12y^3}$$

Be Careful!

The easiest type of question to mess up is one where you have to use both implicit and explicit differentiation techniques in the same term e.g. when differentiating the product of functions of x and y.

e.g. Find $\frac{dy}{dy}$ if $x^2 \sin y = 3y^2$

When differentiating $x^2 \sin y$ we must use the product rule. When differentiating x^2 we needn't worry about implicit differentiation. But when differentiating $\sin y$ we need to multiply by $\frac{dy}{dx}$. Thus:

$$2x\sin y + x^2\cos y\frac{dy}{dx} = 6y\frac{dy}{dx}$$
$$\frac{dy}{dx}(6y - x^2\cos y) = 2x\sin y$$
$$\frac{dy}{dx} = \frac{2x\sin y}{6y - x^2\cos y}$$