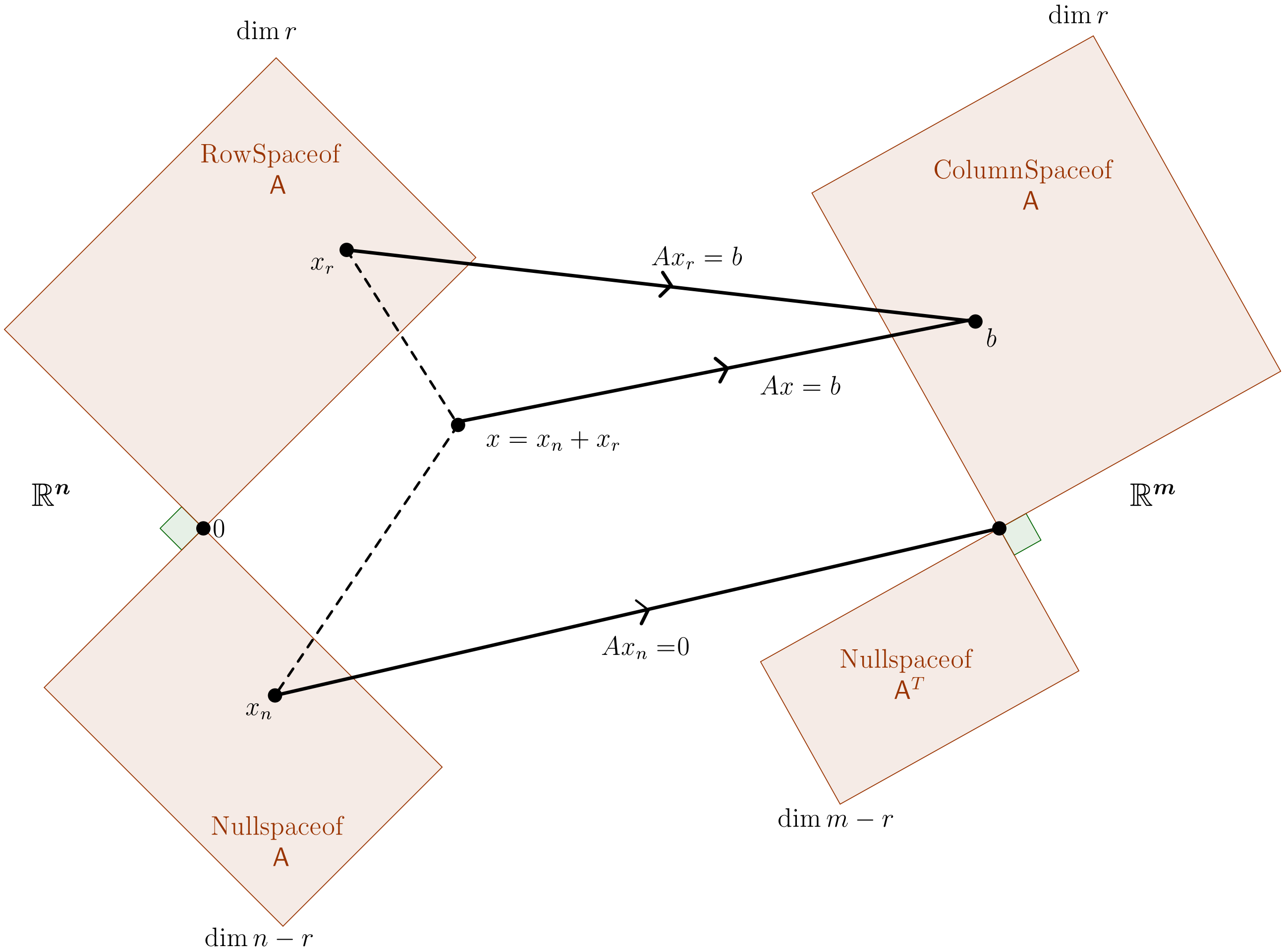
Mathematics for Artificial Intelligence & Computer Engineering I

AICE1004 Lecture notes in the style of *Tractatus Logico-Philosophicus*

$$\top \quad \bot \quad \neg \; \forall \; \exists \quad \implies \quad \iff \quad \vdash \quad \mathbb{N} \;\; \mathbb{Z} \;\; \mathbb{Q} \;\; \mathbb{R} \;\; \mathbb{C}$$

$$\forall X \Big[X \neq \varnothing \implies \exists f : X \to \bigcup X \;\; \forall A \in X \Big(f(A) \in A \Big) \Big]$$

$$\Big\{ \lim_{x \to a} f(x) \Big\} \;\coloneqq\; \big(\forall \epsilon > 0\big) \big(\exists \delta > 0\big) \Big(|x - a| < \delta \implies |f(x) - f(a)| < \epsilon \Big)$$



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Chapter 1 Foundation of Mathematics

1.1 Logic and Proofs

1.1.1 Terminology in mathematics

1.1.1.1. **Axiom**: statement assumed to be true

1.1.1.2. **Definition**: precise meaning of something

1.1.1.3. **Theorem**: a statement that has been proved to be true

- **Lemma**: a small theorem used to prove other theorem
- **Corollary**: a small theorem that follows from proven theorem

1.1.1.4. **Proof**: a step-by-step argument that establishes, based on logic, that something is true

1.1.1.5. **Short forms**

Abb.	full	Abb.	full (Latin)	English
$\Vdash$	theorem	i.e.,	id est	that is
Def	definition	e.g.	exempli gratia	for example
Prop	proposition	etc.	et cetera	and other similar things
max	maximum	QED	quod erat demonstrandum	it was proved
min	minimum		a priori	what comes before
sup	supremum		a posteriori	what comes after
inf	inf		per se	essentially
LHS	left-hand side		verbatim	word-for-word
RHS	right-hand side	iff		if and only if
s.t.	such that, subject to			

1.1.1.6. Common symbols

	LaTeX	meaning		LaTeX	meaning		LaTeX	meaning
$\forall$	`\forall`	for all	$=$	`=`	equality	$\neq$	`\neq`	not equal
$\exists$	`\exists`	there exists	$\equiv$	`\equiv`	identity	$<$	`<`	less than
$\iff$	`\iff`	if and only if	$\coloneqq$	`\coloneqq`	defined	$\leq$	`\leq`	$=$ or $<$
$\square$	`\square`	end of proof	$\mathbb{N}$	`\mathbb{N}`	natural number	$\mathbb{R}$	`\mathbb{R}`	real number

1.1.1.7. Math symbols `https://tug.ctan.org/info/symbols/comprehensive/symbols-a4.pdf`

1.1.2 Axiom (Optional)

1.1.2.1. Axiom (=postulate = assumption) is a statement that is **taken to be true**.

1.1.2.2. Axiom $\approx$ "atom" in chemistry: the building block of everything.

1.1.2.3. Axiom can be intuitive. **E.g.** the first 4 axioms in Euclid's *The Elements*.

1.1.2.4. Axiom can be non-intuitive. **E.g.** Axiom of Choice. A stackexchange thread

1.1.2.5. Axiom doesn't care if you agree

1.1.2.6. **Axiom of Logic**

- **Identity** $A = A$. "Something is what it is and isn't what it isn't. Everything that exists has a specific nature."
  E.g. I am me. I am not you. I am not my dog. My dog is a dog. You are not my dog. You, my dog, and I are all separate. This is not a matter of opinion.
- **Non-Contradiction** $A \neq \neg A$. "Something cannot be both true and false simultaneously in the same context."
  E.g. A square has interior angles that add up to 360 degrees. A triangle has interior angles that add up to 180 degrees. A square-triangle cannot exist because an object cannot simultaneously have interior angles summing to both 180 and 360 degrees. This is not a matter of opinion. According to the axioms of geometry (Euclid's The Elements), this is a fact.
- **Excluded Middle** Every statement is either A or $\neg A$. "A statement is either true or false, with no middle ground."
  E.g. A cat is either dead or alive. This isn't a matter of opinion.
- **Transitivity** If $A = B$ and $B = C$, then $A = C$. "The properties of one premise must carry over to the other premises."
  E.g. A has less money than B. B has less money than C. Therefore, A has less money than C.

1.1.2.7. **Axioms of $\mathbb{N}$ (Peano)**

B1. For all $x \in \mathbb{N}, x = x$ — reflexivity

B2. For all $x, y \in \mathbb{N}$ if $x = y$, then $y = x$ — symmetry

B3. For all $x, y, z \in \mathbb{N}$ if $x = y$ and $y = z$ then $x = z$ — transitivity

P1. 0 is an integer. — $0 \in \mathbb{N}$

P2. If x is an integer, the successor of x is an integer. — If $x \in \mathbb{N}$, then $x + 1 \in \mathbb{N}$

- successor := the integer after x
- The $+1$ is represented by a function $S(x)$, S stands for successor function
- Recursive Definition of Addition
  R1. $a + 0 = a$
  R2. $a + S(b) = S(a + b)$

  Addition is the higher order version of successor.
  Multiplication is defined by Addition. Multiplication is the higher order version of addition.
  Exponentiation is defined by Multiplication. Exponentiation is the higher order version of multiplication.
  Then you have Tetration. Tetration is the higher order version of Exponentiation.
  $\uparrow, \uparrow\uparrow, \ldots$ Graham's number $3 \uparrow\uparrow\uparrow\uparrow 3$, black hole in your brain

P3. 0 is not the successor of an integer. — $\nexists x \in \mathbb{N}$ such that $0 = x + 1$.

P4. Two numbers of which the successors are equal are themselves equal. — If $x + 1 = y + 1$, then $x = y$

P5. If a set S of integers contains 0 and also the successor of every integer in S, then S has every integers

P3, P4, P5 can written compactly using the successor operator S

$$
\begin{aligned}
&P3 \quad \forall n \neg(0 = Sn) \\
&P4 \quad \forall n \forall m \Big(\neg(n = m) \implies \neg(Sn = Sm)\Big) \\
&P5 \quad \Big(0 \in F \wedge \forall n\big(n \in F \implies Sn \in F\big)\Big) \implies \forall n (n \in F)
\end{aligned}
$$

For P5

- P5 is known as Mathematical Induction.
- F is a set of numbers, so $P5$ is a 2nd-order logic notation.
- The 1st-order logic way to express P5 is $\Big(\psi(0) \wedge \forall n\big(\psi(n) \implies \psi(Sn)\big)\Big) \implies \forall n\ \psi(n)$

- **E.g.** Prove 1+1=2 by Peano Axiom

$$
\begin{aligned}
1 + 1 &= 1 + S(0) && \text{1 is the successor of 0} \\
&= S(1 + 0) && (R2) \\
&= S(1) && (R1) \\
&= 2 && \text{2 is the successor of 1}
\end{aligned}
$$

1.1.2.8. **Zermelo-Fraenkel Set Theory**

- Georg Cantor in 1870: A set is a collection of well-defined objects. — Naive set theory
- Russell in 1901: Cantor you are wrong. Let $R = \{x \mid x \notin x\}$, then $R \in R \iff R \notin R$ — Russell's paradox
  Why paradox: because the answer to $R \overset{?}{\in} R$ is both YES and NO, this contradicts to the Axiom of Excluded Middle.
- **Zermelo-Fraenkel Set Theory With Axiom of Choice**
  - Extensionality — if X, Y have same elements, then $X = Y$
    $\forall X \forall Y \big[\forall z (z \in X \iff z \in Y) \implies X = Y\big]$
  - Pairs — for all X, Y there exists a set $\{X, Y\}$
    $\forall X \forall Y \exists Z \forall w (w \in Z \iff w = X \vee w = Y)$
  - Union — for all X, we can form $Y = \bigcup X$
    $\forall X \exists Y \forall Z \big[Z \subset Y \iff \forall w (w \in Z \wedge w \in X)\big]$
  - Power set — for all X, there exists $Y = P(X)$ the set of all subset of X
    $\forall X \exists Y \forall Z \big[Z \subset Y \iff \forall w (w \in Z \implies w \in X)\big]$
  - Infinity — there exists an infinite set
    $\exists X \big[\varnothing \in X \wedge \forall y (y \in X \implies \bigcup\{y, \{y\}\} \in X)\big]$
  - Separation — if ψ is a property with parameter p, then for all X there exists $Y = \{u \in X : Y(u, p)\}$
    $\forall p \big[\forall X \exists Y \forall u (u \in Y \iff u \in X \wedge \psi(u, p))\big]$
  - Replacement $\forall p$ — if F is a function with parameter p, then for all X there is a set $Y = F(X) = \{F(x) : x \in X\}$
    $\Big[\forall X \exists! Y\, F(X, Y, p) \implies \forall W \exists V \forall r \Big(r \in V \iff s\big(s \in W \wedge F(s, r, p)\big)\Big)\Big]$
  - Regularity — all nonempty set has a minimal element (no set may contain itself)
    $\forall X \Big[X \neq \varnothing \implies \exists Y \Big(Y \subset X \wedge \forall z \big(z \in X \implies z \notin Y\big)\Big)\Big]$ — this one kills Russell's paradox

  Zermelo-Fraenkel set theory does not allow for the existence of a universal set (a set containing all sets) nor for unrestricted comprehension, thereby avoiding Russell's paradox

- Axiom of Choice — all family of non-empty sets has a choice function
  $\forall X\Big[X \neq \varnothing \implies \exists f : X \to \bigcup X \;\; \forall A \in X\Big(f(A) \in A\Big)\Big]$
- **ZFC** := Zermelo-Fraenkel Set Theory + Axiom of Choice
  Why ZFC is important: it is *a* foundation of *many (not all)* math
  - This is why can we say that $(\frac{2x}{2})^2 = x^2$
  - This is why we have algebra manipulation
  - This is why can we say that $p \implies q \equiv (\neg p) \vee q$
  - Replacement means instead of using A to describe X, we can use B to describe X.

  ZFC is not saying math "is" set.
  ZFC is saying math is "isomorphic to" sets

1.1.2.1. **Axiom of** $\mathbb{R}$
R1 (Field) $(\mathbb{R}, +, \times)$ is a field
R2 (Ordered) $(\mathbb{R}, +, \times, \leq)$ is ordered
R3 (Complete) All non-empty subset F of $\mathbb{R}$ with an upper bound in $\mathbb{R}$ has a least upper bound in $\mathbb{R}$

1.1.2.2. **The world is incomplete**

- Incompleteness Theorem (Godel, 1931) "This theorem is not provable."
  - "There are statements which are true but which can never be proved."
  - "Mathematics is incomplete."
  - "Mathematical truth cannot be defined in mathematics."
- Undefinability Theorem in Logic (Tarski, 1933) "This statement is not true."
  - "Logic is incomplete."
  - "Arithmetical truth cannot be defined in arithmetic."
- Undecidability Theorem (Church, 1936, Turing, 1937) "This program does not halt."
  - "There are problems in mathematics which cannot be solved by algorithm"
  - "Computation is incomplete."
  - "Computational truth (yes-no decision) cannot be defined in computation."

1.1.3 Proposition

Learning Objectives

- Understand proposition, truth table
- Apply NOT, AND, OR, Tautology, Contradiction, Implication to examples.
- Solve problems using truth table

1.1.3.1. Why logic: we understand math starting from logic.

1.1.3.2. **Def** A *proposition* P is a statement / logic variable that takes the truth value *true* or[1] *false*, and $\underbrace{\text{no others}}_{\text{Law of Excluded Middle}}$.

- Def means Definition, we write $A := B$ to denote "A is defined by B"
- Truth value: 1/T/true/I, 0/F/false/O
- We only said P "has a truth value either T or F", we didn't say
  - What exactly is the truth value of P $\begin{cases} \text{the truth value can be false} \\ \text{the truth value can be true} \end{cases}$.
  - We know the truth value now $\begin{cases} \text{the truth value known now} \\ \text{the truth value not yet known} \begin{cases} \text{can be known after 1 day} \\ \text{can be known after 2 days} \\ \vdots \\ \text{can never be known in the future} \end{cases} \end{cases}$

1.1.3.3. **E.g.** of proposition

- I am your father. — it is false and we know it now

[1] exclusive or

- $1+1=2$ — it is true and you know it many years ago
- $\sin 2\theta = 2\sin\theta\cos\theta$ — it is true and you know it in trigonometry
- (Axiom of Choice) $\forall X \left(\emptyset \notin X \implies \exists f : X \to \bigcup X \,\forall A \in X\, (f(A) \in A)\right)$
  - it is a proposition, it is true in ZFC theory and now you know it
  - this example is to show that a sentence is a proposition or not does not care you know or not knowing it
  - it is a proposition, and its true value is unknown (in fact indeterminable)
- There are two roots in $x^2+1=0$.
  - it is a proposition, it is false in $\mathbb{R}$, true in $\mathbb{C}$
  - this example is to show that the truth value of a proposition is context-dependent (i.e., is enough information given?)
- You will die tomorrow. — a proposition doesn't mean we need to know the true value now
- London Bridge is falling down — we don't know now, we need to go to London to know
- Donald Trump will win the coming US president election — we don't know now
- There is another Earth in the universe — we don't know, just wait until someone solves it
- $\mathbb{P} = \mathbb{NP}$ (the hardest mathematics problem). — we don't know, you get Fields medal if you solve it.
- Your hair is NOT blue.
  - If "you" is specified, this is a proposition with negation
  - If "you" is not specified that it reads as "X's hair is NOT blue", then this is NOT a proposition but a predicate

1.1.3.4. **E.g.** of non-proposition

- I think. — this sentence provides no truthfulness so this is not a proposition
- Hi there. — this sentence provides no truthfulness so this is not a proposition
- $3+2$ — this sentence provides no truthfulness so this is not a proposition
- Do you love sushi? — question is not a proposition
- I am lying. — Liar paradox: this is not a proposition by Law of excluded Middle.

1.1.3.5. **Short notation** We use a symbol (like P) to represent a proposition.

$$P \ : \ \text{The sum } 1+2+\ldots+n = \frac{n(n+1)}{2}$$

Then when we say P, it means "The sum of one plus two plus three, plus plus plus and so on, until plus n, is equal to n times n plus 1 and then divided by 2". Using symbol makes it short.

- Important: here we assume n is a given number and we know the value of n.
- We write $P(n)$ to specify the value of n, now it is not a proposition, it becomes a predicate (next subsection)

1.1.3.6. **Operator** We build new proposition from given one using operator

- Unary operator $F(P)$
- Binary operator $F(P,Q)$
- Ternary operator $F(P,Q,R)$, higher-order operator $F(P_1,\ldots,P_n)$

1.1.3.7. **Logic operator**

- (Unary) NOT $\neg$ — ! or $\sim$ in programming
- (Unary) Identity Id — Id is like "do nothing"
- (Unary) Tautology $\top$ — also denoted by I, looks like "1" in math
- (Unary) Contradiction $\perp$ — also denoted by O, looks like "0" in math
- (Binary) AND $\wedge$ — also called conjunction
- (Binary) OR $\vee$ — also called disjunction
- (Binary) XOR $\veebar$
- (Binary) NAND $\uparrow$
- (Binary) NOR $\downarrow$
- (Binary) Implication / conditional / If $\implies$
- (Binary) Only if $\impliedby$
- (Binary) Bi-implication / biconditional / if and only if / IFF $\iff$
- (Ternary) "If A then B otherwise C"

1.1.3.8. **E.g.** of composition of proposition using operator

- I am not happy. $\neg$
- He is tall and he wears glasses. $\wedge$
- I go to school by bus or on foot. $\vee$
  - inclusive or: either one or both options can be true.
  - I might go to school by bus for the full trip (I have to walk zero step in the whole trip or I teleport to bus station)
  - I might go to school on foot for the full trip
  - I might go to school by both means because I do not live next to bus station, so first I walk to bus station, then I take bus
- Restaurant lunch menu: "beef burger xor chicken burger" $\veebar$
  - exclusive or: either one but not both
- If $n \in \mathbb{N}$ is even, then n^2 is divisible by 4. $\implies$
- $n \in \mathbb{N}$ is even if and only if n^2 is even. $\iff$
- It is what it is. $\top$ (Tautology)
- To be or not to be. $\boldsymbol{T} \vee \neg(T) = T \vee F = \top$ (Tautology)
- My sister is jealous of me because I'm an only child. $\perp$ (contradiction)

1.1.3.9. **Tautology** is a always-true-proposition. A **contradiction** is a always-false-proposition.

- (Liar paradox) I am a liar is a "contradiction" in English, but it is not a contradiction in logic because this sentence does not even belongs to proposition
- Liar paradox $\notin$ proposition $\implies$ Liar paradox $\neq$ contradiction (in propositional logic)

1.1.3.10. A list of logic symbols

Symbol	Name	Meaning	Remark
$\perp$	contradiction	always wrong	
$\top$	tautology	always true	
$\wedge$	conjunction	and	
$\vee$	disjunction	or	
$\neg, \sim, !$	negation	not	
$\implies$	material implication	implies, conditional	ex falso quodlibet
$\iff, \equiv$	bi-conditional	if and only if	
$\exists$	existential quantifier	there exists	
$\forall$	universal quantifier	for all	
$\vdash$	turnstile	entails, proves	
$\models$	double turnstile	entails, therefore	

At university level we do not use $\because$ and $\therefore$.

1.1.3.11. **Exercises**

- Give a proposition
- Give a proposition with AND.
- Give a proposition with OR.
- Give a proposition with NOT.
- Give a proposition with IF ... THEN.
- Give a tautology.
- Give a contradiction.

1.1.3.12. **E.g.** IFF propositions (You will learn these later)

- n is even if and only if $n \equiv 0 \mod 2 \iff$ the remainder of n divided by 2 is 0.
- Sets A, B are equivalent if and only if A is a subset of B and B is a subset of A. See 1.2.1.15.
- A function $f(x)$ is continuous at a point $x = a$ if and only if the limit of $f(x)$ as x approaches a is equal to $f(a)$.
- A matrix $\boldsymbol{A}$ is invertible if and only if its determinant is non-zero. See 3.2.1.31.

1.1.3.13. **Unary operator**

operator	not	identity	tautology	contradiction
P	$\neg P$	$\mathrm{Id}P$	$\top P$	$\perp P$
1	0	1	1	0
0	1	0	1	0
As binary	$P \uparrow P$	$P \wedge 1$ $P \vee 0$	$P \vee 1$	$P \wedge 0$

- Truth table is invented by Ludwig Wittgenstein, PhD student of Bertrand Russell
  - Tractatus Logico-Philosophicu
  - "Whereof one cannot speak, thereof one must be silent"
- Unary means the operator works on single variable
  - Oneness: "uni", "uno", "une", "unus"
  - "University" = "one-verse"
  - "uniform" = "one-form"
  - "Univariate statistic" = "one-variable statistics"
  - "Univariate calculus" = "single-variable calculus"
- $\neg, \mathrm{Id}, \top, \perp$ are all the 4 possibilities you have for defining a unary operator.
  - We have two choices: 1 or 0
  - We have $2^2 = 4$ possible unary outcomes
- Boolean algebra
  - We only have $\{0, 1\}$.
  - $\neg P = 1 - P$
    We define subtraction "$-$" as
    * $\neg 1 = 0 = 1 - 1$
    * $\neg 0 = 1 = 1 - 0$
  - $\mathrm{Id}P = 1 \cdot P = \min\{1, P\}$
  - $\mathrm{Id}P = 0 + P = \max\{0, P\}$
  - $\top P = 1 + P = \max\{1, P\} \equiv 1$
  - $\perp P = 0 \cdot P = \min\{0, P\} \equiv 0$
- $\uparrow$ is NAND (NOT AND), will talk about it later

1.1.3.14. **E.g.** (**Negation**) B : The color of the door is black. $\neg B$: The color of the door is not black.
It is wrong to say $\neg B$: The color of the door is white, because "not black" can be yellow, blue, green, red, white ...

1.1.3.15. **E.g.** (**Negation**) D: I support Donald Trump. $\neg D$: I do not support Donald Trump.

- In logic, $\neg D$ only tells the statement D is negated. This negation does not imply any specific reason why D is negated.
- Therefore, "I do not support Donald Trump." only indicates the absence of support.
- It does not necessarily mean "I hate Trump" or "I don't care." Those are *additional interpretations* that go beyond the logical negation.

1.1.3.16. **False dichotomy / false dilemma**

- P: $x = 1$, $\neg P$: $x \neq 1$ where x can be any number that is not 1
- Law of excluded middle
  - the Law of excluded middle = The truth value of a proposition is either true or false
  - Either P or $\neg P$
  - "Excluded middle" is on the truth value of P
  - "Excluded middle" is not on the content of P
  - $\{$truth value of $P\} \neq \{$content of $P\}$

1.1.3.17. **E.g.** (**Negation**)

- C : The set $\mathcal{C}$ is compact and convex
- $\neg C$: The set $\mathcal{C}$ is not compact or not convex.
- This example shows that we can work with logic without really knowing the meaning of the terms. Not knowing the meaning of "convex" and "compact" has nothing to do with forming the proposition $\neg C$.

1.1.3.18. **Binary operators**

P	Q	AND $P \wedge Q$	OR $P \vee Q$	XOR $P \underline{\vee} Q$	implication $P \implies Q$
1	1	1	1	0	1
1	0	0	1	1	0
0	1	0	1	1	1 (ex falso quodlibet)
0	0	0	0	0	1 (ex falso quodlibet)

- Binary operator: works on two variables P, Q
- AND is also called **conjunction**
  Boolean algebra of $P \wedge Q$ is $\min\{P, Q\}$
  - $1 \times 1 = 1 = \min\{1, 1\}$
  - $1 \times 0 = 0 = \min\{1, 0\}$
  - $0 \times 1 = 0 = \min\{0, 1\}$
  - $0 \times 0 = 0 = \min\{0, 0\}$
- OR is also called **disjunction**
  Boolean algebra of $P \vee Q$ is $\max\{P, Q\}$
  - $1 + 1 = 1 = \max\{1, 1\}$
  - $1 + 0 = 1 = \max\{1, 0\}$
  - $0 + 1 = 1 = \max\{0, 1\}$
  - $0 + 0 = 0 = \max\{0, 0\}$
- XOR = exclusive or. "OR" in English is more likely to be exclusive or.
  - "Would you like coffee or tea?" = Coffee XOR tea
  - "He will arrive in a minute or two" = He will arrive in a minute XOR two
  - English using inclusive or: "Oranges or lemons are a good source of vitamin C."
- Ex falso quodlibet: 'from falsehood, anything follows'

1.1.3.19. Order of precedence

- $\neg$ has precedence over other operator
- For example, $\neg P \vee Q = (\neg P) \vee Q$

  To avoid confusion, write bracket. I.e., write $\neg(P \wedge Q)$ or $(\neg P) \wedge Q$ instead of $\neg P \wedge Q$.

1.1.3.20. **E.g.** Find the truth table for $(P \wedge Q) \vee (P \wedge R)$

- Step 1. Identify the number of cases
  - Each variable has 2 possibilities (T or F)
  - There are 3 variables.
  - There are $2^3 = 8$ possibilities
- Step 2. Write down all combinations of truth value

P	Q	R	$P \wedge Q$	$P \wedge R$
1	1	1		
1	1	0		
1	0	1		
1	0	0		
0	1	1		
0	1	0		
0	0	1		
0	0	0		

- Step 3. Work from inside to out

P	Q	R	$P \wedge Q$	$P \wedge R$
1	1	1	1	1
1	1	0	1	0
1	0	1	0	1
1	0	0	0	0
0	1	1	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0

P	Q	R	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
1	1	1	1	1	1
1	1	0	1	0	1
1	0	1	0	1	1
1	0	0	0	0	0
0	1	1	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	0	0	0

1.1.3.21. **Exercise** construct the truth table for the proposition $\big((\neg P) \vee Q\big) \wedge (\neg R)$

Solution

P	Q	R	$\big((\neg P) \vee Q\big) \wedge (\neg R)$
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

1.1.3.22. **Implication** $P \implies Q$

$$\underbrace{P}_{\substack{\text{antecedent}\\ \text{hypothesis}}} \implies \underbrace{Q}_{\text{consequent}}$$

- Material conditional
- "If P then Q", or "P implies Q"
- **if** I am a millionaire, **then** everyone get's an A.
  - P = I am a millionaire
  - Q = everyone get's an A
- ex falso quodlibet
  "vacuous truth"
  "true, but unimportant"
  "from falsehood, anything follows"
- No freedom: Logic implication is not the same as English implication
  "If I am a mouse then I chase cats." means if I am a mouse, I MUST chase cats, I do not have the "free will" not to chase case, "having freedom/option" is not what the logical implication referring to
  - If you mean "I can choose not to do Q if I have P", this is not $P \implies Q$
  - In logic, $P \implies Q$ means, if P, then MUST Q
- $P \implies Q$
  if I am a millionaire, **then** everyone get's an A.
- $\neg P \vee Q$
  Either I am **not** a millionaire, **or** everyone get's an A.
  - $\neg P$ = I am **not** a millionaire
  - Q = everyone get's an A

1.1.3.23. "I think" is not a proposition in 1.1.3.4.. The statement "I think, therefore I am" (I think $\implies$ I am) is NOT a proposition.

1.1.3.24. English of $P \implies Q$

- These are all the same
  - If P then Q
  - P implies Q
  - P only if Q
  - P is sufficient for Q
  - Q is necessary for P
  - Q provided that P
  - Q whenever P
  - Q if P
- **E.g.**
  - H: "happy"
  - F: "fly"
  - If happy, then fly $\qquad H \implies F$
  - Fly if happy $\qquad H \implies F$
  - Fly only if happy $\qquad F \implies H$
  - Fly if and only if happy $\qquad (F \implies H) \wedge (H \implies F)$, also known as $F \iff H$
  - happy implies not fly $\qquad H \implies \neg F$
  - happy is sufficient for fly $\qquad H \implies F$
  - fly is necessary for happy $\qquad H \implies F$
  - happy is necessary for fly $\qquad F \implies H$ or $H \impliedby F$
- **E.g.**
  - S: "study hard"
  - G: "good grade"
  - If study hard, then good grade $\qquad S \implies G$
  - Good grade if study hard $\qquad S \implies G$
  - Study hard is sufficient for good grade $\qquad S \implies G$

- Note: in reality it is possible "if genius, then good grade", "if cheating, then good grade". But now what we are considering is a hypothetical logical world that these are impossible.

1.1.3.25. Necessity and sufficiency in $P \implies Q$

- H: "have one million"
- R: "rich"
- have one million is sufficient to be rich $H \implies R$
- have one million is necessary for being rich $R \implies H$ or $H \impliedby R$

1.1.3.26. $P \implies Q$ is used intensively in mathematics

- Continuity is a necessary for differentiability
- If a sequence is monotone and bounded, then it converges

1.1.3.27. **Converse** $P \implies Q$ is not the same as $Q \implies P$

- $P \implies Q$ and $Q \implies P$ have different truth values.

P	Q	$P \implies Q$	$Q \implies P$
1	1	1	1
1	0	0	1 (ex falso quodlibet)
0	1	1 (ex falso quodlibet)	0
0	0	1 (ex falso quodlibet)	1 (ex falso quodlibet)

- $Q \implies P$ is called the **converse** of $P \implies Q$
- **Example of converse**
  - $(P \implies Q)$: If it is raining, then we will get wet P=rain, Q=wet
  - $(Q \implies P)$: If we get wet, then it is raining P=wet, Q=raining
  - We could get wet for many reasons (swimming) when it is not raining.

1.1.3.28. Four variations of $P \implies Q$

original statement	if P, then Q
contrapositive	if $\neg Q$, then $\neg P$
converse	if Q, then P
inverse	if $\neg P$, then $\neg Q$

1.1.3.29. **E.g.** Complicate proposition built using simple propositions.

- R: "it rains"
- W: "ground is wet"
- D: "it is dangerous"
- H: "I am happy"
- $\Big((R \implies W) \implies (W \implies D)\Big) \implies H$
  "If it rains, then the ground will be wet, then it is true that if the ground is wet, it is dangerous, and then it is true that I am happy." Ok this sentence doesn't make sense and this is just an example.

R	W	D	H	$P := R \implies W$	$Q := W \implies D$	$(P \implies Q) \implies H$
T	T	T	T	T	T	T
T	T	T	F	T	T	F
T	T	F	T	T	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	T	F	F	T	T
T	F	F	T	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	T	F	T	T	F
F	T	F	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	T	T
F	F	T	F	T	T	F
F	F	F	T	T	T	T
F	F	F	F	T	T	F

1.1.3.30. **Exercises**

- Give an implication proposition
- Give a converse, inverse and contrapositive

1.1.3.31. **IFF : if and only if** $P \iff Q$

- {if and only if} is a lazy way of stating that the proposition is true in two directions.
- P if and only if Q $\qquad P \iff Q$
- $(P \iff Q) \;=\; (P \implies Q) \wedge (P \impliedby Q)$
  - $P \implies Q$ $\qquad P$ is sufficient to Q
  - $P \impliedby Q$ $\qquad Q$ is necessary for P

P	Q	$P \implies Q$	$P \impliedby Q$	$P \iff Q$
1	1	1	1	1
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

1.1.3.32. **E.g. of bi-conditional**

- P :triangle is a right-angle triangle
- Q :the sides satisfy $a^2 + b^2 = c^2$
- $P \iff Q$, i.e., if P is true, then Q is true, and if P is false then Q is false, vice versa

1.1.3.33. **E.g. of IFF (Real analysis)** Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.

- This is in fact how you tell two (real) numbers are equal.
- **Proof**
  - We have to prove two directions
  - P : $\{a$ equals $b\}$
  - Q : { for all $\epsilon > 0$ it follows that $|a - b| < \epsilon$}.
  - The proof of $P \implies Q$: by direct proof
    * If $a = b$, then $|a - b| = 0$
    * Since $\epsilon > 0$ so we have $0 < \epsilon$ regardless of what ϵ is chosen
    * So we have $|a - b| < \epsilon$
  - The proof of $Q \implies P$: by contradiction
    * we assume $a \neq b$ (i.e., $\neg P$)
    * Because $a \neq b$, then $|a - b| \neq 0$, we let $|a - b| =: \epsilon_0$
    * We have $|a - b| < \epsilon$ regardless of what ϵ is chosen, such ϵ includes ϵ_0, so we will have
      $$|a - b| < \epsilon_0 \;\text{ AND }\; |a - b| = \epsilon_0$$
      which cannot both be true (a contradiction).
    * The initial assumption $a \neq b$ is false.
    * Therefore $a = b$
- This example generalizes to the $\epsilon - \delta$ definition of limit, continuity, differentiability in calculus!

1.1.3.34. **Equivalence of proposition** Two logic statements are equivalent if they have the same truth values.

1.1.3.35. **Theorem (Equivalence of implication and contrapositive)** $(P \implies Q) \equiv \Big((\neg Q) \implies (\neg P)\Big)$
Proof (by exhaustion)

P	Q	$\neg P$	$\neg Q$	$P \implies Q$	$(\neg Q) \implies (\neg P)$
1	1	0	0	1	1 (ex falso quodlibet)
1	0	0	1	0	0
0	1	1	0	1 (ex falso quodlibet)	1 (ex falso quodlibet)
0	0	1	1	1 (ex falso quodlibet)	1

- $(P \implies Q)$ and $(\neg Q) \implies (\neg P)$ have the same truth values, they are equivalent.
- The symbol $\equiv$ means equivalence.
- What we did in the proof is to list all possible truth values of P and Q, and this method of proof is called proof by exhaustion / proof by brute force.

- $(\neg Q) \implies (\neg P)$ is called the **contrapositive** of $(P \implies Q)$

1.1.3.36. **Exercises**

- Show that $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ using proof by exhaustion
- Show that $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ using proof by exhaustion
- Show that $(P \vee Q) \wedge \neg(P \wedge Q) \equiv P \underline{\vee} Q$ using proof by exhaustion

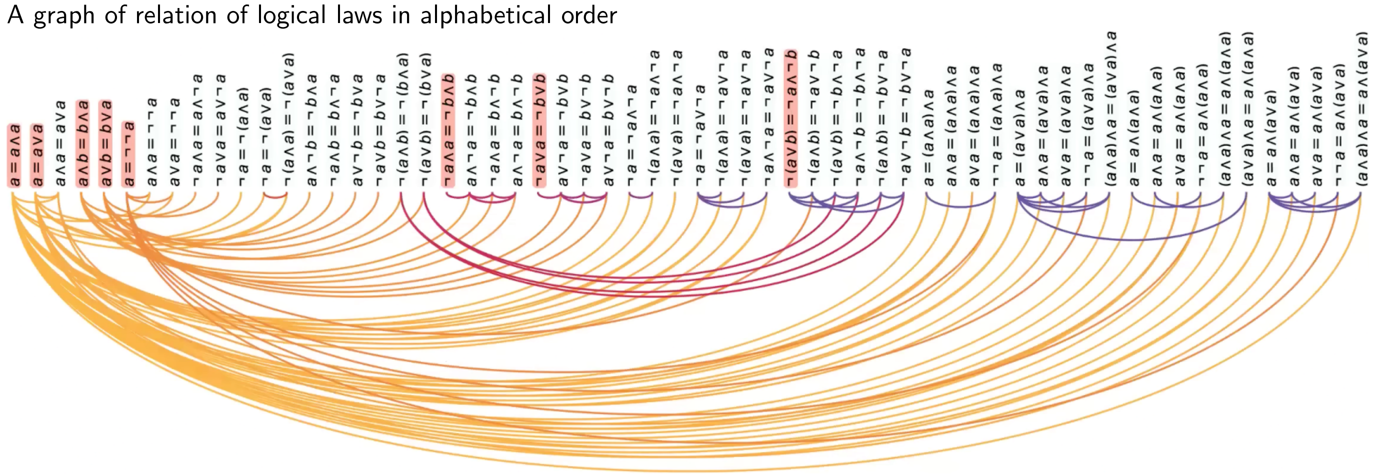
1.1.3.37. **Theorem (Logical equivalences)**

Identity	$P \wedge 1 \equiv P$ $P \vee 0 \equiv P$	**Domination**	$P \wedge 0 \equiv 0$ $P \vee 1 \equiv 1$
Idempotent	$P \wedge P \equiv P$	**Double negation**	$\neg(\neg P) \equiv P$
Commutative	$P \wedge Q \equiv Q \wedge P$ $P \vee Q \equiv Q \vee P$	**Associative**	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
Distributive	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	**De Morgan**	$\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$ $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$

- **Exercises** prove these by exhaustion
- **Example (Double negation is not fundamental)** We show that double negation law can be proved by other laws.

$$
\begin{aligned}
\neg(\neg P) &\equiv (\neg(\neg P)) \wedge 1 && \text{identity law} \\
&\equiv (\neg(\neg P)) \wedge (P \vee \neg P) && P \vee \neg P \equiv 1 \text{ by definition of } \vee \\
&\equiv (\neg(\neg P)) \wedge (\neg P \vee P) && \text{commutative law on the second bracket} \\
&\equiv \Big((\neg(\neg P)) \wedge \neg P\Big) \vee \Big((\neg(\neg P)) \wedge P\Big) && \text{distribution law: distribute } \neg(\neg P) \\
&\equiv O \vee \Big((\neg(\neg P)) \wedge P\Big) && \\
&\equiv \Big(\neg P \wedge P\Big) \vee \Big((\neg(\neg P)) \wedge P\Big) && O = P \wedge \neg P \text{ by definition of } \wedge \\
&\equiv \Big(P \wedge \neg P\Big) \vee \Big((\neg(\neg P)) \wedge P\Big) && \text{commutative law on the first bracket} \\
&\equiv \Big(P \wedge \neg P\Big) \vee \Big(P \wedge (\neg(\neg P))\Big) && \text{commutative law on the second bracket} \\
&\equiv P \wedge \Big((\neg P) \vee (\neg(\neg P))\Big) && \text{distribution law} \\
&\equiv P \wedge 1 && P \vee \neg P \equiv 1 \text{ by definition of } \vee \\
&\equiv P && \text{domination law}
\end{aligned}
$$

1.1.3.38. A graph of relation of logical laws in alphabetical order



- Source: Stephen Wolfram's talk "Computational Irreducibility, Minds, and Machine Learning"
  ISC Summer School on Large Language Models: Science and Stakes, June 7 2024
- The meaning of the graph
  - some "law" ("formula") can be proved by others, these formula
  - these laws provide no new information and knowledge
  - therefore those that are not depending on others are more "fundamental"
  - those fundamental laws are highlighted in red

1.1.3.39. **Def** NAND (Not AND) $P \uparrow Q$

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$P \uparrow Q$
1	1	1	0	0
1	0	0	1	1
0	1	0	1	1
0	0	0	1	1

1.1.3.40. **Theorem** Any logical function can be constructed from one single binary operator NAND.

- NOT $P = P$ NAND P
- P AND $Q =$ NOT (P NAND Q)
- P NOR $Q =$ (NOT P) AND (NOT Q)
- P OR $Q =$ NOT(P NOR Q)
- So we can build AND, OR, NOT using only NAND, which can be used to build all logical function.

1.1.3.41. **Def** An *conjecture* is a proposition which one claims to be true.

- $3n + 1$ conjecture
- $P \neq NP$ conjecture
- Multiplication of any two $n \times n$ matrices can be done in $n^{\omega + o(1)}$ operations where $\omega = 2$.

1.1.3.42. **Corollary (Prove by contradiction)**: we can prove conjectures by contradiction.

- Assume we know P is true and we want to prove that {then Q is true}.
- What we can do instead is
  - assume $\underbrace{Q\text{(what we want to prove) is not true}}_{\neg Q}$
  - prove that $\{Q$ is not true$\}$ gives $\underbrace{\text{the assumption } P \text{ is not true}}_{\neg P}$

  More in the proof section.

1.1.3.43. **Normal form** is used for quickly check a proposition is true or false

- **disjunction normal form (DNF)** is a disjunction(OR) of conjunction(AND).
  E.g. $(\neg P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (P \wedge Q)$
- **conjunction normal form (CDNF)** is a conjunction(AND) of disjunction(OR).
  E.g. $(\neg P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (P \vee Q)$

1.1.3.44. **SAT (Satisfiability)** What values of P, Q will make $(P \vee Q) \wedge (\neg P \vee \neg Q)$ true? Answer: $P = 1$ and $Q = 0$.

- SAT problem is like "solve equation $x + y = 1$", but now we work with Boolean variable (true or false)
- **Daily life example**
  - John can only meet either on Monday, Wednesday or Thursday.
  - Catherine cannot meet on Wednesday.
  - Anne cannot meet on Friday
  - Peter cannot meet neither on Tuesday nor on Thursday
  - Question: When can the meeting take place?
  - We solve it as
  $$(\text{Mon} \vee \text{Wed} \vee \text{Thu}) \wedge (\neg\text{Wed}) \wedge (\neg\text{Fri}) \wedge (\neg\text{Tue} \wedge \neg\text{Thu})$$
    The meeting must take place on Monday.
    This example is also gives one reason for "why the heck do we learn proposition logic"
- ⊩**(Cook–Levin)** SAT is a NP-(complete) problem.
  In English: SAT is a "hard" problem.
  If you find a method that can solve any SAT problem "fast", you get Fields medal / Nobel prize / Abel prize.

1.1.3.45. **Exercises**

- Write down the truth table of $((\neg P) \vee Q) \wedge (\neg R)$
- Write down the truth table of $(P \vee \neg Q) \vee (Q \wedge R)$
- Solve $\neg(P \wedge Q) \stackrel{?}{\equiv} (\neg P) \vee (\neg Q)$
- Solve $(P \vee Q) \vee ((\neg P) \wedge (\neg Q)) \stackrel{?}{\equiv} I$. recall I denotes tautology
- Solve $(P \wedge Q) \wedge ((\neg P) \vee (\neg Q)) \stackrel{?}{\equiv} O$. recall O denotes contradiction
- Simplify $((\neg R) \wedge P \wedge Q) \vee ((\neg R) \wedge (\neg P) \wedge Q)$ to $(\neg R) \wedge Q$
- Prove $\neg(P \implies Q) \equiv P \wedge (\neg Q)$
- Prove $P \implies Q \equiv (\neg P) \vee Q$
- Prove $P \iff Q \equiv ((\neg P) \vee Q) \wedge ((\neg Q) \vee P)$
- Rewrite $(P \implies Q) \implies Q$ without $\implies$

1.1.4 Predicate

Learning Objectives

- Understand $\forall, \exists$ and their negation
- Apply $\forall, \exists$ and their negation to examples.

1.1.4.1. Consider proposition: P : The sum $1 + 2 + ... + n = \frac{n(n+1)}{2}$.

Then when we say P, it means "The sum of one plus two plus three, plus plus plus and so on, until plus n, is equal to n times n plus 1 and then divided by 2". Using symbol makes it short.

- Important: here we assume n is a given number and we know the value of n.
- We write $P(n)$ to specify the value of n, now it is not a proposition, it becomes a predicate
- **E.g.** predicate
  $Q(x) : x$ is even
  $R(x, y) : x < y$
  $S(x, y, z) : x^2 + y^2 = z^2$ that x, y, z are integers
  (Fermat's Last Theorem) $F(x, y, z, n) : x^n + y^n = z^n$ that x, y, z, n are integers has no solution if $n \geq 3$.

1.1.4.2. **Def** A *predicate* is a proposition-valued function of logic variables.

1.1.4.3. **Def** *Universe of discourse* / *Domain of discourse* $:=$ the set of all truth values of the logic variables

1.1.4.4. **E.g. Predicate vs proposition** $P(x) : x$ is handsome.

- $P(x)$ has the truth value depends on x, y
- $P(x)$ is not a proposition, who are you referring to?
- $P(\text{Andersen})$ is a proposition, and has the truth value true (of course).

1.1.4.5. **E.g. Predicate vs proposition** $P(x, y)$: "$x + 2 = y$" is a predicate.

- $P(x, y)$ has the truth value depends on x, y
- $P(x, y)$ is not a proposition
- $P(1, 1)$ is a proposition, and has the truth value false.

1.1.4.6. **E.g.** $P(n)$: "n is prime"

n	1	2	3	4	5	6	7	8	9	10	11	...
$P(n)$	F	T	T	F	T	F	T	F	F	F	T	...

If you know everything about $P(n)$ you get Fields's medal.
This example shows that there are predicates that we do not know everything

1.1.4.7. You can construct new predicates from given ones like proposition using connectives.
E.g. $Q(x, y, z) = P(x) \wedge R(y, z)$

1.1.4.8. **Universal quantification (a way to convert predicate to proposition)**

- Given a predicate $P(x)$ on some universe of discourse.
- The *universal quantification of* $P(x)$ is the proposition: "$P(x)$ is true for all x in the universe of discourse".
- We write $\forall x P(x)$, say "for all $x, P(x)$"
- Let $Q = \forall x P(x)$,
  - then proposition Q is true if $P(x)$ is true for every single x
  - then proposition Q is false if there is an x for which $P(x)$ is false
    - we don't care if there are more than one x for which $P(x)$ is false

1.1.4.9. **E.g.** $A(x)$: "$x = 1$", $B(x)$: "$x > 2$", $C(x)$: "$x < 2$", and the universe of discourse is $\{1, 2, 3\}$.
Then $\forall x (C(x) \implies A(x))$ is true.
$\forall x (C(x) \vee B(x))$ is false.

1.1.4.10. **Existential quantification (a way to convert predicate to proposition)**

- Given a predicate $P(x)$ on some universe of discourse.
- The *existential quantification of* $P(x)$ is the proposition: "$P(x)$ is true for some x in the universe of discourse".
- We write $\exists x P(x)$, say "for some $x, P(x)$"
- Let $Q = \exists x P(x)$, then proposition Q is true if $P(x)$ is true for some x

- we don't care if there are more than one x for which $P(x)$ is true

- Let $Q = \exists x P(x)$, then proposition Q is false if there for all x that $P(x)$ is false

1.1.4.11. **E.g.** "All cats eat meat".

- $Cx \equiv$ "x is a cat"
- $Mx \equiv$ "x eats meat"

  "All cats eat meat" $\equiv \forall x\big(Cx \implies Mx\big)$

- The statement says nothing about the existence of a cat in the universe of all animal
- The statement only says "if a cat exists (more than one), then it will have certain properties"

1.1.4.12. **E.g.** "There exists a number less than 7" can be written as $\exists x \ : \ x < 7$

1.1.4.13. **E.g.** "If function f is continuous, then f is differentiable"

- $C(f)$: f is continuous
- $D(f)$: f is differentiable
- Fact: there is a function that is continuous and is not differentiable, so $(\exists f)\neg\Big(C(f) \implies D(f)\Big)$
- Ok now I tell you what f are: $|x|, \displaystyle\sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$

1.1.4.14. **E.g. Eudoxus-Archimedean property** $\forall x \, \exists m(x < m \ \wedge \ m \in \mathbb{N})$

- Historically, Archimedes refers to $x \in \mathbb{Q}$ where $x = \dfrac{a}{b}$.
  "given two positive integers a and b, there is an integer m such that $bm > a$"
- Implication of Archimedean property: the natural number is unbounded.
  "There is no ceiling in $\mathbb{N}$"

1.1.4.15. Quantifier negation

- $\neg(\forall x P(x))$ is the same as $\exists x(\neg P(x))$
  Example: $\neg\{$all primes are odd$\}$ is "there exists 2, which is prime, but not odd"
- $\neg(\exists x P(x))$ is the same as $\forall x(\neg P(x))$
  Example: $\neg\{$some primes larger than 2 are even$\}$ is "all primes larger than 2, are odd"

1.1.4.16. **E.g.** Descartes: I think therefore I am.
Formalize it as : ∀x: think(x) $\implies$ exist(x)
Since $A \implies B$ is equivalent to $\neg(A \wedge \neg B)$
The negation of Descartes's statement is

∃x: think(x) and not exists(x)

$$
\begin{aligned}
\forall x : A(x) \implies B(x) \qquad & \text{Descartes's statement} \\
\neg\Big(\forall x : A(x) \implies B(x)\Big) \qquad & \text{negation of Descartes's statement} \\
\neg\Big(\forall x : \neg\big(A(x) \wedge \neg B(x)\big)\Big) \qquad & A \implies B \ \equiv \ \neg(A \wedge \neg B) \\
\neg(\forall x) : \neg\neg\big(A(x) \wedge \neg B(x)\big) \qquad & \text{distribute } \neg \text{ inside the bracket} \\
\neg(\forall x) : A(x) \wedge \neg B(x) \qquad & \neg\neg = \mathrm{Id} \text{ (identity law in Theorem 1.1.3.37.)} \\
\exists x : A(x) \wedge \neg B(x) \qquad & \neg\forall = \exists \text{ (in quantifier negation 1.1.4.15.)}
\end{aligned}
$$

In English: the negation of Descarte's statement is: there is a person think, but that person doesn't exist

Exercise: now work on the converse, inverse and contrapositive

1.1.4.17. Nested Quantifier

- $\forall x \forall y P(x, y) = T$ — $P(x, y) = T$ for all x and y
- $\forall x \exists y P(x, y) = T$ — for all x there exists y such that $P(x, y) = T$
- $\exists x \forall y P(x, y) = T$ — there exists x such that $P(x, y) = T$ for all y
- $\exists x \exists y P(x, y) = T$ — there exists x and there exists y such that $P(x, y) = T$

1.1.4.18. **E.g.** Let $L(x, y)$ be the sentence "x loves y", and let the universe of discourse be all people.

- $\forall x, L(x, \text{Alice})$ Everybody loves Alice.
- $\neg\Big(\forall x, L(x, \text{Bob})\Big) \equiv \exists x, \neg L(x, \text{Bob})$ Not everybody loves Bob.
- $\forall x, \exists y, L(x, y)$ Everybody loves somebody.
- $\exists y, \forall x, L(x, y)$ There is somebody that everybody loves.
- $\exists y, \neg L(\text{Bob}, y)$ There is somebody that Bob does not love.
- $\exists y, \forall x, \neg L(x, y)$ There is somebody that no one loves.
- $\forall x, L(x, x)$ Everyone loves himself or herself.
- $\exists x, \neg L(x, x)$ There is somebody that does not love himself or herself.
- $\exists x, \forall y, \Big[L(x, y) \implies x = y\Big]$ There is somebody who loves no one besides himself or herself.

1.1.4.19. **E.g.** Write "roses are red" in predicate logic.
- rose(x): "x is a rose"
- red(x): "x is red"
- $\forall x\Big(\text{rose}(x) \implies \text{red}(x)\Big)$

yellow sunflower	$F \implies F$	T
red lily	$T \implies F$	T
red rose	$T \implies T$	T
white rose	$T \implies F$	F

1.1.4.20. **E.g.** (Optional) **(Zermelo's theorem in game theory)** In finite two-person games of perfect information in which the players move alternately and in which chance does not affect the decision making process, if draw is impossible, then one of the two players can force a win.

- Terms
  - Two players: $1, 2$
  - Perfect information: every player knows all the moves that have been made by all players up to that point
  - Finite: finite number of moves
  - Alternate moves: player-1 move, player-2 move, player-1 move, player-2 move
  - Chance does not affect the decision making process: no chance
  - Draw is impossible: the win function $W \in \{0, 1\}$, no others
  - Can force a win: have a winning strategy
- Notation
  - Let $x_1, x_2, x_3, \cdots$ be the decision made by player-1, and $y_1, y_2, y_3, \cdots$ be the decision made by player-2.
  - Let $W_1(x_1, y_1, x_2, y_2, \cdots)$ be the win function of player-1, and $W_2(x_1, y_1, x_2, y_2, \cdots)$ be the win function of player-2.
  - There is no draw means that both W_1 and W_2 can take values $\{0, 1\}$ where 0 means lose and 1 means win
  - Player 1's win means Player 2 loses, and Player 2's win means Player 1 loses, hence $W_2 = 1 - W_1$.
- Using the notation, Zermelo's theorem is $\exists x_1 \forall y_1 \exists x_2 \forall y_2 \cdots \exists x_n \forall y_n W_1(x_1, y_1, x_2, y_2, \cdots, x_n, y_n) = 1$, or we swap x and y for W_2

1.1.5 Sequent (Optional)

Learning Objectives

- Understand basic deductive argument forms

1.1.5.1. **Def** A *valid argument* is a finite set of propositions $P_1, P_2, ..., P_r$ called *premises*, together with a proposition C called *conclusion*, such that the proposition $(P_1 \wedge P_2 \wedge ... \wedge P_r) \implies C$ is a tautology.

- We call C the *logical consequence* of the premises
- We write $P_1, P_2, ..., P_r \vdash C$. That is, premise$_1$, premise$_2$, ..., premise$_r$ $\vdash$ conclusion, This is called linear notation.
- $\vdash$ is called *turnstile*

If an argument is not valid then we say that it is invalid.

1.1.5.2. **E.g.** $A, A \implies B \vdash B$

- premise$_1$ is A
- premise$_1$ is $A \implies B$
- there are only two premises
- conclusion is B

1.1.5.3. Another notation system called Gentzen system.

$$\frac{\text{premise}_1 \qquad \text{premise}_2}{\text{conclusion}}$$

1.1.5.4. **E.g.** $A, A \implies B \vdash B$

$$\frac{A \qquad A \implies B}{B}$$

1.1.5.5. Linear system use $\vdash$, and Gentzen system uses a horizontal line and also spacing.

1.1.5.6. There is another system called Fitch system, we don't care.

1.1.5.7. **E.g.** $\begin{cases} P_1 = \text{"Amy graduates"} \\ P_3 = \text{"Amy gets a job"} \end{cases}$. Now we consider the argument
"If Amy graduates then she gets a job".
"Amy graduates".
"Therefore Amy gets a job".

- This argument is symbolised as

$$P_1 \implies P_3,\ P_1 \vdash P_3 \qquad \text{or} \qquad \frac{P_1 \implies P_3 \qquad P_1}{P_3}$$

- To prove this argument is valid, we need to show that $\Big((P_1 \implies P_3) \wedge P_1\Big) \implies P_3$ is a tautology

P_1	P_3	$P_1 \implies P_3$	$(P_1 \implies P_3) \wedge P_1$	$\Big((P_1 \implies P_3) \wedge P_1\Big) \implies P_3$
1	1	1	1	1
1	0	0	0	1
0	1	1	0	1
0	0	0	0	1

It is a tautology so the argument is valid.

1.1.5.8. **E.g.** $\begin{cases} P_1 = \text{"Amy graduates"} \\ P_2 = \text{"Bob gets a job"} \\ P_4 = \text{"Bob gets a job"} \\ P_5 = \text{"Bob earns money"} \end{cases}$. Now we consider the argument
"If Bob graduates then he gets a job".
"Bob does not get a job".
"Therefore Bob does not graduate".

- This argument is symbolised as

$$P_2 \implies P_4,\ (\neg P_4) \vdash (\neg P_2) \qquad \text{or} \qquad \frac{P_2 \implies P_4 \qquad \neg P_4}{\neg P_2}$$

- **Exercise** Verify $\Big((P_2 \implies P_4) \wedge (\neg P_4)\Big) \implies (\neg P_2)$ is a tautology

1.1.5.9. **E.g.** $\begin{cases} P_1 = \text{"Amy graduates"} \\ P_2 = \text{"Bob gets a job"} \end{cases}$. Now we consider the argument
"Either Amy or Bob graduates".
"Bob does not graduate".
"Therefore Amy graduate".

- This argument is symbolised as

$$(P_1 \vee P_2),\ (\neg P_2) \vdash P_1 \qquad \text{or} \qquad \frac{P_1 \vee P_2 \qquad \neg P_2}{P_1}$$

- **Exercise** Verify $\Big((P_1 \vee P_2) \wedge (\neg P_2)\Big) \implies P_1$ is a tautology

1.1.5.10. **E.g.** $\begin{cases} P_1 = \text{"Amy graduates"} \\ P_2 = \text{"Bob gets a job"} \\ P_4 = \text{"Bob gets a job"} \\ P_5 = \text{"Bob earns money"} \end{cases}$.

Now we consider the argument
"If Bob graduates then he gets a job".
"If Bob get a job then he get money".
"Therefore if Bob graduates then he earns money".

- This argument is symbolised as

$$(P_2 \implies P_4),\ (P_4 \implies P_5) \vdash P_2 \implies P_5 \qquad \text{or} \qquad \frac{P_2 \implies P_4 \qquad P_4 \implies P_5}{P_2 \implies P_5}$$

- **Exercise** Verify $\Big((P_2 \implies P_4) \wedge (P_4 \implies P_5)\Big) \implies (P_2 \implies P_5)$ is a tautology

1.1.5.11. **List of deductive argument forms / deductive reasoning / deductive inferences**

Name / Latin name	Gentzen's sequent expression	proposition in linear notation
Affirming antecedent/ modus ponens	$\dfrac{P \implies Q \qquad P}{Q}$	$\Big((P \implies Q) \wedge P\Big) \implies Q$
Contrapositive/ modus tollens	$\dfrac{P \implies Q \qquad \neg Q}{\neg P}$	$\Big((P \implies Q) \wedge (\neg Q)\Big) \implies (\neg P)$
hypothetical syllogism	$\dfrac{P \implies Q \qquad Q \implies R}{P \implies R}$	$\Big((P \implies Q) \wedge (Q \implies R)\Big) \implies (P \implies R)$
disjunctive syllogism/ modus tollendo ponens	$\dfrac{P \vee Q \qquad \neg P}{Q}$	$\Big((P \vee Q) \wedge (\neg P)\Big) \implies Q$
Contradiction/ reductio ad absurdum	$\dfrac{P \implies \perp \qquad P}{\neg P}$	$\Big((P \implies \perp) \wedge P\Big) \implies \neg P$

1.1.5.12. **Exercise**

- Verify $P \implies Q, Q \vee P \ \vdash \ (\neg Q) \vee (\neg P)$ is invalid
- Verify $P \implies (S \implies (\neg R)), P \implies R, P \ \vdash \ (\neg S)$ is invalid
- Verify $(P \vee Q) \implies S, Q \implies S \ \vdash \ S$ is invalid

1.1.5.13. **Def (Soundness)** An argument is *sound* if $A \vdash B \implies A \models B$

1.1.5.14. **Def (Completeness)** An argument is *complete* if $A \models B \implies A \vdash B$

1.1.5.15. Gentzen System Rules of Inference

- Conjunction Introduction ($\wedge$-I) $\qquad \dfrac{A \quad B}{A \wedge B}$ ($\wedge$-I)
- Conjunction Elimination ($\wedge$-E) $\qquad \dfrac{A \wedge B}{A}$ ($\wedge$-E$_1$) $\quad \dfrac{A \wedge B}{B}$ ($\wedge$-E$_2$)
- Disjunction Introduction ($\vee$-I) $\qquad \dfrac{A}{A \vee B}$ ($\vee$-I$_1$) $\quad \dfrac{B}{A \vee B}$ ($\vee$-I$_2$)
- Disjunction Elimination ($\vee$-E) $\qquad \dfrac{A \vee B \quad \frac{A}{C} \quad \frac{B}{C}}{C}$ ($\vee$-E)
- Implication Introduction ($\rightarrow$-I) $\qquad \dfrac{\frac{A}{B}}{A \rightarrow B}$ ($\rightarrow$ -I)
- Implication Elimination ($\rightarrow$-E) $\qquad \dfrac{A \quad A \rightarrow B}{B}$ ($\rightarrow$ -E)

1.1.6 Proof

Learning Objectives

- Understand proof by brute force, contradiction, contrapositive and induction
- Apply proof by brute force, contradiction, contrapositive and induction in examples.

1.1.6.1. Proof := a step-by-step argument-chain that establishes, logically with certainty, that something (statement/proposition) is true. Each statement in the argument-chain must be:

- a definition
- an axiom
- a previously-proved statement
- a logical consequence of some conjunction of previous statements

1.1.6.2. ⊩**(Godel 1931; Church 1936; Turing 1937)** There is no systematic method for finding proofs.

1.1.6.3. Doing proof = the hardest thing in the world.

1.1.6.4. Types of proof

- Proof by construction / example
- Proof by cases / exhaustion / brute force
- Proof by contradiction
- Proof by contrapositive
- Proof by induction
- Probabilistic proof
- Combinatorial proof
- Algebraic proof
- Analytic proof
- Topological / geometric proof

1.1.6.5. **Proof by example**

- For showing something exists
- ⊩There exists an even prime number.
  Proof: 2 is an even prime number.
- This is also a proof by example that there are proofs by example.

1.1.6.6. **E.g.** "All birds can fly" is false.
Prove by counter example: penguin can't fly

1.1.6.7. **E.g.** "All continuous function is differentiable" is false. (We will learn this later)
Prove by counter example: $|x|$

1.1.6.8. **Proof by cases / Proof by exhaustion / brute force**

- ⊩If $n \in \mathbb{Z}$, $2 \leq n \leq 5$, then $n^2 + 2$ is not a multiple of 4
  Proof:
  case $n = 2$: $n^2 + 2 = 6$, and $4 \nmid 6$
  case $n = 3$: $n^2 + 2 = 11$, and $4 \nmid 11$
  case $n = 4$: $n^2 + 2 = 18$, and $4 \nmid 18$
  case $n = 5$: $n^2 + 2 = 27$, and $4 \nmid 27$
- **E.g.** The set $A = \{1, 2, 3\}$ is equivalent to the set $B = \{3, 2, 1\}$
  Proof: we show $A \subset B$ and $B \subset A$
  case $1 \in A$: $1 \in B$
  case $2 \in A$: $2 \in B$
  case $3 \in A$: $3 \in B$
  case $3 \in B$: $3 \in A$
  case $2 \in B$: $2 \in A$
  case $1 \in B$: $1 \in A$
  The first three line shows $A \subset B$, the last three lines show $B \subset A$, hence $A = B$
- **E.g.** (Master level: Conjugate of norm is the indicator function on unit ball of dual norm)
  `https://angms.science/doc/CVX/conjugateNormIcDual.pdf`
- **E.g.** (Master level: Subdifferential of norm)
  `https://angms.science/doc/CVX/SubdifferentialOfNorm.pdf`
- To show two values X, Y are equal (i.e., $X = Y$)
  - Shows $X \geq Y$
  - Shows $Y \geq X$

  E.g. (Master level: nuclear norm of a matrix is a norm `https://angms.science/doc/LA/KyFanNorm.pdf`)

1.1.6.9. **Proof by contradiction**

- $\Big((P \implies \perp) \wedge P\Big) \implies \neg P$
- We want to prove P, we start with $\neg P$ and arrive at an contrdiction
- ⊩$\sqrt{3}$ is irrational.
  Proof:
  (1) For the purpose of contradiction, assume $\sqrt{3}$ is rational.
  (2) By (1) we have $\sqrt{3} = p/q$ for $p, q \in \mathbb{Z}$ co-primes
  (3) From (2) we have $\sqrt{3} = p/q \implies p = \sqrt{3}q \implies p^2 = 3q^2 \implies p^2/3 = q^2$
  (4) Fact: if a is prime and a divides p^2, then a divides p
  (5) On $p^2/3 = q^2$, use (4) we have 3 is a factor of p
  (6) By (5) 3 is a factor of p, so p can be written as $p = 3c$ for some integer c
  (7) Put $p = 3c$ in (6) to $p^2/3 = q^2$ in (3) gives $3c^2 = q^2 \implies c^2 = q^2/3$
  (8) Apply fact (4) on $c^2 = q^2/3$ from (7) means 3 is a factor of q

$$(9) \begin{cases} 3 \text{ is a factor of } p \text{ from (5)} \\ 3 \text{ is a factor of } q \text{ from (8)} \end{cases} \quad \text{contradicts with (2) that } p, q \text{ co-prime (no common factor)}$$

So by contradiction we have $\neg(1)$, therefore $\sqrt{3}$ is irrational.

- ⊩**Euclid** There are infinitely many prime numbers.
- ⊩**Cantor** The set of all integers is uncountable. We will come back to this in Set Theory.
- **E.g.** e is irrational
  4-minute youtube `https://www.youtube.com/watch?v=mP90N_w85XQ`

1.1.6.10. **Proof by contrapositive**

- $\Big((P \implies Q) \wedge (\neg Q)\Big) \implies (\neg P)$
- Pythagoras Theorem: A triangle is right-angled at A if $a^2 = b^2 + c^2$
- Contrapositive: if $a^2 \neq b^2 + c^2$ then the triangle is not right-angled at A

1.1.6.11. **E.g.** Use contrapositive to prove X : If n^2 is even, then n is even.

- Understand the proposition X
  - it is in the form of $P \implies Q$
  - P is "n^2 is even"
  - Q is "n is even"
  - The contrapositive of $P \implies Q$ is $(\neg Q) \implies (\neg P)$
- Now we prove the proposition X using **proof by contrapositive**
  - First we build the contrapositive of X, call it Y
  - $Y : (\neg Q) \implies (\neg P)$
  - $(\neg Q)$: "n is not even"
  - $(\neg Q)$: "n is odd"
  - $(\neg P)$: "n^2 is not even"
  - $(\neg P)$: "n^2 is odd"
  - So our "task" is to show that "if n is odd, then n^2 is odd"
- The proof
  1. **(Assume n is odd)** By definition, an odd number can be written as $n = 2k + 1$ for some integer k.
  2. **(Square n)** $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$, where $m = 2k^2 + 2k$ is an integer.
  3. Since n^2 is of the form $2m + 1$, it is odd.
  4. Therefore, we have shown that if n is odd, then n^2 is odd. By the contrapositive, this proves that if n^2 is even, then n is even.

1.1.6.12. **E.g.** Use direct prove to show that X' : If n is even, then n^2 is even.

- Note that X' is different from X
- Here is the direct proof of X'

  1 **(Assume n is even)** $n = 2k$ for some integer k
  2 **(Show n^2 is even)** $n^2 = (2k)^2 = 4k^2 = 2(2k^2) = 2m$ for some integer m
  3 So we have showed that $\{n \text{ is even }\} \implies \{n^2 \text{ is even }\}$

1.1.6.13. **Proof by Mathematical Induction**

- The principle of mathematical induction is indeed the fifth axiom in the Peano Axioms in 1.1.2.7.
  - Axiom is assumed to be true: this is why mathematical induction "works"
  - Fact: Peano Axioms is not the most fundamental, it can be proved by ZFC. We write ZFC ⊢ Peano Axioms
    - The symbol ⊢ is in 1.1.5

1.1.6.14. **E.g.** Prove $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ is true for all $n \in \mathbb{N}$

- Recall that we are now not specifying n, so this is a predicate
- We add a universal quantifier $\forall n$ to turn this is predicate to a proposition
- Therefore we are proving a proposition
- Here is the steps

Step 1. Base case.
You prove the base case is true.
Here we have $1 = 1$ for $n = 1$ so the base case is true

Step 2. Induction hypothesis.
You assume the case at k is true.
We are not specifying the k here: we are having a predicate
In this example we have $1 + 2 + 3 + \cdots + k = \dfrac{k(k+1)}{2}$

Step 3. Prove the $k+1$ case is true.
We are actually proving the universal quantified predicate $(\forall k)\Big((\text{case } k \text{ is true}) \implies (\text{case } k+1 \text{ is true})\Big)$
In this example we want to show $1 + 2 + 3 + \cdots + k + k + 1 = \dfrac{k+1(k+1+1)}{2}$

$$\underbrace{1 + 2 + 3 + \cdots + k}_{= \frac{k(k+1)}{2} \text{ by hypothesis}} + k + 1 = \frac{k(k+1)}{2} + k + 1 = (k+1)\Big(\frac{k}{2} + 1\Big) = (k+1)\Big(\frac{k}{2} + \frac{2}{2}\Big) = (k+1)\frac{k+2}{2}$$

So by the principle of mathematical induction, the statement is true for all integer n

1.1.6.15. **Bad proof** $P \implies Q$ Ex falso quodlibet: from falsehood, you can prove anything.

- $P \implies Q$ always true if P is false, regardless of Q

$$\begin{array}{rll} & 2 + 2 = 5 & \\ \text{therefore} & 4 = 5 & \\ \text{therefore} & 1 = 2 & \\ \text{now,} & \Big|\Big\{\text{Andersen, your father}\Big\}\Big| = 2 & \\ \text{therefore} & \Big|\Big\{\text{Andersen, your father}\Big\}\Big| = 1 & \text{because } 2 = 1 \\ \text{therefore} & \text{Andersen} = \text{your father} & |\{a, b\}| = 1 \implies a = b \end{array}$$

In fact, if $2 = 1$, then Andersen can be anything: UK president, the strongest man, the most handsome person

1.1.6.16. **E.g. Why you cannot divided by zero**

Proof. Assume division by zero is allowed.

Let a be any number.

By assumption, we can write $a \div 0 = b$ for some number b.

By the definition of division, this means: $a = b \times 0$

Since $b \times 0 = 0$ for any b, we have $a = 0$

Since we let a be *any* number, then $a = 0$ implies that

$$\text{any non-zero number } a \text{ must be equal to zero } \begin{cases} \text{this is a contradiction if your number system has non-zero } (a \neq 0) \\ \text{this is ok if your number system only has zero} \end{cases}$$

□

Note: if your number system only has zero, then division by zero is allowed.

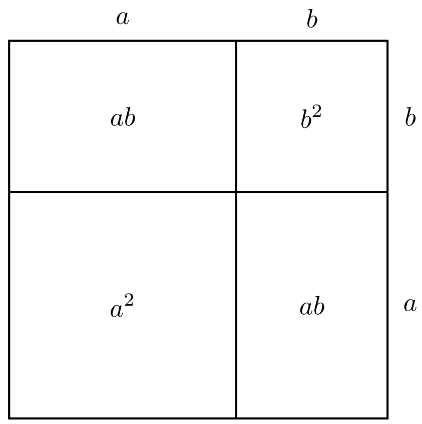
BUT you wouldn't be able to express other numbers

- $0 + 0 = 0$
- $0 - 0 = 0$
- $0 \times 0 = 0$
- $\dfrac{0}{0} = 0$

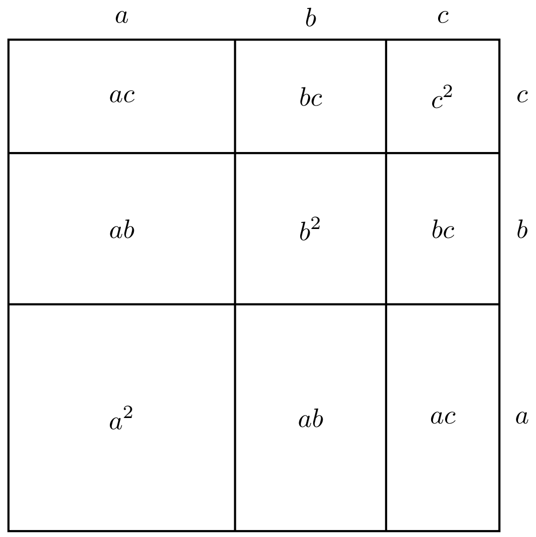
You cannot create other number from the 4 basic arithmetic rules.
In other words, your system is useless.

1.1.6.17. **Proof by geometry**

E.g. $(a+b)^2 = a^2 + 2ab + b^2$



E.g. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$



E.g. "Period 3 implies chaos"

1.1.6.18. Extra video: Introduction to Proof Theorem `https://www.youtube.com/watch?v=grjMRgmjddE`

1.1.7 LEAN4.0 (Optional)

1.1.7.1. What is LEAN 4

- a programming language
- a programming language using dependent type theory
- a programming language using dependent type theory that can be used to formalize mathematics
- a programming language using dependent type theory that can be used to formalize mathematics for proof assistant

1.1.7.2. You may ask what is Type Theory: don't. The fastest way to enter a mental hospital : look at the abyss of mathematics

- ZFC Set Theory
- ETCC Category Theory
- MLTT Type Theory

"When you stare into the abyss, the abyss stares back at you"
`https://www.ma.imperial.ac.uk/~buzzard/lean_together/source/appendix/type_theory.html`
`https://mathoverflow.net/questions/376839/what-makes-dependent-type-theory-more-suitable-than-set-the`

1.1.7.3. How to run LEAN 4

- We will just use `https://live.lean-lang.org`
- You can also install LEAN
- You need to enable JavaScript in your browser to use LEAN 4 Web Editor

1.1.7.4. A LEAN 4 program

```
def add (x y : Int) : Int := x + y
#eval add 1 3
```

- `def` is used to define a new function
- add is the name of the function
- `(x y : Int)` are the parameters of the function, both of type `Int`
- `: Int` after the parameters indicates the return type of the function.
- `:=` is used to define what the function does
- `x + y` is the body of the function, which adds the parameters `x` and `y` together.
- `#eval` is used to evaluate an expression, in this case calling the function add with arguments 1 and 3, you should get 4

1.1.7.5. More `https://leanprover-community.github.io/logic_and_proof/index.html`

1.2 Set and Function

1.2.1 Set

Learning Objectives

- Understand set operations.
- Apply set operations in examples.

1.2.1.1. Why set theory

- it is a foundation of mathematics, see 1.2.6
- it is a foundation of computer science: data structure, database theory, formal language and automata

1.2.1.2. **Two set theories**: naive set theory and Zermelo-Fraenkel set theory.

- We focus on naive set theory.
- Why there are two set theories:
  - Historically, naive set theory is introduced by Cantor
  - Russell in 1901 found that naive set theory contains a paradox.
    Let A be the set of all sets that are not members of themselves. Then we have contradiction.

$$\text{Let } A := \{x \mid x \notin x\}, \text{ then } A \in A \iff A \notin A. \qquad \text{(Russell's paradox)}$$

    What the Russel's paradox says: Cantor's hypothesis that every property $P(x)$ there exists a set $\{x : P(x)\}$ is false. There are property $P(x)$ cannot be expressed by a set.
  - To fix (remove) Russell's paradox from set theory, later Zermelo and Fraenkel developed a new set theory.

1.2.1.3. **Def (Set)** A *set* S is a collection of objects. These objects are called *elements* or *members* of the set.

- Not all set is legal, we ignore those illegal sets for now.

1.2.1.4. Not everything can be a set

- Universal set $\{x : x = x\}, \{x : x \subset x\}$
- Set of $\aleph_0, \aleph_1, \aleph_2, \cdots$

1.2.1.5. **E.g.**

- The set of students enrolled in this course.
- The set of students presented now in the lecture hall.
- The set of students presented now in the lecture hall that like Andersen's teaching.

We see that the three sets are possibly different.

1.2.1.6. **Set notation & membership** We use capital italic letter $S, A, B, C, ...$ to denote sets, we use small italic letter $a, b, c, ..., x, y, z$ to denote elements.

- If "a is an element of A", we write $a \in A$.
- If "a is not an element of A", we write $a \notin A$.

1.2.1.7. **Specifying a set** There are two ways to specify a set.

- List out all the elements. For example Vowel $= \{a, e, i, o, u\}$.
  - The elements are separated by commas
  - The elements are enclosed in brace { }
- Using the properties of the element, we write a formula that describes the elements in the set. For example

$$B = \{x : x \text{ is an integer}, x > 0\} \qquad B \text{ is the set of } x \text{ such that } x \text{ is an integer and } x \text{ is greater than zero.}$$

  - The symbol x denotes an arbitrary element of B
  - The colon : means "such that"
  - The comma , means "logic and"

1.2.1.8. **E.g.** The solutions of the quadratic equation $x^2 - 3x + 2 = 0$

- $S_1 = \{1, 2\}$
- $S_2 = \{x : x^2 - 3x + 2 = 0\}$

1.2.1.9. **Important sets** Here are some special sets that worth attention.
Note that we are not giving the definition of these set here. You may think that "integer", "real number" are simple concepts. In fact, it takes a whole module in the pure mathematics degree to learn "how to define integer", "how to construct real number".

Symbol	Set name	Meaning	Example of elements
$\mathbb{N}$	natural number	zero and positive integer	$0, 1, 2, 3, ...$
$\mathbb{Z}$	integer	zero, positive or negative integers	$0, \pm 1, \pm 2, \pm 3, ...$
$\mathbb{Q}$	rational number	fraction of two integer (denominator nonzero)	$\frac{1}{2}, \frac{1}{3}, \frac{22}{7}, ...$
$\mathbb{R}$	real number	rational and irrational number	$\sqrt{2}, \pi, e, i^i, \frac{1+\sqrt{5}}{2}$
$\mathbb{C}$	complex number	any $a + bi$ for $a \in \mathbb{R}, b \in \mathbb{R}$, here $i = \sqrt{-1}$	
$\mathbb{H}$	quaternion	$a + bi + cj + dk$ with $ijk = -1$	
SL(2,$\mathbb{R}$)	special linear group	all the 2-by-2 real-valued matrices with determinant 1	
GL(2,$\mathbb{R}$)	general linear group	all the 2-by-2 real-valued matrices with inverse	

1.2.1.10. **Def (Subset)** Let A, B be sets. We call A is a *subset* of B if every element of A is in B, write $A \subseteq B$.

- $A \subseteq B \iff \forall x(x \in A \implies x \in B)$
- We call B the **superset** of A, write $B \supseteq A$
- If A is not the subset of B we write $A \nsubseteq B$
- Example of subset: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- Any set is a subset of itself: $\forall A : A \subset A$

1.2.1.11. **E.g.** $S = \Big\{a, \{a\}, \{b\}, \{a, b, c\}, \{\{d\}\}\Big\}$

- $a \in S$? Yes
- $b \in S$? No. $b \neq \{b\}$
- $\{a\} \in S$? Yes.
- $\{b\} \in S$? Yes.
- $\{a, b\} \in S$? No
- $d \in S$? No.

- When we talk about subset $\{x_1, x_2, ...\} \subset S$, we ask $1 \overset{?}{\in} S$,
- $\{a\} \subset S$? Yes because $a \in S$
- $\{b\} \subset S$? No because $b \notin S$
- $\{\{b\}\} \subset S$? Yes because $\{b\} \in S$
- $\{a, \{a, b, c\}\} \subset S$? Yes because both $a \in S$ and $\{a, b, c\} \in S$
- d, $\{d\}$ and $\{\{d\}\}$ are all different things

1.2.1.12. **E.g.** ($\star\star$) Suppose
$$\begin{gathered}\left\{x : 0 < x < \frac{1}{1}\right\}\\ \left\{x : 0 < x < \frac{1}{2}\right\}\\ \left\{x : 0 < x < \frac{1}{3}\right\}\\ \vdots\end{gathered}$$
Do these sets share an element in common?

The answer is No. Proof by contradiction.
(1) Suppose the sets share a common element e.
(2) As e belongs to these sets, so e fulfils the condition in the sets, that is, $e > 0$.
(3) $e > 0$ means that there is an integer n that $e > \frac{1}{n}$.
(4) Then by (3) that means $e \notin \left\{x : 0 < x < \frac{1}{n}\right\}$.
(5) We have contradiction between (4) and (1).

1.2.1.13. **Def (Proper subset)** A is a *proper subset* of B if there exist at least one element of B that is not element of A

- $A \subsetneq B \iff \forall x(x \in A \implies x \in B) \;\wedge\; \exists y(y \in B \;\wedge\; y \notin A)$

1.2.1.14. **Def Empty set** $\varnothing$

- $\varnothing := \{\}$ contains nothing
- $\varnothing$ is a subset of any set S
- $\varnothing$ is a proper subset of any nonempty set S

1.2.1.15. **Def Equality of sets** Two sets A, B are *equal* if and only if they have the same elements.

- $A = B \iff \forall x(x \in A \iff x \in B)$

  If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

- $A = B \iff A \subseteq B \wedge B \subseteq A$

  How to prove equality of set: *pick-a-point* method; take an arbitrary element $u \in A$. Use any known true statement on properties of A and B to prove $u \in B$

1.2.1.16. **E.g.** $E = \{x : x^2 - 3x + 2 = 0\}$, $F = \{2, 1\}$, $G = \{1, 2, 2, 1\}$ are equal sets: $E = F = G$. A set remains the same if its elements are repeated or rearranged.

1.2.1.17. **E.g.** $E = \{x : x^4 = 16, x \text{ is odd}\}$. Then $E = \varnothing$.

1.2.1.18. **E.g.** ($\star\star$) Let $X = \{\varnothing\}$.

- $\varnothing \subset X$ because empty set is a subset of any set
- $X \subseteq \{\varnothing\}$ because this is the same as saying $X \subseteq X$
- $X \not\subset \varnothing$ because X contains one element and $\varnothing$ contains nothing
- By $\varnothing \subset X$ and $X \not\subset \varnothing$, therefore it does not fulfill the definition of set equality, thus $X \neq \varnothing$
- This example shows that $\varnothing \neq \{\varnothing\}$

1.2.1.19. **Inequality of sets** If $A \neq B$, it means $A \nsubseteq B$ or $B \nsubseteq A$.
That is, either

- A is not a subset of B, but B is a subset of A
- A is a subset of B, but B is not a subset of A
- A is not a subset of B and B is not a subset of A

1.2.1.20. **Def (Power set)** 2^A Let A be a set, the *power set* of A, denoted by 2^A, is the set of all possible subset of A

- In some books the notation $P(A)$ is used. I don't like this notation.
- The power set is called "power" set and the "power" is denoted by the notation 2^A.
- Clearly, $A \subset 2^A$ by definition of power set
- $2^\varnothing = \varnothing$: the power set of empty set is the empty set itself.

1.2.1.21. **E.g.** $A = \{1, 2, 3\}$

$$2^A = \Big\{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, A\Big\}$$

Do not underestimate this example. See if you know how to solve the next one.

1.2.1.22. **E.g.** $A = \{1\}$. Find $B = 2^A$ and $C = 2^B$.

$$B = \Big\{\varnothing, A\Big\} = \Big\{\varnothing, \{1\}\Big\}, \quad C = \Big\{\varnothing, \{\varnothing\}, \{A\}, B\Big\}.$$

How we construct C? The same as the previous example !

1.2.1.23. **Universal set** $\mathcal{U}$

- $\mathcal{U}$ contains *everything*. I.e., it is the opposite to $\varnothing$.
- All sets are subsets of $\mathcal{U}$, or we call $\mathcal{U}$ is a superset of all sets
- If $\varnothing$ is like "0", then $\mathcal{U}$ is like "∞"
- You will get "error" when you compute with infinity. E.g., what is $2^{\mathcal{U}}$? Does $2^{\mathcal{U}} \subset \mathcal{U}$ or $\mathcal{U} \subset 2^{\mathcal{U}}$? Does $\mathcal{U} \subset \mathcal{U}$?
- $\Vdash$**Universal set does not exist.** $\neg(\exists \mathcal{U} \ : \ \forall S : S \in \mathcal{U})$.
  Proof (By contradiction) Suppose $\mathcal{U}$ exists.
  By the Axiom of Separation in the Zermelo-Fraenkel Set Theory 1.1.2.8., we can create a new set R from $\mathcal{U}$ as $R = \{A \in \mathcal{U} : A \notin A\}$.
  By (Russell's paradox), such set R does not exist.
  Thus $R \notin \mathcal{U}$.
  So $\mathcal{U}$ cannot contain everything.
  Contradiction, hence universal set does not exist.

1.2.1.24. **Def (Complement)** $A^c := \{x \,|\, x \notin A\}$. Sometimes we write $\overline{A}$.

1.2.1.25. **E.g.** $S = \{1, 2, 3\}$. List all the subsets of S. Give two sets that are complementary to each other.

$$\text{all the subsets of } S: \ \varnothing,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}$$

$\{1\}$ and $\{2, 3\}$ are complementary to each other.
$\{2\}$ and $\{1, 3\}$ are complementary to each other.
$\{1, 2, 3\}$ and $\varnothing$ are complementary to each other.

1.2.1.26. **Def (Union)** $\cup$ $\quad A \cup B := \{x \,|\, (x \in A) \vee (x \in B)\}$

1.2.1.27. **Def (Intersection)** $\cap$ $\quad A \cap B := \{x \,|\, (x \in A) \wedge (x \in B)\}$

1.2.1.28. The set operators $\cap, \cup$ and the logic operators $\wedge, \vee$ have the similar meaning.

1.2.1.29. **E.g.** $A = \{1, 2, 3, 4\}, B = \{2, 3, 5, 8, 10\}$. Then $A \cup B = \{1, 2, 3, 4, 5, 8, 10\}$ and $A \cap B = \{2, 3\}$

1.2.1.30. **Def (Disjoint)** For two sets X and Y, when $X \cap Y = \varnothing$, we say X and Y are *disjoint*.

1.2.1.31. **Def (Difference / relative complement)** $\setminus$ $\quad A \setminus B := \{x \,|\, (x \in A) \wedge (x \notin B)\}$

- $A \setminus B$ and B are disjoint: $(A \setminus B) \cap B = \varnothing$

1.2.1.32. **Def (Symmetric difference)**Δ $A \Delta B := (A \cup B) \setminus (A \cap B) = \{x \,|\, (x \in A) \underline{\vee} (x \in B)\}$

1.2.1.33. **E.g.** $A = \{1, 2, 3, 4\}, B = \{2, 3, 5, 8, 10\}$. Then

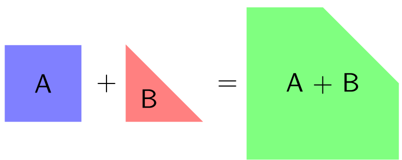
- $A \setminus B = \{1, 4\}$
- $A \setminus B$ and B are disjoint
- $A \Delta B = \{1, 4, 5, 8, 10\}$

1.2.1.34. **Def (Co-disjoint)** For two subsets X and Y of a set E, if $(E \setminus X) \cap (E \setminus Y) = \varnothing$, we say X and Y are *co-disjoint*.

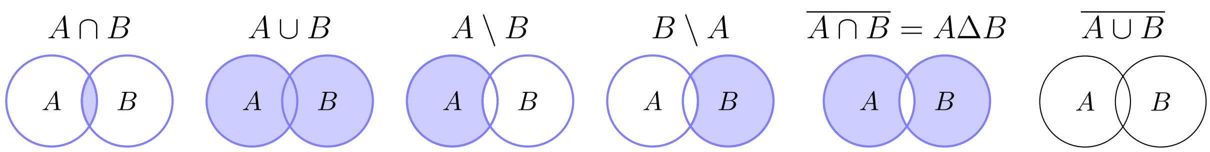
1.2.1.35. **Def (Partition)** A set of k disjoint nonempty subsets B_i of a set E is called a *partition* of E if $B_1 \cup B_2 \cup \ldots \cup B_k = E$.

1.2.1.36. **Def (Minkowski sum)** $A + B = \{a + b \mid (a \in A) \wedge (b \in B)\}$

1.2.1.37. **E.g.** Minkowski sum of a square and a triangle



1.2.1.38. **Venn diagram**



1.2.1.39. ⊩**Set operation laws**

Identity	$A \cup \varnothing = A$ $A \cap U = A$	**Domination**	$A \cap \varnothing = \varnothing$ $A \cup U = U$
Idempotent	$A \cap A = A$ $A \cup A = A$	**Double negation**	$\overline{(\overline{A})} = A$
Commutative	$A \cap B = B \cap A$ $A \cup B = B \cup A$	**Associative**	$(A \cap B) \cap C \equiv A \cap (B \cap C)$ $(A \cup B) \cup C \equiv A \cup (B \cup C)$
Distributive	$A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$	**De Morgan's**	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	**Complement**	$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$

Exercises prove these.

1.2.1.40. **E.g.**

$$\begin{array}{llll} A = \{\} & B = \varnothing & C = \{\varnothing\} & D = \{\varnothing, \{\}\} \\ E = \{\varnothing, \{\varnothing\}\} & F = \{\varnothing, \{\varnothing, \{\varnothing\}\}\} & G = \{\{\varnothing\}\} & H = \{\varnothing, \{1\}\} \\ I = \{1\} & J = \{2\} & K = \{3\} & L = \{1,2\} \\ M = \{1,3\} & N = \{2,3\} & O = \{1,2,3\} & P = \{1, \{1\}\} \\ Q = \{1, \{2\}\} & R = \{1, \{3\}\} & S = \{2, \{1\}\} & T = \{2, \{2\}\} \\ U = \{2, \{3\}\} & V = \{1, \{2,3\}\} & W = \{2, \{1,3\}\} & X = \{3, \{2,3\}\} \\ Y = \{1,2,3,\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\} & & & \end{array}$$

- Determine if $A = B$. $A = B$
- Check if $C \subseteq D$. $C \subseteq D$
- Find $E \cup G$. $E \cup G = \{\varnothing, \{\varnothing\}, \{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$
- Find $L \cap M$. $L \cap M = \{1\}$
- Determine the complement of I in O. $O \setminus I = \{2,3\}$
- Check if P is a proper subset of Q. $P \nsubseteq Q$
- Find the set difference $N - J$. $N - J = \{3\}$
- Determine the symmetric difference between L and N. $L \Delta N = \{1,3\}$
- Check if H is a member of Y. $H \notin Y$
- Find the union of V and W. $V \cup W = \{1, \{2,3\}, 2, \{1,3\}\}$

1.2.2 Tuple

1.2.2.1. Tuple = another word for'list'

1.2.2.2. Ordered:

- Think of subject-verb-object in English: I love you $\neq$ you love me
- I owe you $1000£ \neq$ you owe me $1000£$
- 3 apples, 4 oranges $\neq$ 4 apples, 3 oranges, $\begin{bmatrix} 3 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

1.2.2.3. monad: one thing $\begin{cases} \text{age} \\ \text{flat rent} \end{cases}$

1.2.2.4. pairs: two things $\begin{cases} \text{coordinate in a x-y plane(x,y)} \\ \text{body information (height, weight)} \end{cases}$

1.2.2.5. triple: three things

- coordinate in a x-y-z space (x, y, z)

1.2.2.6. quadruple, quintuple, or n-tuple

1.2.2.7. Why cares about tuple?

- A group is a 2-tuple
- A graph is a 2-tuple
- The physical spacetime in classical physics is a 4-tuple
- Deterministic finite automaton is a 5-tuple
- A Pushdown automaton is a 6-tuple
- A Turing machine is a 7-tuple $\mathbb{M} = (Q, \Gamma, b, \Sigma, \delta, q_0, F)$.
- Vector in $\mathbb{R}^n$ is a n-tuple $(x_1, x_2, ..., x_n)$

1.2.2.8. **Def (Ordered pair)** For sets, $\{a, b\} = \{b, a\}$ since we do not care about order in the set. Sometimes order is important, when that is the case, i.e., $(a, b) \neq (b, a)$, we call (a, b) *ordered pairs*.

- We can also define ordered triples (a, b, c)

- Can we express ordered pairs using only set? Yes.

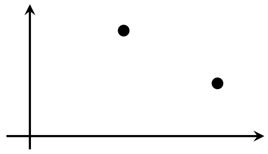
$$(a,b) = \{\{a,b\},\{a\}\}$$

See the discussion here

1.2.2.9. **E.g.** Let $A = B = \mathbb{R}$, then $(1,2)$ and $(2,1)$ are ordered pairs.
We represent them as points in the plane $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$.

We call these "coordinate"

We can write $(1,2)$ as vertically as $\begin{bmatrix}1\\2\end{bmatrix}$, which we call it "vector"



1.2.2.10. **Def (Direct product)** The direct product of two sets A, B is the set of ordered pairs (a,b) with $a \in A$ and $b \in B$

1.2.2.11. **Def (Cartesian product)** is a special kind of direct product. $A \times B = \{(a,b) : a \in A \wedge b \in B\}$.

- We often write $A \times A$ as A^2. Similarly, we write $A \times A \times A$ as A^3

1.2.2.12. **E.g.** $A = \{d,e,f\}, B = \{e,f,g\}$. Then

$$A \times B \;=\; \begin{Bmatrix} (d,e) & (d,f) & (d,g) \\ (e,e) & (e,f) & (e,g) \\ (f,e) & (f,f) & (f,g) \end{Bmatrix} \;\neq\; \begin{Bmatrix} (e,d) & (e,e) & (e,f) \\ (f,d) & (f,e) & (f,f) \\ (g,d) & (g,e) & (g,f) \end{Bmatrix} \;=\; B \times A$$

1.2.2.13. **E.g.** $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} =$ plane

$$\mathbb{R}^2 = \Big\{(x,y) \;:\; x,y \in \mathbb{R}\Big\}$$

1.2.2.14. **E.g.** $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \times \mathbb{R} =$ 3D cube = plane $\times$ line

$$\mathbb{R}^3 = \Big\{\Big((x,y),z\Big) \;:\; x,y,z \in \mathbb{R}\Big\} = \Big\{(x,y,z) \;:\; x,y,z \in \mathbb{R}\Big\}$$

1.2.3 Relation

Learning Objectives

- Understand terminology in relation.
- Apply terminology in relation in examples.

1.2.3.1. Why relation: relation used for $\begin{cases}\text{database}\\ \text{functional programming}\\ \text{object-oriented programming}\end{cases}$

1.2.3.2. What is relation: given x, y in a set, is x, y related or not related?

1.2.3.3. **E.g.** Relation that you already know $\leq$

- $3 \leq 5$ is true, we write $3 \leq 5$
- $7 \leq 5$ is false, we do not write $7 \leq 5$
- here $\leq$ is the symbol represents the relation, and x, y are two objects, so we call such relation an *binary relation*

1.2.3.4. **E.g.** Consider "like" as a relation.

- Peter likes David is not the same as David likes Peter
- This example shows that the *order* of the two object in a relation is important
- Relation can be represented by a *directed graph*

1.2.3.5. **Def (Binary Relation)** A *binary relation*, denoted as R, on a set A consists of A and a set of ordered pairs from $A \times A$. Let $x \in A$ and $y \in A$, when (x,y) is in the set $A \times A$, we write xRy. If (x,y) is in the set $A \times A$, we write $x \not R y$.

1.2.3.6. From the Law of excluded middle (in logic), we have xRy xor $x \not R y$. That is, either xRy true $x \not R y$ false, or xRy false $x \not R y$ true, nothing else.

1.2.3.7. Order matters. Relation is defined by ordered pair, so xRy is not yRx

1.2.3.8. You can also define unary relation and ternary relation

1.2.3.9. **E.g.** Let $A = \{1, 2, 3, 4, 5\}$. Let R as $=$. List the set of ordered pair of the relation structure (A, R).

$$\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$$

1.2.3.10. **E.g.** Let $A = \{1, 2, 3, 4\}$. Let R as $<$. List the set of ordered pair of the relation structure (A, R).

$$\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

1.2.3.11. **E.g.** Let $A = \{1, 2\}$. Let R as $\subset$. List the set of ordered pair of the relation structure $(2^A, R)$, where 2^A is the power set of A.

$$2^A = \Big\{\varnothing, \{1\}, \{2\}, \{1, 2\}\Big\}$$

$$\Big\{(\varnothing, \varnothing),\ (\varnothing, \{1\}),\ (\varnothing, \{2\}),\ (\varnothing, \{1, 2\}),\ (\{1\}, \{1\}),\ (\{1\}, \{1, 2\}),\ (\{2\}, \{2\}),\ (\{2\}, \{1, 2\}),\ (\{1, 2\}, \{1, 2\})\Big\}$$

1.2.3.12. **E.g.** Let $A = (x, y) \in \mathbb{R}^2$. Let R as $\leq$. Then (A, R), is the region above the line $y = x$ in $\mathbb{R}^2$.

1.2.3.13. **E.g.** Let $C = \forall x \exists y (xRy)$. Does the following binary relation R satisfy C?

- R defined on $\mathbb{R}$ by xRy if and only if $x > y$
- R defined on $\mathbb{N}$ by xRy if and only if x divides y
- R defined on $2^{\mathbb{N}}$ by xRy if and only if $x \subset y$

Answer: all true

1.2.3.14. **Def (Relation on two sets)** A *binary relation*, denoted as R, on two sets A, B consists of set of ordered pairs from a subset of $A \times B$. Let $x \in A$ and $y \in B$, when (x, y) is in the set $A \times B$, we write xRy. If (x, y) is in the set $A \times B$, we write $x \not R y$.
That is, a *binary relation* from A to B is a subset of $A \times B$.

1.2.3.15. **Def (Domain)** Given a relation R from A to B. The domain of R is the set $\big\{x \mid (x, y) \in R\big\}$

1.2.3.16. **Def (Range)** Given a relation R from A to B. The range of R is the set $\big\{y \mid (x, y) \in R\big\}$

1.2.3.17. **Def (Inverse)** Given a relation R from A to B. The inverse of R, denoted as R^{-1} is the set $\big\{(y, x) \mid (x, y) \in R\big\}$

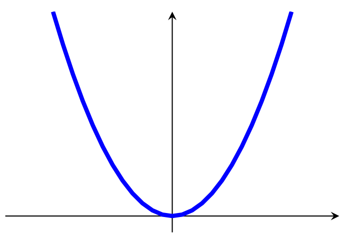
1.2.3.18. **E.g.** $A = \{1, 2, 3\}, B = \{1, 2, 3\}$ and $R = \{(1, 2), (1, 3), (2, 3)\}$. Then

- domain of R is $\{1, 2\}$, which is a subset of A
- range of R is $\{2, 3\}$, which is a subset of B
- $R^{-1} = \{(2, 1), (3, 1), (3, 2)\}$
- Actually R is $<$ and R^{-1} is $>$

1.2.3.19. **E.g.** $A = \{1, 2, 3\}, B = \{0, 1, 4, 9, 16\}$ and $R = \{(1, 1), (2, 4), (3, 9)\}$. Then

- domain of R is A
- range of R is $\{1, 4, 9\}$, which is a subset of B
- $R^{-1} = \{(1, 1), (4, 2), (9, 3)\}$
- Actually R is $(\cdot)^2$ and R^{-1} is $\sqrt{\cdot}$

1.2.3.20. **E.g.** Quadratic function



- $A = \mathbb{R}, B = \mathbb{R}$ and $R = \cdot^2$.
- Then we have the $y = x^2$ you learn in high school.
- dom$R$ is all the real number
- range$R$ is all the nonnegative real number, which is a subset of B. In other words, not all elements in B is used in the range of R

1.2.3.21. **Def (Reflexive)** A relation R is called *refexive* if for all $x \in A$ and $x \in B$ we have xRx.

$$(\forall x \in A)(\forall x \in B)\ :\ xRx$$

1.2.3.22. To prove R is not reflexive, we need to show there is a x that $x \not R x$.

1.2.3.23. **Def (Symmetric)** A relation R is called *symmetric* if for all $x \in A$ and $y \in B$ that xRy we have yRx.

$$(\forall x \in A)(\forall y \in B)\ :\ xRy \implies yRx$$

1.2.3.24. To prove R is not symmetric, we need to show there are x, y that xRy but $y\not{R}x$.

1.2.3.25. **Def (Transitive)** A relation R is called *transitive* if for all $x \in A$, $y \in X$, $y \in B$ and $z \in B$ that xRy and yRz, we have xRz.

$$(\forall x \in A)(\forall y \in A)(\forall y \in B)(\forall z \in B)\ :\ (xRy) \wedge (yRz) \implies xRz$$

1.2.3.26. To prove R is not symmetric, we need to show there are x, y, z that xRy, yRz but $x\not{R}z$.

1.2.3.27. **E.g.** $A = \mathbb{N}$ and R is $<$. Then relation R is not reflexive, not symmetric, but R is transitive.

1.2.3.28. **E.g.** $A = \mathbb{N}$ and R is $\leq$. Then relation R is reflexive, transitive but not symmetric.

1.2.3.29. **E.g.** For any set A, the relation R defined by $\subset$ is reflexive, transitive but not symmetric.

1.2.3.30. **Def (equivalence)** A relation R is called *equivalance* if it is reflexive, symmetric and transitive.

1.2.3.31. For an equivalance relation R, we usually write $\sim$ instead of R

1.2.3.32. What's the point of relation and equivalence relation? The idea is that "we can treat a bunch of numbers, seeming different, as the same number". For example,

$$\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \dots$$

are all the "same number".

Another example is the "clock number"

$$1 = 13 = 25 = ..., \quad 2 = 14 = 26 = ..., 3 = 15 = 27 = ...$$

We call these "equivalence" that we define the equivalent class.

1.2.3.33. **Def (equivalence class)** Given an equivalence relation R, we define equivalence class of a as

$$[a] \ = \ \left\{ s \mid sRa \right\}.$$

1.2.3.34. **E.g.** For $A = \mathbb{N}$ and R defined by $|x| = |y|$. Then the equivalence classes are

$$\{0\},\ \{1, -1\},\ \{2, -2\},\ ...$$

1.2.3.35. **E.g.** For $A = \mathbb{N}$ and R defined by $x = y \pmod{12}$. Then the equivalence classes are

$$\{0, 12, 24, ...\},\ \{1, 13, 25, ...\},\ \{2, 14, 26, ...\},\ ...\ \{11, 23, 35, ...\}$$

1.2.3.36. **Property of equivalence class, 1: equivalence** If two elements a, b ($a \neq b$) are related by an equivalence relation R on a set A, then their equivalence class are equal. That is, $[a] = [b]$.
Proof Suppose $a, b \in A$ and aRb. Now consider an element s that $s \in [a]$.

$$\begin{aligned} s \in [a] \quad &\implies \quad sRa \text{ by the definition of } [a] \\ &\implies \quad sRb \text{ by the transitivity of } R \text{ since } sRa \text{ and } aRb \\ &\implies \quad s \in b \text{ by the definition of } [b] \end{aligned}$$

So all elements of $[a]$ belongs to $[b]$.
Similarly, all elements of $[b]$ belongs to $[a]$, hence by the definition of set equivalence, we have $[a] = [b]$.

1.2.3.37. **Property of equivalence class , 2: uniqueness** If R is an equivalence relation on a set A, then every element of A belongs to exactly one equivalence class.
Proof Suppose $a, b, c \in A$ and $c \in [a] \cap [b]$.

$$\begin{aligned} c \in [a]\ \wedge\ c \in [b] \quad &\implies \quad cRa\ \wedge\ cRb \text{ by the definition of } [a] \text{ and } [b] \\ &\implies \quad aRc\ \wedge\ bRc \text{ by symmetry of } R \text{ (an equivalence relation is symmetric)} \\ &\implies \quad aRb \text{ by transitivity of } R \\ &\implies \quad [a] = [b] \text{ by the previous property} \end{aligned}$$

1.2.3.38. **Property of equivalence classes, 3: disjoint** Equivalence classes are disjoint.
Proof We prove the contrapositive: if $[a] \cap [b] \neq \varnothing$ then $[a] = [b]$.
If $[a] \cap [b] \neq \varnothing$ then there exists an element $e \in A$ with $e \in [a] \cap [b]$.
Then aRe and bRe.

By symmetry of R, we have eRb.
By transitivity of R on aRe and eRb, we have aRb, thus $[a] = [b]$.

1.2.3.39. We summarize the three properties of equivalence class as a theorem.
$\Vdash$Let R be an equivalence relation in a set A, and let $[a]$ be the equivalence class of $a \in A$. Then:

- For all $a \in A$, $a \in [a]$ existence
- $[a] = [b]$ if and only if aRb uniqueness
- If $[a] \neq [b]$ then $[a]$ and $[b]$ are disjoint disjoint

Here are the proof.

- Since R is reflexive, so aRa for all $a \in A$. Therefore $a \in [a]$
- See 1.2.3.37.
- See 1.2.3.38.

1.2.3.40. **Def (Quotient)** The set of equivalence classes of A, denoted by A/R, is call the *quotient* of A by R:

$$A/R \;=\; \Big\{[a] \,:\, a \in A\Big\}$$

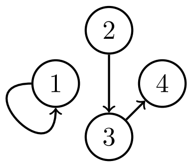
- do not confuse with set complement $A \setminus B$

1.2.3.41. **Def (Partitions)** A partition of a set S is a set of subsets of S such that each element of S is in exactly one of the subsets.

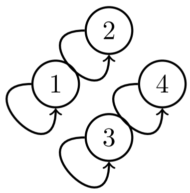
1.2.3.42. $\Vdash$**Fundamental theorem of equivalence relation** If R be an equivalence relation in A, the quotient set A/R is a partition of A.

1.2.3.43. **Relation as directed graph**. Relation can be represented by directed graph.

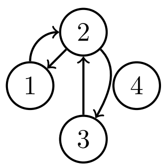
Suppose $A = \Big\{1, 2, 3, 4\Big\}$ and $R = \Big\{(1, 1), (2, 3), (3, 4)\Big\}$



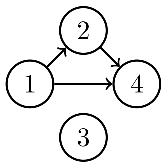
Suppose $A = \Big\{1, 2, 3, 4\Big\}$ and $R = \Big\{(1, 1), (2, 2), (3, 3), (4, 4)\Big\}$
R is reflexive

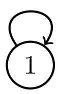


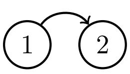
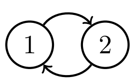
Suppose $A = \Big\{1, 2, 3, 4\Big\}$ and $R = \Big\{(1, 2), (2, 1), (2, 3), (3, 2)\Big\}$
R is symmetric

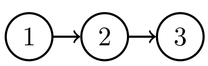
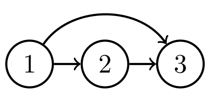


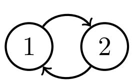
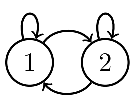
Suppose $A = \Big\{1, 2, 3, 4\Big\}$ and $R = \Big\{(1, 2), (2, 1), (2, 3), (3, 2)\Big\}$
R is transitive



Reflexivity: whenever I see  I will see 

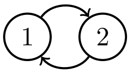
Symmetry: whenever I see  I will see 

Transitivity: whenever I see  I will see 

Transitivity: whenever I see  I will see  (this example is also an equivalence relation)

1.2.3.44. **Def (antisymmetric)** A relation R is called *antisymmetric* if for all $x \in A$ and $y \in B$ that if xRy and yRx we have $x = y$.

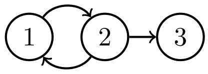
$$(\forall x \in A)(\forall y \in B) \ : \ xRy \wedge yRx \implies x = y$$

Antisymmetry: I never see 

1.2.3.45. Antisymmetric does not mean "not symmetric"

1.2.3.46. This example is neither symmetric nor antisymmetric

$$R = \{(1,2),(2,1),(2,3)\}$$



1.2.3.47. **E.g.**

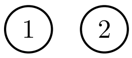
- $\leq$ on $\mathbb{R}$ is antisymmetric relation.
  $a \leq b$ and $b \leq a$ gives $a = b$
- $\subseteq$ on 2^S is antisymmetric relation.
  $A \subseteq B$ and $B \subseteq A$ gives $A = B$
- Divisibility on $\mathbb{N}$ is antisymmetric relation.

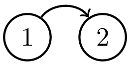
1.2.3.48. **Def (partial order)** A binary relation R on a set A is an *partial order* if it is

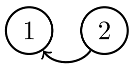
- reflexive
- antisymmetric
- transitive

1.2.3.49. **Def (total order)** A binary relation R on a set A is an *total order* if it is

- a partial order
- for all a, b, either aRb or bRa

Total order: whenever I see 

I will see 

or 

1.2.3.50. **Least element** A least element in an relation is an element ℓ such that ℓRx for all $x \in R$

1.2.3.51. **Def (Well-order)** A well-order relation R on a set A is a total order relation such that, for any nonempty $S \subseteq A$, there exists an $s \in S$ such that aRs for all $a \in S$

$$R \text{ is well-ordered on } A \iff (\forall S \neq \varnothing, S \subseteq A)(\exists s \in S \ \text{s.t.} \ aRs \ \forall a \in S)$$

1.2.3.52. $\Vdash$**(Zermelo's Well-Ordering)** If Axiom of Choice holds, then every set is well-orderable.

1.2.3.53. $\Vdash$**(Kuratowski-Zorn Lemma)** Let S be a partially ordered set. If S contains upper bounds for every totally ordered subset, then S must contains at least one maximal element.

1.2.3.54. $\Vdash$Kuratowski-Zorn Lemma is equivalent to the Well-Ordering Theorem and the Axiom of Choice. In other words,

$$\text{Kuratowski-Zorn Lemma holds} \iff \text{Zermelo's Well-Ordering Theorem holds} \iff \text{the Axiom of Choice holds}$$

1.2.3.55. **Def (Composition)** Consider three sets A, B, C and two relations U, V. If U is a relation from A to B, and V is a relation from B to C, then the relation from A to C is defined by all ordered pair that $(a, c) \in A \times C$ such that there exists $b \in B$ that $(a, b) \in U$ and $(b, c) \in V$. We denote the composition of U and V as $V \circ U$.

$$V \circ U := \Big\{(x, y) \ : \ x \in A, y \in C; \exists b \in B \, \text{s.t.} \, (x, b) \in U, (b, y) \in V\Big\}$$

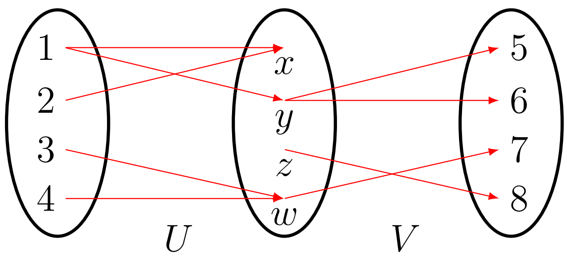
1.2.3.56. Note that we write $V \circ U$, not $U \circ V$

1.2.3.57. **E.g.**

- $A = \{1, 2, 3, 4\}$
- $B = \{x, y, z, w\}$
- $C = \{5, 6, 7, 8\}$
- $U = \Big\{(1, x), (1, y), (2, x), (3, w), (4, w)\Big\}$
- $V = \Big\{(y, 5), (y, 6), (z, 8), (w, 7)\Big\}$

Then $V \circ U = \{(1, 5), (1, 6), (3, 7), (4, 7)\}$.

It is better to see it by picture: composition refers to all the possible paths from set A to set C



1.2.3.58. **E.g.** Let U, V be relation in $\mathbb{R}$ defined as

$$U = \Big\{(x,y) \ : \ x^2+y^2=1\Big\}, \quad V = \Big\{(y,z) \ : \ 2y+3z=4\Big\},$$

Then $V \circ U$ is

$$V \circ U = \Big\{(x,z) \ : \ 4x^2+9z^2-24z+12=0\Big\}$$

How do you get this: eliminate y in both equations.

1.2.3.59. **Summary of relation** A relation R on a set X is

name	definition	example
reflexive	xRx for all $x \in X$	"is equal to", "is a subset of", "divides"
symmetric	$xRy \implies yRx$ for all $x \in X, y \in X$	"is equal to", "is adjancent to"
anti-symmetric	$xRy \wedge yRx \implies x = y$ for all $x \in X, y \in X$	"divides", "$\leq$"
transitive	$xRy \wedge yRz \implies xRz$ for all $x \in X, y \in X, z \in X$	"is a subset of", "implies", "dividies"
equivalence	reflexive, symmetric and transitive	"is equal to", "congruence modulo n on $\mathbb{Z}$"
partial order	reflexive, anti-symmetric and transitive	"divides", "divides" on $\mathbb{N}$
total order	partial order, and xRy or yRx for all $x \in X, y \in X$	"$\leq$" on $\mathbb{Z}$
well order	total order, and each subset of X has a least element	"$\leq$" on $\mathbb{N}$

1.2.4 Function

Learning Objectives

- Understand terminology in function: dom, codom, range, injective, surjective, bijective
- Apply terminology in function in examples.

1.2.4.1. **Def (Function)** Given two sets X, Y, a function $f : X \to Y$ is a binary relation R between X and Y that

- for all $x \in X$ there exists a $y \in Y$ such that xRy
- Any xRy and xRz implies $y = z$

1.2.4.2. Function is also called a map.

1.2.4.3. **Notation of function** A function f that maps a set V to W is denoted by $f : V \to W$. An element $v \in V$ is then mapped to an element $w \in W$, which is written as $f : v \mapsto w$ or $f(v) = w$.

1.2.4.4. Terms

- The set X is called the domain of f, we write $\mathrm{dom} f$
- The set of Y is called the codomain of f, we write $\mathrm{codom} f$
- The graph of the function, denoted as Gra f, is defined by

$$\text{Gra} f := \big\{(x, f(x)) \ : \ x \in R\big\}$$

- The range of f is defined as

$$f[X] = \text{range } f := \{f(x) \ : \ x \in X\}$$

1.2.4.5. **Equality of function** Two functions $f : X \to Y$ and $g : X \to Y$ are called equal, written $f = g$, iff $f(x) = g(x)$ for all $x \in X$. In other words, they have the same graph.

1.2.4.6. **E.g.** Consider two functions f, g defined on $\mathbb{R}$ as

$$f(x) = \frac{x-3}{x-4}, \qquad g(x) = \frac{(x-2)(x-3)}{(x-2)(x-4)}$$

The two functions are not equal since $f(2) = \dfrac{1}{2} \neq g(2) =$ undefined.
The two functions are equal if you restrict the domain from $\mathbb{R}$ to $\mathbb{R} \setminus \{2, 4\}$

1.2.4.7. **E.g.** $f(x) = x^5$, $g(x) = x^3|x|^2$. Then f, g are equal

1.2.4.8. **What's the big deal of function equality**: any "function" can have *more-than-one* expression. Nothing said that a function can only be expressed in one way!

1.2.4.9. **E.g.** Sine function has many representations

- $f_1(x) = \sin x$

- $f_2(x) = x - \dfrac{x^3}{5} + \dfrac{x^5}{5!} - \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$
- $f_3(x) = \dfrac{e^{ix} - e^{ix}}{2i}$
- $f_4(x) = x \displaystyle\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$
- $f_5(x) = \displaystyle\int_0^x \cos(t)\,\mathrm{d}t$

1.2.4.10. **E.g. (Fourier series of square wave)** Consider two functions f, g defined on $\mathbb{R}$ as

$$f(x) = \begin{cases} 1 & x \in [2n\pi, (2n+1)\pi] \\ -1 & x \in [(2n+1)n\pi, 2(n+1)\pi] \end{cases} \qquad g(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin(2n+1)x$$

The two functions are equal.

1.2.4.11. **E.g. (Fourier series of absolute value)** Over the interval $[-\pi, \pi]$,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}$$

1.2.4.12. **Composition of function** Given a function $f : X \to Y$ and $g : Y \to Z$, then the function composition $g \circ f$, or written $g(f(x))$, is defined as a composition relation from X to Y.

1.2.4.13. **Restriction** Given a function $f : X \to Y$ and a set $A \subset X$, the restriction of f on A, denoted as $f|_A$, is defined as a function from A to Y

$$f|_A \coloneqq \{f(a) \mid a \in A\}$$

1.2.4.14. **Extension** Given a function $f : X \to Y$ and a set $X \subset A$. If f is the restriction of $g : A \to Y$ on X, then g is called the extension of f on A.

1.2.4.15. **Injective function**

- also called 1-1
- all distinct $x \in X$ has mapped to something distinct in Y (uniqueness of mapping)
- distinct elements in X has distinct image: $f(x) = f(x') \implies x = x'$
- here we say nothing on Y: possibly some y in Y mapped by none

1.2.4.16. **Surjective function**

- also called onto
- all $y \in Y$ has mapped from something in X (existence of mapping)
- all $y \in Y$ are images of some element in X: $f\forall y \in Y \implies \exists x \in X, y = f(x)$
- $f[X] = Y$
- here we say nothing on X: possibly some x in X mapped to none

1.2.4.17. **Bijective function**

- injective and surjective
- also called 1-1 correspondence
- if $f : X \to Y$ is bijective between sets X and Y, then f associates every element of Y with a unique element of X (at most one element of X because it is 1-1, and at least one element of X because it is onto)

1.2.4.18. Important table

f	What definition said	What definition didn't say
non-function	one x maps to multiple y	
Function	one x maps to one y	multiple x can map to the same y
injective	x's do not share y	all y has to be mapped, some y can be mapped by no one
surjective	every y is mapped to by at least one x	multiple x can map to the same y

1.2.4.19. **Identity function** The diagonal $\Delta_X \subset X \times X$ is called he identity function on X. We write 1_X or Id_X. We have $\mathrm{Id}_X(x) = x$ for all $x \in X$.

1.2.4.20. Clearly, $\mathrm{Id}_Y \circ f = f \circ \mathrm{Id}_X = f$

1.2.4.21. The relation between a function and its inverse: $f \circ f^{-1} = \mathrm{Id}_Y$ and $f^{-1} \circ f = \mathrm{Id}_X$

1.2.4.22. **Existence of inverse function** If $f : X \to Y$ and $g : Y \to X$ satisfy $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$, then $f^{-1} : B \to A$ exists and $g = f^{-1}$.

1.2.4.23. **Inverse of bijection.** If $f : X \to Y$ is bijective, then

- it has an inverse $f^{-1} : B \to A$
- $f^{-1} : B \to A$ is also bijective

1.2.4.24. **Composition of bijection is transitive.** If $f : X \to Y$ is bijective and $g : Y \to Z$ is bijective, then

- $h = g \circ f = g(f) : X \to Z$ is also a bijection

1.2.4.25. Graphical test of function

- Vertical Line Test: This test is used to determine if a graph represents a function. If any vertical line intersects the graph at more than one point, then the graph does not represent a function.

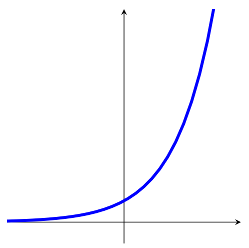
1.2.4.26. Graphical test of $1 \to 1$

- Horizontal Line Test: For a function to be injective, any horizontal line should intersect the graph at most once. If a horizontal line intersects the graph more than once, the function is not injective.

1.2.4.27. Graphical test of onto

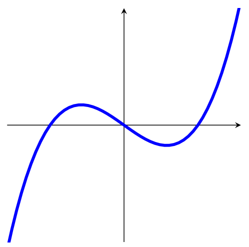
- Horizontal Line Test: This test is used to determine if a function is surjective. For a function to be surjective, every horizontal line should intersect the graph at least once. This means that every possible output value (in the codomain) is covered by the function.

1.2.4.28. **E.g.** $f(x) = e^x$



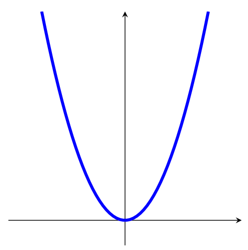
- $\mathrm{dom} f = X = \mathbb{R}$, codom $f = Y = \mathbb{R}$, range$f = \mathbb{R}_+$
- range of f is all the nonnegative real number, which is a subset of Y. In other words, not all elements in Y is used in the range
- The function is $1 \to 1$ (injective)
  $\iff$ $\forall$ horizontal line contains at most one point of graph$f$
- The function is not onto (surjective)
  $\iff$ $\exists$ horizontal line does not contain a point of graph$f$

1.2.4.29. **E.g.** $f(x) = x^3 - x$



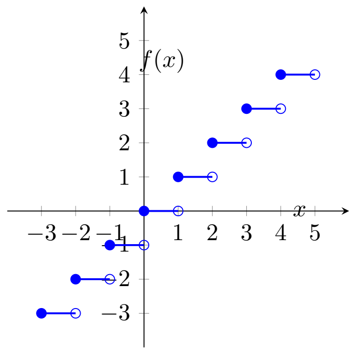
- $\mathrm{dom} f = X = \mathbb{R}$, codom $f = Y = \mathbb{R}$, range$f = \mathbb{R}_+$
- range of f is all the positive real number, which is a subset of Y.
- The function is not one-one (injective)
  $\iff$ $\exists$ horizontal lines cross over more than one point of graph$f$
- The function is onto (surjective)
  $\iff$ all horizontal lines contains one (or more) point of graph$f$

1.2.4.30. **E.g.** $f(x) = x^2$



- $\mathrm{dom} f = X = \mathbb{R}$, codom $f = Y = \mathbb{R}$, range$f = \mathbb{R}$
- range of f is all the real number, which is Y.
- The function is not one-one (injective)
  $\iff$ $\exists$ horizontal lines cross over more than one point of graph$f$
- The function is not onto (surjective)
  $\iff$ $\exists$ horizontal lines contains zero point of graph$f$

1.2.4.31. **E.g.** $f(x) = \lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \geq x\}$



- dom$f = X = \mathbb{R}$, codom $f = \mathbb{Z} \subset \mathbb{R}$, range$f = \mathbb{Z}$
- range of f is all the integer.
- function is not one-one (injective)
  $\iff$ $\exists$ horizontal lines cross over more than one point of graph$f$
- function is onto (surjective) if codom $f = \mathbb{Z}$
- function is not onto (surjective) if codom $f = \mathbb{R}$

1.2.4.32. **Def (Multi-variable function)** A n-variable function $f : \mathbb{R}^n \to \mathbb{R}$ maps n-dimensional real numbers to a real number.

- $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R}$
- In dimension-2, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
- We call it function of vector variable
- **E.g.** $f(x, y) = x^2 + y^2, g(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$

1.2.4.33. **Def (Multi-variable vector-valued function)** A n-variable vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$ maps n-dimensional real numbers to m-dimensional real numbers.

- **E.g.** $f(x, y) = (2x, 3y), g(x_1, x_2, x_3) = (2x_1x_2, 2x_1x_3, 2x_2x_3)$

1.2.4.34. **Def (Function over function space)** Consider a n-variable function $f : \mathbb{R}^n \to \mathbb{R}$ where $n \to \infty$, the set of space of square-integrable functions is called a Hilbert space, denoted as L^2

- **E.g.** $f(g) = \displaystyle\int_a^b g(x)\mathrm{d}x$ is $f : L^2 \to \mathbb{R}$, where L^2 is all square-integrable $f : \mathbb{R}^n \to \mathbb{R}$ where $n \to \infty$
- **E.g.** $\mathrm{D}(g)\Big|_{x=a} = f(g)\Big|_{x=a} = \dfrac{\mathrm{d}}{\mathrm{d}x}g(x)\Big|_{x=a}$ is $f : L^2 \to \mathbb{R}$
- **E.g.** Laplace transform
- **E.g.** Fourier transform

1.2.4.35. ⊩**Cantor-Schroder-Bernstein**

1.2.5 Cardinality

Learning Objectives

- Understand cardinality of set.
- Apply cardinality of set in examples.

1.2.5.1. This section contains lots of crazy but interesting things.

1.2.5.2. **Def (Cardinality)** For any set X we denote its cardinality by $|X|$ or $\mathrm{card}(X)$.

- Cardinality of a set can be $\begin{cases} \text{a finite integer, e.g. the sets } X = \{0,1\}, Y = \{3,4,5\}, Z = \{z \in \mathbb{N} \mid z^2 = 1\}. \\ \text{countable infinite } \aleph_0\text{, e.g. the sets } \mathbb{N} = \{0,1,2,...\}, \mathbb{Z} := \{\pm z | z \in \mathbb{N}\}, \mathbb{E} := \{2z | z \in \mathbb{N}\} \\ \text{uncountable infinite } \aleph_1\text{, e.g. the sets } [0,1] := \{r \in \mathbb{R} \,|\, 0 \leq r \leq 1\} \end{cases}$
- note that $|X|$ and X are two different things

1.2.5.3. **Def (Singleton)** For a set X, if $|X| = 1$, we call X a *singleton*.

- The set has exactly one element.

1.2.5.4. **Example of cardinality**

- $A = \{\}$, then $|A| = 0$. It means A contains nothing.
- $A = \{\varnothing\}$, then $|A| = 1$. It means A contains one thing, the empty set.
- $A = \{1\}$, then $|A| = 1$. It means A contains one thing, the element 1.
- $A = \{1, \{2, 3\}\}$, then $|A| = 2$. It means A contains two things, the element 1 and the set $\{2, 3\}$.
- $A = \{1, \varnothing\}$, then $|A| = 2$. It means A contains two things, the element 1 and the empty set.

- $A = \{1, 1, 1\}$, then $|A| = 1$. It means A contains one (unique) thing, the element 1.
- $A = \{1, 2, 3\}$, then $|A| = 3$.
- $A = \big\{1, \{2, 3\}, \{2, 2, 3\}\big\}$, then $|A| = 2$.
- $A = \big\{x \in \mathbb{Z} : x^2 \leq 4\big\}$, then $|A| = 5$ since $A = \{-2, -1, 0, 1, 2\}$.
- $A = \mathbb{N}$, then $|A| = \aleph_0$, known as countable infinite.
- $A = \{x \in \mathbb{R} : x \in [0, 1]\}$, then $|A| = \aleph_1$, known as uncountable infinite.

1.2.5.5. For two sets A, B, having $|A| = |B|$ doesn't mean $A = B$. For example $A = \{\text{Andersen}\}, B = \{\text{your father}\}$

1.2.5.6. **Combinatorics**

- **Inclusion-Exclusion principle** $|A \cup B| = |A| + |B| - |A \cap B|$
- **Product rule** $|A \times B| = |A| \cdot |B|$
- $|2^A| = 2^{|A|}$
- More on combinatorics: see my slide `https://angms.science/doc/COMP1215/COMP1215_Combinatorics.pdf`

1.2.5.7. Given two sets A, B. We said A is 1-to-1 to B, if there is a bijection $f : A \to B$

- i.e., $\exists$ a bijective function f that maps from A to B
- we write $|A| = |B|$ if A is one-to-one to B

1.2.5.8. **Def (Countable infinite)** We call the size of A is countable infinite if $|A| = |\mathbb{N}| =: \aleph_0$.

1.2.5.9. **E.g.** The set of even integers has the same cardinality of $\mathbb{N}$ $\begin{array}{c|ccccc} \mathbb{N} & 1 & 2 & 3 & 4 & \dots \\ \mathbb{E} & 2 & 4 & 6 & 8 & \dots \end{array}$

1.2.5.10. **E.g.** The set of odd integers has the same cardinality of $\mathbb{N}$ $\begin{array}{c|ccccc} \mathbb{N} & 1 & 2 & 3 & 4 & \dots \\ \mathbb{E} & 1 & 3 & 5 & 7 & \dots \end{array}$

1.2.5.11. **E.g.** The set of squares $S = \{n^2 \ : \ n \in \mathbb{N}\}$ has the same cardinality of $\mathbb{N}$ $\begin{array}{c|ccccc} \mathbb{N} & 1 & 2 & 3 & 4 & \dots \\ \mathbb{E} & 1 & 4 & 9 & 16 & \dots \end{array}$

1.2.5.12. **E.g.** The set of 15th-power integer $S = \{n^{15} \ : \ n \in \mathbb{N}\}$ has the same cardinality of $\mathbb{N}$ $\begin{array}{c|ccccc} \mathbb{N} & 1 & 2 & 3 & 4 & \dots \\ \mathbb{E} & 1 & 2^{15} & 3^{15} & 4^{15} & \dots \end{array}$

1.2.5.13. **E.g.** The set $S = \{n^{n!} \ : \ n \in \mathbb{N}\}$ has the same cardinality of $\mathbb{N}$ $\begin{array}{c|ccccc} \mathbb{N} & 1 & 2 & 3 & 4 & \dots \\ \mathbb{E} & 1 & 2^2 & 3^6 & 4^{24} & \dots \end{array}$

1.2.5.14. **E.g.** The set $S = \{n^{n^{n^{n!}}} \ : \ n \in \mathbb{N}\}$ has the same cardinality of $\mathbb{N}$

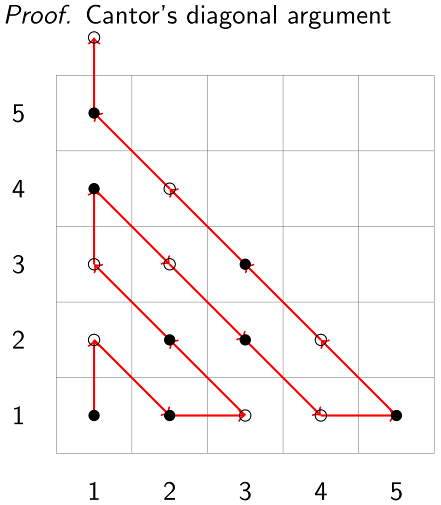
1.2.5.15. **E.g.** The set of prime numbers has the same number of elements as in $\mathbb{N}$

Proof. The set of primes, denoted by P, is a subset of $\mathbb{N}$
If $A \subset B$ then $|A| \leq |B|$
Hence $|P| \leq |\mathbb{N}| = \aleph_0$
Euclid proved that there are infinitely many primes, so $<$ is false, and we have $|P| = \aleph_0$ $\square$

1.2.5.16. **E.g.** (The 2D grids has the same number of points as a line grid)
$\mathbb{N} \times \mathbb{N}$ has the same cardinality of $\mathbb{N}$

- This sounds crazy, all the possible (i, j) has the same number of all possible (i).

Proof. Cantor's diagonal argument



$\mathbb{N}$	1	2	3	4	5	6	...
$\mathbb{N} \times \mathbb{N}$	(1,1)	(2,1)	(1,2)	(1,3)	(2,2)	(3,1)	...

$\square$

Instead of Cantor's diagonal proof, you can also consider Peano curve.

1.2.5.17. **E.g.** The set $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ has the same cardinality of $\mathbb{N}$
I.e., in a 3D space, all the integer coordinate points has the same cardinality as the number of integer coordinate points in a straight line

1.2.5.18. **E.g.** The set $\underbrace{\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{m\in\mathbb{N}}$ has the same cardinality of $\mathbb{N}$

1.2.5.19. **E.g.** Computer program is a finite string of characters. The set of all possible programs is countably infinite.

Proof. 1. The set of all finite strings over a finite alphabet is countable.

2. The union of countably many countable sets is also countable.

3. Since each set of strings of a given length is countable, and there are countably many possible lengths, the set of all possible programs is countable.

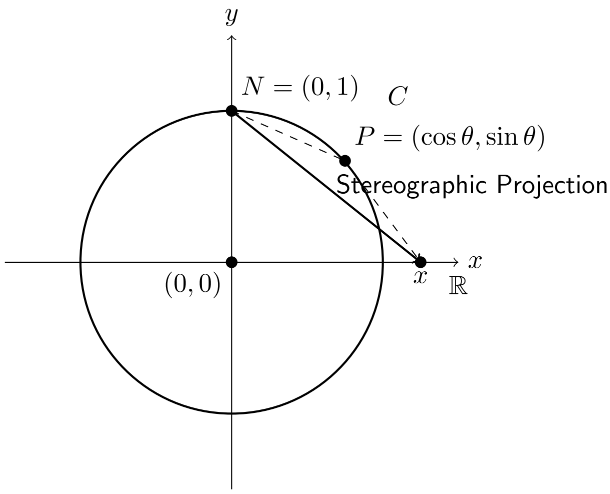
□

1.2.5.20. **E.g.** A straight line and a circle has the same number of points.

Proof. To show that a circle and a line have the same cardinality, we need to establish a bijective (one-to-one and onto) correspondence between the points on a circle and the points on a line.

Mapping Points from the Circle to the Line Consider the unit circle C centered at $(0, 0)$ with radius 1. Any point on the circle can be represented as $(\cos\theta, \sin\theta)$ where θ ranges from 0 to 2π.
We will map points on the circle to points on the real line $\mathbb{R}$ using a stereographic projection. The stereographic projection maps each point on the circle (except the north pole) to a unique point on the real line.
Stereographic Projection Let the north pole of the circle be $N = (0, 1)$. For any other point $P = (\cos\theta, \sin\theta)$ on the circle, draw a line from N to P. This line intersects the real line $y = 0$ at a unique point x.
The formula for the stereographic projection is:

$$f(\cos\theta, \sin\theta) = \frac{\cos\theta}{1 - \sin\theta}.$$



Proving the map is Bijective **Injective (One-to-One)**: Suppose $f(\cos\theta_1, \sin\theta_1) = f(\cos\theta_2, \sin\theta_2)$. Then,

$$\frac{\cos\theta_1}{1 - \sin\theta_1} = \frac{\cos\theta_2}{1 - \sin\theta_2}.$$

This implies $\cos\theta_1(1 - \sin\theta_2) = \cos\theta_2(1 - \sin\theta_1)$. Since $\cos\theta$ and $\sin\theta$ are uniquely determined by θ, it follows that $\theta_1 = \theta_2$, so the mapping is injective.

Surjective (Onto): For any point x on the real line, there exists a point $(\cos\theta, \sin\theta)$ on the circle such that

$$x = \frac{\cos\theta}{1 - \sin\theta}.$$

Solving for $\cos\theta$ and $\sin\theta$, we can find the corresponding point on the circle. Thus, the mapping is surjective.

Since there is a bijective (one-to-one and onto) correspondence between the points on the circle and the points on the real line, the circle and the line have the same cardinality.

Both the circle and the line have the cardinality of the continuum, which is the same as the cardinality of the real numbers $\mathbb{R}$, and thus they are uncountably infinite.

□

1.2.5.21. **E.g.** Two concentric circles with different radius have the same number of points.

Proof. Let C_1 be the circle with radius r_1 and center at the origin $(0, 0)$. Let C_2 be the circle with radius r_2 and the same center $(0, 0)$.

Parameterization Any point on C_1 can be represented as $(r_1\cos\theta, r_1\sin\theta)$ where θ ranges from 0 to 2π.

Similarly, any point on C_2 can be represented as $(r_2\cos\theta, r_2\sin\theta)$ where θ ranges from 0 to 2π.

Showing a One-to-One Correspondence We can map each point on C_1 to a point on C_2 using the same angle θ. Define the mapping $f : C_1 \to C_2$ by

$$f(r_1 \cos\theta, r_1 \sin\theta) = (r_2 \cos\theta, r_2 \sin\theta).$$

Showing the map is bijective

Injective (One-to-One): If $f(r_1 \cos\theta_1, r_1 \sin\theta_1) = f(r_1 \cos\theta_2, r_1 \sin\theta_2)$, then

$$(r_2 \cos\theta_1, r_2 \sin\theta_1) = (r_2 \cos\theta_2, r_2 \sin\theta_2).$$

This implies $\theta_1 = \theta_2$, so the mapping is injective.

Surjective (Onto): For any point $(r_2 \cos\theta, r_2 \sin\theta)$ on C_2, there exists a point $(r_1 \cos\theta, r_1 \sin\theta)$ on C_1 such that

$$f(r_1 \cos\theta, r_1 \sin\theta) = (r_2 \cos\theta, r_2 \sin\theta).$$

Thus, the mapping is surjective.

Since there is a bijective (one-to-one and onto) correspondence between the points on C_1 and C_2, the two circles have the same cardinality. □

1.2.5.22. ⊩**Cantor** All the real numbers in the interval $[0, 1]$ contains more element than all the natural numbers.

- We write the cardinality of $[0, 1] \in \mathbb{R}$ as $\aleph_1$
- Cantor showed that $\aleph_0 := |\mathbb{N}| < \aleph_1 = |[0, 1]|$

Proof. **Cantor's Diagonal Argument** Assume, for the sake of contradiction, that the set is countable. Then we can list all real numbers in this interval as follows:

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\ldots \\ x_2 &= 0.a_{21}a_{22}a_{23}\ldots \\ x_3 &= 0.a_{31}a_{32}a_{33}\ldots \\ &\vdots \end{aligned}$$

where a_{ij} are the digits of the ith number in the jth decimal place.

Now, construct a new number y such that its i-th digit b_i is a_{ij}. Then y differs from each x_i, so y cannot be in the list, contradicting our assumption that we have listed all real numbers between 0 and 1. Hence, the set of real numbers between 0 and 1 is uncountable, and we conclude that: $\aleph_0 < \aleph_1$. □

1.2.5.23. **The hardest problem in mathematics that is impossible to solve: Continuum hypothesis**
$2^{\aleph_0} \stackrel{?}{=} \aleph_1$
⊩$2^{\aleph_0} \stackrel{?}{=} \aleph_1$ is undecidable

- $\left\{2^{\aleph_0} \stackrel{?}{=} \aleph_1 \text{ is undecidable}\right\}$ is a proposition and it has the truth value T
- $\left\{2^{\aleph_0} = \aleph_1\right\}$ is a proposition but we cannot tell its truth value

1.2.5.24. **E.g.** $|\mathbb{N}^{\aleph_0}| \le |(2^{|\mathbb{N}|})^{\aleph_0}| = |2^{|\mathbb{N}\times\mathbb{N}|}| = |2^{|\mathbb{N}|}| = |2^{\aleph_0}| < \aleph_1$

1.2.5.25. (Interesting) Let $S = \{\aleph_0, \aleph_1, \aleph_2, ...\}$

- We cannot talk about $|S|$ because it is not well-defined
- S itself is not even a set!

1.2.5.26. **E.g.** The set of all possible program has a cardinality is smaller than the number of points on the curve $y = x^2$.

1.2.6 Set in Mathematics (Optional)

1.2.6.1. Integer number can be defined by set: the John von Neumann ordinal

Number	Set
0	$\lvert\varnothing\rvert$
1	$\lvert\{\varnothing\}\rvert$
2	$\lvert\{\varnothing,\{\varnothing\}\}\rvert$
3	$\left\lvert\left\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\right\}\right\rvert$
4	$\left\lvert\left\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\},\left\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\right\}\right\}\right\rvert$

- $\varnothing=\{\}$, and $A=\varnothing, B=\{\varnothing\}$ and $C=\{\{\varnothing\}\}$ are different things. $|A|=0, |B|=|C|=1$
- Once we have natural numbers, we can define
  - $\mathbb{Z}$ by defending subtraction as inverse operation to addition
  - $\mathbb{Q}$ as ration of integer
  - $\mathbb{R}$ by *Construction of the real numbers*, this is a whole course called real analysis in maths department
  - $\mathbb{C}$ by *Field extension*, this is a whole course called complex analysis in maths department

1.2.6.2. Topology can be defined by set: point-set topology

- Given a set X
  - (A1) We assume $X \neq \varnothing$ — X is a nonempty set
- From X we have power set of X, denoted as 2^X, (or $P(X)$ in some books)
- **Def (Topology)** If $\mathcal{T} \subseteq 2^X$ for $X \neq \varnothing$ such that
  - (T1) $\varnothing \in \mathcal{T}$ — empty set is in $\mathcal{T}$
  - (T2) $X \in \mathcal{T}$ — X itself is in $\mathcal{T}$
  - (T3) If $S_i \in \mathcal{T}$ then $\bigcap_{i=1}^{n} S_i \in \mathcal{T}$ — any finite intersection of members of $\mathcal{T}$ is in $\mathcal{T}$
  - (T4) If $S_i \in \mathcal{T}$ then $\bigcup_{i=1}^{\infty} S_i \in \mathcal{T}$ — any union of members of $\mathcal{T}$ is in $\mathcal{T}$

  Then $(X,\mathcal{T})$ is called a *topological space*, or simply *topology*.
- With topology we can define the concept of open, closed and *continuous function*
  - All member of $\mathcal{T}$ is called an *open set*
  - A set is closed it its complement is open
  - Let $(\mathcal{T},X)$ and $(\mathcal{S},Y)$ be topological spaces. A function $f:X\to Y$ is said to be *continuous* if each open subset of $V\subset\mathcal{S}$ of Y, the set $f^{-1}(V)$ is open subset of $\mathcal{T}$ of X

1.2.6.3. Probability is set : σ-algebra

- Given a set X
  - (A1) We assume $X \neq \varnothing$ — X is a nonempty set
- From X we have power set of X, denoted as $P(X)$, (or 2^X in some books)
- **Def (σ-algebra)** If $\mathcal{A} \subseteq P(X)$ for $X \neq \varnothing$ such that
  - (T1) $\varnothing \in \mathcal{A}$ — empty set is in $\mathcal{A}$
  - (T2) $X \in \mathcal{A}$ — X itself is in $\mathcal{A}$
  - (T3) If $S \in \mathcal{A}$ then $S^c := X \setminus S \in \mathcal{A}$ — any complement of members of $\mathcal{A}$ is in $\mathcal{A}$
  - (T4) If $S_i \in \mathcal{A}$ then $\bigcup_{i=1}^{\infty} S_i \in \mathcal{A}$ — any union of members of T is in T

  Then $(X,\mathcal{A})$ is called a σ*-algebra*.
- With σ-algebra we can define the concept of "size" / "volume"
  - All member of $\mathcal{A}$ is called an $\mathcal{A}$-measurable set
  - Lebesgue integration: solving the issues of Riemann integral
  - Formal definition of probability — Why do we need sigma-algebras to define probability spaces?

1.2.6.4. Matroid is set

- Given a set X called ground set
  - (A1) We assume $X \neq \varnothing$ — X is a nonempty set
- From X we have power set of X, denoted as $P(X)$, (or 2^X in some books)

- **Def (σ-algebra)** If $\mathcal{A} \subseteq P(X)$ for $X \neq \varnothing$ such that
  - (T1) $\varnothing \in \mathcal{A}$ — empty set is in $\mathcal{A}$
  - (T2) If $S \in \mathcal{A}$ and $T \subset S$, then $T \in \mathcal{A}$ — downward-closed property.
  - (T3) If $S \in \mathcal{A}$, $T \in \mathcal{A}$ and $|T| > |S|$, then there is $b \in B \setminus A$ with $A \cup \{b\} \in \mathcal{A}$ — augmentation property

  Then $(X, \mathcal{A})$ is called a *matriod*.
- Matriod is a useful language for studying network, graph and combinatorial objects.

1.2.6.5. Graph of a function is set

- Given a set X called domain
- Given a set Y called codomain
- Given a function $f : X \to Y$ called function
- **Def (Graph of function)** The graph a function $f : X \to Y$ is

$$\text{graph} f := \Big\{(x, y) \in X \times Y \mid y = f(x)\Big\}.$$

  Application of graph$f$: if f is continuous $\iff$ graph$f$ is continuous
- **Def (epigraph of function)** The epigraph a function $f : X \to Y$ is

$$\operatorname{epi} f := \Big\{(x, t) \in X \times \mathbb{R} \mid t \geq f(x)\Big\}.$$

  Application of $\operatorname{epi} f$: if f is convex $\iff$ $\operatorname{epi} f$ is a convex set

Chapter 2 Calculus and analysis

2.1 Pre-calculus

2.1.1 Number

Learning Objectives

- Re-look at high-school mathematics as a revision

2.1.1.1. This subsection is the summary of a typical 36-hour module in Pure Mathematics called *the Construction of the real numbers*. You do not need to know all the proof, you just need to know the idea. You can think of this subsection as "a history of number system"

2.1.1.2. The currently oldest record of the number "1" is on a Sumerian clay tablet, 3000 to 2001 BC

2.1.1.3. In the ancient time, people do not accept the concept of 0. The first recorded 0 appeared in Mesopotamia around 3 B.C.

2.1.1.4. The set of natural number $\mathbb{N} := \{0, 1, 2, ...\}$.

2.1.1.5. In the ancient time, people do not accept the concept of negative integer.

- The Chinese was known to be the first to use negative number around 200 BC
- Note that despite using negative integer, at that time Chinese did not have 0

2.1.1.6. The modern zero was invented in India. Around 6th century AD, Aryabhata and Brahmagupta regularly use zero in their calculation.

2.1.1.7. The set of integer $\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$

- Z comes from the German word Zahlen ("numbers")

2.1.1.8. **Basic (group) properties of natural number (positive integer)**

(P1)	addition is associative	$a + (b + c) = (a + b) + c$
(P2)	existence of additive identity	$a + 0 = 0 + a = a$
(P3)	existence of additive inverse	$a + (-a) = (-a) + a = 0$
(P4)	addition is commutative	$a + b = b + a = 0$
(P5)	multiplication is associative	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
(P6)	existence of multiplicative identity	$a \cdot 1 = 1 \cdot a = a;\quad 1 \neq 0$
(P7)	existence of multiplicative inverse	$a \cdot a^{-1} = a^{-1} \cdot a = 1,$ for $a \neq 0$
(P8)	multiplication is commutative	$a \cdot b = b \cdot a = 0$
(P9)	multiplication is distributive	$a \cdot (b + c) = a \cdot b + a \cdot c$
(P10)	trichotomy law	$\forall a \in \mathbb{Z}$, one and only one of the following holds $\begin{cases} \text{(i)} & a = 0 \\ \text{(ii)} & a \in \mathbb{Z}_+ \\ \text{(iii)} & -a \in \mathbb{Z}_+ \end{cases}$
(P11)	addition is closed	$(a \in \mathbb{Z}_+) \wedge (b \in \mathbb{Z}_+) \implies a + b \in \mathbb{Z}_+$
(P12)	multiplication is closed	$(a \in \mathbb{Z}_+) \wedge (b \in \mathbb{Z}_+) \implies a \cdot b \in \mathbb{Z}_+$

2.1.1.9. **E.g.** $x^2 - y^2 = (x - y)(x + y)$.
You learnt in high school that $(x^2 - y^2) = (x - y)(x + y)$.
We can prove this based on (P1)-(P12).

Proof

$$\begin{array}{rcll} (x-y)(x+y) & = & (x+(-y))(x+y) & \text{we write } -y = +(-y) \\ & = & x(x+y) + (-y)(x+y) & \text{by P9 with } b = x, c = (-y), a = (x+y) \\ & = & xx + xy + (-y)x + (-y)y & \text{by P9 twice on two terms separately} \\ & = & x^2 + xy + -yx + -y^2 & \\ & = & x^2 + xy + -xy + -y^2 & \text{by P8 on } yx = xy \\ & = & x^2 + 0 + -y^2 & \text{by P4 with } a = xy, b = -xy \\ & = & x^2 + -y^2 & \text{by P2 with } a = x^2 + y^2 \\ & = & x^2 - y^2 & \end{array}$$

The point of this example is to illustrate **what is a proof**: we make argument based on given information / facts

2.1.1.10. **E.g.** We can make use of the result 2.1.1.9. to derive new result.

$$x^2 = y^2 \implies x = y \ \veebar \ x = -y. \qquad \text{That is, if } x^2 = y^2, \text{ then } x = y \text{ xor } x = -y.$$

Proof Put $x^2 = y^2$ into 2.1.1.9. gives

$$\begin{array}{rcl|l} x^2 - y^2 & = & (x-y)(x+y) & 2.1.1.9. \\ y^2 - y^2 & = & (x-y)(x+y) & x^2 = y^2 \\ 0 & = & (x-y)(x+y) & \text{P3} \\ \iff & & (x-y) = 0 \ \underline{\vee} \ (x+y) = 0 & \\ \iff & & x = y \ \text{ XOR } \ x = -y & \end{array}$$

2.1.1.11. After $\mathbb{Z}$, people found that ratio of integers give fraction $\longrightarrow \mathbb{Q}$

2.1.1.12. The set of rational number $\mathbb{Q} := \left\{ \dfrac{m}{n} \ : \ m, n \in \mathbb{Z}, n \neq 0 \right\}$

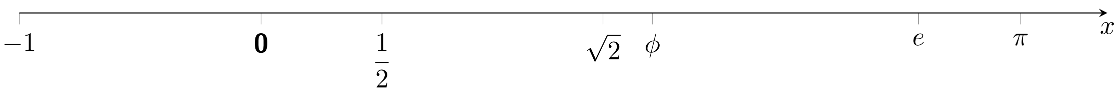
- we know have fraction: we can represent ratio
- Archimedean property: for all rational numbers r, there is an integer n such that $n > r$
  It basically means "no number is too large or too small"
- Proof. $r \in \mathbb{Q}$ so it can be represented as $\dfrac{p}{q}$ for two integers p, q with $q \neq 0$
  Since p is integer, so there exists an integer m that $m > p$.
  $n := mq > pq > p/q =: r$. $\square$

2.1.1.13. Someone found that not all numbers are rational, for example $\sqrt{2}$ is irrational, and that person was thrown into the sea.

- in fact, square root of any prime number is irrational

2.1.1.14. $\mathbb{R}$ includes $\mathbb{Q}$ and irrational number, where $\mathbb{Q}$ includes $\mathbb{Z}$ and fraction.

2.1.1.15. The real number line is a representation of $\mathbb{R}$: on a horizontal line, each point on the line corresponds to an element $x \in \mathbb{R}$ and every $x \in \mathbb{R}$ corresponds to a point on the line.



2.1.1.16. $(\mathbb{R}, \leq)$ is totally ordered
The binary relation $\leq$ on the set $\mathbb{R}$ satisfies the following for all a, c, b in $\mathbb{R}$

- $a \leq a$ (reflexive relation)
- If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitive relation)
- If $a \leq b$ and $b \leq a$ then $a = c$ (antisymmetric relation)
- Either $a \leq b$ or $b \leq a$ and no others (connected relation)

2.1.1.17. $\mathbb{R}$ is complete
Roughly, real numbers have "no hole", contrary to $\mathbb{Q}$ having holes
(actually $\mathbb{Q}$ itself are the holes: algebraic number vs transcendental number).
It takes you a full course (at least 36 hours) to learn how to prove $\mathbb{R}$ is complete.

2.1.1.18. π

- $\dfrac{1}{\pi} := \dfrac{2\sqrt{2}}{9801} \lim\limits_{N\to\infty} \displaystyle\sum_{i=1}^{N} \frac{(4k)!(1103 + 26390k)}{k!^4 (396^{4k})}$
- $\pi \approx 3.14159265$
- π is irrational
- π is transcendental

2.1.1.19. e

- $e := \lim\limits_{N\to\infty} \displaystyle\sum_{i=1}^{N} \frac{1}{n!}$
- $e := \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n$
- $e \approx 2.71828$
- e is the base of the natural log (we will know why when we talk about differentiation of log function)
- e is irrational
- e is transcendental

2.1.1.20. We (all human) actually know nothing about numbers

- there are more transcendental numbers on the real line, in fact the "holes" on the real line are algebraic numbers
- We don't know is $\gamma := \lim_{K\to\infty}\left(\sum_{k=1}^{K}\frac{1}{k} - \ln k\right)$ rational or irrational
- We don't know are $\pi \pm e, \pi e, \pi^e, e^\pi, e^e, \pi^\pi, \frac{\pi}{e}$ transcendental or not

2.1.1.21. The set of Complex number $\mathbb{C} := \{a + \sqrt{-1}b \ : \ a, b \in \mathbb{R}\}$

- $i := \sqrt{-1}$
- $a =$Re(z)
- $b =$Im(z)
- $z = a + bi, \overline{z} = a - b\sqrt{-1}$
- Re$(z) = \dfrac{\overline{z}+z}{2}$
- Im$(z) = \dfrac{\overline{z}+z}{2i}$
- $\overline{\overline{z}} = z$
- $|z|^2 = z\overline{z} = a^2 + y^2 \geq 0$
- Polar form $z = re^{i\theta}$

2.1.1.22. **Basic topology: interval, bound, open and close**

- Open interval $(a, b) := \{x \,|\, a < x < b, x \in \mathbb{R}\}$
  The two ending points a, b not included
- Closed interval $[a, b] := \{x \,|\, a \leq x \leq b, x \in \mathbb{R}\}$
  The two ending points a, b included
- Half-open interval $[a, b) := \{x \,|\, a \leq x < b, x \in \mathbb{R}\}$
- Half-open interval $(a, b] := \{x \,|\, a < x \leq b, x \in \mathbb{R}\}$
- $\mathbb{R} = (-\infty, +\infty) = \{x \,|\, -\infty < x < +\infty, x \in \mathbb{R}\}$
- A set $A \subset \mathbb{R}$ is open if for any $x \in A$, there exists a real number $\epsilon > 0$ such that any point $x \in \mathbb{R}$ that is distance $< \epsilon$ from x is also contained in A
- **Def (Open interval)** $D \subset \mathbb{R}$ is open if $(\forall x \in D)(\exists \epsilon > 0)\Big((x - \epsilon, x + \epsilon) \subset D\Big)$
- **Def (Closed interval)** $D \subset \mathbb{R}$ is closed if $D^c := \{x \in \mathbb{R} \ : \ x \notin D\}$ is open
- **Def (Bounded interval)** $D \subset \mathbb{R}$ is bounded if there exists $M \in \mathbb{R}$ such that $(\forall x \in D)(|x| < M)$
- **Def (Bounded interval)** $D \subset \mathbb{R}$ is compact if D is closed and bounded.
- **E.g.** for $a < b \in \mathbb{R}$

D	open	closed	bounded
$\varnothing$	yes	yes	yes, $M \in \mathbb{R}$
(a, b)	yes	no	yes, $M = \max\{\lvert a\rvert, \lvert b\rvert\}$
$[a, b]$	no	yes	yes, $M = \max\{\lvert a\rvert, \lvert b\rvert\} + \epsilon$
$(a, b], [a, b)$	no	no	yes, $M = \max\{\lvert a\rvert, \lvert b\rvert\} + \epsilon$
$(a, +\infty$	yes	no	no
$[a, +\infty)$	no	yes	no
$(-\infty, +\infty)$,	yes	yes	no
$(-\infty, a) \cup [b, +\infty)$	no	no	no

2.1.1.23. **Other number systems (not in exam)**

- Quaternion $\mathbb{H} := \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \ : \ a, b, c, d \in \mathbb{R}\}$
  - $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$
  - H stands for Hamilton
  - Hamilton walked on a bridge in Dublin and then suddenly sparked the idea **ijk** $= -1$, then he cut the formula on the stone of the bridge (did he broke the law of vandalism?)
  - quaternion is very useful in robotics and computer graphics
- Hyper-real
- Surreal
- Dual number
- p-adic number

2.1.2 Polynomial

2.1.2.1. **Linear equation** $a \neq 0$, $ax + b = 0$

- a is the slope of the line
- $-b$ is the y-intercept

2.1.2.2. **Quadratic equation** $a \neq 0, ax^2 + bx + c = 0 \iff x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- conjugate root occurs in pairs
- real roots if $b^2 \geq 4ac$
- complex roots if $b^2 < 4ac$
- double roots if $b^2 = 4ac$

2.1.2.3. **Single-variable polynomial** $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$

- $\deg p = n$

2.1.2.4. ⊩**Gauss's Fundamental theorem of algebra** Degree-n polynomial with complex coefficients has n complex roots

2.1.2.5. ⊩**Abel-Galois** There is no algebraic solutions for polynomial equation with degree above or equal to 4.

2.1.2.6. **Conic section** A quadratic equation in two variables

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Later (after two months) you will learn how to write this in matrix form

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^\top \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 \qquad \iff \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x} = 0$$

- $\det \boldsymbol{A} = 0$, the conic is degenerate
- $\det \boldsymbol{A} \neq 0$, $\det \boldsymbol{M}_{33} < 0$, the conic is a hyperbola
- $\det \boldsymbol{A} \neq 0$, $\det \boldsymbol{M}_{33} = 0$, the conic is a parabola
- $\det \boldsymbol{A} \neq 0$, $\det \boldsymbol{M}_{33} > 0$, the conic is an ellipse
- $\det \boldsymbol{A} \neq 0$, $\det \boldsymbol{M}_{33} > 0$, $A = C$ and $B = C$, the conic is an circle

2.1.3 Rational Function and Partial Fraction

2.1.3.1. **Def (Rational function)** A rational function

$$r(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0 \forall x$$

is a fraction where both the numerator and the denominator are polynomials.

2.1.3.2. **Def (Proper rational function)** $r(x)$ is proper if $\deg(p(x)) \leq \deg(q(x))$.

2.1.3.3. If $r(x)$ is not proper, it can be expressed as a polynomial plus a proper fraction.

2.1.3.4. PFD (Partial Fraction Decomposition) is a method used to break down rational functions into simpler fractions

2.1.3.5. **E.g.** $\dfrac{2x+3}{(x-1)(x+2)}$

- Write the rational function as a sum of fractions with unknown coefficients. $\dfrac{2x+3}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$
- Multiply both sides of the equation by the common denominator to clear the fractions. $2x + 3 = A(x+2) + B(x-1)$
- Solve for the Coefficients
  - Set up a system of equations $\begin{cases} A + B = 2 \\ 2A - B = 3 \end{cases}$
  - Solve the system of equations (Linear Algebra) gives $A = \dfrac{5}{3}$ and $B = \dfrac{1}{3}$.
- $\dfrac{2x+3}{(x-1)(x+2)} = \dfrac{5/3}{x-1} + \dfrac{1/3}{x+2}$

2.1.3.6. **Repeated Linear Factors**

- If the denominator has repeated linear factors, include terms for each power of the factor.
- Example: $\dfrac{3x+5}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$

2.1.3.7. **Irreducible Quadratic Factors**

- If the denominator has irreducible quadratic factors, include terms with linear numerators.
- Example: $\dfrac{4x^2+3x+2}{(x^2+1)(x-2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-2}$

2.1.4 Sum, splitting and telescoping

2.1.4.1. **Def (Summation sign)** $\displaystyle\sum_{k=k_0}^{k_1} a_k$ means $a_{k_0} + a_{k_0+1} + a_{k_0+2} + \cdots + a_{k_1}$

2.1.4.2. **E.g.** $\displaystyle\sum_{k=1}^{3} a = a + a + a = 3a$

2.1.4.3. **E.g.** $\displaystyle\sum_{k=1}^{3} a_k = a_1 + a_2 + a_3$

2.1.4.4. **E.g.** $\displaystyle\sum_{k=3}^{5} \ln(a_k) = \ln(a_3) + \ln(a_4) + \ln(a_5)$

2.1.4.5. **E.g.** $\displaystyle\sum_{k=2}^{4} \frac{1}{k^2}\sin(kx) = \frac{1}{2^2}\sin(2x) + \frac{1}{3^2}\sin(3x) + \frac{1}{4^2}\sin(4x)$

2.1.4.6. **E.g.** Find $\displaystyle\sum_{k=1}^{100} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$

- you can never compute this by simple addition
- Using partial fraction, we have $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$
- Now we have $\displaystyle\sum_{k=1}^{100}\left(\frac{1}{k} - \frac{1}{k+1}\right)$
- Telescoping: terms cancel out.

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{100} - \frac{1}{101}\right) = 1 - \frac{1}{101}$$

  So,

$$\sum_{k=1}^{100} \frac{1}{k(k+1)} = 1 - \frac{1}{101} = \frac{100}{101}$$

- The advanced topic on this is called Gosper splitting.

2.1.5 Trigonometry

2.1.5.1. Basic Trigonometric Functions $\quad \sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

2.1.5.2. Reciprocal Trigonometric Functions $\quad \csc\theta = \dfrac{1}{\sin\theta} \quad \sec\theta = \dfrac{1}{\cos\theta} \quad \cot\theta = \dfrac{1}{\tan\theta}$

2.1.5.3. Pythagorean Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$

2.1.5.4. Angle Sum and Difference Formulas

- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

2.1.5.5. Double Angle Formulas

- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
- $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

2.1.5.6. Half Angle Formulas

- $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$
- $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$
- $\tan\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$

2.1.5.7. Product-to-Sum Formulas

- $\sin\alpha\sin\beta = \dfrac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$
- $\cos\alpha\cos\beta = \dfrac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$
- $\sin\alpha\cos\beta = \dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$

2.1.5.8. Sum-to-Product Formulas

- $\sin\alpha + \sin\beta = 2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$
- $\sin\alpha - \sin\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$
- $\cos\alpha + \cos\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$
- $\cos\alpha - \cos\beta = -2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$

2.1.5.9. Power formula

- $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
- $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
- $\cos^3\theta = \dfrac{3\cos\theta+\cos 3\theta}{4}$
- $\sin^3\theta = \dfrac{3\sin\theta-\sin 3\theta}{4}$

2.1.5.10. Euler formula $e^{ix} = \cos x + i\sin x$

- $e^{ix} = \cos x + i\sin x$
- $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$
- $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

2.1.5.11. Hyperbolic functions (you see them in ODEs)

- $\sinh x = \dfrac{e^x - e^{-x}}{2} = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots$
- $\cosh x = \dfrac{e^x + e^{-x}}{2} = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots$
- $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
- $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
- $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
- $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$
- $\cosh ix = \cos x$
- $\cos ix = \cosh x$
- $\sinh ix = i\sin x$
- $\sin ix = i\sinh x$
- $\sinh x = -\sinh(-x)$
- $\cosh x = \cosh(-x)$
- $\tanh x = -\tanh(-x)$
- $\operatorname{sech} x = \operatorname{sech}(-x)$
- $\operatorname{csch} x = -\operatorname{csch}(-x)$
- $\coth x = -\coth(-x)$
- $\cosh^2 x - \sinh^2 x = 1$

2.1.5.12. Hyperbolic functions

$$\sinh x = \frac{2\tanh(x/2)}{1-\tanh^2(x/2)} = \frac{\tanh x}{\sqrt{1-\tanh^2 x}} \qquad \cosh x = \frac{1+\tanh^2(x/2)}{1-\tanh^2(x/2)} = \frac{1}{\sqrt{1-\tanh^2 x}}$$

$$\tanh x = \sqrt{1-\operatorname{sech}^2 x} \qquad \operatorname{sech} x = \sqrt{1-\tanh^2 x}$$

$$\coth x = \sqrt{\operatorname{cosech}^2 x + 1} \qquad \operatorname{cosech} x = \sqrt{\coth^2 x - 1}$$

$$\sinh\frac{x}{2} = \sqrt{\frac{\cosh x - 1}{2}} \qquad \cosh(x/2) = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\tanh\frac{x}{2} = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$$\sinh(2x) = 2\sinh x\cosh x \qquad \tanh(2x) = \frac{2\tanh x}{1+\tanh^2 x}$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

$$\sinh(3x) = 3\sinh x + 4\sinh^3 x \qquad \cosh 3x = 4\cosh^3 x - 3\cosh x$$

$$\tanh(3x) = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$$

$$\sinh(x\pm y) = \sinh x\cosh y \pm \cosh x\sinh y$$

$$\cosh(x\pm y) = \cosh x\cosh y \pm \sinh x\sinh y$$

$$\tanh(x\pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x\tanh y}$$

$$\sinh x + \sinh y = 2\sinh\frac{1}{2}(x+y)\cosh\frac{1}{2}(x-y) \qquad \cosh x + \cosh y = 2\cosh\frac{1}{2}(x+y)\cosh\frac{1}{2}(x-y)$$

$$\sinh x - \sinh y = 2\cosh\frac{1}{2}(x+y)\sinh\frac{1}{2}(x-y) \qquad \cosh x - \cosh y = 2\sinh\frac{1}{2}(x+y)\sinh\frac{1}{2}(x-y)$$

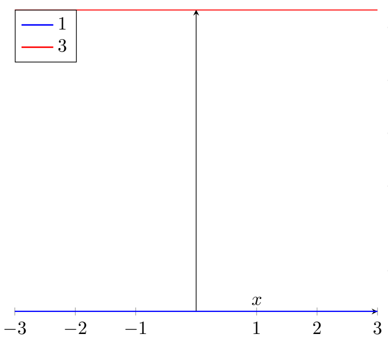
$$\sinh x \pm \cosh x = \frac{1\pm\tanh(x/2)}{1\mp\tanh(x/2)} = \mathrm{e}^{\pm x}$$

$$\tanh x \pm \tanh y = \frac{\sinh(x\pm y)}{\cosh x\cosh y}$$

$$\coth x \pm \coth y = \pm\frac{\sinh(x\pm y)}{\sinh x\sinh y}$$

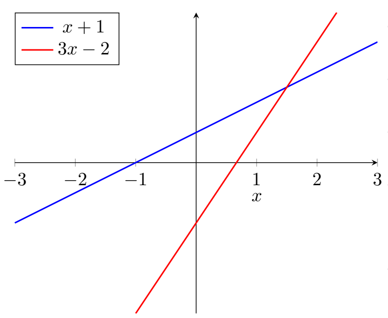
2.1.6 Elementary function

2.1.6.1. Constant function $y \equiv c$



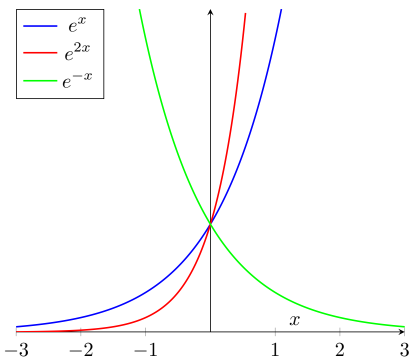
- $\operatorname{graph} f = \{(x,y)\ :\ x\in\mathbb{R}, y = c\}$
- $\operatorname{dom} f = X = \mathbb{R}$,
- codom $f = Y = \mathbb{R}$
- range $f \in \mathbb{R}$
- f is not $1\to 1$ (injective)
  $\iff$ some horizontal line contains more than one point of graph f
- f is not onto (surjective)
  $\iff$ some horizontal line does not contain a point of graph f

2.1.6.2. Linear function $y = mx + c$



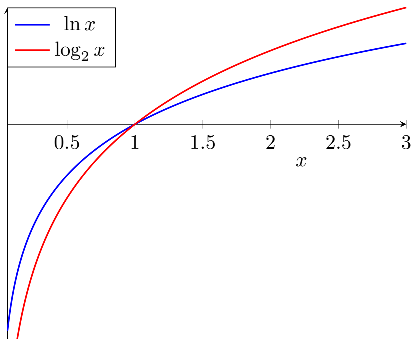
- $\operatorname{graph} f = \{(x,y)\ :\ x\in\mathbb{R}, y = c\}$
- $\operatorname{dom} f = X = \mathbb{R}$,
- codom $f = Y = \mathbb{R}$
- range $f \in \mathbb{R}$
- f is $1\to 1$ (injective)
  $\iff$ each horizontal line contains does not contain than one point of graph f
- f is onto (surjective)
  $\iff$ each horizontal line does not contain a point of graph f

2.1.6.3. Exponential function $e^x, a^x = e^{\ln a^x} = e^{\ln a}e^x$



- graph$f = \{(x, y) \; : \; x \in \mathbb{R}, y = e^x\}$
- dom$f = X = \mathbb{R}$,
- codom $f = Y = \mathbb{R}$
- range$f = \mathbb{R}_+$
- f is $1 \to 1$ (injective)
  $\iff$ each horizontal line does not contain more than one point of graph$f$
- f is not onto (surjective)
  $\iff$ some horizontal line does not contain a point of graph$f$

2.1.6.4. Logarithm: $\log_a x, \ln x$



- graph$f = \{(x, y) \; : \; x \in \mathbb{R}_+, y = \ln x\}$
- dom$f = X = \mathbb{R}_+$, $\ln$ is undefined in $\mathbb{R}$ for $x \leq 0$
- codom $f = Y = \mathbb{R}$
- range$f = \mathbb{R}$
- f is $1 \to 1$ (injective)
  $\iff$ each horizontal line does not contain more than one point of graph$f$
- f is onto (surjective)
  $\iff$ each horizontal line does contain at least one point of graph$f$

2.1.6.5. Finite compositions of elementary functions

2.1.7 Inequality

2.1.7.1. **Absolute value** $\forall a \in \mathbb{Z}$, $|a| := \begin{cases} a & a \geq 0 \\ -a & a \leq 0 \end{cases}$

2.1.7.2. ⊩**Triangle inequality** $\forall (a \in \mathbb{Z})(b \in \mathbb{Z}), |a + b| \leq |a| + |b|$

- **Exercise** Proof triangle inequality by brute force: consider all the four possible cases of (a, b)
- $|x| = \sqrt{x^2}$, where $\sqrt{a}$ is the positive square root of a and $a \in \mathbb{Z}_+$
- **Exercise** Proof triangle inequality using square root and $x^2 < y^2 \implies x < y$ for x, y nonnegative

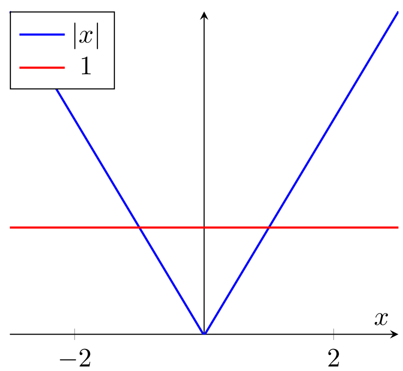
$$\begin{array}{rcl|l} (|a+b|)^2 & = & (a+b)^2 & |x|^2 = x^2 \\ & = & a^2 + 2ab + b^2 & \text{expand quadratic expression} \\ & = & |a|^2 + 2ab + |b|^2 & |x|^2 = x^2 \\ & \leq & |a|^2 + 2|a \cdot b| + |b|^2 & x \leq |x| \\ & = & |a|^2 + 2|a| \cdot |b| + |b|^2 & |x \cdot y| = |x| \cdot |y \\ & = & (|a| + |b|)^2 & \text{quadratic expression} \end{array}$$

Hence

$$(|a+b|)^2 \leq (|a| + |b|)^2 \quad \overset{x^2<y^2 \implies x<y}{\Longrightarrow} \quad |a+b| \leq |a| + |b|$$

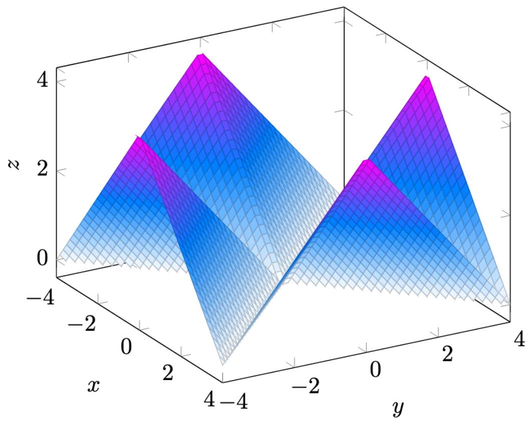
2.1.7.3. Lemma

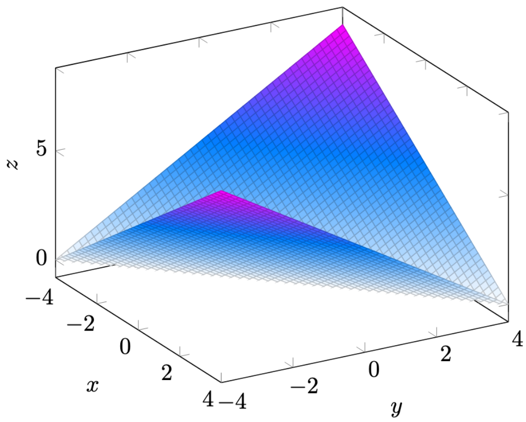
- $|x| \leq a \iff x \leq -a$ and $x \geq a$
- $|x| \geq 0$, prove by case
- $-|x| \leq x \leq |x|$, prove by case
- $|x| \leq y \iff -y \leq |x| \leq y$, prove by case



2.1.7.4. Reverse triangle inequality $\Big||x| - |y|\Big| \leq |x - y|$

$$\begin{array}{rcllr}
|b| & \leq & |a| + |b - a| & \text{triangle inequality} & 1 \\
|a| & \leq & |b| + |a - b| & \text{triangle inequality} & 2 \\
|b| - |a| & \leq & |b - a| & \text{rearrange 1} & 3 \\
|b| - |a| & \leq & |a - b| & \text{rearrange 3} & 4 \\
|a| - |b| & \leq & |a - b| & \text{rearrange 2} & 5 \\
-(|b| - |a|) & \leq & |a - b| & \text{rearrange 5} & 6
\end{array}$$





2.1.7.5. **Max and Min**

$$\max(a, b) = \frac{a + b + |a - b|}{2} \qquad \min(a, b) = \frac{a + b - |a - b|}{2}$$

Can you image how $\max(x, y) = 5$ looks like?

2.1.7.6. **Floor and ceiling**

2.1.7.6.1. The floor function $\lfloor\,\rfloor$ of a real number x, denoted by $\lfloor x \rfloor$, is defined as the greatest lower bound in $\mathbb{Z}$ of x.

$$\lfloor x \rfloor \;=\; \max\{m \in \mathbb{Z} \mid m \leq x\}. \tag{Floor}$$

2.1.7.6.2. The ceiling function $\lceil\,\rceil$ of a real number x, denoted by $\lceil x \rceil$, is defined as the smallest upper bound in $\mathbb{Z}$ of x.

$$\lfloor x \rfloor \;=\; \min\{m \in \mathbb{Z} \mid x \leq m\}. \tag{Floor}$$

2.1.7.6.3. **E.g.**

- $\lfloor 3.14 \rfloor = 3$, $\lceil 3.14 \rceil = 4$
- $\lfloor 3 \rfloor = 3$, $\lceil 3 \rceil = 3$
- $\lfloor -2.001 \rfloor = -3$, $\lceil -2.001 \rceil = -2$

2.1.7.6.4. Whenever we see a mathematical object, the first thing in mind should be the following questions: does it exist? is it unique?

2.1.7.6.5. ⊩**Floor function exists and is unique** $\forall x \in \mathbb{R}, \exists! n \in \mathbb{Z}\,\text{s.t.}\, n \leq x < n + 1$.

2.1.7.6.6. Further reading

- Wiki
- A stackexchange discussion
- a blog

2.1.7.7. ⊩**Triangle inequality** Given $x, y \in \mathbb{R}$, then

$$|x + y| \leq |x| + |y| \tag{Triangle-IQ}$$

2.1.7.8. ⊩**QM-AM-GM-HM inequalities** Given $x, y \in \mathbb{R}_+$, then

$$0 < \frac{2}{\frac{1}{x} + \frac{1}{y}} \leq \sqrt{xy} \leq \frac{x + y}{2} \leq \sqrt{\frac{x^2 + y^2}{2}} \tag{QM-AM-GM-HM-IQ}$$

2.1.7.9. ⊩**Cauchy-Schwarz inequality** Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then

$$|\boldsymbol{x}^\top \boldsymbol{y}| \leq \|\boldsymbol{x}\| \|\boldsymbol{y}\|$$

2.1.7.10. ⊩**Minkowski inequality** (Triangle inequality in higher diemension) Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then

$$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$$

2.1.7.11. ⊩**Exponential inequality** For all real x

$$1 + x \leq e^x$$

2.2 Limit and continuity

2.2.1 Limit of sequence

Learning Objectives

- Understand the definition of limit
- Apply the definition of limit
- Understand and apply operations in limit in examples

2.2.1.1. **Def (Infinite sequence of real numbers)** An infinite sequence of real numbers is a function $f : \mathbb{N} := \{1, 2, 3, ...\} \to \mathbb{R}$

2.2.1.2. We write $\{a_n\}_{n\in\mathbb{N}}, \{a_n\}$ or simply just a_n to denote a sequence $a_1, a_2, ...$

2.2.1.3. **E.g.**

- $a_n = 2, 4, 6, ..., 2n, ...$
- Fibonacci's sequence $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) = 1, 1, 2, 3, 5, 8, 13, ...$

2.2.1.4. **Def (Limit of sequence)**

$$\left\{\lim_{n\to\infty} a_n = L\right\} \triangleq (\forall \epsilon > 0)(\exists N \in \mathbb{N})\Big[n > N \implies |a_n - L| < \epsilon\Big]$$

- We write $\lim_{n\to\infty} a_N = L$ to represent a sequence a_n converges to L
- $L \in \mathbb{R}$ is a finite real number $(L \neq -\infty, L \neq \infty)$
- We say $\lim_{n\to\infty} a_N = L$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such taht for all $n > N$ we have $|a_n - L| < \epsilon$

2.2.1.5. **E.g.**

- $a_n = \frac{1}{n^2} = 1, \frac{1}{4}, \frac{1}{9}, ...$ converges to 0
- $a_n = \frac{n}{n+1} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...$ converges to 1
- $a_n = \left(1 + \frac{1}{n}\right)^n = 2, \frac{9}{4}, \frac{64}{27}, \frac{625}{256}, ...$ converges to $e \approx 2.71828$
- $a_n = 1 - \frac{1}{10^n} = 0.9, 0.99, 0.999, ...$ converges to 1
- $a_n = (-1)^n = -1, 1, -1, 1, ...$ diverges

2.2.1.6. **E.g. (0.9999...=1)** Prove $a_n = 1 - \frac{1}{10^n} = 0.9, 0.99, 0.999, ...$ converges to 1

- $|a_n - L| < \epsilon$ translates to

$$\begin{aligned}
\left|1 - \frac{1}{10^n} - 1\right| < \epsilon \iff \left|-\frac{1}{10^n}\right| < \epsilon \quad &\iff \quad \frac{1}{10^n} < \epsilon \\
&\iff \quad 10^n > \frac{1}{\epsilon} \\
&\iff \quad n \log_{10} 10 > -\log_{10} \epsilon \\
&\iff \quad n > -\log_{10} \epsilon
\end{aligned}$$

- You give me $\epsilon = 0.01$, then I give you $N = 2$ that $a_{n>N} = a_{n>2} = \{0.999, 0.9999, 0.99999, ...\}$ all have difference with 1 (the limit I claim) smaller than ϵ (your requirement)
- You give me $\epsilon = 0.001$, then I give you $N = 3$ that $a_{n>N} = a_{n>3} = \{0.9999, 0.99999, ...\}$ all have difference with 1 (the limit I claim) smaller than ϵ (your requirement)
- for any challenge ϵ given by you, I can always give a N that $|a_n - L| < \epsilon$

2.2.1.7. **E.g.** Prove $a_n = a^n$ converges to 0 if $0 < a < 1$.

- $|a_n - L| < \epsilon$ translates to

$$\begin{aligned} \left|a^n - 0\right| < \epsilon &\iff a^n < \epsilon && a > 0, n \in \mathbb{N} \text{ so absolute sign can be removed} \\ &\iff n \ln a < \ln \epsilon \\ &\iff n > \frac{\ln \epsilon}{\ln a} && \text{since } a < 1 \text{ so } \ln a < 0 \text{ so division flip sign} \end{aligned}$$

for any challenge ϵ given by you, I can always give a $N = \left\lceil \frac{\ln \epsilon}{\ln a} \right\rceil$ that $|a_n - L| < \epsilon$

2.2.1.8. **E.g.** Prove $a_n = \dfrac{1}{n}$ converges to 0 for all $n \in \mathbb{N}$.

- $|a_n - L| < \epsilon$ translates to

$$\begin{aligned} \left|\frac{1}{n} - 0\right| < \epsilon &\iff \frac{1}{n} < \epsilon && n \in \mathbb{N} \text{ so absolute sign can be removed} \\ &\iff n > \frac{1}{\epsilon} \end{aligned}$$

for any challenge ϵ given by you, I can always give a $N = \left\lceil \frac{1}{\epsilon} \right\rceil$ that $|a_n - L| < \epsilon$

2.2.1.9. **Def (Monotonic increasing sequence)** a_n is monotonic increasing if for all $m < n$ we have $a_m \leq a_n$.

2.2.1.10. **Def (Strictly increasing sequence)** a_n is strictly increasing if for all $m < n$ we have $a_m < a_n$.

2.2.1.11. **Def (Monotonic decreasing sequence)** a_n is monotonic decreasing if for all $m < n$ we have $a_m \geq a_n$.

2.2.1.12. **Def (Strictly decreasing sequence)** a_n is strictly decreasing if for all $m < n$ we have $a_m > a_n$.

2.2.1.13. **Def (Bounded sequence)** a_n is bounded if there exists real number M such that $|a_b| < M$ for all $n \in \mathbb{N}$.

2.2.1.14. **E.g.**

a_n	bounded	monotonic	convergent
$\frac{1}{n}$	yes, $M = 1$	yes, decreasing	yes, to 0
$1 - \frac{1}{10^n}$	yes, $M = 1$	yes, increasing	yes, to 1
$(-1)^n$	yes, $M = 1$	no	no
n	no	yes	no
$(-1)^n n$	no	no	no

2.2.1.15. ⊩Convergent $\implies$ Bounded
If a_n is a convergent sequence, i.e., $\lim_{n\to\infty} a_n = L$ for some $L \in \mathbb{R}$, then a_n is also bounded, i.e., there exists $M > 0$ that $|a_n| < M$.

Proof. Let $\{a_n\}$ be a convergent sequence. By definition, there exists a limit $L \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists a positive integer N such that for all $n \geq N$, we have $|a_n - L| < \epsilon$.

Choose $\epsilon = 1$. Then, there exists a positive integer N such that for all $n \geq N$,

$$|a_n - L| < 1.$$

This implies

$$L - 1 < a_n < L + 1 \quad \text{for all} \quad n \geq N.$$

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |L| + 1\}$. Then, for all $n \in \mathbb{N}$,

$$|a_n| \leq M.$$

Therefore, the sequence $\{a_n\}$ is bounded. □

2.2.1.16. Caution: Bounded $\not\Longrightarrow$ Convergent

2.2.1.17. ⊩Bounded and monotonic $\implies$ Convergent

Proof. Out of scope, it requires the completeness of $\mathbb{R}$, which is based on Heine-Borel theorem from topology. □

2.2.1.18. Lazy notation: we write $a_n \to a$ to represent $\lim_{n\to\infty} a_n = a$

2.2.1.19. ⊩**Algebraic limit theorem for sequences** Let $a_n \to a$ and $b_n \to b$ and assume $c \in \mathbb{R}$, then

- $a_n \pm b_n \to a \pm b$
- $ca_n \to ca$
- $a_n b_n \to ab$
- $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$ if $b \neq 0$ and each $b_n \neq 0$

Proof. We will prove each part separately. We will make use of inequality of absolute value.

2.2.1.19.1. Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, for any $\epsilon > 0$, there exist integers N_1 and N_2 such that for all $n > N_1$, $|a_n - a| < \dfrac{\epsilon}{2}$ and for all $n > N_2$, $|b_n - b| < \dfrac{\epsilon}{2}$. Let $N = \max(N_1, N_2)$. Then for all $n > N$

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |a_n - a + b_n - b| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Thus, $\lim_{n\to\infty} (a_n + b_n) = a + b$.
For $\lim_{n\to\infty} (a_n - b_n) = a - b$, the proof is similar.

2.2.1.19.2. For any $\epsilon > 0$, there exists an integer N such that for all $n > N$, $|a_n - a| < \dfrac{\epsilon}{|c|}$. Then for all $n > N$, we have:

$$|c \cdot a_n - c \cdot a| = |c| \cdot |a_n - a| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon$$

Thus, $\lim_{n\to\infty} (c \cdot a_n) = c \cdot a$.

2.2.1.19.3. For any $\epsilon > 0$, there exist integers N_1 and N_2 such that for all $n > N_1$, $|a_n - a| < \sqrt{\epsilon}$ and for all $n > N_2$, $|b_n - b| < \sqrt{\epsilon}$. Let $N = \max(N_1, N_2)$. Then for all $n > N$, we have:

$$|a_n \cdot b_n - a \cdot b| = |a_n \cdot b_n - a_n \cdot b + a_n \cdot b - a \cdot b| \leq |a_n| \cdot |b_n - b| + |a| \cdot |a_n - a| < (|a| + \sqrt{\epsilon}) \cdot \sqrt{\epsilon} + |a| \cdot \sqrt{\epsilon} = \epsilon$$

Thus, $\lim_{n\to\infty} (a_n \cdot b_n) = a \cdot b$.

2.2.1.19.4. If $b \neq 0$, for any $\epsilon > 0$, there exists an integer N such that for all $n > N$, $|b_n - b| < \dfrac{|b|}{2}$ and $|a_n - a| < \dfrac{\epsilon|b|^2}{2}$. Then for all $n > N$, we have:

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{a_n b - a b_n}{b_n b}\right| = \left|\frac{a_n b - ab + ab - a b_n}{b_n b}\right| \leq \frac{|a_n - a||b| + |a||b - b_n|}{|b_n||b|} < \frac{\frac{\epsilon|b|^2}{2} + |a|\frac{|b|}{2}}{|b|\frac{|b|}{2}} = \epsilon$$

Thus, $\lim_{n\to\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{a}{b}$.

□

2.2.1.20. **E.g.** Let $a_n \to 0$ then $a_n b_n \to 0$ is generally not true. For $a_n = \dfrac{1}{n}$ and $b_n = n$, we have $a_n \to 0$ but $a_n b_n \equiv 1 \neq 0$

- Why it is wrong: for the algebraic limit theorem $a_n b_n \to ab$ to hold, we need $a_n \to a$ and $b_n \to b$. Here $b_n \to b$ is not satisfied, $b_n = n$ is a diverging sequence, and we have said nothing about limit of diverging sequence

2.2.1.21. **E.g.** Let $a_n \to 0$ and $b_n \to b$, then $a_n b_n \to 0$ is true by the algebraic limit theorem.

2.2.1.22. **E.g.** $|a_n|$ is convergent $\not\Longrightarrow$ a_n is convergent
$a_n = (-1)^n$ is not convergent, but $|a_n| = |(-1)^n| \equiv 1$ is convergent.

2.2.1.23. **E.g.** a_n is convergent $\Longrightarrow$ $|a_n|$ is convergent. To be exact, $a_n \to a \implies |a_n| \to |a|$
Proof. We show that if

$$\left\{\lim_{n\to\infty} a_n = a\right\} \triangleq (\forall\epsilon > 0)(\exists N \in \mathbb{N})\Big[n > N \implies |a_n - a| < \epsilon\Big]$$

Then we have

$$\left\{\lim_{n\to\infty} |a_n| = |a|\right\} \triangleq (\forall\epsilon > 0)(\exists N \in \mathbb{N})\Big[n > N \implies ||a_n| - |a|| < \epsilon\Big]$$

Here is the key step: by reverse triangle inequality $||a_n| - |a|| < |a_n - a|$, then by assumption $|a_n - a| < \epsilon$, we have $||a_n| - |a|| < \epsilon$

2.2.1.24. How to do mathematics: do not believe in your intuition, try the true or false questions in the MadBookPro

2.2.1.25. **E.g.** $\lim\limits_{n\to\infty} \dfrac{7n+1}{8n-2}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{7n+1}{8n-2} = \lim_{n\to\infty} \frac{7n+1}{8n-2}\cdot\frac{1/n}{1/n} &= \lim_{n\to\infty} \frac{7-\frac{1}{n}}{8-\frac{2}{n}} \\
&= \frac{\lim\limits_{n\to\infty}\left(7-\dfrac{1}{n}\right)}{\lim\limits_{n\to\infty}\left(8-\dfrac{2}{n}\right)} && \text{this step we used the 4th Algebraic limit theorem 2.2.1.19.} \\
&= \frac{\lim\limits_{n\to\infty} 7 - \lim\limits_{n\to\infty}\dfrac{1}{n}}{\lim\limits_{n\to\infty} 8 - \lim\limits_{n\to\infty}\dfrac{2}{n}} && \text{this step we used the 1st Algebraic limit theorem 2.2.1.19.} \\
&= \frac{7 - \lim\limits_{n\to\infty}\dfrac{1}{n}}{8 - \lim\limits_{n\to\infty}\dfrac{2}{n}} && \text{limit of constant sequence is the constant itself} \\
&= \frac{7 - \lim\limits_{n\to\infty}\dfrac{1}{n}}{8 - 2\cdot\lim\limits_{n\to\infty}\dfrac{1}{n}} && \text{this step we used the 2nd Algebraic limit theorem 2.2.1.19.} \\
&= \frac{7-0}{8-2\cdot 0} && \text{by Example (2.2.1.8.)} \\
&= \frac{7}{8}
\end{aligned}$$

2.2.1.26. **E.g.** $\lim\limits_{n\to\infty} \dfrac{1}{n^2}$

$$\begin{aligned}
= \lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{n} &= \lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n} && \text{the 3rd Algebraic limit theorem 2.2.1.19.} \\
&= 0\cdot 0 = 0
\end{aligned}$$

2.2.1.27. **E.g.** $\lim\limits_{n\to\infty} \dfrac{n^2-2n+7}{4n^2+5n-3}$

$$= \lim_{n\to\infty}\frac{1-2\frac{1}{n}+7\frac{1}{n^2}}{4+5\frac{1}{n}-3\frac{1}{n^2}} = \lim_{n\to\infty}\frac{1}{4} \quad \text{using Algebraic limit theorem 2.2.1.19., Examples 2.2.1.8., 2.2.1.26.}$$

2.2.1.28. **E.g.** $\lim\limits_{n\to\infty} \dfrac{3n-\sqrt{4n^2+7}}{4n+\sqrt{25n^2-9}}$

$$= \lim_{n\to\infty}\frac{3-\sqrt{4+\dfrac{7}{n^2}}}{4+\sqrt{25-\dfrac{9}{n^2}}} = \frac{3-\sqrt{4}}{4+\sqrt{25}} = \frac{3-2}{4+5} = \frac{1}{9}$$

2.2.1.29. **E.g.** $\lim\limits_{n\to\infty} n-\sqrt{n^2-4n+1}$

$$= \lim_{n\to\infty}(n-\sqrt{n^2-4n+1})\frac{n+\sqrt{n^2-4n+1}}{n+\sqrt{n^2-4n+1}} = \lim_{n\to\infty}\frac{n^2-n^2+4n-1}{n+\sqrt{n^2-4n+1}} = \lim_{n\to\infty}\frac{4-\frac{1}{n}}{1+\sqrt{1-\frac{4}{n}+\frac{1}{n^2}}} = 2$$

2.2.1.30. **E.g.** $\lim\limits_{n\to\infty} \dfrac{\ln(n^7+1)}{\ln(n^5-2)}$

$$= \lim_{n\to\infty}\frac{\ln\left(n^7(1+\frac{1}{n^7})\right)}{\ln\left(n^5(1-\frac{2}{n^5})\right)} = \lim_{n\to\infty}\frac{7\ln n+\ln\left(1+\frac{1}{n^7}\right)}{5\ln n+\ln\left(1-\frac{2}{n^5}\right)} = \frac{7+\frac{\ln\left(1+\frac{1}{n^7}\right)}{\ln n}}{5+\frac{\ln\left(1-\frac{2}{n^5}\right)}{\ln n}} = \frac{7+\lim\limits_{n\to\infty}\dfrac{\ln\left(1+\frac{1}{n^7}\right)}{\ln n}}{5+\lim\limits_{n\to\infty}\dfrac{\ln\left(1-\frac{2}{n^5}\right)}{\ln n}} = \frac{7}{5}$$

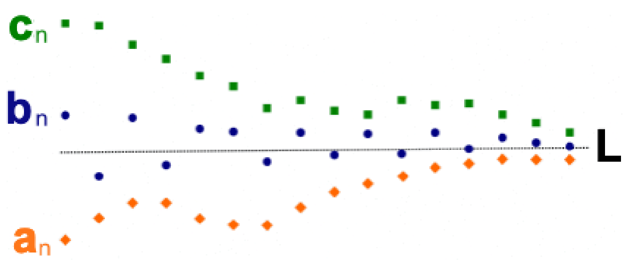
2.2.1.31. ⊩**Squeeze Theorem** If sequences $a_n \le b_n \le c_n$ for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then b_n is convergent and $\lim_{n\to\infty} b_n = L$

Proof

$$
\begin{array}{rlll}
|a_n - L| < \epsilon & \text{whenever } n > N_1 & \text{convergent of } a_n & (1)\\
|c_n - L| < \epsilon & \text{whenever } n > N_2 & \text{convergent of } c_n & (2)\\
-\epsilon < a_n - L < \epsilon & \text{whenever } n > N_1 & \text{rearrange (1)} & (3)\\
-\epsilon < c_n - L < \epsilon & \text{whenever } n > N_2 & \text{rearrange (2)} & (4)\\
L-\epsilon < a_n < L+\epsilon & \text{whenever } n > N_1 & \text{rearrange (3)} & (5)\\
L-\epsilon < c_n < L+\epsilon & \text{whenever } n > N_2 & \text{rearrange (4)} & (6)\\
L-\epsilon < a_n \le b_n \le c_n < L+\epsilon & \text{whenever } n > \max\{N_1,N_2\} & \text{by assumption and (5),(6)} & (7)\\
L-\epsilon < b_n < L+\epsilon & \text{whenever } n > N_3 & \text{let } N_3 := \max\{N_1,N_2\} & (8)\\
-\epsilon < b_n - L < \epsilon & \text{whenever } n > N_3 & \text{rearrange (8)} & (9)\\
|b_n - L| < \epsilon & \text{whenever } n > N_3 & \text{rearrange (9)} & (10)
\end{array}
$$

Therefore $|b_n - L| < \epsilon$ for all $n > N_3$ gives $\lim_{n\to\infty} b_n = L$. □

2.2.1.32. Illustration



2.2.1.33. **E.g.** $\lim_{n\to\infty} \frac{2^n}{n!} = 0$

Proof For $n \ge 3$

$$0 < \frac{2^n}{n!} = 2\Big(\frac{2}{2}\cdot\frac{2}{3}\cdot\frac{2}{4}\cdot\frac{2}{n-1}\Big)\cdot\frac{2}{n} \le 2\cdot\frac{2}{n} = \frac{4}{n}.$$

Since $\lim_{n\to\infty} \frac{4}{n} = 0$. By squeeze theorem $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

2.2.1.34. ⊩**Limited of bounded sequence** If a_n is bounded and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n b_n = 0$

Proof

a_n is bounded $\iff \exists M \in \mathbb{R}$ s.t. $(\forall n \in \mathbb{N})(-M < a_n < M)$	we are using the assumption	1
$(\forall n \in \mathbb{N})(-M\lvert b_n\rvert < a_n b_n < M\lvert b_n\rvert)$	we use $-M < a_n < M$ on $a_n b_n$	2
$\lim_{n\to\infty} b_n = 0 \implies \lim_{n\to\infty} \lvert b_n\rvert = 0$	by example 2.2.1.23.	3
$\lim_{n\to\infty} (\pm M\lvert b_n\rvert) = \pm M \lim_{n\to\infty} \lvert b_n\rvert$	by the 2nd algebraic limit theorem 2.2.1.19.	4
$= \pm M \cdot 0$	we just showed $\lim_{n\to\infty} \lvert b_n\rvert = 0$ in 3	5
$= 0$		6
$\lim_{n\to\infty} a_n b_n = 0$	using 2, 6 and squeeze theorem	

2.2.1.35. **E.g.** $\lim_{n\to\infty} \frac{\sqrt{n}+(-1)^n}{\sqrt{n}-(-1)^n}$.

Since $(-1)^n$ is bounded and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, then $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$.

$$\lim_{n\to\infty} \frac{\sqrt{n}+(-1)^n}{\sqrt{n}-(-1)^n} = \lim_{n\to\infty} \frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}} = \frac{1+\lim_{n\to\infty}\frac{1}{\sqrt{n}}}{1-\lim_{n\to\infty}\frac{1}{\sqrt{n}}} = \frac{1+0}{1-0} = 1$$

2.2.1.36. ⊩**Monotone convergent theorem** If a_n is bounded and monotonic, then a_n is convergent.

2.2.1.36.1. If $\{a_n\}$ is increasing and bounded above, then $\{a_n\}$ converges to $\sup\{a_n\}$.

2.2.1.36.2. If $\{a_n\}$ is decreasing and bounded below, then $\{a_n\}$ converges to $\inf\{a_n\}$.

Proof. We will prove the first part. The proof for the second part is similar.

Assume $\{a_n\}$ is an increasing sequence and bounded above. Let $L = \sup\{a_n\}$. We need to show that $\lim_{n\to\infty} a_n = L$.

For any $\epsilon > 0$, since $L - \epsilon$ is not an upper bound of $\{a_n\}$, there exists an integer N such that $a_N > L - \epsilon$. Since $\{a_n\}$ is increasing, for all $n \geq N$, we have:

$$a_n \geq a_N > L - \epsilon$$

Also, since L is an upper bound of $\{a_n\}$, we have $a_n \leq L$ for all n. Therefore, for all $n \geq N$, we have:

$$L - \epsilon < a_n \leq L$$

This implies:

$$|a_n - L| < \epsilon$$

Hence, $\lim\limits_{n\to\infty} a_n = L$.

The proof for a decreasing sequence bounded below follows similarly by considering the infimum.

2.2.2 Limit of series

2.2.2.1. **Def (Series)** A series is a sum of a sequence.

- **Def (Finite series)** A finite series of $K < \infty$ term is of a sequence is $S_K = \sum_{i=1}^{K} a_k = a_1 + a_2 + \cdots + a_K$
- **Def (Infinite series)** An infinite series of a sequence is $S_\infty = \lim\limits_{K\to\infty} \sum_{k=1}^{K} a_k = a_1 + a_2 + \cdots$

2.2.2.2. **Def (Partial sum)** Given a series $S_K = \sum_{i=1}^{K} a_k$, a partial sum $s_n := \sum_{i=1}^{n} a_k$

2.2.2.3. Finite arithmetic series $S_n = a + (a+d) + (a+2d) + \cdots + [a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$

- the sequence here is $a, a+d, a+2d, ...$

2.2.2.4. Finite geometric series $s_{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r}$

- the sequence here is $a, ar, ar^2, ...$

2.2.2.5. Infinite geometric series $S_\infty = \frac{a}{1-r}$,

- the sequence here is $a, ar, ar^2, ...$
- $|r| < 1$

2.2.2.6. **Def (sin)** $\sin x := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

2.2.2.7. **Def (cos)** $\cos x := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

2.2.2.8. **Def (Exp)** $\exp x = e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

2.2.2.9. **Def (Convergence of infinite series)** We call S_∞ is convergent if the sequence of its partial sum converge.

- We have a $S_\infty = a_1 + a_2 + \cdots$
- Now we construct a sequence $\left\{a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \cdots\right\}$
- We call S_∞ converges if the sequence we constructed converge.

2.2.2.10. **E.g.** $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$

The n-th partial sum of this series is $s_n = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$.

This is a geometric series with $a = 1$ and $r = \frac{1}{2}$.

The sum of geometric series is $s_n = \dfrac{a(1-r^{n+1})}{1-r}$. For $a = 1$ and $r = \dfrac{1}{2}$, we get $S_n = \dfrac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2\left(1-\left(\dfrac{1}{2}\right)^{n+1}\right)$.

As $n \to \infty$, $\left(\dfrac{1}{2}\right)^{n+1}$ approaches 0. Therefore, $\lim\limits_{n\to\infty} S_n = 2\,(1-0) = 2$.

Thus $\sum\limits_{k=0}^{\infty} \left(\dfrac{1}{2}\right)^k = 2$.

2.2.2.11. ⊩If $\sum\limits_{k=1}^{\infty} a_k$ is convergent, then $\lim\limits_{k\to\infty} a_k = 0$.

Proof. Let $\sum\limits_{k=1}^{\infty} a_k$ be a convergent series. By definition, the sequence of partial sums

$$S_N = \sum_{k=1}^{N} a_k$$

converges to a limit S as $N \to \infty$. That is,

$$\lim_{N\to\infty} S_N = S.$$

We need to show that $\lim\limits_{n\to\infty} a_n = 0$.

Consider the sequence of partial sums S_{N-1} and S_N. We have

$$S_N = S_{N-1} + a_N.$$

Taking the limit as $N \to \infty$ on both sides, we get

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} (S_{N-1} + a_N).$$

Since $\{S_N\}$ converges to S, $\{S_{N-1}\}$ also converges to S (as $N-1 \to \infty$ is essentially the same as $N \to \infty$). Therefore,

$$S = S + \lim_{N\to\infty} a_N.$$

Subtracting S from both sides, we obtain

$$0 = \lim_{N\to\infty} a_N.$$

Thus, $\lim\limits_{n\to\infty} a_n = 0$. □

2.2.2.12. Caution: the converse is not true. $\lim\limits_{k\to\infty} \dfrac{1}{k} = 0$ but $\sum\limits_{k=1}^{\infty} \dfrac{1}{k}$ is divergent.

2.2.2.13. **E.g.** In $\sum\limits_{k=0}^{\infty} \left(\dfrac{1}{2}\right)^k = 1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots = 2$, the a_k is getting smaller.

2.2.2.14. Why do we care about convergence of series

- It is related to function convergence, we will see it in the next section
- It is related to integral.

  E.g. Let $f(x)$ be a continuous, positive, and decreasing for $x \geq 1$, and let $a_n = f(n)$. Consider the series $\sum\limits_{n=1}^{\infty} a_n$.

  Proof:

  1. *If the integral converges, then the series converges:*

  Assume $\int_1^{\infty} f(x)\,dx$ converges. Then, for any integer N,

  $$\int_N^{\infty} f(x)\,dx < \infty.$$

  Since $f(x)$ is decreasing, we have

  $$\int_N^{N+1} f(x)\,dx \geq f(N+1).$$

Summing these inequalities from N to ∞, we get

$$\sum_{n=N+1}^{\infty} a_n \leq \int_N^\infty f(x)\,dx < \infty.$$

Thus, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

2. *If the series converges, then the integral converges:*

Assume $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Then, for any integer N,

$$\sum_{n=N}^{\infty} a_n < \infty.$$

Since $f(x)$ is decreasing, we have

$$\int_N^{N+1} f(x)\,dx \leq f(N).$$

Summing these inequalities from N to ∞, we get

$$\int_N^\infty f(x)\,dx \leq \sum_{n=N}^{\infty} a_n < \infty.$$

Thus, $\displaystyle\int_1^\infty f(x)\,dx$ converges.

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\displaystyle\int_1^\infty f(x)\,dx$ converges.

2.2.3 Limit of function

2.2.3.1. **Def (Limit of function)** Let $f(x) : \mathbb{R} \to \mathbb{R}$. We say that $L \in \mathbb{R}$ is a limit of $f(x)$ at $x = a$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. Compactly,

$$\lim_{x\to a} f(x) = L \quad \Longleftrightarrow \quad (\forall \epsilon > 0)(\exists \delta > 0)\Big(|x - a| < \delta \implies |f(x) - L| < \epsilon\Big)$$

2.2.3.2. Caution

- f has a limit $\neq$ that limit is in the range of f
- limit of f at $x = a$ exists has nothing to do with f exists at $x = a$, in fact f can be undefined at $x = a$
- limit of f at $x = a$ may not exists.
- If a limit exists, it is unique

2.2.3.3. **E.g.** (**limit of f at $x = a$ exists has nothing to do with f exists at $x = a$**) $f(x) = \dfrac{x^2 - 1}{x - 1}$

- Simplify the function: $f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x-1)(x+1)}{x - 1}$. For $x \neq 1$, this simplifies to: $f(x) = x + 1$
- Find the limit as x approaches 1 : $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x + 1) = 2$
- Check the range of $f(x)$: The function $f(x) = x + 1$ for $x \neq 1$ has a range of all real numbers except at $x = 1$, where the function is undefined.
- In this case, the limit of $\lim_{x\to 1} f(x) = 2$, which is outside the range of f because f is not defined at $x = 1$.

2.2.3.4. **E.g.** $\lim_{x\to 2}(3x + 1) = 7$

Proof. We need to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|3x + 1 - 7| < \epsilon$.

Let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|3x + 1 - 7| < \epsilon$.

First, simplify the expression inside the absolute value: $|3x + 1 - 7| = 3|x - 2|$

We need $3|x - 2| < \epsilon$, which is equivalent to: $|x - 2| < \dfrac{\epsilon}{3}$

Thus, we can choose $\delta = \dfrac{\epsilon}{3}$. Now, if $0 < |x - 2| < \delta$, then: $|x - 2| < \dfrac{\epsilon}{3}$

Multiplying both sides by 3, we get:

$$3|x-2| < \epsilon \quad \iff \quad |3x+1-7| < \epsilon.$$

□

2.2.3.5. ⊩**Uniqueness of limit** If a limit exists, it is unique.

Proof by contradiction. Assume that f has two different limits L and M as x approaches a.

- **Assumption**: $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$ with $L \neq M$.
- **Definition of Limit**: By the definition of a limit, for any $\epsilon > 0$, there exists a $\delta_1 > 0$ such that:

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \frac{\epsilon}{2}$$

Similarly, there exists a $\delta_2 > 0$ such that:

$$0 < |x-a| < \delta_2 \implies |f(x) - M| < \frac{\epsilon}{2}$$

- **Choosing** δ: Let $\delta = \min(\delta_1, \delta_2)$. Then for $0 < |x-a| < \delta$, both conditions hold:

$$|f(x) - L| < \frac{\epsilon}{2} \quad \text{and} \quad |f(x) - M| < \frac{\epsilon}{2}$$

- **Triangle Inequality**: Using the triangle inequality, we get:

$$|L - M| = |L - f(x) + f(x) - M| \leq |L - f(x)| + |f(x) - M|$$

Since $|L - f(x)| < \frac{\epsilon}{2}$ and $|f(x) - M| < \frac{\epsilon}{2}$, we have:

$$|L - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

- **Contradiction**: Since ϵ can be made arbitrarily small, the only way $|L-M| < \epsilon$ can hold for all $\epsilon > 0$ is if $|L-M| = 0$, which means $L = M$. Therefore, the assumption that $L \neq M$ leads to a contradiction, proving that the limit of a function as x approaches a is unique.

2.2.3.6. ⊩**Equivalence of limit of function and limit of sequence** For $f : \mathbb{R} \to \mathbb{R}$. $\lim_{x\to a} f(x) = L$ if and only if for any sequence x_n with $\lim_{n to \infty} x_n = a$, we have $\lim_{x\to a} f(x_n) = L$

2.2.3.7. ⊩**Algebraic limit theorem for function** Let $f(x)$ and $g(x)$ be functions and $c \in \mathbb{R}$, then

- $\lim_{x\to a} \big(f(x) + g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$
- $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Proof.

2.2.3.7.1. As $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, for any $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all x with $0 < |x-a| < \delta_1$, $|f(x) - L| < \frac{\epsilon}{2}$ and for all x with $0 < |x-a| < \delta_2$, $|g(x) - M| < \frac{\epsilon}{2}$. Let $\delta = \min(\delta_1, \delta_2)$. Then for all x with $0 < |x-a| < \delta$, we have:

$$\Big|\big(f(x) + g(x)\big) - (L+M)\Big| \;\leq\; |f(x) - L| + |g(x) - M| \;<\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.$$

Thus, $\lim_{x\to a} (f(x) + g(x)) = L + M$.
The proof for $\lim_{x\to a} (f(x) - g(x)) = L - M$ is similar.

2.2.3.7.2. For any $\epsilon > 0$, there exists $\delta > 0$ such that for all x with $0 < |x-a| < \delta$, $|f(x) - L| < \frac{\epsilon}{|c|}$. Then for all x with $0 < |x-a| < \delta$, we have:

$$\Big|c \cdot f(x) - c \cdot L\Big| \;=\; |c| \cdot |f(x) - L| \;<\; |c| \cdot \frac{\epsilon}{|c|} \;=\; \epsilon.$$

Thus, $\lim_{x\to a} (c \cdot f(x)) = c \cdot L$.

2.2.3.7.3. For any $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all x with $0 < |x-a| < \delta_1$, $|f(x) - L| < \sqrt{\epsilon}$ and for all x with $0 < |x-a| < \delta_2$, $|g(x) - M| < \sqrt{\epsilon}$. Let $\delta = \min(\delta_1, \delta_2)$. Then for all x with $0 < |x-a| < \delta$, we have:

$$\begin{aligned} |f(x)\cdot g(x) - L\cdot M| &= |f(x)\cdot g(x) - f(x)\cdot M + f(x)\cdot M - L\cdot M| \\ &\leq |f(x)|\cdot|g(x) - M| + |L|\cdot|f(x) - L| \\ &< (|L| + \sqrt{\epsilon})\cdot\sqrt{\epsilon} + |L|\cdot\sqrt{\epsilon} \\ &= \epsilon \end{aligned}$$

Thus, $\lim\limits_{x\to a}(f(x)\cdot g(x)) = L\cdot M$.

2.2.3.7.4. If $M \neq 0$, for any $\epsilon > 0$, there exists $\delta > 0$ such that for all x with $0 < |x-a| < \delta$, $|g(x) - M| < \dfrac{|M|}{2}$ and $|f(x) - L| < \dfrac{\epsilon|M|^2}{2}$. Then for all x with $0 < |x-a| < \delta$, we have: $\left|\dfrac{f(x)}{g(x)} - \dfrac{L}{M}\right| = \left|\dfrac{f(x)M - Lg(x)}{g(x)M}\right|$, then

$$= \left|\frac{f(x)M - LM + LM - Lg(x)}{g(x)M}\right| \leq \frac{|f(x)-L||M| + |L||M-g(x)|}{|g(x)||M|} < \frac{\frac{\epsilon|M|^2}{2} + |L|\frac{|M|}{2}}{|M|\frac{|M|}{2}} = \epsilon$$

Thus, $\lim\limits_{x\to a}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{L}{M}$.

2.2.3.8. **E.g.** Let $f(x) = \dfrac{\ln(2e^{4x}+3)}{\ln(3e^{2x}+5)}$, find $\lim\limits_{x\to\infty} f(x)$

$$\lim_{x\to\infty}\frac{\ln(2e^{4x}+3)}{\ln(3e^{2x}+5)} = \lim_{x\to\infty}\frac{\ln(e^{4x}(2+3e^{-4x}))}{\ln(e^{2x}(3+5e^{-2x}))} = \lim_{x\to\infty}\frac{4x+\ln(2+3e^{-4x})}{2x+\ln(3+5e^{-2x})} = \lim_{x\to\infty}\frac{4+\frac{\ln(2+3e^{-4x})}{x}}{2+\frac{\ln(3+5e^{-2x})}{x}} = \frac{4}{2} = 2$$

2.2.3.9. ⊩**Squeeze theorem for limit of function** If functions $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ on a neighborhood of a, and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then the limit of $g(x)$ at $x = a$ exsits and $\lim\limits_{x\to a} g(x) = L$

Proof. Let $\epsilon > 0$ be given. Since $\lim\limits_{x\to a} f(x) = L$, there exists a $\delta_1 > 0$ such that for all x with $0 < |x-a| < \delta_1$, we have:

$$|f(x) - L| < \epsilon$$

This implies:

$$L - \epsilon < f(x) < L + \epsilon$$

Similarly, since $\lim\limits_{x\to a} h(x) = L$, there exists a $\delta_2 > 0$ such that for all x with $0 < |x-a| < \delta_2$, we have:

$$|h(x) - L| < \epsilon$$

This implies:

$$L - \epsilon < h(x) < L + \epsilon$$

Let $\delta = \min(\delta_1, \delta_2)$. Then for all x with $0 < |x-a| < \delta$, we have:

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$$

Therefore:

$$L - \epsilon < g(x) < L + \epsilon$$

This implies $|g(x) - L| < \epsilon$, hence, $\lim\limits_{x\to a} g(x) = L$. □

2.2.3.10. **E.g.** $\lim\limits_{x\to 0}\dfrac{e^x - 1}{x} = 1$
For all $-1 < x < 1$ with $x \neq 0$

$$\frac{e^x - 1}{x} \overset{(2.2.2.8.)}{=} \frac{1 + x + \dfrac{x^2}{2!} + \cdots - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots$$

Here is a less-obvious part

We want to prove for $x \in (-1, 1)$ and for all integers $n \geq 2$,

$$\frac{x^n}{(n+1)!} \leq \frac{x^2}{2^n}.$$

Consider the ratio of the LHS to the RHS

$$\frac{\frac{x^n}{(n+1)!}}{\frac{x^2}{2^n}} = \frac{x^n \cdot 2^n}{(n+1)! \cdot x^2} = \frac{x^{n-2} \cdot 2^n}{(n+1)!} \leq \frac{|x^{n-2}| \cdot 2^n}{(n+1)!}.$$

Since $x \in (-1, 1)$, we have $|x| < 1$. Therefore, $|x^{n-2}| < 1$ for $n \geq 2$, now

$$\frac{\frac{x^n}{(n+1)!}}{\frac{x^2}{2^n}} \leq \frac{2^n}{(n+1)!}.$$

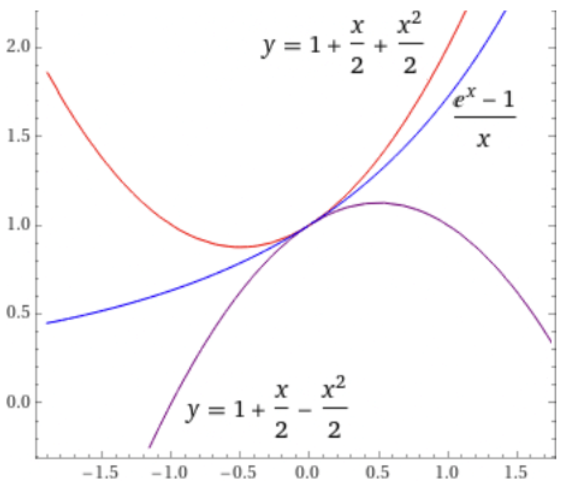
Now, $(n+1)! \geq 2^n$ (this is an execrise of mathematical induction in the MadBookPro), so

$$\frac{\frac{x^n}{(n+1)!}}{\frac{x^2}{2^n}} \leq \frac{2^n}{(n+1)!} \leq 1.$$

Therefore $\dfrac{x^n}{(n+1)!} \leq \dfrac{x^2}{2^n}$.

Now

$$\begin{aligned}
\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \left(\frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots\right) &\leq 1 + \frac{x}{2!} + \left(\frac{x^2}{4} + \frac{x^2}{8} + \frac{x^2}{16} + \cdots\right) \\
&\leq 1 + \frac{x}{2!} + \frac{x^2}{4}\underbrace{\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)}_{=2 \text{ by } 2.2.2.10.} \\
&= 1 + \frac{x}{2!} + \frac{x^2}{2}
\end{aligned}$$



Now we also have

$$\begin{aligned}
\frac{e^x - 1}{x} &= 1 + \frac{x}{2!} + \left(\frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots\right) \\
&\geq 1 + \frac{x}{2!} - \left(\frac{x^2}{4} + \frac{x^2}{8} + \frac{x^2}{16} + \cdots\right) \\
&= 1 + \frac{x}{2!} - \frac{x^2}{2}
\end{aligned}$$

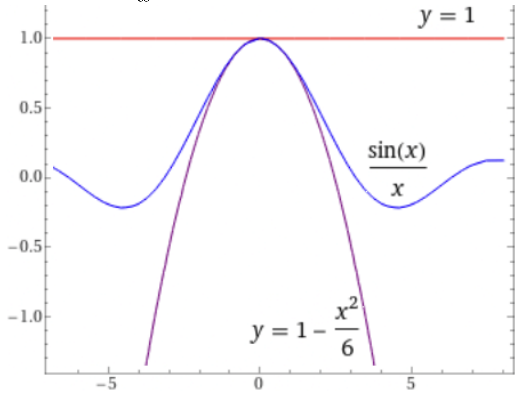
$$\lim_{x \to 0} 1 + \frac{x}{2!} + \frac{x^2}{2} = \lim_{x \to 0} 1 + \frac{x}{2!} - \frac{x^2}{2} = 1$$

therefore by squeeze theorem $\displaystyle\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

2.2.3.11. **E.g.** $\displaystyle\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$

- This example illustrate skills in evaluating limit using other limit.
- First let $y = \ln(x+1)$, thus $e^y = 1 + x$ and $x = e^y - 1$
- $x \to 0$ and $y \to 0$, so $\displaystyle\lim_{x \to 0} \frac{\ln(x+1)}{x} = \lim_{y \to 0} \frac{y}{e^y - 1}$
- $\displaystyle\lim_{y \to 0} \frac{y}{e^y - 1} = \lim_{y \to 0} \frac{1}{\frac{e^y - 1}{y}}$
- By the previous example $\displaystyle\lim_{y \to 0} \frac{1}{\frac{e^y-1}{y}} = 1 \neq 0$, so we can use the algebraic limit theorem $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\displaystyle\lim_{x \to a} g(x) \neq 0$
- $\displaystyle\lim_{y \to 0} \frac{1}{\frac{e^y-1}{y}} = \frac{\lim_{y \to 0} 1}{\lim_{y \to 0} \frac{e^y - 1}{y}} = \frac{1}{1} = 1$, hence $\displaystyle\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$

2.2.3.12. **E.g.** $\lim_{x\to 0}\frac{\sin x}{x}=1$



$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots$$

$$\frac{\sin x}{x}=1-\left(\frac{x^2}{3!}-\frac{x^4}{5!}\right)-\left(\frac{x^6}{7!}-\frac{x^8}{9!}\right)-\cdots\le 1$$

$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\left(\frac{x^4}{5!}-\frac{x^6}{7!}\right)+\left(\frac{x^8}{9!}-\frac{x^{10}}{11!}\right)+\cdots\ge 1-\frac{x^2}{3!}$$

$$\lim_{x\to 0}1=\lim_{x\to 0}1-\frac{x^2}{3!}=1.$$

By squeeze theorem $\lim_{x\to 0}\frac{\sin x}{x}=1$

2.2.3.13. **E.g.** $\lim_{x\to 0}\frac{1-\cos x}{x\tan x}$

$$=\lim_{x\to 0}\frac{1-\cos x}{x\frac{\sin x}{\cos x}}\frac{1+\cos x}{1+\cos x}=\lim_{x\to 0}\frac{(1-\cos^2 x)\cos x}{x\sin x(1+\cos x)}=\lim_{x\to 0}\frac{\sin x\cos x}{x(1+\cos x)}=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\cos x}{1+\cos x}=\frac{1}{2}$$

2.2.3.14. **E.g.** $\lim_{x\to+\infty}\frac{x^k}{e^x}=0$ for any integer k

2.2.3.15. **E.g.** $\lim_{x\to+\infty}\frac{(\ln x)^k}{x}=0$ for any integer k

2.2.3.16. **E.g.** (Tricky technique) $\lim_{x\to 0}\frac{e^{2x}-1}{\ln(1+3x)}$

$$\lim_{x\to 0}\frac{e^{2x}-1}{\ln(1+3x)}=\lim_{x\to 0}\frac{e^{2x}-1}{\ln(1+3x)}\frac{2}{3}\frac{3x}{2x}=\frac{2}{3}\lim_{x\to 0}\frac{e^{2x}-1}{2x}\frac{3x}{\ln(1+3x)}=\frac{2}{3}\left(\lim_{x\to 0}\frac{e^{2x}-1}{2x}\right)\left(\lim_{x\to 0}\frac{3x}{\ln(1+3x)}\right)=\frac{2}{3}$$

What is the skill here: creating terms from nothing so that we can make use of previously know results.

2.2.3.17. **Def (Continuity of function at a point)** A function $f:\mathbb{R}\to\mathbb{R}$ is said to be continuous at a point $x=a$ if the following three conditions are satisfied:

2.2.3.17.1. $f(a)$ is defined (it is not $\varnothing$ nor ∞ since $\infty\notin\mathbb{R}$ and $\varnothing\notin\mathbb{R}$).

2.2.3.17.2. $\lim_{x\to a}f(x)$ exists.

2.2.3.17.3. $\lim_{x\to a}f(x)=f(a)$.

In other words, f is continuous at a if:

$$\lim_{x\to a}f(x)=f(a)\notin\{\varnothing,\pm\infty\}$$

Or

$$(\forall\epsilon>0)(\exists\delta>0)\Big(|x-a|<\delta\implies|f(x)-f(a)|<\epsilon\Big)$$

2.2.3.18. **Def (Continuity of function at an interval)** A function $f:\mathbb{R}\to\mathbb{R}$ is said to be continuous at an interval I if f is continuous at every $x\in I$.

Here are some topology concept on the interval

- I can be closed
- I can be opened
- I can be half-open/half-closed

2.2.3.19. **E.g.** $f(x)=2x+3$ is continuous at $x=1$

Proof. To prove that $f(x)=2x+3$ is continuous at $x=1$, we need to show that for every $\epsilon>0$, there exists a $\delta>0$ such that if $0<|x-1|<\delta$, then $|f(x)-f(1)|<\epsilon$.

1. Let $\epsilon>0$ be given.

2. We need to find $\delta>0$ such that if $0<|x-1|<\delta$, then $|f(x)-f(1)|<\epsilon$.

First, compute $f(1)$:

$$f(1)=2(1)+3=5.$$

Next, consider $|f(x) - f(1)|$:

$$|f(x) - f(1)| = |(2x + 3) - 5| = |2x - 2| = 2|x - 1|.$$

We want $|f(x) - f(1)| < \epsilon$:

$$2|x - 1| < \epsilon.$$

Divide both sides by 2:

$$|x - 1| < \frac{\epsilon}{2}.$$

Thus, we can choose $\delta = \dfrac{\epsilon}{2}$.

3. Therefore, if $0 < |x - 1| < \delta$, then

$$|x - 1| < \delta = \frac{\epsilon}{2} \implies 2|x - 1| < \epsilon \implies |f(x) - f(1)| < \epsilon.$$

This completes the proof that $f(x) = 2x + 3$ is continuous at $x = 1$. □

2.2.3.20. If every times we need to do this $\epsilon - \delta$ proof, it will be tedious, hence we show that some standard functions are continuous, and then we make use of their result.

2.2.3.21. ⊩**Polynomial is Continuous**

Proof. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function, where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real coefficients.

To prove that $f(x)$ is continuous, we need to show that for any $c \in \mathbb{R}$,

$$\lim_{x \to c} f(x) = f(c).$$

We start by using the definition of the limit for each term in the polynomial:

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right).$$

Using the properties of limits, we can separate the limit of a sum into the sum of the limits:

$$\lim_{x \to c} f(x) = a_n \lim_{x \to c} x^n + a_{n-1} \lim_{x \to c} x^{n-1} + \cdots + a_1 \lim_{x \to c} x + a_0 \lim_{x \to c} 1.$$

Since the limit of a constant is the constant itself and the limit of x^k as x approaches c is c^k, we have:

$$\lim_{x \to c} f(x) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0.$$

Notice that this is exactly $f(c)$:

$$f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0.$$

Therefore $\lim_{x \to c} f(x) = f(c)$.

This shows that $f(x)$ is continuous at any point $c \in \mathbb{R}$. As c was arbitrary, $f(x)$ is continuous everywhere on $\mathbb{R}$. □

2.2.3.22. **E.g.** $f(x) = 2x + 3$ is continuous at $x = 1$

Proof. $2x + 3$ is a polynomial, which is continuous everywhere, including $x = 1$. □

2.2.3.23. ⊩**Algebraic theorem for continuous functions** Let $f(x)$ and $g(x)$ be continuous functions at $D \subset \mathbb{R}$ and $c \in \mathbb{R}$, then

- $f(x) + g(x)$ is continuous
- $cf(x)$ is continuous
- $f(x)g(x)$ is continuous
- $\dfrac{f(x)}{g(x)}$ is continuous at points $g(x) \neq 0$

Proof. We prove the first one, for the others see textbooks (they are too long so I don't type here).

Since f and g are continuous at $x \in D$, say $x = a$ for $a \in D$. For any $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that:

$$|x-a| < \delta_1 \implies |f(x) - f(a)| < \frac{\epsilon}{2}$$

$$|x-a| < \delta_2 \implies |g(x) - g(a)| < \frac{\epsilon}{2}$$

Let $\delta = \min(\delta_1, \delta_2)$. Then, if $|x-a| < \delta$,

$$|(f(x)+g(x)) - (f(a)+g(a))| \le |f(x)-f(a)| + |g(x)-g(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, $f+g$ is continuous at $x = a$. ☐

2.2.3.24. **E.g.** Let $f(x)$ and $g(x)$ be continuous functions, then $f(x) - g(x)$ is continuous.

Proof. g is continuous and $-g$ is continuous by item 2 in the algebraic theorem for continuous functions. Now by the item 1 in the algebraic theorem for continuous functions, $f + (-g)$ is continuous. ☐

2.2.3.25. **E.g.** $f(x) = 2x + 3$ is continuous at $x = 1$

Proof. Let $g(x) = 2x$ and $h(x) = 3$, a constant function.

We now view $f(x)$ as $g(x) + h(x) = 2x + 3$

Then $g(x)$ is continuous at $x = 1$ because it is a polynomial.

$h(x) = 3$ is continuous.

By the item 1 in the algebraic theorem for continuous functions, $g(x) + h(x) = 2x + 3$ is continuous at $x = 1$. ☐

2.2.3.26. **⊩Limit of composition** For functions $f(u)$ and $u = g(x)$, if $f(u)$ is continuous and the limit of g at $x = a$ exists, then $\lim\limits_{x\to a} f\big(g(x)\big) = f\big(\lim\limits_{x\to a} g(x)\big)$

2.2.3.27. **⊩(Composite function is continuous)** If f is continuous at c and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.

Proof. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. Assume that f is continuous at $c \in A$ and g is continuous at $f(c)$.

We need to show that $g \circ f$ is continuous at c. That is, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $|x - c| < \delta$, then $|(g \circ f)(x) - (g \circ f)(c)| < \epsilon$.

Since g is continuous at $f(c)$, for every $\epsilon > 0$, there exists a $\delta_1 > 0$ such that if $|y - f(c)| < \delta_1$, then $|g(y) - g(f(c))| < \epsilon$.

Since f is continuous at c, for $\delta_1 > 0$, there exists a $\delta_2 > 0$ such that if $|x - c| < \delta_2$, then $|f(x) - f(c)| < \delta_1$.

Let $\delta = \delta_2$. Then, if $|x - c| < \delta$, we have $|f(x) - f(c)| < \delta_1$, and thus $|g(f(x)) - g(f(c))| < \epsilon$.

Therefore, $|(g \circ f)(x) - (g \circ f)(c)| = |g(f(x)) - g(f(c))| < \epsilon$.

This completes the proof that $g \circ f$ is continuous at c. ☐

2.2.3.28. **E.g.** e^x is continuous everywhere in $\mathbb{R}$

$$\lim_{x\to a} e^x = \lim_{h\to 0} e^{a+h} = \lim_{h\to 0} e^a e^h = e^a \lim_{h\to 0} e^h = e^a \cdot 1 = e^a.$$

As $a \in \mathbb{R}$ so e^x is continuous everywhere in $\mathbb{R}$

2.2.3.29. **E.g.** $\cos, \sin$ are continuous on $\mathbb{R}$ and $\ln$ is continuous on $\mathbb{R}_+$

2.2.3.30. **E.g.** $f(x) = 2x + 3$ is continuous at $x = 1$

Proof. Let $g(x) = 2x$ and $h(x) = x + 1$.

We now view $f(x)$ as $(h \circ g)(x) = h(2x) = 2x + 1$

Then $g(x)$ is continuous at $x = 1$ because it is a polynomial.

$h(x)$ is continuous at $x = 2$ because it is also a polynomial.

So $h \circ g$ is continuous at $x = 1$. ☐

2.2.3.31. **Def (High-school's left/right definition of continuity of function)** A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = a$ if $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = f(a)$, where $\lim\limits_{x\to a^+} f(x)$ is called limit from the right-hand side and $\lim\limits_{x\to a^-} f(x)$ is called limit from the left-hand side

2.2.3.32. ⊩**The $\epsilon - \delta$ definition is equivalent to the high-school's left/right definition**

- $\epsilon - \delta$ definition: f is continuous at a point c if $(\forall\epsilon > 0)(\exists\delta > 0)\Big(|x - c| < \delta \implies |f(x) - f(c)| < \epsilon\Big)$
- High-school definition of continuity of function
  The left-hand limit of f as x approaches c is L if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $c - \delta < x < c$, then $|f(x) - L| < \epsilon$.
  The right-hand limit of f as x approaches c is R if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $c < x < c+\delta$, then $|f(x) - R| < \epsilon$.
- Top show $\iff$, we show $\implies$ and $\impliedby$
- $\epsilon - \delta$ **continuity** $\implies$ **Left/Right Limits**
  $(\forall\epsilon > 0)(\exists\delta > 0)\Big(|x - c| < \delta \implies |f(x) - f(c)| < \epsilon\Big)$
  For the left-hand limit, consider x approaching c from the left ($x < c$). Since $|x-c| < \delta$ implies $c-\delta < x < c$, we have $|f(x) - f(c)| < \epsilon$. Thus, the left-hand limit of f as x approaches c is $f(c)$.
  For the right-hand limit, consider x approaching c from the right ($x > c$). Since $|x - c| < \delta$ implies $c < x < c + \delta$, we have $|f(x) - f(c)| < \epsilon$. Thus, the right-hand limit of f as x approaches c is $f(c)$.
  Therefore, the left-hand limit and the right-hand limit both equal $f(c)$.
- $\epsilon - \delta$ **continuity** $\impliedby$ **Left/Right Limits**
  Assume the left-hand limit and the right-hand limit of f as x approaches c both equal $f(c)$.
  For every $\epsilon > 0$, there exists a $\delta_1 > 0$ such that if $c - \delta_1 < x < c$, then $|f(x) - f(c)| < \epsilon$.
  Similarly, for every $\epsilon > 0$, there exists a $\delta_2 > 0$ such that if $c < x < c + \delta_2$, then $|f(x) - f(c)| < \epsilon$.
  Let $\delta = \min(\delta_1, \delta_2)$. Then for $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$ whether x approaches c from the left or the right.
  Therefore, f is continuous at c.

2.2.3.33. LHS/RHS limit are useful for piecewise functions.

2.2.3.34. **Def (Piecewise function)** A function f is piecewise if $f(x) = \begin{cases} g(x) & x \geq a \\ h(x) & x < a \end{cases}$ for some function g and h

2.2.3.35. **E.g.** $f(x) = \begin{cases} 2x - 1 & x < 2 \\ a & x = 2 \\ x^2 + b & x > 2 \end{cases}$ is continuous at $x = 2$, find a, b

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} (2x - 1) = 3 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (x^2 + b) = 4 + b \\ f(2) &= a \end{aligned}$$

Since f is continuous at $x = 2$, thus $3 = 4 + b = a$ gives $a = 3, b = -1$

2.3 Differentiation

2.3.1 Differentiability

Learning Objectives

- Understand the definition of differentiation
- Understand and apply operations in differentiation in examples

2.3.1.1. In differentiation, the first question we ask is "is the function differentiable"? This issue is known as *differentiability*.

2.3.1.2. **Def (Differentiability)**

A function $f(x) : \mathbb{R} \to \mathbb{R}$ is said to be differentiable at $x = a$ if the limit $f'(a) = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ exists.

2.3.1.3. **E.g.** $f(x) = x^2$, then $f'(1) = 2$

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x\to 1} x + 1 = 2$$

2.3.1.4. **Def (Differentiability in h notation)**

A function $f(x) : \mathbb{R} \to \mathbb{R}$ is said to be differentiable at $x = a$ if the limit $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a + h) - f(a)}{h}$ exists.

2.3.1.5. **E.g.** $f(x) = x^2$, then $f'(1) = 2$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh-h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh-h^2}{h} \\ &= \lim_{h\to 0} 2x - h = 2x - \lim_{h\to 0} h = 2x - 0 = 2x \implies f'(1) = 2 \end{aligned}$$

2.3.1.6. **Def (Differentiability in $\epsilon - \delta$ form)**
A function $f(x) : \mathbb{R} \to \mathbb{R}$ is said to be differentiable at $x = a$ if there exists a real number L such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x-a| < \delta \implies \left| \frac{f(x)-f(a)}{x-a} - L \right| < \epsilon.$$

The number $L = f'(a)$.

2.3.1.7. **E.g.** $f(x) = x^2$, then $f'(1) = 2$

We need to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$0 < |x-1| < \delta \implies \left| \frac{x^2-1}{x-1} - 2 \right| < \epsilon$$

Simplify the right hand side, we have:

$$\left| \frac{f(x)-f(a)}{x-a} - L \right| < \epsilon \iff \left| \frac{x^2-1}{x-1} - 2 \right| < \epsilon \iff |x+1-2| < \epsilon \iff |x-1| < \epsilon$$

So whenever you give me an $\epsilon > 0$, that exists a $\delta > 0$, which we can choose $\delta = \epsilon$, such that there exists $L = 2$ that

$$0 < |x-a| < \delta \implies \left| \frac{f(x)-f(a)}{x-a} - L \right| < \epsilon$$

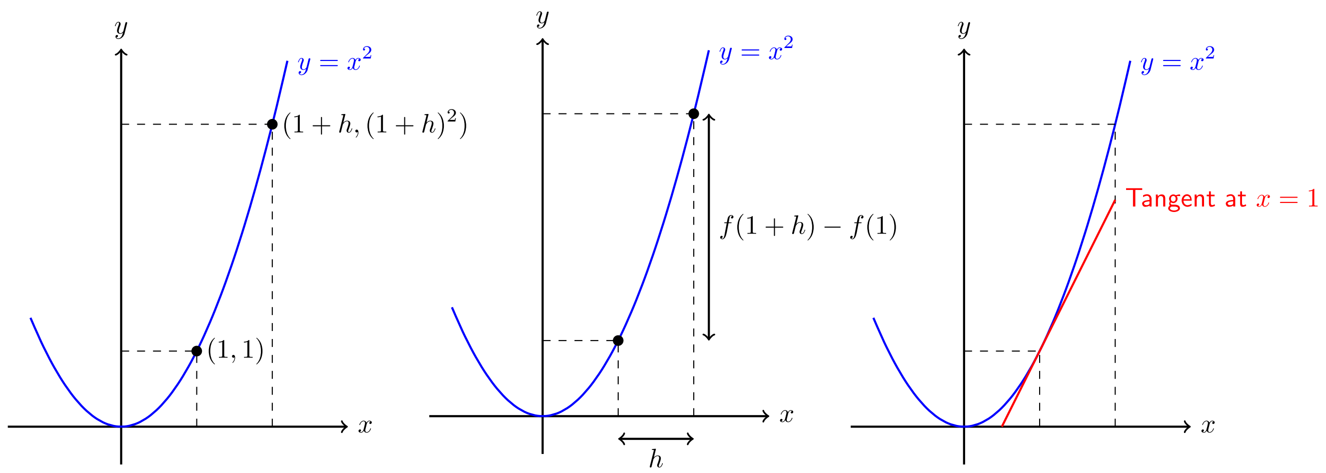
is always true.

2.3.1.8. **Def (q-Differentiability)**
Let $f : \mathbb{R} \to \mathbb{R}$. We say $f(x)$ is q-differentiable at $x = a$ if the limit $f'_q(a) = \lim_{q\to 1} \frac{f(qa)-f(a)}{qa-a}$ exists.
q stands for "quantum". q-Calculus is (relatively) a new subject.

2.3.1.9. **E.g.** $f(x) = x^2$, then $f'_q(1) = 2$

$$f'_q(1) = \lim_{q\to 1} \frac{f(q1)-f(1)}{q1-1} = \lim_{q\to 1} \frac{f(q)-1}{q-1} = \lim_{q\to 1} \frac{q^2-1}{q-1} = \cdots = 2$$

2.3.1.10. The picture of differentiation of x^2



2.3.1.11. $\Vdash f(x)$ differentiable at $x = a \implies f(x)$ is continuous at $x = a$

Proof.

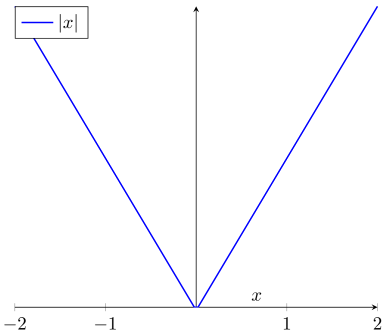
$$\begin{aligned}\lim_{x\to a}\Big(f(x)-f(a)\Big) &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)(x-a) && \text{tricky step: create terms}\\ &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\lim_{x\to a}(x-a) && \text{lim can be distributed becoz } f(x) \text{ differentiable at } x=a \text{ and lim of both ( ) exists}\\ &= f'(a)\cdot 0 \;=\; 0\end{aligned}$$

So now we have $\lim_{x\to a}\Big(f(x)-f(a)\Big)=0 \iff \lim_{x\to a} f(x)=f(a) \iff f$ is continuous at $x=a$ □

2.3.1.12. **E.g.**

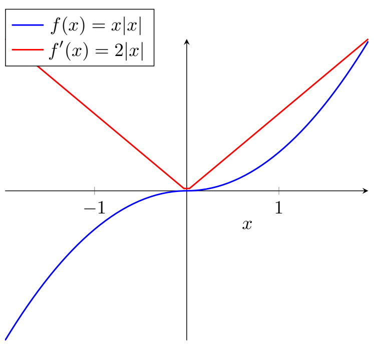
- $f(x)=e^x,\ f'(0)=\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\dfrac{e^x-1}{x}\overset{2.2.3.10.}{=}1$
- $f(x)=\ln x,\ f'(1)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\dfrac{\ln(1+h)}{h}\overset{2.2.3.11.}{=}1$
- $f(x)=\sin x,\ f'(0)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\dfrac{\sin h-\sin 0}{h}=\lim_{h\to 0}\dfrac{\sin h}{h}\overset{2.2.3.12.}{=}1$

2.3.1.13. ⊩Absolute value $|x|$ is not differentiable at $x=0$.



- We will use the left/right definition of limit
- $\lim_{h\to 0^-}\dfrac{f(h)-f(0)}{h}=\lim_{h\to 0^-}\dfrac{-h}{h}=\lim_{h\to 0^-}-1=-1$
- $\lim_{h\to 0^+}\dfrac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\dfrac{h}{h}=\lim_{h\to 0^+}1=1$
- $\lim_{h\to 0^-}\dfrac{f(h)-f(0)}{h}\neq\lim_{h\to 0^+}\dfrac{f(h)-f(0)}{h}$ so $|x|$ is not differentiable at $x=0$.

2.3.1.14. ⊩$|x|x$ is differentiable



- If $x<0,\ f(x)=-x^2,\ f'(x)=-2x$
- If $x>0,\ f(x)=x^2,\ f'(x)=2x$
- $\lim_{h\to 0^-}\dfrac{f(h)-f(0)}{h}=\lim_{h\to 0^-}\dfrac{-h^2}{h}=0$
- $\lim_{h\to 0^+}\dfrac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\dfrac{h^2-0}{h}=0$
- $\lim_{h\to 0^-}\dfrac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\dfrac{f(h)-f(0)}{h}$
  so $x|x|$ is differentiable at $x=0$ and $f'(0)=0$
- $f'(x)=\begin{cases}-2x & x<0\\ 0 & x=0\\ 2x & x>0\end{cases}$, so $f'(x)=2|x|$

2.3.1.15. **Def (First derivative)** Let $y=f(x)$ be a differentiable function on (a,b). The first derivative of $f(x)$ is the function on (a,b) defined by $\dfrac{dy}{dx}=f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

2.3.1.16. ⊩$\underbrace{\text{Continuously differentiable}}_{\text{derivative is continuous}}$ $\implies$ differentiable $\implies$ continuous $\implies$ limit exists $\implies$ limit exists from both sides

2.3.2 Differentiation

2.3.2.1. ⊩Algebraic theorem of derivative: let u,v be differentiable functions of x, and let $c\in\mathbb{R}$, then

- $(u+v)'=u'+v'$ — sum rule
- $(cu)'=cu'$, c real number — scalar product
- $(uv)'=u'v+uv'$ — product rule
- $\left(\dfrac{u}{v}\right)'=\dfrac{u'v-uv'}{v^2}$ if $v(x)\neq 0$ — quotient rule

Proof. We do it one-by-one

- sum rule

$$
\begin{aligned}
(u+v)' &= \lim_{h\to 0}\frac{\Big(u(x+h)+v(x+h)\Big)-\Big(u(x)+v(x)\Big)}{h} = \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}+\frac{v(x+h)-v(x)}{h}\\
&= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}+\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}\\
&= u'+v'
\end{aligned}
$$

- scalar product $(cu)' = \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h} = c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} = cu'$.

- product rule $(uv)' = \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h}$

$$
\begin{aligned}
&= \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x+h)+u(x)v(x+h)-u(x)v(x)}{h} && \text{tricky step: creating terms}\\
&= \lim_{h\to 0}\left(\frac{u(x+h)v(x+h)-u(x)v(x+h)}{h}+\frac{u(x)v(x+h)-u(x)v(x)}{h}\right)\\
&= \lim_{h\to 0}\left(v(x+h)\frac{u(x+h)-u(x)}{h}+u(x)\frac{v(x+h)-v(x)}{h}\right) && \text{factorizing terms}\\
&= \left(\lim_{h\to 0}v(x+h)\frac{u(x+h)-u(x)}{h}\right)+\left(\lim_{h\to 0}u(x)\frac{v(x+h)-v(x)}{h}\right) && \text{distribute lim into sum}\\
&= \left(\lim_{h\to 0}v(x+h)\right)\left(\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}\right)+\left(\lim_{h\to 0}u(x)\right)\left(\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}\right) && \text{distribute lim into product}\\
&= v(x)\left(\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}\right)+u(x)\left(\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}\right) = u'v+uv'
\end{aligned}
$$

- quotient rule $\left(\frac{u}{v}\right)' = \lim_{h\to 0}\frac{\frac{u(x+h)}{v(x+h)}-\frac{u(x)}{v(x)}}{h}$

$$
\begin{aligned}
&= \lim_{h\to 0}\frac{u(x+h)v(x)-u(x)v(x+h)}{hv(x+h)v(x)} && \text{simplify the fraction}\\
&= \lim_{h\to 0}\frac{u(x+h)v(x)-u(x)v(x)+u(x)v(x)-u(x)v(x+h)}{hv(x+h)v(x)} && \text{tricky step: creating terms}\\
&= \lim_{h\to 0}\left(\frac{v(x)(u(x+h)-u(x))}{hv(x+h)v(x)}-\frac{u(x)(v(x+h)-v(x))}{hv(x+h)v(x)}\right)\\
&= \lim_{h\to 0}\left(\frac{u(x+h)-u(x)}{h}\cdot\frac{v(x)}{v(x+h)v(x)}-\frac{v(x+h)-v(x)}{h}\cdot\frac{u(x)}{v(x+h)v(x)}\right)\\
&= \frac{u'(x)v(x)}{v(x)^2}-\frac{u(x)v'(x)}{v(x)^2} = \frac{u'v-uv'}{v^2}
\end{aligned}
$$

□

2.3.2.2. **E.g.** $\frac{d}{dx}x^n = nx^{n-1}$

Proof-1. Using $a^n-b^n = (a-b)\Big(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+a^2b^{n-3}+ab^{n-2}+b^{n-1}\Big)$

$$
\begin{aligned}
\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n-x^n}{h} &= \lim_{h\to 0}\frac{(x+h-x)\Big((x+h)^{n-1}+(x+h)^{n-2}x+\cdots+x^{n-1}\Big)}{h}\\
&= \lim_{h\to 0}(x+h)^{n-1}+(x+h)^{n-2}x+\cdots+x^{n-1}\\
&= x^{n-1}+x^{n-2}x+\cdots+x^{n-1} = nx^{n-1}
\end{aligned}
$$

Proof-2. Using product rule

$$\begin{aligned}\frac{d}{dx}x^n = \frac{d}{dx}x\cdot x^{n-1} = x\frac{d}{dx}x^{n-1} + x^{n-1}\frac{d}{dx}x &= x\frac{d}{dx}x^{n-1} + x^{n-1}\Big(\lim_{h\to 0}\frac{(x+h)-x}{h}\Big) = x\frac{d}{dx}x^{n-1} + x^{n-1}\\ &= x\Big(x\frac{d}{dx}x^{n-2} + x^{n-2}\Big) + x^{n-1} = x^2\frac{d}{dx}x^{n-2} + 2x^{n-1}\\ &= x^3\frac{d}{dx}x^{n-3} + 3x^{n-1}\ \cdots\ = nx^{n-1}\end{aligned}$$

Note. Proof-1 & Proof-2 both only work for $n \in \mathbb{N}$.

2.3.2.3. **E.g.** $\frac{d}{dx}e^x = e^x$

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x \underbrace{\lim_{h\to 0}\frac{e^h-1}{h}}_{=1,\text{ see } 2.2.3.10.} = e^x$$

2.3.2.4. **E.g.** $\frac{d}{dx}\ln x = \frac{1}{x}$, $x > 0$

$$\frac{d}{dx}\ln x = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln(1+\frac{h}{x})}{h}$$

Now to proceed, we make use of the fact that $\lim_{x\to 0} cf(x) = c\lim_{x\to 0} f(x)$

Let $k = \frac{h}{x}$. So if $h \to 0$ then $k \to 0$ for $x > 0$.

$$\begin{aligned}\lim_{h\to 0}\frac{\ln(1+\frac{h}{x})}{h} &= \lim_{k\to 0}\frac{\ln(1+k)}{kx} && \text{change of variable}\\ &= \lim_{k\to 0}\frac{\ln(1+k)}{k}\cdot\frac{1}{x}\\ &= \frac{1}{x}\lim_{k\to 0}\frac{\ln(1+k)}{k} && \lim_{x\to 0} cf(x) = c\lim_{x\to 0} f(x)\\ &= \frac{1}{x} && 2.2.3.11.\end{aligned}$$

2.3.2.5. Trigonometric and hyperbolic formula

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \cos x & \frac{d}{dx}(\sinh x) &= \cosh x\\ \frac{d}{dx}(\cos x) &= -\sin x & \frac{d}{dx}(\cosh x) &= \sinh x\\ \frac{d}{dx}(\tan x) &= \sec^2 x & \frac{d}{dx}(\tanh x) &= \operatorname{sech}^2 x\\ \frac{d}{dx}(\sec x) &= \sec x\tan x & \frac{d}{dx}(\operatorname{sech} x) &= -\operatorname{sech} x\tanh x\\ \frac{d}{dx}(\cot x) &= -\operatorname{cosec}^2 x & \frac{d}{dx}(\coth x) &= -\operatorname{cosech}^2 x\\ \frac{d}{dx}(\operatorname{cosec} x) &= -\operatorname{cosec} x\cot x & \frac{d}{dx}(\operatorname{cosech} x) &= -\operatorname{cosech} x\coth x\end{aligned}$$

2.3.2.6. ⊩**Chain rule** Let f and g be functions such that f is differentiable at $g(x)$ and g is differentiable at x. The composition $h(x) = f(g(x))$ is differentiable at x and that $h'(x) = f'(g(x))\cdot g'(x)$.

Proof. By the definition of the derivative, we have:

$$f'(g(x)) = \lim_{u\to g(x)}\frac{f(u)-f(g(x))}{u-g(x)}, \qquad g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

We need to show that:

$$h'(x) = \lim_{h\to 0}\frac{h(x+h)-h(x)}{h} = \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}$$

Using the substitution $u = g(x+h)$, we get:

$$h'(x) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)} \cdot \frac{g(x+h)-g(x)}{h} \quad \text{this step is ok as } g(x+h)-g(x) \neq 0 \text{ before we take } \lim$$

As $h \to 0$, $g(x+h) \to g(x)$. Therefore, we can rewrite the limit as:

$$h'(x) = \left(\lim_{u\to g(x)} \frac{f(u) - f(g(x))}{u - g(x)}\right) \cdot \left(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right)$$

By the definition of the derivatives of f and g, we have:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

□

2.3.2.7. **E.g.**

- $\dfrac{\mathrm{d}}{\mathrm{d}x}u^n = nu^{n-1}\dfrac{\mathrm{d}}{\mathrm{d}x}u$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}e^u = e^u\dfrac{\mathrm{d}}{\mathrm{d}x}u$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln u = \dfrac{1}{u}\dfrac{\mathrm{d}}{\mathrm{d}x}u$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin u = \cos u\dfrac{\mathrm{d}}{\mathrm{d}x}u$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos u = -\sin u\dfrac{\mathrm{d}}{\mathrm{d}x}u$

2.3.2.8. **E.g.**

- $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin^3 x = 3\sin^2 x\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x = 3\sin^2 x\cos x$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}e^{\sqrt{x}} = e^{\sqrt{x}}\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{(\ln x)^2} = -\dfrac{2}{(\ln x)^3}\dfrac{\mathrm{d}}{\mathrm{d}x}\ln x = \dfrac{-2}{x(\ln x)^3}$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(\cos 2x) = \dfrac{1}{\cos 2x}(-\sin 2x)2 = -2\tan 2x$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\sec 2x}{\ln x} = \dfrac{\ln x(2\sec 2x\tan 2x) - \sec 2x\frac{1}{x}}{(\ln x)^2} = \dfrac{\sec 2x(2\boldsymbol{x}\tan 2x\ln x - 1)}{x(\ln x)^2}$

2.3.2.9. **Def (Implicit function)** An implicit function is an equation of the form $F(x,y) = 0$.

- $y = f(x)$ is called explicit function
- Implicit function is a larger class of function that can be used to define function can do not have explicit form
- An implicit function may not define a function

2.3.2.10. **E.g.** $x^2 - xy - xy^2 = 0$ is an implicit function.

2.3.2.11. ⊩(**Implicit function differentiation**) For implicit function $F(x,y) = 0$, then $\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

- $\dfrac{\partial F(x,y)}{\partial x}$ is the partial derivative of F with respect to x and treating y as constant.
- Let $F(x,y)$ be a function of two variables. The partial derivative of F with respect to x at the point (a,b) is

$$\frac{\partial F}{\partial x}(a,b) = \lim_{h\to 0}\frac{F(a+h,b) - F(a,b)}{h}$$

Similarly, the partial derivative of F with respect to y at the point (a,b) is defined as:

$$\frac{\partial F}{\partial y}(a,b) = \lim_{h\to 0}\frac{F(a,b+h) - F(a,b)}{h}$$

2.3.2.12. For $x^2 - xy - xy^2 = 0$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

$$\begin{aligned} 2x - (y + xy') - (y^2 + 2xyy') &= 0 \\ xy' + 2xyy' &= 2x - y - y^2 \\ y' &= \frac{2x - y - y^2}{x + 2xy} \end{aligned}$$

2.3.2.13. **E.g.** For $\cos(xe^y) + x^2\tan y = 0$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\begin{aligned} -\sin(xe^y)(e^y + xe^y y') + 2x\tan y + x^2\sec^2 y \cdot y' &= 0 \\ y' &= \frac{e^y\sin(xe^y) - 2x\tan y}{x^2\sec^2 y - xe^y\sin(xe^y)} \end{aligned}$$

2.3.2.14. ⊩**Inverse function derivative** Suppose $f(y)$ is differentiable with $f'(y) \neq 0$ for all y. Then $f^{-1}(x)$ is differentiable and $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

Proof. First $f(f^{-1}(x)) = x$. By chain rule $f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$. □

2.3.2.15. **E.g.** Let $\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $\frac{\mathrm{d}y}{\mathrm{d}x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

Proof.

$$\begin{aligned} y &= \sin^{-1} x \\ \sin y &= x \\ \cos y\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos y} \\ &= \frac{1}{\sqrt{1-\sin^2 y}} \qquad \cos y \geq 0\ \forall y \in [-\pi/2, \pi/2] \\ &= \frac{1}{1-x^2} \end{aligned}$$

□

2.3.2.16. **E.g.** Let $\sin^{-1} : [-1, 1] \to [0, \pi]$. Then $\frac{\mathrm{d}y}{\mathrm{d}x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

2.3.2.17. **E.g.** Let $\tan^{-1} : \mathbb{R} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $\frac{\mathrm{d}y}{\mathrm{d}x}\tan^{-1} x = \frac{1}{\sqrt{1+x^2}}$

2.3.2.18. **Feynman's differentiation technique** $\frac{\mathrm{d}}{\mathrm{d}x}u^v = u^v v'\ln u + u^{v-1}vu'$

$$\begin{aligned} y &= u^v \\ \ln y &= v\ln u \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= v'\ln u + \frac{v}{u}u' \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= yv'\ln u + y\frac{v}{u}u' \\ &= u^v v'\ln u + u^{v-1}vu' \end{aligned}$$

2.3.2.19. **E.g.** $\frac{d}{dx}\Big(\frac{2+x\sin x}{e^x+x^2}\Big)^{\cos x}$

$$\begin{aligned} f &= \Big(\frac{2+x\sin x}{e^x+x^2}\Big)^{\cos x} \\ \ln f &= \cos x\ln\Big(\frac{2+x\sin x}{e^x+x^2}\Big) = \cos x\ln\Big(2+x\sin x\Big) - \cos x\ln\Big(e^x+x^2\Big) \\ \frac{1}{f}f' &= -\sin x\ln\Big(2+x\sin x\Big) + \cos x\frac{\sin x + x\cos x}{2+x\sin x} + \sin x\ln\Big(e^x+x^2\Big) - \cos x\frac{e^x+2x}{e^x+x^2} \\ f' &= f\cdot\Bigg(-\sin x\ln\Big(2+x\sin x\Big) + \cos x\frac{\sin x + x\cos x}{2+x\sin x} + \sin x\ln\Big(e^x+x^2\Big) - \cos x\frac{e^x+2x}{e^x+x^2}\Bigg) \end{aligned}$$

2.3.2.20. **Def (Second derivative)** $\frac{\mathrm{d}^2y}{\mathrm{d}x} \coloneqq \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}$.

- Notation: y'' or $y^{(2)}$
- We also write y as $y^{(0)}$

2.3.2.21. **E.g.** $\frac{\mathrm{d}^2}{\mathrm{d}x^2}x^3 = 6x$

2.3.2.22. ⊩**(Leinbiz's Rule)**

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ and $k! = k(k-1)(k-2)\cdots 1$ for $k \in \mathbb{N}$

2.3.2.23. **E.g.**

- $(uv)^{(0)} = uv$
- $(uv)^{(1)} = u'v + v'u$
- $(uv)^{(2)} = u''v + 2u'v' + uv''$
- $(uv)^{(3)} = u'''v + 3u''v + 3u'v'' + v'''$

2.3.2.24. **E.g.** $(x^4 \sin x)^{(2)}$

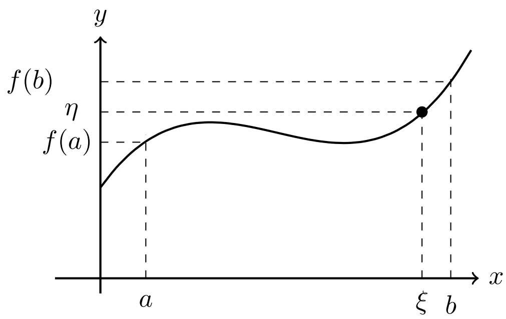
$$\begin{aligned} (x^4 \sin x)^{(2)} &= (x^4)^{(2)} \sin x + 2(x^4)^{(1)}(\sin x)^{(1)} + x^4(\sin x)^{(2)} \\ &= 4(x^3)' \sin x + 2 \cdot 4x^3 \cos x + x^4(\cos x)' \\ &= 12x^2 \sin x + 8x^3 \cos x - x^4 \sin x \end{aligned}$$

2.3.3 Advanced topics

2.3.3.1. We are not touching all the proof in this section because of time constraint.

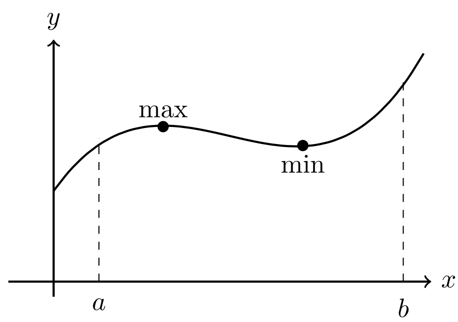
2.3.3.2. ⊩**Intermediate value theorem**

If $f : \mathbb{R} \to \mathbb{R}$ is continuous on a closed and bounded interval $[a, b]$, then for any number $\eta \in \big(f(a), f(b)\big)$, there exists $\xi \in (a, b)$ such that $f(\xi) = \eta$



2.3.3.3. ⊩**Extreme value theorem**

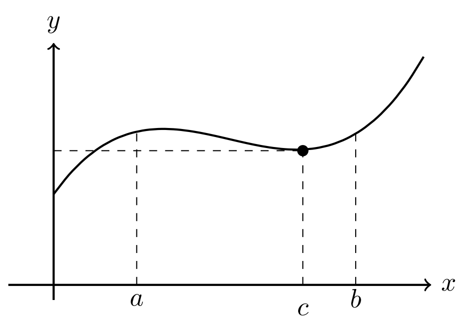
If $f : \mathbb{R} \to \mathbb{R}$ is continuous on a closed and bounded interval $[a, b]$, then there exists $\alpha, \beta \in [a, b]$ such that for any $x \in [a, b]$, we have $f(\alpha) \le f(x) \le f(\beta)$



2.3.3.4. ⊩**Rolle's Theorem**

$f : \mathbb{R} \to \mathbb{R}$ is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists at least one point c in (a, b) such that:

$$f'(c) = 0$$

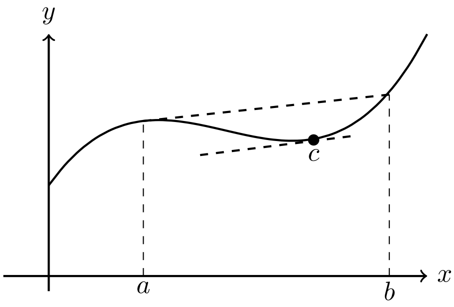


2.3.3.5. ⊩**Lagrangian's Mean Value Theorem**

Lagrange's MVT generalizes Rolle's Theorem. It states that if f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one point c in (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Rolle's Theorem can be seen as a special case of Lagrange's MVT where $f(a) = f(b)$.

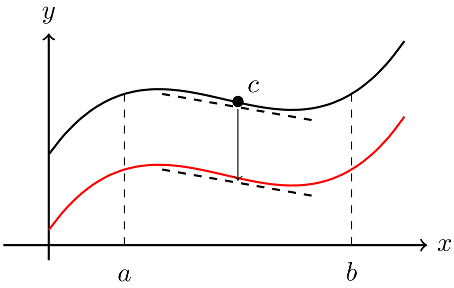


2.3.3.6. ⊩**Cauchy's Mean Value Theorem**

Cauchy's Mean Value Theorem is a further generalization of Lagrange's MVT. It states that if two functions f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $g'(x) \neq 0$ for all x in (a, b), then there exists at least one point c in (a, b) such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

By setting $g(x) = x$, Cauchy's MVT reduces to Lagrange's MVT.



2.3.3.7. ⊩**L'Hospital's Rule** L'Hospital's Rule can be derived using Cauchy's MVT. L'Hospital's Rule states that if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\pm\infty$, and f and g are differentiable near a with $g'(x) \neq 0$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right-hand side exists.

2.3.3.8. **E.g.** Indeterminate forms

$$\infty - \infty \quad \frac{\infty}{\infty} \quad \frac{0}{0} \quad 0^{\infty} \quad \infty^{0}$$

2.3.3.9. **E.g.** Indeterminate form $\frac{0}{0}$

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x - x\cos x}{x^3} &= \lim_{x\to 0} \frac{\cos x + x\sin x - \cos x}{3x^2} \quad LH \\
&= \lim_{x\to 0} \frac{\sin x}{3x} = \frac{1}{3} \underbrace{\lim_{x\to 0} \frac{\sin x}{x}}_{=1 \text{ by } 2.2.3.12.} = \frac{1}{3}
\end{aligned}$$

2.3.3.10. **E.g.** Indeterminate form $\frac{\infty}{\infty}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(1+x^4)}{\ln(1+x^2)} &= \lim_{x\to\infty} \frac{\frac{4x^3}{1+x^4}}{\frac{2x}{1+x^2}} \quad LH \\
&= 2\lim_{x\to\infty} \frac{x^2(1+x^2)}{1+x^4} = 2\lim_{x\to\infty} \frac{x^2+x^4}{1+x^4} = 2\lim_{x\to\infty} \frac{\frac{1}{x^2}+1}{\frac{1}{x^4}+1} = 2\frac{0+1}{0+1} = 2
\end{aligned}$$

2.3.3.11. **E.g.** Indeterminate form $\infty - \infty$

$$\begin{aligned}
\lim_{x\to 1} \frac{1}{\ln x} - \frac{1}{x-1} = \lim_{x\to 1} \frac{(x-1) - \ln x}{(x-1)\ln x} &= \lim_{x\to 1} \frac{1 - \frac{1}{x}}{\frac{x-1}{x} + \ln x} \quad LH \\
&= \lim_{x\to 1} \frac{x-1}{x - 1 + x\ln x} \\
&= \lim_{x\to 1} \frac{1}{1 + 1 + \ln x} \quad LH \\
&= \frac{1}{2}
\end{aligned}$$

2.3.3.12. **E.g.** Indeterminate form ∞^0

$$\begin{aligned}
&\lim_{x\to\infty}(1+2x)^{\frac{1}{3\ln x}} \\
\ln\left(\lim_{x\to\infty}(1+2x)^{\frac{1}{3\ln x}}\right) &= \lim_{x\to\infty}\frac{\ln(1+2x)}{3\ln x} \\
&= \lim_{x\to\infty}\frac{2/(1+2x)}{3/x} && LH \\
&= \lim_{x\to\infty}\frac{2x}{3(1+2x)} = \lim_{x\to\infty}\frac{2}{3(1/x+2)} = \frac{1}{3} \\
\lim_{x\to\infty}(1+2x)^{\frac{1}{3\ln x}} &= e^{1/3}
\end{aligned}$$

2.3.3.13. **E.g.** Indeterminate form 0^0

$$\begin{aligned}
\lim_{x\to 0^+} x^x &= \lim_{x\to 0^+}\exp\Big(x\ln(x)\Big) \\
&= \exp\Big(\lim_{x\to 0^+} x\ln(x)\Big) && \text{by 2.2.3.26.} \\
&= \exp\Big(\lim_{x\to 0^+}\frac{\ln(x)}{\frac{1}{x}}\Big) \\
&\overset{LH}{=} \exp\Big(\lim_{x\to 0^+}\frac{1/x}{\frac{-1}{x^2}}\Big) \\
&= \exp\Big(\lim_{x\to 0^+} -x\Big) = e^0 = 1
\end{aligned}$$

2.3.3.14. **E.g.** Wrong use of LH

$$\lim_{x\to\infty}\frac{5x-2\cos^2 x}{3x+\sin^2 x} \overset{LH}{=} \lim_{x\to\infty}\frac{5-2\sin x\cos x}{3+\sin x\cos x} \overset{LH}{=} \lim_{x\to\infty}\frac{2(-\sin^2 x+\cos^2 x)}{-\sin^2 x+\cos^2 x} = 2$$

Why wrong: the limit on top and limit on bottom both do not exist.

Correct way

$$\lim_{x\to\infty}\frac{5x-2\cos^2 x}{3x+\sin^2 x} = \lim_{x\to\infty}\frac{5-2\dfrac{\cos^2 x}{x}}{3+\dfrac{\sin^2 x}{x}}$$

Both $\displaystyle\lim_{x\to\infty}\frac{\cos^2 x}{x} = \lim_{x\to\infty}\frac{\sin^2 x}{x} = 0$ because of squeeze theorem, hence the limit is 5/3

2.3.3.15. Relationship Between Mean Value Theorems and L'Hospital's Rule

- **Rolle's Theorem** is a special case of **Lagrange's MVT**.

- **Lagrange's MVT** is a special case of **Cauchy's MVT**.

- **L'Hospital's Rule** is derived using **Cauchy's MVT**.

Thus, we can visualize the hierarchy as:

$$\text{Rolle's Theorem} \subset \text{Lagrange's MVT} \subset \text{Cauchy's MVT}$$

2.3.4 Taylor series

2.3.4.1. The idea of expansion: given a function $f(x)$, you "expand" f by writing f as a sum $f(x) = a(x)+b(x)+c(x)+d(x)+\cdots$

- You already know some expansions in number
  - $2 = \displaystyle\sum_{k=0}^{\infty}\frac{1}{2^k} = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$
  - $e = \displaystyle\sum_{k=0}^{\infty}\frac{1}{k!} = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$
- Here are some "not so trivial" expansion
  - (Euler) $\displaystyle\frac{\pi^2}{6} = \sum_{k=1}^{\infty}\frac{1}{k^2} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$

- $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$
- (Ramanujan) $\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$

There are people spent their whole life on these things, they are called analytic number theorist.

2.3.4.2. Taylor's series is the "function version" of expansion
If $f(x)$ is n-times differentiable, then, at $x = a$, it has a Taylor series

$$f(x) = f(a+u) + u\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x=a} + \frac{u^2}{2!}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\Big|_{x=a} + \frac{u^3}{3!}\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\Big|_{x=a} + \cdots$$

with $u = x - a$

2.3.4.3. **E.g.**

- Taylor polynomial for e^x for x near 0

$$e^x \approx \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

- Taylor polynomial for $\cos(x)$ for any x

$$\cos(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!}$$

- Taylor polynomial for $\sin(x)$ for any x

$$\sin(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

- Taylor polynomial for $\ln(1+x)$ for $x \in (-1, 1]$

$$\ln(1+x) \approx \sum_{k=1}^{n} \frac{(-1)^{k+1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} x^n}{n}$$

- Taylor polynomial for $\frac{1}{1-x}$ for x near 0 (this is also known as geometric series)

$$\frac{1}{1-x} \approx \sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 + \cdots + x^n$$

- (Hard) Taylor polynomial for $\sqrt{1+x}$ for x near 0

$$\sqrt{1+x} \approx \sum_{k=0}^{n} \binom{1/2}{k} x^k = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots + \binom{1/2}{n} x^n$$

where $\binom{1/2}{k} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(k-1)\right)}{k!}$ is the generalized bionomial coefficient

2.3.4.4. **E.g.**

- Taylor polynomial for $e^x atx = c$

$$e^x \approx \sum_{k=0}^{n} \frac{(x-c)^k}{k!} e^c = e^c \left(1 + (x-c) + \frac{(x-c)^2}{2!} + \frac{(x-c)^3}{3!} + \cdots + \frac{(x-c)^n}{n!}\right)$$

- Taylor polynomial for $\cos(x)$ at $x = c$

$$\cos(x) \approx \sum_{k=0}^{n} \frac{(-1)^k (x-c)^{2k}}{(2k)!} \cos(c) = \cos(c) - \frac{(x-c)^2}{2!} \sin(c) + \frac{(x-c)^4}{4!} \cos(c) - \cdots + \frac{(-1)^n (x-c)^{2n}}{(2n)!} \cos(c)$$

- Taylor polynomial for $\sin(x)$ at $x = c$

$$\sin(x) \approx \sum_{k=0}^{n} \frac{(-1)^k (x-c)^{2k+1}}{(2k+1)!} \sin(c) = \sin(c) + (x-c)\cos(c) - \frac{(x-c)^3}{3!}\sin(c) + \cdots + \frac{(-1)^n (x-c)^{2n+1}}{(2n+1)!}\sin(c)$$

- Taylor polynomial for $\ln(1+x)$ at $x = c$

$$\ln(1+x) \approx \sum_{k=1}^{n} \frac{(-1)^{k+1}(x-c)^k}{k} \ln(1+c) = \ln(1+c) + (x-c) - \frac{(x-c)^2}{2} + \frac{(x-c)^3}{3} - \cdots + \frac{(-1)^{n+1}(x-c)^n}{n}$$

- Taylor polynomial for $\dfrac{1}{1-x}$ at $x = c$

$$\frac{1}{1-x} \approx \sum_{k=0}^{n} (x-c)^k \frac{1}{1-c} = \frac{1}{1-c} + (x-c) + (x-c)^2 + (x-c)^3 + \cdots + (x-c)^n$$

- Taylor polynomial for $\sqrt{1+x}$ at $x = c$

$$\sqrt{1+x} \approx \sum_{k=0}^{n} \binom{1/2}{k} (x-c)^k \sqrt{1+c} = \sqrt{1+c} + \frac{1}{2}(x-c) - \frac{1}{8}(x-c)^2 + \frac{1}{16}(x-c)^3 - \cdots + \binom{1/2}{n}(x-c)^n$$

where $\binom{1/2}{k} = \dfrac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(k-1)\right)}{k!}$ is the generalized binomial coefficient

2.3.4.5. Extra content, not in exam

- Existence of Taylor's series
- Error term in Taylor's series
- Taylor's Remainder Theorem

2.3.4.6. **Berstein's polynomial**

$$B_n(f, x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) b_j^n(x)$$

Bernstein basis polynomial

$$b_j^n(x) = \binom{n}{j} x^j (1-x)^{n-j}$$

2.3.4.7. ⊩**Universal Approximation Theorem using Bernstein polynomials** Let f be a continuous function on the interval $[0, 1]$.

As n approaches infinity, the Bernstein polynomial $B_n(f, x)$ converges uniformly to $f(x)$ on $[0, 1]$:

$$\lim_{n\to\infty} B_n(f, x) = f(x).$$

2.4 Integration

2.4.1 Indefinite Integration

Learning Objectives

- Understand antiderivative
- Understand and apply operations in indefinite integral in examples
- Understand definite integral

Primitive function

2.4.1.1. The task: suppose we know f is the derivative of a function F, find F.

- This problem is the inverse problem of differentiation.
- f is called the derivative of F, we write $F'(x) = f(x)$
- F is called the primitive function or antiderivative of f, we write $F(x) + C = \displaystyle\int f(x)\mathrm{d}x$

- The following terms are the same: primitive function, antiderivative, inverse derivative, primitive integral, indefinite integral

2.4.1.2. **Def (Primitive function)** On an interval I, if $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$, we call $F(x)$ the primitive function of f on I.

2.4.1.3. When we look at an mathematical object, we immediately should think of two things: (1) when does it exists, and (2) is it uniquely exists? The following theorem give a negative result to the latter.

2.4.1.4. ⊩**Primitive function not unique** If $F(x)$ is the primitive function of $f(x)$ on I, then every function $F(x) + C$ where C is a constant, is a primitive function of f.

- The theorem tells that if we have $F(x)$, we have all the whole class of primitive functions of $f(x)$.
- Since $C \in \mathbb{R}$ has infinitely many possible values, there are infinitely many primitive functions.

Proof. Let $G(x)$ be an arbitrary primitive function of f. Then by the definition of primitive function 2.4.1.2., we have $\frac{\mathrm{d}}{\mathrm{d}x}G(x) = f(x)$. By assumption in the theorem we have $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$, hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(G(x) - F(x)\Big) \;=\; \frac{\mathrm{d}}{\mathrm{d}x}G(x) - \frac{\mathrm{d}}{\mathrm{d}x}F(x) \;=\; f(x) - f(x) \;\equiv\; 0, \quad \forall x \in I.$$

Now by the formula $\frac{\mathrm{d}}{\mathrm{d}x}C \equiv 0$, we have $G(x) - F(x) = C$, and thus $G(x) = F(x) + C$. □

2.4.1.5. **Def (Indefinite integral)** If $F(x)$ is a primitive function of f, then all the primitive functions of f has the general form $F(x) + C$, which we call it the indefinite integral / antiderivative of f, and we write

$$\int f(x)\mathrm{d}x = F(x) + C,$$

where $\int$ is call the integral symbol, function $f(x)$ is called the integrand, the symbol $\mathrm{d}x$ is called the differential of the variable x, and C is called integration constant.

2.4.1.6. Do all functions have antiderivative? No. So we need to have a theorem to tell under what condition a function f has antiderivative.

2.4.1.7. ⊩**Existence of antiderivative** If f is a continuous function on the interval $[a, b]$, then we can find a function F defined on $[a, b]$ such that $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$ for all $\boldsymbol{x} \in [a, b]$.

ACtually this is the *Fundamental theorem of calculus*: $F(x) = \int_a^x f(t)\mathrm{d}t$, not our focus now but later.

2.4.1.8. **E.g.** The function $f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ has no antiderivative.

2.4.1.9. Some functions have antiderivative but not elementary. For example, $f(x) = e^{-x^2}$, you cannot find an elementary function $F(x)$ that $\frac{\mathrm{d}}{\mathrm{d}x}F(x)$ gives you e^{-x^2}, but the antiderivative of e^{-x^2} exsits.

Here are some functions with no antiderivative.

$$\sqrt{1 - x^4}, \quad e^{-x^2}, \quad e^{e^x}, \quad \frac{e^{-x}}{x}, \quad \sin(x^2), \quad \cos(x^2), \quad \frac{1}{\ln x}, \quad \ln(\ln x)$$

2.4.1.10. Now we discussed $\begin{cases} \text{What is the definition of antiderivative} \\ \text{The uniqueness of antiderivative} \\ \text{The existence of antiderivative} \end{cases}$.

What's next: how to actually calculate antiderivative / indefinite integral.

2.4.1.11. **(What makes integration harder than differentiation)** Notice that for the definition of indefinite integral 2.4.1.5., we are using f to define F.

- It is an indirect definition. We are not defining F directly, we definite F using another object.
- The definition 2.4.1.5. tells nothing on how to compute antiderivaitive.

2.4.1.12. **Derivative and antiderivative are inverse to each other.**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\mathrm{d}x\right) = f(x). \qquad \text{(Integrate first, then derivative, the two cancel out)}$$

$$\int\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)\mathrm{d}x = f(x)+C. \qquad \text{(Derivative first, then integrate, the two cancel out with a } +C\text{)}$$

2.4.1.13. **Antiderivative is a linear operator.** On the same interval I, if $f(x)$, $g(x)$ have antiderivative, then

$$\int\big(af(x)+bg(x)\big)\mathrm{d}x \;=\; a\int f(x)\mathrm{d}x + b\int g(x)\mathrm{d}x, \qquad \text{where } a,b \text{ not both zero.}$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(a\int f(x)\mathrm{d}x + b\int g(x)\mathrm{d}x\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(a\int f(x)\mathrm{d}x\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(b\int g(x)\mathrm{d}x\right) && \Big|\ \text{derivative is linear}\\
&= a\frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\mathrm{d}x\right)+b\frac{\mathrm{d}}{\mathrm{d}x}\left(\int g(x)\mathrm{d}x\right) && \Big|\ \frac{\mathrm{d}}{\mathrm{d}x}c\phi(x) = c\frac{\mathrm{d}}{\mathrm{d}x}\phi(x)\\
&= af(x)+bg(x) && \Big|\ \text{by 2.4.1.12.}
\end{aligned}$$

So by Definition 2.4.1.2., $\frac{\mathrm{d}}{\mathrm{d}x}\left(a\int f(x)\mathrm{d}x + b\int g(x)\mathrm{d}x\right)$ is the primitive function of $af(x)+bg(x)$. ☐

2.4.1.14. **E.g. What you learnt in high-school is wrong** $\int \frac{1}{x}\mathrm{d}x \neq \ln|x|+C$.

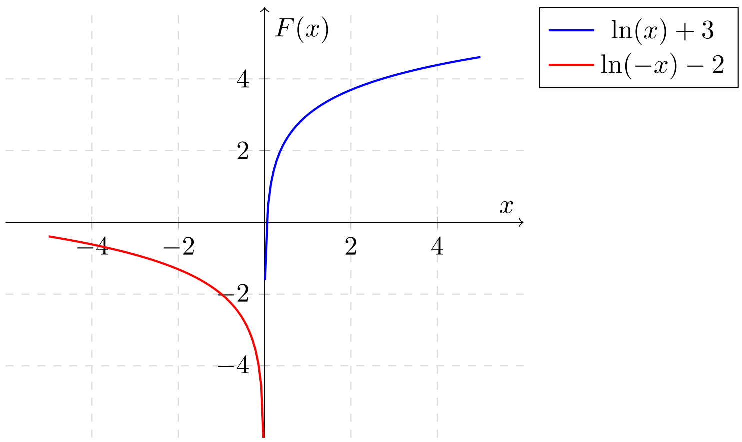
This $F(x)$ also give $\frac{1}{x}$ after $\frac{\mathrm{d}}{\mathrm{d}x}$

$$F(x) = \begin{cases}\ln(x)+3 & x>0\\ \ln(-x)-2 & x<0\end{cases}$$

In general

$$F(x) = \begin{cases}\ln(x)+C_1 & x>0\\ \ln(-x)-C_2 & x<0\end{cases}$$

Nothing forces $C_1 = C_2$. They can be different. Why $\ln|x|+C$ is wrong: it assumes $C_1 = C_2$, thus it is only *a subset* of all the possible primitive functions



Revision of differentiation: $\frac{\mathrm{d}}{\mathrm{d}x}\ln(-x) = -\frac{\mathrm{d}}{\mathrm{d}(-x)}\ln(-x) = -\frac{\mathrm{d}}{\mathrm{d}u}\ln(u) = -\frac{1}{u} = -\frac{1}{-x} = \frac{1}{x}$

2.4.1.15. **Basic formula of indefinite integral**

1 $\displaystyle\int x^n\mathrm{d}x = \begin{cases}\frac{1}{n+1}x^{n+1}+C, & (n\neq -1)\\ \begin{cases}\ln(x)+C_1 & x>0\\ \ln(-x)-C_2 & x<0\end{cases}, & (n=-1)\end{cases}$

5 $\displaystyle\int e^x\mathrm{d}x = e^x+C$

6 $\displaystyle\int \sin x\mathrm{d}x = -\cos x + C$

7 $\displaystyle\int \cos x\mathrm{d}x = \sin x + C$

8 $\displaystyle\int \sec^2 x\mathrm{d}x = \tan x + C$

9 $\displaystyle\int \csc^2 x\mathrm{d}x = -\cot x + C$

10 $\displaystyle\int \sec x\tan x\mathrm{d}x = \sec x + C$

11 $\displaystyle\int \csc x\cot x\mathrm{d}x = -\csc x + C$

12 $\displaystyle\int \frac{1}{1-x^2}\mathrm{d}x = \sin^{-1}x + C$

13 $\displaystyle\int \frac{1}{1+x^2}\mathrm{d}x = \tan^{-1}x + C$

2.4.1.16. **E.g.** $\int x^3 \mathrm{d}x = \frac{1}{4}x^4 + C$

2.4.1.17. **E.g.** $\int 2^x \mathrm{d}x = \frac{2^x}{\ln 2} + C$

2.4.1.18. **E.g.** $\int (e^x + 2\sin x)\mathrm{d}x = \int e^x \mathrm{d}x + 2\int \sin x \mathrm{d}x = e^x - \cos x + C$

2.4.1.19. **E.g.** $\int x\Big(\sqrt{x} - \frac{2}{x^2}\Big)\mathrm{d}x = \int \Big(x^{3/2} - \frac{2}{x}\Big)\mathrm{d}x = \int x^{3/2}\mathrm{d}x - 2\int \frac{1}{x}\mathrm{d}x = \frac{x^{3/2+1}}{\frac{3}{2}+1} - 2\ln|x| + C = \frac{2}{5}x^{5/2} - 2\ln|x| + C$

2.4.1.20. **E.g.** $\int \frac{x^2}{1+x^2}\mathrm{d}x = \int \frac{1+x^2-1}{1+x^2}\mathrm{d}x = \int 1\mathrm{d}x - \int \frac{1}{1+x^2}\mathrm{d}x = x - \tan^{-1} x + C$

2.4.1.21. **E.g.** $\int \cot^2 \mathrm{d}x = \int (\csc^2 - 1)\mathrm{d}x = -\cot x - x + C$

2.4.1.22. **E.g.** $\int \frac{1}{\sin^2 x \cos^2 x}\mathrm{d}x = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\mathrm{d}x = \int \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}\mathrm{d}x = -\cot x + \tan x + C$

2.4.1.23. **E.g.** $\int \frac{1-\cos x}{1-\cos 2x}\mathrm{d}x$

$$\int \frac{1-\cos x}{1-\cos 2x}\mathrm{d}x = \int \frac{1-\cos x}{2\sin^2 x}\mathrm{d}x = \frac{\int \Big(\csc^2 x - \cos x \csc^2 x\Big)\mathrm{d}x}{2} = \frac{-\cot x - \int \cot x \csc x \mathrm{d}x}{2} = \frac{-\cot x + \csc x}{2} + C$$

2.4.1.24. **Change of variable** Let $f(x)$ is continuous. Suppose $x = g(t)$ has continuous derivative, then

$$\int f\big(g(t)\big)g'(t)\mathrm{d}t \;=\; \int f(g(t))dg(t) \overset{x=g(t)}{=} \int f(x)\mathrm{d}x.$$

2.4.1.25. **E.g.** $\int x\sqrt{x^2+4}\mathrm{d}x$

Let $t = x^2 + 4$, then $\mathrm{d}t = 2x\mathrm{d}x$. $\int x\sqrt{x^2+4}\mathrm{d}x = \frac{1}{2}\int 2x\sqrt{x^2+4}\mathrm{d}x = \frac{1}{2}\int \sqrt{t}\mathrm{d}t = \frac{1}{3}t^{3/2} + C = \frac{(x^2+4)^{3/2}}{3} + C$

2.4.1.26. **E.g.** $\int 2xe^{x^2}\mathrm{d}x = \int e^{x^2}\mathrm{d}x^2 \overset{t=x^2}{=} \int e^t \mathrm{d}t = e^t + C \overset{t=x^2}{=} e^{x^2} + C$

2.4.1.27. **E.g.** $\int \frac{\ln x}{x}\mathrm{d}x = \int \ln x d\ln x \overset{t=\ln x}{=} \int t\mathrm{d}t = \frac{1}{2}t^2 + C \overset{t=\ln x}{=} \frac{(\ln x)^2}{2} + C$

2.4.1.28. **E.g.** $\int \frac{\tan^{-1} x}{1+x^2}\mathrm{d}x = \int \tan^{-1} x \mathrm{d}\tan^{-1} x = \frac{1}{2}(\tan^{-1} x)^2 + C$

2.4.1.29. **E.g.** $\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{-1}{\cos x}\mathrm{d}\cos x = -\ln|\cos x| + C$

2.4.1.30. **E.g.** $\int \frac{1}{1+\sqrt{x}}\mathrm{d}x$.
Let $t = \sqrt{x}$ hence $x = t^2$ and $\mathrm{d}x = 2t\mathrm{d}t$, we have
$\int \frac{1}{1+\sqrt{x}}\mathrm{d}x \overset{t=\sqrt{x}}{=} \int \frac{1}{1+t}\underbrace{2t\mathrm{d}t}_{\mathrm{d}x} = \int 2 - \frac{2}{1+t}\mathrm{d}t = 2t - 2\ln(1+t) + C \overset{t=\sqrt{x}}{=} 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C.$

2.4.1.31. **E.g.** $\int \frac{1}{\sqrt{e^x-1}}\mathrm{d}x$. Let $t = \sqrt{e^x - 1} > 0$ hence $x = \ln(1+t^2)$ and $\mathrm{d}x = 2t\frac{1}{1+t^2}\mathrm{d}t$, we have
$\int \frac{1}{\sqrt{e^x-1}}\mathrm{d}x \overset{t=\sqrt{e^x-1}>0}{=} \int \frac{1}{t}\underbrace{\frac{2t}{1+t^2}\mathrm{d}t}_{\mathrm{d}x} = 2\int \frac{\mathrm{d}t}{1+t^2} = 2\tan^{-1} t + C \overset{t=\sqrt{e^x-1}>0}{=} 2\tan^{-1}\sqrt{e^x-1} + C.$

2.4.1.32. $\int \frac{1}{a^2-x^2}\mathrm{d}x = \int \frac{1}{a(1-(\frac{x}{a})^2)}\mathrm{d}x = \int \frac{1}{1-(\frac{x}{a})^2}d\frac{x}{a} \overset{t=x/a}{=} \sin^{-1} t + C \overset{t=x/a}{=} \sin^{-1}\frac{x}{a} + C$

2.4.1.33. $\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x, a > 0.$

This one is a little bit complicated. First

$$\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \int \frac{\mathrm{d}x}{\sqrt{a^2(1-\frac{x^2}{a^2})}} = \int \frac{\mathrm{d}x}{|a|\sqrt{1-(\frac{x}{a})^2}} \overset{a\geq 0}{=} \int \frac{\mathrm{d}x}{a\sqrt{1-(\frac{x}{a})^2}}.$$

Let $t = \frac{x}{a}$, thus $x = at$ and $\mathrm{d}x = a\mathrm{d}t$

$$\int \frac{\mathrm{d}x}{a\sqrt{1-(\frac{x}{a})^2}} = \int \frac{a\mathrm{d}t}{a\sqrt{1-t^2}} = \int \frac{\mathrm{d}t}{\sqrt{1-t^2}}.$$

Now let $t = \sin\theta$, then $\mathrm{d}t = \cos\theta d\theta$

$$\int \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \int \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta d\theta}{\sqrt{\cos^2\theta}} = \int \frac{\cos\theta d\theta}{\cos\theta} = \int d\theta = \theta + C.$$

Now we go back from θ to t and then to x.

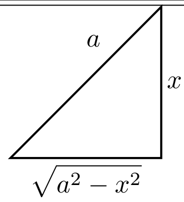
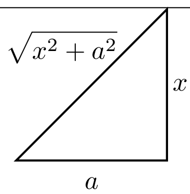
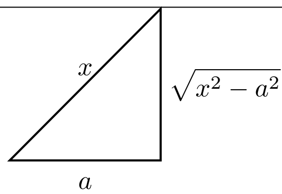
$$\theta + C \overset{t=\sin\theta}{=} \sin^{-1} t + C \overset{t=x/a}{=} \sin^{-1}\frac{x}{a} + C.$$

So in the whole problem we used change of variable twice.

2.4.1.34. Change of variable for square root: for $\sqrt{ax+b}$, let $t = \sqrt{ax+b}$ and thus $x = \frac{t^2-b}{a}$ and $\mathrm{d}x = \frac{2t\mathrm{d}t}{a}$

2.4.1.35. Trigonometric change of variable

- If you see $a^2 - x^2$, let $x = a\sin\theta$ for using $\sin^2\theta + \cos^2\theta = 1$
- If you see $a^2 + x^2$, let $x = a\tan\theta$ for using $\tan^2\theta + 1 = \sec^2\theta$
- If you see $x^2 - a^2$, let $x = a\sec\theta$ for using $\tan^2\theta + 1 = \sec^2\theta$

Expression	Substitution	$\mathrm{d}x$	Trigonometric ratios
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$\mathrm{d}x = a\cos\theta\mathrm{d}\theta$	 $\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$ $\sin\theta = \frac{x}{a}$ $\tan\theta = \frac{x}{\sqrt{a^2-x^2}}$
$\sqrt{x^2+a^2}$	$x = a\tan\theta$	$\mathrm{d}x = a\sec^2\theta\mathrm{d}\theta$	 $\cos\theta = \frac{a}{\sqrt{a^2+x^2}}$ $\sin\theta = \frac{x}{\sqrt{a^2+x^2}}$ $\tan\theta = \frac{x}{a}$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\mathrm{d}x = a\sec\theta\tan\theta\mathrm{d}\theta$	 $\cos\theta = \frac{a}{x}$ $\sin\theta = \frac{\sqrt{x^2-a^2}}{x}$ $\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$

2.4.1.36. Trigonometric change of variable

$$\begin{aligned}
\int \sin^k x\cos^{2m+1} x\mathrm{d}x &= \int \sin^k x\cos^{2m} x\cos x\mathrm{d}x \\
&= \int \sin^k x\cos^{2m} x\mathrm{d}\sin x && \cos x\mathrm{d}x = \mathrm{d}\sin x \\
&= \int \sin^k x\big(\cos^2 x\big)^m\mathrm{d}\sin x \\
&= \int \sin^k x\big(1-\sin^2 x\big)^m\mathrm{d}\sin x && \cos^2 x + \cos^2 x = 1 \\
&= \int u^k(1-u^2)^m\mathrm{d}u && \sin x = u \\
&= \int u^k\Big(1-\binom{m}{1}u^2+\binom{m}{2}u^4+\cdots\Big)\mathrm{d}u && \text{Binomial expansion} \\
&= \frac{1}{k+1}u^{k+1}-\frac{1}{k+3}\binom{m}{1}u^{k+3}+\frac{1}{k+5}\binom{m}{2}u^{k+5}+\cdots+C \\
&= \frac{1}{k+1}\sin^{k+1}x-\frac{1}{k+3}\binom{m}{1}\sin^{k+3}x+\frac{1}{k+5}\binom{m}{2}\sin^{k+5}x+\cdots+C
\end{aligned}$$

Side note

$$\int x^a(1-x^2)^b\mathrm{d}x = \frac{x^{a+1}}{a+1}\,{}_2F_1\Big(\frac{a+1}{2},-b;\frac{a+3}{2};x^2\Big)$$

$${}_2F_1\Big(p,q;r;z\Big) = 1+\frac{pq}{r}\frac{z}{1!}+\frac{p(p+1)q(q+1)}{r(r+1)}\frac{z^2}{2!}+\cdots \qquad \text{(hypergeometric function)}$$

2.4.1.37. Trigonometric change of variable $\int \cos 3x\cos 2x\mathrm{d}x$

$$\begin{aligned}
&= \frac{1}{2}\int(\cos x+\cos 5x)\mathrm{d}x \qquad \cos A\cos B = \frac{\cos(A-B)+\cos(A+B)}{2} \\
&= \frac{1}{2}\sin x+\frac{1}{10}\sin 5x+C
\end{aligned}$$

2.4.1.38. Trigonometric change of variable $\int \csc x\mathrm{d}x = \int \frac{1}{\sin x}\mathrm{d}x$

$$= \int\frac{\sin x}{\sin^2 x}\mathrm{d}x = \int\frac{1}{\cos^2 x-1}\mathrm{d}\cos x = \int\frac{1}{u^2-1}\mathrm{d}u = \int\frac{1}{(u-1)(u+1)}\mathrm{d}u$$

$$= \int\frac{1}{u-1}-\frac{1}{u+1}\mathrm{d}u = \frac{1}{2}\Big(\ln|u-1|-\ln|u+1|\Big)+C = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+C = \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|+C$$

2.4.1.39. Trigonometric change of variable $\int \csc x\mathrm{d}x = \int \frac{1}{\sin x}\mathrm{d}x$

$$= \int\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\mathrm{d}x = \int\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\frac{\frac{1}{\cos^2\frac{x}{2}}}{\frac{1}{\cos^2\frac{x}{2}}}\mathrm{d}x = \int\frac{\sec^2\frac{x}{2}}{\tan\frac{x}{2}}d\frac{x}{2} = \int\frac{1}{\tan\frac{x}{2}}d\tan\frac{x}{2} = \ln\left|\tan\frac{x}{2}\right|+C$$

2.4.1.40. Trigonometric change of variable using complex number $\int \csc x \mathrm{d}x = \int \frac{1}{\sin x}\mathrm{d}x$

$$= \int \frac{1}{\dfrac{e^{ix}-e^{-ix}}{2i}}\mathrm{d}x \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$= 2\int \frac{1}{e^{ix}-e^{-ix}}dix = 2\int \frac{1}{e^{u}-e^{-u}}\mathrm{d}u = \int \frac{1}{e^{u}-e^{-u}}\frac{e^u}{e^u}\mathrm{d}u = 2\int \frac{1}{e^{2u}-1}de^u$$

$$= 2\int \frac{1}{s^2-1}ds = \ln\left|\frac{s-1}{s+1}\right| + C = \ln\left|\frac{e^{ix}-1}{e^{ix}+1}\right| + C = \ln\left|\frac{e^{ix}-1}{e^{ix}+1}\frac{e^{-ix/2}}{e^{-ix/2}}\frac{\frac{1}{2}}{\frac{1}{2}}\right| + C$$

$$= \ln\left|\frac{\dfrac{e^{ix/2}-e^{-ix/2}}{2}}{\dfrac{e^{ix/2}+e^{-ix/2}}{2}}\right| + C = \ln\left|\frac{\dfrac{e^{ix/2}-e^{-ix/2}}{2i}}{\cos\frac{x}{2}i}\right| + C = \ln\left|\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}i}\right| + C$$

$$= \ln\left|\tan\frac{x}{2}\cdot\frac{1}{i}\right| + C = \ln\left|\tan\frac{x}{2}\cdot\frac{1}{i}\cdot\frac{i}{i}\right| + C$$

$$= \ln\left|\tan\frac{x}{2}\cdot(-1)\cdot i\right| + C = \ln\left|\tan\frac{x}{2}\right| + C \qquad |a+bi| = \sqrt{a^2+b^2} \overset{a=0}{=} \sqrt{b^2} = |b|$$

2.4.1.41. Trigonometric change of variable, $\int \frac{1}{x^2+1}\mathrm{d}x$
Let $x = \tan t$, thus $x^2+1 = \sec^2 t$ and $\mathrm{d}x = \sec^2 t\mathrm{d}t$.

$$\int \frac{1}{x^2+1}\mathrm{d}x = \int \frac{\sec^2 t}{\sec^2 t}\mathrm{d}t = \int \mathrm{d}t = t + C = \tan^{-1} x + C$$

In general

$$\int \frac{1}{x^2+a^2}\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

2.4.1.42. Trigonometric change of variable, $\int \frac{1}{\sqrt{x^2+1}}\mathrm{d}x$
Let $x = \tan t$, thus $x^2+1 = \sec^2 t$ and $\mathrm{d}x = \sec^2 t\mathrm{d}t$.

$$\int \frac{1}{\sqrt{x^2+1}}\mathrm{d}x = \int \frac{\sec^2 t}{\sec t}\mathrm{d}t = \int \sec t\mathrm{d}t = \int \sec t\cdot\frac{\sec t+\tan t}{\sec t+\tan t}\mathrm{d}t = \int \frac{\sec^2 t+\sec t\tan t}{\sec t+\tan t}\mathrm{d}t$$

Let $u = \sec t + \tan t$, then $\mathrm{d}u = (\sec^2 t + \sec t\tan t)\mathrm{d}t$

$$= \int \frac{1}{u}\mathrm{d}u = \ln|u| + C = \ln\left|\sec t + \tan t\right| + C$$

By $x = \tan t$ we have $\sec t = \sqrt{1+\tan^2 t} = \sqrt{1+x^2}$, hence

$$\int \frac{1}{\sqrt{x^2+1}}\mathrm{d}x = \ln\left|\sqrt{1+x^2}+x\right| + C$$

2.4.1.43. **Integration by parts** Let u, v both differentiable, then

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u \qquad \int uv'\mathrm{d}x = uv - \int u'v\mathrm{d}x$$

2.4.1.44. **E.g.** $\int xe^x \mathrm{d}x$

$$\begin{aligned} u &= x \\ v &= e^x \\ \mathrm{d}v &= e^x \mathrm{d}x \\ \mathrm{d}u &= \mathrm{d}x \\ \int xe^x \mathrm{d}x &= \int u \mathrm{d}v \\ &= xe^x - \int e^x \, \mathrm{d}x \\ &= xe^x - e^x + C \\ &= e^x(x-1) + C \end{aligned}$$

2.4.1.45. **E.g.** $\int x \sin(x) \, \mathrm{d}x$

$$\begin{aligned} u &= x \\ v &= -\cos x \\ \mathrm{d}v &= \sin x \mathrm{d}x \\ \mathrm{d}u &= \mathrm{d}x \\ \int x \sin(x) \mathrm{d}x &= -x\cos(x) + \int \cos(x) \, \mathrm{d}x \\ &= -x\cos(x) + \sin(x) + C \end{aligned}$$

2.4.1.46. **E.g.** $\int x^2 \ln(x) \, \mathrm{d}x$

$$\begin{aligned} u &= \ln x \\ v &= \frac{x^3}{3} \\ \mathrm{d}v &= x^2 \mathrm{d}x \\ \mathrm{d}u &= \frac{1}{x} \mathrm{d}x \\ \int x^2 \ln(x) \, \mathrm{d}x &= \int u \, \mathrm{d}v \\ &= \frac{x^3}{3} \ln(x) - \frac{1}{3} \int x^2 \, \mathrm{d}x \\ &= \frac{x^3}{3} \ln(x) - \frac{x^3}{9} + C \end{aligned}$$

2.4.1.47. **E.g.** $\int \ln(x) \, \mathrm{d}x$

$$\begin{aligned} \int \ln(x) \, \mathrm{d}x &= x \ln x - \int x \mathrm{d} \ln x \\ &= x \ln x - \int \mathrm{d}x \\ &= x \ln x - x + C \end{aligned}$$

2.4.1.48. **Integration of rational function**

$$\int \frac{1}{(u-1)(u+1)} \mathrm{d}u = \int \frac{1}{u-1} - \frac{1}{u+1} \mathrm{d}u = \frac{1}{2}\Big(\ln|u-1| - \ln|u+1|\Big) + C = \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C$$

- Partial fraction $\dfrac{1}{(u-1)(u+1)} = \dfrac{1}{u-1} - \dfrac{1}{u+1}$

2.4.1.49. **E.g.** $\dfrac{1}{(1-x)(1-2x)} = \dfrac{?}{1-x} + \dfrac{?}{1-2x}$

$$\frac{1}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x} \quad \Longleftrightarrow \quad 1 = A(1-2x) + B(1-x).$$

Put $x=1$ gives $A=-1$. Put $x=\frac{1}{2}$ gives $B=2$, so $\frac{1}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{2}{1-2x}$.

2.4.1.50. **E.g.** $\frac{17x-53}{x^2-2x-15} = \frac{?}{x-5} + \frac{?}{x+3}$

$$\frac{17x-53}{x^2-2x-15} = \frac{17x-53}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3} \iff 17x-53 = A(x+3)+B(x-5).$$

Put $x=5$ gives $A=4$ and put $x=-3$ gives $B=13$. Hence $\frac{17x-53}{x^2-2x-15} = \frac{4}{x-5} + \frac{13}{x+3}$

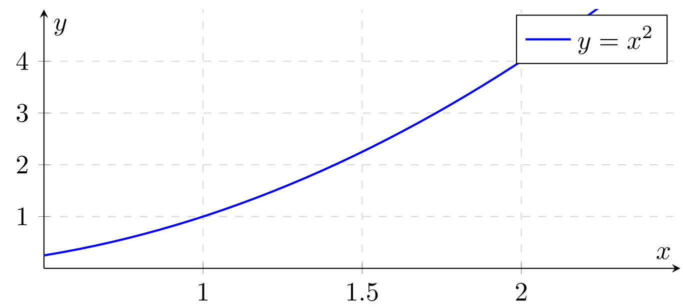
2.4.1.51. **E.g.** $\frac{10+35}{(x+4)^2} = \frac{10}{x+4} + \frac{-5}{(x+4)^2}$

$$\frac{10x+35}{(x+4)^2} = \frac{A}{x+4} + \frac{B}{(x+4)^2} \iff 10x+35 = A(x+4)+B.$$

Put $x=-4$ gives $B=-5$ and put $x=0$ with $B=-5$ gives $A=10$.

2.4.2 Definite Integration

2.4.2.1. We will focus on $f(x)=x^2$

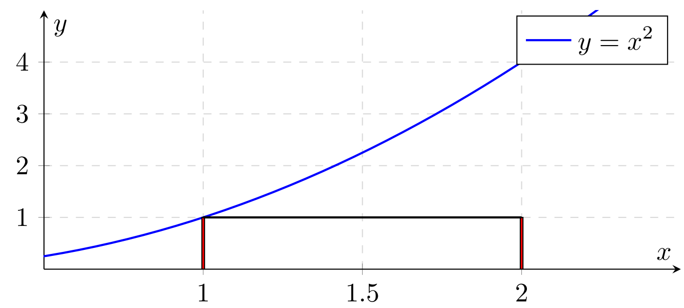


We want to find the area under the curve from $x=1$ to $x=2$.
We write it as

$$\int_1^2 x^2 \mathrm{d}x$$

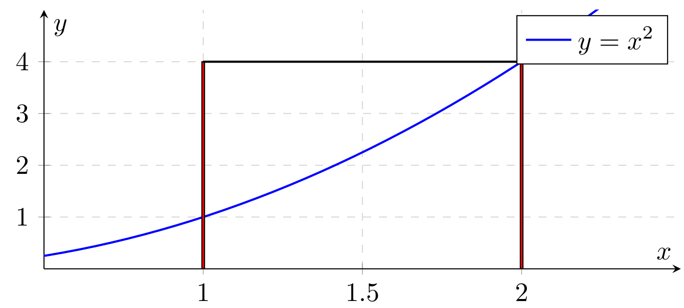
This is known as definite integral
We are expecting the area is a number, so this is "definite"

2.4.2.2. We know how to find the area of rectangle, so we start with rectangle approximation.



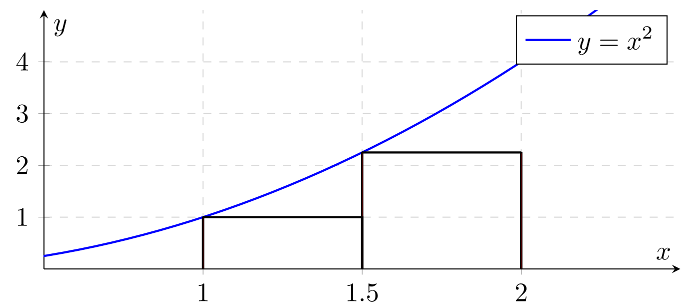
The rectangle has area $1\times 1 = 1$
The rectangle is below the curve, so area of rectangle is smaller than the area under the curve, thus

$$1 < \int_1^2 x^2 \mathrm{d}x$$



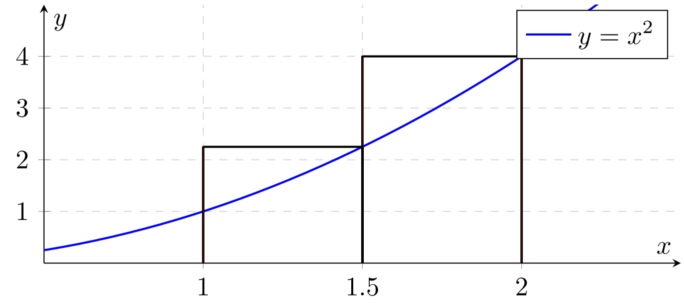
The rectangle has area $4\times 1 = 4$
The rectangle is above the curve, so area of rectangle is larger than the area under the curve, thus

$$1 < \int_1^2 x^2 \mathrm{d}x < 4$$



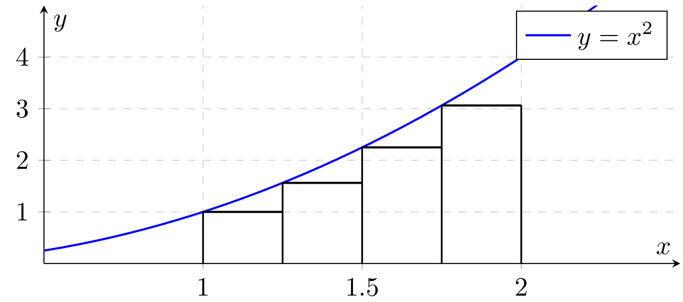
The rectangle-left has area $1\times 0.5 = 0.5$
The rectangle-right has area $1.5^2\times 0.5 = 1.125$
The rectangle is below the curve, so the area of all the rectangles is smaller than the area under the curve, thus

$$1.625 < \int_1^2 x^2 \mathrm{d}x < 4$$



The rectangle-left has area $1.5^2 \times 0.5 = 1.125$
The rectangle-right has area $2^2 \times 0.5 = 2$
The rectangle is above the curve, so the area of all the rectangles is grater than the area under the curve, thus

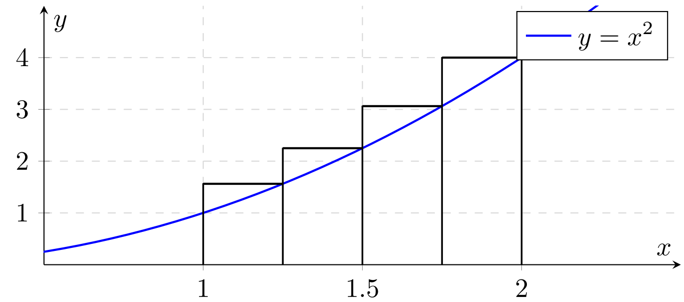
$$1.625 < \int_1^2 x^2 \mathrm{d}x < 3.125$$



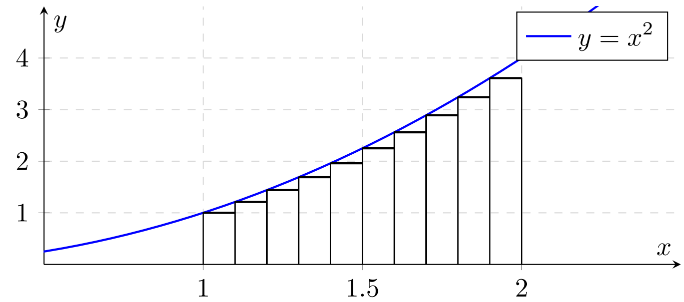
$$\begin{array}{r} 1^2 \cdot 0.25 \\ +1.25^2 \cdot 0.25 \\ +1.5^2 \cdot 0.25 \\ +1.75^2 \cdot 0.25 \end{array} < \int_1^2 x^2 \mathrm{d}x < 3.125$$

$$(1^2 + 1.25^2 + 1.5^2 + 1.75^2)0.25 < \int_1^2 x^2 \mathrm{d}x < 3.125$$

$$h = 0.25, \quad \sum_{i=1}^{4} (1 + (i-1)h)^2 h < \int_1^2 x^2 \mathrm{d}x < 3.125$$



$$h = 0.25, \; 1.96875 < \int_1^2 x^2 \mathrm{d}x < 2.71875$$



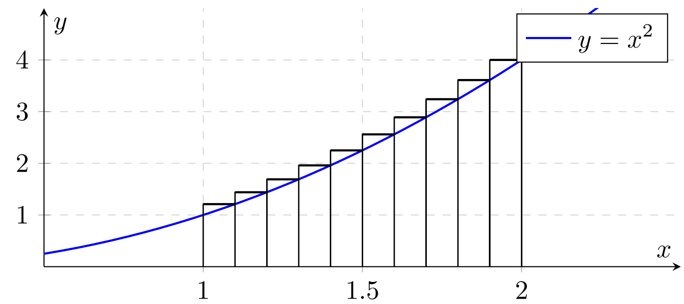
$$N = 10, \quad b = 2, \quad a = 1, \quad h = \frac{b-a}{N} = 0.1$$

$$\sum_{i=1}^{N} (a + (i-1)h)^2 h < \int_1^2 x^2 \mathrm{d}x < 2.71875$$

WolframAlpha code:

```
Sum[(x^2) * (0.1), {x, 1, 1.9, 0.1}]
```

$$2.185 < \int_1^2 x^2 \mathrm{d}x < 2.71875$$



$$N = 10, \quad b = 2, \quad a = 1, \quad h = \frac{b-a}{N} = 0.1$$

$$2.185 < \int_1^2 x^2 \mathrm{d}x < \sum_{i=1}^{N} (a + (i+1)h)^2 h$$

WolframAlpha code:

```
Sum[(x^2) * (0.1), {x, 1.1, 2, 0.1}]
```

$$2.185 < \int_1^2 x^2 \mathrm{d}x < 2.485$$

2.4.2.3. **(Revision) Squeeze Theorem** If sequences $a_n \leq b_n \leq c_n$ for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then b_n is convergent and $\lim_{n\to\infty} b_n = L$

- In the example, $\{a_n\}$ is the area under the curve for increasing number of rectangles

$$\{a_1, a_2, a_3, \cdots\} \;=\; \Big\{1,\; 1.625,\; 1.96875,\; \cdots\Big\}$$

- In the example, $\{c_n\}$ is the area above the curve for increasing number of rectangles

$$\{c_1, c_2, c_3, \cdots\} \;=\; \Big\{4,\; 3.125,\; 2.71875,\; \cdots\Big\}$$

- In the example, $\{b_n\}$ is the constant sequences

$$\{b_1, b_2, b_3, \cdots\} = \left\{\int_1^2 x^2 \mathrm{d}x,\ \int_1^2 x^2 \mathrm{d}x,\ \int_1^2 x^2 \mathrm{d}x,\ \cdots\right\}$$

- The Upper bound is usually denoted as $U(x)$ and the lower bound is usually denoted as $L(x)$. Now what we did is basically the function version of squeeze theorem: if $L(x) \le I(x) \le U(x)$ for all x, and $\lim_{x\to a} U(x) = \lim_{x\to a} L(x) = c$, then $\lim_{x\to a} I(x) = c$

2.4.2.4. **Riemann integral**

$$\int_a^b f(x)\mathrm{d}x := \lim_{N\to\infty} \sum_{i=1}^N f(a+ih)h, \qquad h = \frac{b-a}{N}$$

For $f(x) = x^2$ with $a = 1, b = 2$

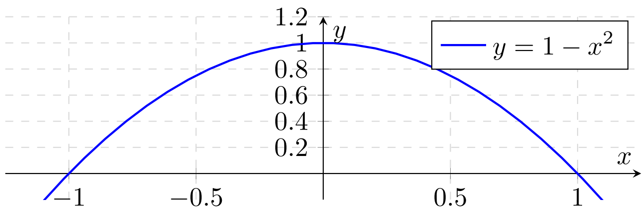
$$\int_1^2 x^2 \mathrm{d}x = \lim_{N\to\infty} \sum_{i=1}^N \left(1 + \frac{i}{N}(2-1)\right)^2 \frac{2-1}{N} = \lim_{N\to\infty} \sum_{i=1}^N \left(1+\frac{i}{N}\right)^2 \frac{1}{N} = \lim_{N\to\infty} \sum_{i=1}^N \left(1 + \frac{2i}{N} + \frac{i^2}{N^2}\right)\frac{1}{N}$$

$$\begin{aligned}
&= \lim_{N\to\infty} \sum_{i=1}^N \left(\frac{1}{N} + \frac{2i}{N^2} + \frac{i^2}{N^3}\right) = \lim_{N\to\infty} \left(\sum_{i=1}^N \frac{1}{N} + \sum_{i=1}^N \frac{2i}{N^2} + \sum_{i=1}^N \frac{i^2}{N^3}\right) \\
&= \lim_{N\to\infty} \left(\frac{1}{N}\sum_{i=1}^N 1 + \frac{2}{N^2}\sum_{i=1}^N i + \frac{1}{N^3}\sum_{i=1}^N i^2\right) \\
&= \lim_{N\to\infty} \left(\frac{1}{N}\cdot N + \frac{2}{N^2}\frac{N(N+1)}{2} + \frac{1}{N^3}\frac{N(N+1)(2N+1)}{6}\right) \\
&= 1 + \lim_{N\to\infty} \frac{N+1}{N} + \frac{1}{6}\lim_{N\to\infty} \frac{(N+1)(2N+1)}{N^2} \\
&= 1 + \lim_{N\to\infty} \frac{1+\frac{1}{N}}{1} + \frac{1}{6}\lim_{N\to\infty} \frac{(1+\frac{1}{N})(2+\frac{1}{N})}{1} = 1+1+\frac{2}{6} = \frac{7}{3}
\end{aligned}$$

In fact

$$\int_1^2 x^2 \mathrm{d}x = \frac{x^3}{3}\bigg|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8-1}{3} = \frac{7}{3} \approx 2.3333...$$

2.4.2.5. We now focus on $f(x) = 1 - x^2$



We want to find

$$\int_{-1}^1 (1-x^2)\mathrm{d}x$$

2.4.2.6. **Lebesgue integral (not in exam)**

- Riemann integral = draw vertical rectangles below/above the curve
  - we "chop" the x-axis, and then from x we find $y = f(x+h)$, lastly sum up the rectangle
  - we "start" at x-axis, look into y-axis
- Lebesgue integral = draw horizontal rectangles below/above the curve
  - we "chop" the y-axis, and then from y we find the pre-image x, lastly sum up the rectangle
  - we "start" at y-axis, look into x-axis

2.4.2.7. Lebesgue Integral of $\displaystyle\int_{-1}^1 (1-x^2)\mathrm{d}x$

- **Chopping y-axis** Divide range$f$ into N intervals. In chopping x-axis we used h, now in chopping y-axis, we use the symbol ϵ.
  Chop y-axis into intervals of length ϵ:

$$[0,\epsilon), [\epsilon, 2\epsilon), \ldots, [k\epsilon, (k+1)\epsilon), \ldots, [(N-1)\epsilon, N\epsilon)$$

The interval starts at 0 because we know the smallest y-value of f is 0. In general, if the range of y is $[c, d]$, then

$$\epsilon = \frac{d-c}{N}, \qquad [c, c+\epsilon), [c+\epsilon, c+2\epsilon), \dots, [c+(N-1)\epsilon, c+N\epsilon)$$

- **Finding preimage in x-axis** For each interval on the y-axis, we find the corresponding preimage on the x-axis.

  The preimage of an interval $[k\epsilon, (k+1)\epsilon)$ is the set of all x such that $k\epsilon \leq 1-x^2 < (k+1)\epsilon$.

  Here is how the x interval derived

$$\begin{array}{rcccl} k\epsilon & \leq & 1-x^2 & \leq & (k+1)\epsilon \\ k\epsilon - 1 & \leq & -x^2 & \leq & (k+1)\epsilon - 1 \\ -\Big(k\epsilon - 1\Big) & \geq & x^2 & \geq & -\Big((k+1)\epsilon - 1\Big) \\ 1 - k\epsilon & \geq & x^2 & \geq & 1-(k+1)\epsilon \\ 1-(k+1)\epsilon & \leq & x^2 & \leq & 1-k\epsilon \end{array}$$

  Here is tricky:

  - we need to assume $1-(k+1)\epsilon > 0$ because otherwise we cannot take square-root
  - $\sqrt{x^2} = |x|$, don't oversee it and just write $\sqrt{x^2} = x$

$$\sqrt{1-(k+1)\epsilon} \quad \leq \quad |x| \quad \leq \quad \sqrt{1-k\epsilon}$$

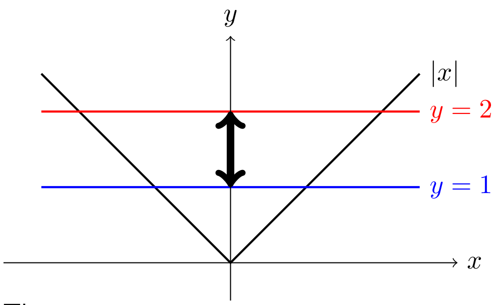
  Now we have two inequalities

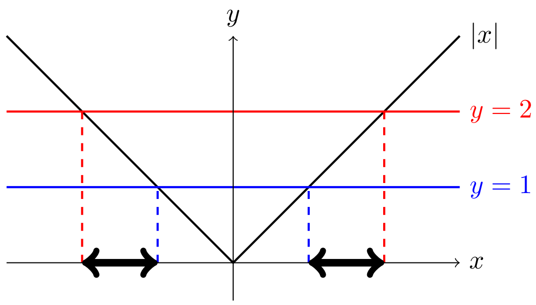
$$\begin{array}{rclcl} \sqrt{1-(k+1)\epsilon} & \leq & |x| & \iff & x \leq -\sqrt{1-(k+1)\epsilon}, \quad x \geq \sqrt{1-(k+1)\epsilon} \\ |x| & \leq & \sqrt{1-k\epsilon} & \iff & -\sqrt{1-k\epsilon} \leq x \leq \sqrt{1-k\epsilon} \end{array}$$

  Hence

$$x \in [-\sqrt{1-k\epsilon}, -\sqrt{1-(k+1)\epsilon}] \cup [\sqrt{1-(k+1)\epsilon}, \sqrt{1-k\epsilon}]$$

  Better use the graph to explain





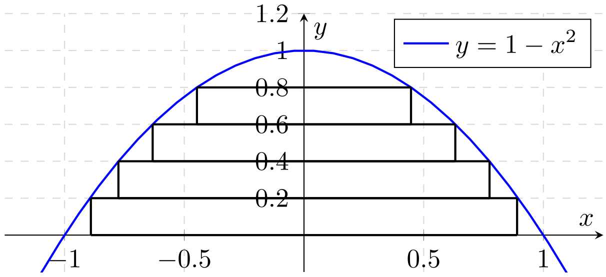
  Thus

  The preimage of $y \in [k\epsilon, (k+1)\epsilon)$ is $\quad x \in [-\sqrt{1-k\epsilon}, -\sqrt{1-(k+1)\epsilon}] \ \cup \ [\sqrt{1-(k+1)\epsilon}, \sqrt{1-k\epsilon}]$

- **Summation** The Lebesgue integral is then the sum of the *measures* of the preimages multiplied by the value of the function within each interval:

$$\int_{-1}^{1} (1-x^2)\,\mathrm{d}x := \sum_{k=0}^{\infty} \epsilon \cdot \mu\Big(\{x \in [-1,1] \mid k\epsilon \leq 1-x^2 < (k+1)\epsilon\}\Big)$$

  - μ denotes the Lebesgue measure of the preimage sets.
  - the term *measures* is a (fancy version) of "length"
    * in fact, it takes a 36-hour course to learn what is "length" in mathematics
    * this course is called Measure Theory / Advanced Real Analysis
    * we will just take "measure" as "length" in daily life
  - $\mu\Big(\{x \in [-1,1] \mid k\epsilon \leq 1-x^2 < (k+1)\epsilon\}\Big)$ means: "length of x-axis such that for $x \in [-1,1]$ and $k\epsilon \leq 1-x^2 < (k+1)\epsilon$ holds"
  - For small ϵ, this sum approximates the integral.
  - Picture



2.4.2.8. **E.g.** Not all functions are Riemann integrable. The function $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is not Riemann integrable, but it is Lebesgue integrable.

2.4.2.9. ⊩Let f and g be integrable functions on $I = [a, b]$ with $a < b$. Let $c \in I$ and k be a constant.

- $\int_a^b (f(x) + g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$
- $\int_a^b kf(x)\mathrm{d}x = k\int_a^b f(x)\mathrm{d}x$
- $\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$
- $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$

2.4.2.10. ⊩**(Fundamental Theorem of Calculus)**

- It links differentiation and integration
- First Part: If f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

- Second Part: If f is continuous on $[a, b]$, then the function F defined by:

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

for x in $[a, b]$, is continuous on $[a, b]$, differentiable on (a, b), and $F'(x) = f(x)$.

2.4.2.11. **E.g.** $\int_1^3 (x^3 - 4x + 5)\mathrm{d}x$

First, find the antiderivative:

$$\int (x^3 - 4x + 5)\,\mathrm{d}x = \frac{x^4}{4} - 2x^2 + 5x + C$$

Evaluate it from 1 to 3, ignore th C

$$\left[\frac{x^4}{4} - 2x^2 + 5x\right]_1^3 = \left(\frac{3^4}{4} - 2 \cdot 3^2 + 5 \cdot 3\right) - \left(\frac{1^4}{4} - 2 \cdot 1^2 + 5 \cdot 1\right) = 14$$

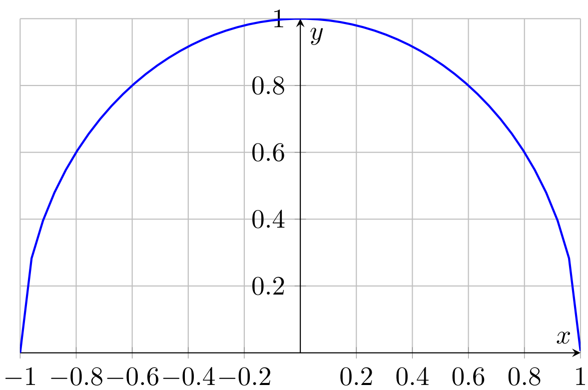
2.4.3 Advanced Integration techniques

2.4.3.1. Using geometry

Suppose we want to find $\int_{-1}^1 \sqrt{1 - x^2}\mathrm{d}x$

Using geometry,

$$\begin{aligned} & \int_{-1}^1 \sqrt{1 - x^2}\mathrm{d}x \\ = & \frac{\text{area of circle with radius 1}}{2} \\ = & \frac{\pi 1^2}{2} \\ = & \frac{\pi}{2} \end{aligned}$$

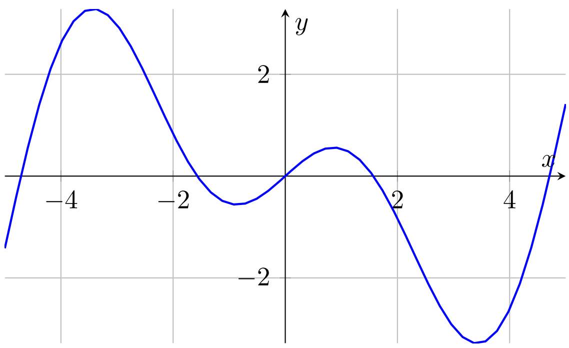


2.4.3.2. Symmetry and Even/Odd function

Suppose we want to find $\displaystyle\int_{-5}^{5} x^7 \cos x \mathrm{d}x$

Using geometry, $x\cos x$ is an odd function, hence

$$\begin{aligned}\int_{-5}^{5} x\cos x\mathrm{d}x &= \Big(\int_{-5}^{0}+\int_{0}^{5}\Big)x\cos x\mathrm{d}x\\ &= \int_{-5}^{0} x\cos x\mathrm{d}x+\int_{0}^{5} x\cos x\mathrm{d}x\\ &= 0\end{aligned}$$



2.4.3.3. t-substitution

2.4.3.4. Reduction formula

2.4.3.5. Complex integration

Chapter 3 Linear algebra

3.1 Operations in linear algebra

3.1.1 Vectors

3.1.1.1. **Notation of vector**

- In book we use **bold italic small font** to represent vector, e.g. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}$
- In handwriting we put an arrow on top of the letter, e.g. $\vec{a}, \vec{x}$, or we just write a, x without the arrow

3.1.1.2. (This is wrong) A vector is an ordered n-tuple

- n is called the *dimension* of the vector
- do not call n the *length* or *size*, it is confusing. The length of a vector is actually another concept.
- Shorthand: we call a vector with dimension n a n-vector

  This understanding, seems okay, is actually wrong. Treating vector as a "list of number" is missing the idea that "vector is an element of a vector space". The concept of vector space belongs to the abstract linear algebra and therefore hard to understand for beginner, so we now "assume" this understanding is correct.

3.1.1.3. **Column vector** are vectors written as a vertical array

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 0.0 \\ \pi \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 0.0 & \pi & -2 \end{bmatrix}^\top$$

- $\boldsymbol{x}$ is a 4-vector, or 4-by-1 column vector
- the numbers inside the array are called the *element* / *entries* of the vector
- the 4th element of $\boldsymbol{x}$ is -2, we write $x_4 = -2$
- $\begin{bmatrix} 1 & 0.0 & \pi & -2 \end{bmatrix}$ is a 1-by-4 **row vector**
- $\top$ is called transpose (do not confuse with tautology in logic)

3.1.1.4. **Equality of vectors** Like the equality of sets, a n-vector $\boldsymbol{x}$ and a m-vector $\boldsymbol{y}$ are equal if $\begin{cases} m = n \\ x_i = y_i \ \forall i \end{cases}$

3.1.1.5. A 1-vector a is called *scalar*

- $a^\top = a$

3.1.1.6. The entries of vector can take $\mathbb{R}$ or $\mathbb{C}$ or other values

- $\mathbb{R}^n$: n-dimensional real-valued vector
- $\mathbb{C}^n$: n-dimensional complex-valued vector
- $\{0,1\}^n$: n-dimensional binary vector / truth vector / logic vector / Boolean vector

3.1.1.7. **Zero vector** and **one vector** in n-dimension is denoted by $\mathbf{0}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ and $\mathbf{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

3.1.1.8. **Standard unit vector** $\boldsymbol{e}_i$ is a vector with $e_i = 1$ and $e_j = 0 \ \forall j \neq i$. E.g., in $\mathbb{R}^3$, $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

3.1.1.9. **Sparsity** is the number of nonzero in a vector. E.g., $\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \|\boldsymbol{x}\|_0 = 1 \ , \boldsymbol{y} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \|\boldsymbol{y}\|_0 = 2.$

3.1.1.10. **What vector represents**

- n-vector is ordered n-tuple. So n-vector can represent anything that has n attributes
- If $n = 3$, then 3-vector represents physical quantities: force, location, displacement, velocity, acceleration
- 3-vector also represents color (R,G,B).
  - Many 3-vectors represent a bunch of pixel $\implies$ we have an digital image

  - Many images $\implies$ a video
- A 24-vector can represents the number of people waiting in a bus station in each hour.
- Time series. A day has 1440 minutes. A 1440-vector can represents your heart beat rate in the whole day.
- Actually, an infinite dimensional vector is called *function*

3.1.1.11. **Vector addition** Given two n-vector $\boldsymbol{x}, \boldsymbol{y}$, the vector sum $\boldsymbol{x}+\boldsymbol{y}$ is defined by a vector $\boldsymbol{z}$ where $z_i = x_i + y_i$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}}_{\boldsymbol{x}+\boldsymbol{y}} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}}_{\boldsymbol{z}}$$

Example $\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}}_{\boldsymbol{x}+\boldsymbol{y}}, \quad \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\boldsymbol{x}+\boldsymbol{y}}$

3.1.1.12. Vector subtraction $\boldsymbol{x}-\boldsymbol{y} = \boldsymbol{x}+(-\boldsymbol{y})$. That is, subtraction is just additio: flip the sign of $\boldsymbol{y}$ and then add it to $\boldsymbol{x}$

3.1.1.13. **Properties of vector addition: an Abelian group**
Consider $\mathbb{R}^n$ and $+$, the set $(\mathbb{R}^n, +)$ is a Abelian group.
For any vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in $\mathbb{R}^n$, we have

- **Closure** $\boldsymbol{a}+\boldsymbol{b} \in \mathbb{R}^n$
  Meaning: two vectors in $\mathbb{R}^n$ after addition still stay within $\mathbb{R}^n$
- **Associativity** $\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c}) = (\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}$
  Meaning: we can write $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ without confusion, because whether you do the first sum first or the second sum first, you'll get the same result.
- **Identity** There is $\boldsymbol{0}$ such that $\boldsymbol{a}+\boldsymbol{0} = \boldsymbol{0}+\boldsymbol{a} = \boldsymbol{a}$.
  Meaning: in the perspective of addition, there is a special element in $\mathbb{R}^n$ that addition with such element has no effect. Such element is $\boldsymbol{0}$.
- **Inverse** For any $\boldsymbol{a}$, there is an element $\boldsymbol{b}$ such that $\boldsymbol{a}+\boldsymbol{b} = \boldsymbol{b}+\boldsymbol{a} = \boldsymbol{0}$. In fact, $\boldsymbol{b} = -\boldsymbol{a}$.
  Meaning: there is an additive inverse for any vector.
- **Commutativity** $\boldsymbol{a}+\boldsymbol{b} = \boldsymbol{b}+\boldsymbol{a}$
  Meaning: the order of summands doesn't matter.

3.1.1.14. **Geometric interpretation of vector addition** If you learn vector in high school, you know that a 2-vector is a vector representing an arrow in the 2D plane. Then to add two vectors $\boldsymbol{a}, \boldsymbol{b}$, you put $\boldsymbol{b}$ to the tip of $\boldsymbol{a}$, and then travel from the root of $\boldsymbol{a}$ to the tip of $\boldsymbol{b}$.

3.1.1.15. After addition, we talk about multiplication. There are several vector multiplications.

- Scalar-vector multiplication
- Vector-vector inner product
- Vector-vector cross product
- Vector-vector exterior product
- Vector-vector geometric product
- Vector-vector tensor product

3.1.1.16. **Scalar-vector product** Given a scalar $c \in \mathbb{R}$ and a vector $\boldsymbol{v} \in \mathbb{R}^n$, the product $c\boldsymbol{v}$ is just scaling every elements in $\boldsymbol{v}$ by c.

$$3\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}, \qquad \frac{1}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \qquad -1\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

Geometric interpretation

- If $c > 1$, the product $c\boldsymbol{v}$ is to "elongate" $\boldsymbol{v}$ by c amount
- If $c \in [0,1]$, the product $c\boldsymbol{v}$ is to "shrink" $\boldsymbol{v}$ by c amount
- If $c < 0$, the product $c\boldsymbol{v}$ is to "U-turn" $\boldsymbol{v}$, and then scale the length of the arrow by c amount

3.1.1.17. **Scalar multiplication is distributive over vector addition** $\quad c(\boldsymbol{u}+\boldsymbol{v}) = c\boldsymbol{u}+c\boldsymbol{v}$

3.1.1.18. **Linear Combination** If we are given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ and scalars $c_1, c_2, ..., c_n$, the vector

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + ... + c_n\boldsymbol{v}_n \tag{Linear Combination}$$

is called a *linear combination* of $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ with *coefficients* $c_1, c_2, ..., c_n$.

3.1.1.19. **Special cases of linear combination**

- On coefficient $c_1, c_2, ..., c_n$
  - If all $c_i \geq 0$, we have *conic combination*
  - If all $c_i \geq 0$ and $\sum c_i = 1$, we have *convex combination*
- On vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$
  - If $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ are standard basis vector $\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n$, then $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + ... + c_n\boldsymbol{v}_n$ is a representation of a vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ in $\mathbb{R}^n$

3.1.1.20. **Vector-vector inner product**

$$a_1b_1 + a_2b_2 + ... + a_nb_n = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}^\top \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \boldsymbol{a}^\top\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} = \langle \boldsymbol{a}, \boldsymbol{b} \rangle$$

- Inputs are two vectors with same dimension
- Output is a scalar
- We can call that inner product is a $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ mapping

Example

$$\begin{bmatrix} 2 \\ -4 \end{bmatrix}^\top \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (2)(0) + (-4)(1) = -4, \quad \begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix}^\top \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (\pi)(1) + (e)(2) + (\sqrt{2})(3) \approx 12.82, \quad \begin{bmatrix} 3 \end{bmatrix}^\top \begin{bmatrix} 4 \end{bmatrix} = (3)(4) = 12$$

3.1.1.21. **Properties of inner product**
For any vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in $\mathbb{R}^n$ and scalar γ, we have

- **Commutativity** $\boldsymbol{a}^\top\boldsymbol{b} = \boldsymbol{b}^\top\boldsymbol{a}$
  Meaning: the order of product doesn't matter.
- **Associativity with scalar multiplication** $(\gamma\boldsymbol{a})^\top\boldsymbol{b} = \gamma(\boldsymbol{a}^\top\boldsymbol{b})$
  Meaning: we can write $\gamma\boldsymbol{a}^\top\boldsymbol{b}$ without confusion, because whether you do the first way or the second way, you'll get the same result.
- **Distributivity with vector addition** $(\boldsymbol{a} + \boldsymbol{b})^\top\boldsymbol{c} = \boldsymbol{a}^\top\boldsymbol{c} + \boldsymbol{b}^\top\boldsymbol{c}$

Note: $(\mathbb{R}^n, \langle\,,\,\rangle)$ is not a group. E.g.,there is no identity and no inverse in $\langle\,,\,\rangle$.

3.1.1.22. $(\boldsymbol{a} + \boldsymbol{b})^\top(\boldsymbol{c} + \boldsymbol{d}) = \boldsymbol{a}^\top\boldsymbol{c} + \boldsymbol{a}^\top\boldsymbol{d} + \boldsymbol{b}^\top\boldsymbol{c} + \boldsymbol{b}^\top\boldsymbol{d}$
The $+$ on the left are vector additions, The $+$ on the right are scalar additions.

3.1.1.23. **Special inner products** Given two vectors $\boldsymbol{a}, \boldsymbol{b}$. The inner product $\boldsymbol{a}^\top\boldsymbol{b}$ has special meaning if one of the vector is special.

- **Select an element.** If $\boldsymbol{a} = \boldsymbol{e}_i$, then $\boldsymbol{a}^\top\boldsymbol{b}$ is the same as picking the ith element of $\boldsymbol{b}$.
  That is, $\boldsymbol{e}_i^\top\boldsymbol{b} = b_i$.
- **Sum.** If $\boldsymbol{a} = \boldsymbol{1}$, then $\boldsymbol{a}^\top\boldsymbol{b}$ is the same as summing the elements of $\boldsymbol{b}$.
  That is, $\boldsymbol{1}^\top\boldsymbol{b} = b_1 + ... + b_n$.
- **Average.** If $\boldsymbol{a} = \boldsymbol{1}/n$, then $\boldsymbol{a}^\top\boldsymbol{b}$ is the same as taking average of the elements of $\boldsymbol{b}$.
  That is, $(\boldsymbol{1}/n)^\top\boldsymbol{b} = (\boldsymbol{1}^\top)/n\boldsymbol{b} = (b_1 + ... + b_n)/n$.
- **Sum of squares.** If $\boldsymbol{a} = \boldsymbol{b}$, then $\boldsymbol{a}^\top\boldsymbol{b}$ is the sum of squares of the elements of the vector $\boldsymbol{b}$.
  That is $\boldsymbol{b}^\top\boldsymbol{b} = b_1^2 + ... + b_n^2$.

3.1.1.24. Application of inner product

- Counting co-occurrence in combinatorics.
  $\boldsymbol{a}, \boldsymbol{b}$ are 0-1 vectors that label occurrence, then $\boldsymbol{a}^\top \boldsymbol{b}$ gives the number of co-occurrence.
- Expected values.
  $\boldsymbol{a}$ is probability of a random variable $\boldsymbol{b}$, then $\boldsymbol{a}^\top \boldsymbol{b}$ is the expected value of $\boldsymbol{b}$.
- Representing polynomial.

$$p(x) = c_1 + c_2 x + c_3 x^2 + ... + c_n x^{n-1} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \cdots \\ c_n \end{bmatrix}^\top \begin{bmatrix} 1 \\ x \\ x^2 \\ \cdots \\ x^{n-1} \end{bmatrix}$$

- Similarity measure in machine learning

3.1.2 Linear functions

3.1.2.1. A function with n variable input and scalar output is denoted by $f : \mathbb{R}^n \to \mathbb{R}$, explicitly written as $f(\boldsymbol{x}) = f(x_1, x_2, ..., x_n)$.

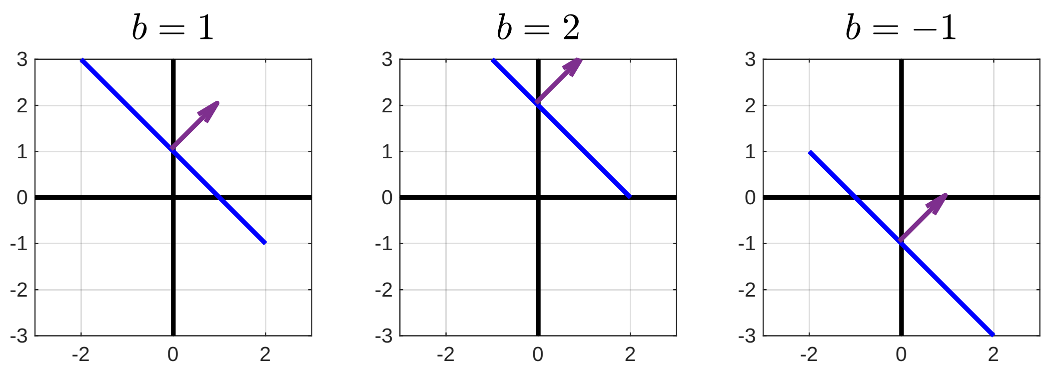
- $f : \mathbb{R}^n \to \mathbb{R}$ means the function maps $\mathbb{R}^n$ (vector) to $\mathbb{R}$ (real number)

3.1.2.2. A special kind of function is inner-product function $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(\boldsymbol{x}) = \boldsymbol{a}^\top \boldsymbol{x}$.

- Example: $f(x_1, x_2) = 2x_1 - 3x_2$

3.1.2.3. The geometry of inner product function is called **hyperplane**: the function $\boldsymbol{a}^\top \boldsymbol{x} = b$ represents a plane that

- has orientation $\boldsymbol{a}$, representing the normal of the plane
- has an "offset value" b, how much the plane is "raised up"
- For example $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $b \in \{1, 2, -1\}$



3.1.2.4. **Linearity** If a function $f(x)$ satisfies $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, then we call f a *linear function*.

3.1.2.5. ⊩**Inner product function is linear.**
Proof We perform direct proof. Let $f(\boldsymbol{z}) = \boldsymbol{a}^\top \boldsymbol{z}$ be a inner product function for any $\boldsymbol{a} \in \mathbb{R}^n$. For any two scalars $\alpha, \beta \in \mathbb{R}$ and any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we have

$$\begin{aligned} f(\alpha \boldsymbol{x} + \beta \boldsymbol{y}) &= \boldsymbol{a}^\top (\alpha \boldsymbol{x} + \beta \boldsymbol{y}) && \text{we are using the definition } f(\boldsymbol{z}) = \boldsymbol{a}^\top \boldsymbol{z} \\ &= \boldsymbol{a}^\top (\alpha \boldsymbol{x}) + \boldsymbol{a}^\top (\beta \boldsymbol{y}) && \text{inner product is distributive by 3.1.1.21.} \\ &= \alpha(\boldsymbol{a}^\top \boldsymbol{x}) + \beta(\boldsymbol{a}^\top \boldsymbol{y}) && \text{inner product is associative with scalar multiplication by 3.1.1.21.} \\ &= \alpha f(\boldsymbol{x}) + \beta f(\boldsymbol{y}) && \text{we are using the definition } f(\boldsymbol{z}) = \boldsymbol{a}^\top \boldsymbol{z} \end{aligned}$$

3.1.2.6. **Corollary [General form of linearity]** If $f(\boldsymbol{x})$ is linear, then

$$f\left(\sum_{i=1}^{k} \alpha_i \boldsymbol{x}_i\right) = \sum_{i=1}^{k} \alpha_i f(\boldsymbol{x}_i)$$

For example, for $k=4$, we have $f(a\boldsymbol{x}+b\boldsymbol{y}+c\boldsymbol{z}+d\boldsymbol{w}) = af(\boldsymbol{x})+bf(\boldsymbol{y})+cf(\boldsymbol{z})+df(\boldsymbol{w})$.
Proof We perform direct proof.

$$\begin{aligned}
f\left(\sum_{i=1}^{k}\alpha_i\boldsymbol{x}_i\right) &= f(a_1\boldsymbol{x}_1+a_2\boldsymbol{x}_2+\ldots+a_k\boldsymbol{x}_k) \\
&= f(a_1\boldsymbol{x}_1)+f(a_2\boldsymbol{x}_2+\ldots+a_k\boldsymbol{x}_k) && \text{by 3.1.2.5.: } \alpha=a_1, \boldsymbol{x}=\boldsymbol{x}_1, \beta=1, \boldsymbol{y}=a_2\boldsymbol{x}_2+\ldots+a_k\boldsymbol{x}_k \\
&= f(a_1\boldsymbol{x}_1)+f(a_2\boldsymbol{x}_2)+f(a_3\boldsymbol{x}_3+\ldots+a_k\boldsymbol{x}_k) && \text{by 3.1.2.5. again} \\
&\vdots \\
&= \sum_{i=1}^{k}\alpha_i f(\boldsymbol{x}_i).
\end{aligned}$$

3.1.2.7. Example of not linear function: $\max$ (it is in fact piece-wise linear)

3.1.2.8. **Affine** A linear function that do not touch the origin is called *affine*

- Linear: $\boldsymbol{a}^\top\boldsymbol{x}$, if we put $\boldsymbol{x}=\boldsymbol{0}$ we get 0
- Affine: $\boldsymbol{a}^\top\boldsymbol{x}+b$, if we put $\boldsymbol{x}=\boldsymbol{0}$ we get b

In daily language people mix up linear and affine.

3.1.2.9. **Application of affine function: linear regression** $\hat{y}=\boldsymbol{x}^\top\boldsymbol{\beta}+c$. Here $\hat{y}$ is *prediction*, $\boldsymbol{x}$ is *regressor* or *variable*. Both $\boldsymbol{\beta}$ and c are called *coefficient*.

3.1.2.10. ⊩**Cauchy-Schwarz inequality** $|\langle\boldsymbol{u},\boldsymbol{v}\rangle|^2 \;\le\; \langle\boldsymbol{u},\boldsymbol{u}\rangle\cdot\langle\boldsymbol{v},\boldsymbol{v}\rangle$. We prove it in Section 3.1.3.

3.1.2.11. **Vector-valued linear function** $f:\mathbb{R}^n\to\mathbb{R}^m$

3.1.2.12. Example $n=4, m=3$,

$$\boldsymbol{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}, \quad f(\boldsymbol{x}) \;=\; \begin{bmatrix}f_1(\boldsymbol{x})\\f_2(\boldsymbol{x})\\f_3(\boldsymbol{x})\end{bmatrix} \;=\; \boldsymbol{A}_{3\times4}\boldsymbol{x}$$

3.1.3 Norm, distance and angle

3.1.3.1. What is the purpose of norm: how we compare the size of vectors

3.1.3.2. **Euclidean norm** $\|\boldsymbol{x}\|=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$

- $\|\boldsymbol{x}\|$ is also denoted as $\|\boldsymbol{x}\|_2$
- Based on inner product, we have $\|\boldsymbol{x}\|=\sqrt{\boldsymbol{x}^\top\boldsymbol{x}}=\sqrt{\langle\boldsymbol{x},\boldsymbol{x}\rangle}$
- Norm-squared is $\|\boldsymbol{x}\|^2=\boldsymbol{x}^\top\boldsymbol{x}=\langle\boldsymbol{x},\boldsymbol{x}\rangle$

3.1.3.3. **E.g.**

- $\boldsymbol{x}=\begin{bmatrix}1\\2\end{bmatrix}$, we have $\|\boldsymbol{x}\|=\sqrt{1^2+2^2}=\sqrt{5}$
- $\boldsymbol{x}=\begin{bmatrix}0\\-2\\1\end{bmatrix}$, we have $\|\boldsymbol{x}\|=\sqrt{0^2+(-2)^2+1^2}=\sqrt{5}$

3.1.3.4. If $\boldsymbol{x}\in\mathbb{R}^1$, then $\|\boldsymbol{x}\|=|x|$

3.1.3.5. **Properties of norm** In mathematics, norm is a function that behaves like the distance from the origin:

- *Nonnegative* $(\forall\boldsymbol{x})\|\boldsymbol{x}\|\ge 0$
- *Definiteness* $\|\boldsymbol{x}\|=\boldsymbol{0}$ if and only if $\boldsymbol{x}=\boldsymbol{0}$
- *Nonnegative homogeneity* $\|\alpha\boldsymbol{x}\|=|\alpha|\cdot\|\boldsymbol{x}\|$
- *Sub-additivity / triangle inequality* $(\forall\boldsymbol{x},\boldsymbol{y})\|\boldsymbol{x}+\boldsymbol{y}\|\ge\|\boldsymbol{x}\|+\|\boldsymbol{y}\|$

3.1.3.6. ⊩**Cauchy-Schwarz inequality** $|\langle\boldsymbol{u},\boldsymbol{v}\rangle| \;\le\; \|\boldsymbol{u}\|\cdot\|\boldsymbol{v}\|$.
Proof. Instead of giving a pretentious short proof, we give a very long but very detailed proof for understanding.

3.1.3.6.1. Consider the vector $\dfrac{\boldsymbol{u}}{\|\boldsymbol{u}\|}-\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ for $\boldsymbol{u}\ne\boldsymbol{0},\boldsymbol{v}\ne\boldsymbol{0}$. The inequality holds if $\boldsymbol{u}=\boldsymbol{0}$ or $\boldsymbol{v}=\boldsymbol{0}$.

3.1.3.6.2. By the nonnegativity of norm (3.1.3.5.), we have $\left\|\dfrac{\boldsymbol{u}}{\|\boldsymbol{u}\|}-\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\right\|\ge 0$.

3.1.3.6.3. Take the squares of the inequality gives $\left\| \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} - \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \right\|^2 \geq 0.$

3.1.3.6.4. The norm-square gives $\left\langle \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} - \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}, \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} - \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \right\rangle \geq 0.$

3.1.3.6.5. Expand the inner product gives $\left\langle \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}, \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} \right\rangle + \left\langle \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}, -\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \right\rangle + \left\langle -\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}, \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} \right\rangle + \left\langle -\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}, -\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \right\rangle \geq 0.$

3.1.3.6.6. Simplify the sign gives $\left\langle \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}, \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} \right\rangle - \left\langle \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}, \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \right\rangle - \left\langle \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}, \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} \right\rangle + \left\langle \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}, \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \right\rangle \geq 0.$

3.1.3.6.7. Apply commutativity (3.1.1.21.) to the negative terms, we have $\left\langle \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}, \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} \right\rangle - 2\left\langle \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}, \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \right\rangle + \left\langle \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}, \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \right\rangle \geq 0.$

3.1.3.6.8. By associativity of scalar multiplication (3.1.1.21.), we can take out the denominator inside the produce and get $\frac{1}{\|\boldsymbol{u}\|^2}\langle \boldsymbol{u}, \boldsymbol{u}\rangle - \frac{2}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \frac{1}{\|\boldsymbol{v}\|^2}\langle \boldsymbol{v}, \boldsymbol{v}\rangle \geq 0.$

3.1.3.6.9. Moving the negative term to the other side gives $\frac{1}{\|\boldsymbol{u}\|^2}\langle \boldsymbol{u}, \boldsymbol{u}\rangle + \frac{1}{\|\boldsymbol{v}\|^2}\langle \boldsymbol{v}, \boldsymbol{v}\rangle \geq \frac{2}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\langle \boldsymbol{u}, \boldsymbol{v}\rangle.$

3.1.3.6.10. Recall $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = \|\boldsymbol{x}\|^2$, hence we can simplify the expression and get $1 + 1 \geq \frac{2}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\langle \boldsymbol{u}, \boldsymbol{v}\rangle$, or $1 \geq \frac{1}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\langle \boldsymbol{u}, \boldsymbol{v}\rangle$

3.1.3.6.11. Multiply the denominator to the whole inequality gives $\|\boldsymbol{u}\|\|\boldsymbol{v}\| \geq \langle \boldsymbol{u}, \boldsymbol{v}\rangle$

3.1.3.7. **One line proof of Cauchy-Schwarz inequality by Jean-Baptiste Hiriart-Urruty**

$$0 \leq \frac{1}{2}\left\| \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} \pm \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \right\|^2 = 1 \pm \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \iff \begin{cases} \langle \boldsymbol{u}, \boldsymbol{v}\rangle \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\| \\ -\langle \boldsymbol{u}, \boldsymbol{v}\rangle \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\| \end{cases}$$

3.1.3.8. ⊩**Euclidean norm satisfies the triangle inequality**

$$\begin{array}{rcll} \|\boldsymbol{u}+\boldsymbol{v}\|^2 & = & \langle \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}\rangle & \\ & = & \|\boldsymbol{u}\|^2 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{v}\|^2 & \text{following same steps as in } 3.1.3.6. \\ & \leq & \|\boldsymbol{u}\|^2 + 2|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| + \|\boldsymbol{v}\|^2 & a \leq |a| \text{ for all real number } a \\ & \leq & \|\boldsymbol{u}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \|\boldsymbol{v}\|^2 & \text{Cauchy-Schawrz inequality 3.1.3.6.} \\ & = & (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)^2 & \\ \|\boldsymbol{u}+\boldsymbol{v}\| & \leq & \|\boldsymbol{u}\| + \|\boldsymbol{v}\| & \text{take square-root} \end{array}$$

3.1.3.9. **Application of triangle inequality**
$\|\boldsymbol{a}+\boldsymbol{b}\| = \|(\boldsymbol{a}-\boldsymbol{c}) + (\boldsymbol{b}-\boldsymbol{c})\| \leq \|\boldsymbol{a}-\boldsymbol{c}\| + \|\boldsymbol{b}-\boldsymbol{c}\|$

3.1.3.10. Norm of sum.

- $\|\boldsymbol{x}+\boldsymbol{y}\| = \sqrt{\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + +2\langle \boldsymbol{x}, \boldsymbol{y}\rangle}$
- $\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\| = \sqrt{\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + \|\boldsymbol{z}\|^2 + 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle + 2\langle \boldsymbol{x}, \boldsymbol{z}\rangle + 2\langle \boldsymbol{y}, \boldsymbol{z}\rangle}$

Exercise: prove these formula

3.1.3.11. Root-mean-squares $\textbf{rms}(\boldsymbol{x}) = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} = \frac{\|\boldsymbol{x}\|}{\sqrt{n}}$

3.1.3.12. **Distance.** The Euclidean distance between two vectors $\boldsymbol{x}, \boldsymbol{y}$ is $\|\boldsymbol{x}-\boldsymbol{y}\|$

- Sometimes we write $\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|$
- Meaning of small $\|\boldsymbol{x}-\boldsymbol{y}\|$ is that $\boldsymbol{x}, \boldsymbol{y}$ are close to each other
- Meaning of zero $\|\boldsymbol{x}-\boldsymbol{y}\|$ is that $\boldsymbol{x}, \boldsymbol{y}$ are the same point
- Meaning of large $\|\boldsymbol{x}-\boldsymbol{y}\|$ is that $\boldsymbol{x}, \boldsymbol{y}$ are very far from each other

3.1.3.13. **Angle.** The angle between two nonzero vectors $\boldsymbol{a}, \boldsymbol{b}$ is defined as $\theta = \cos^{-1}\left(\frac{\boldsymbol{a}^\top \boldsymbol{b}}{\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|}\right) \in [0, \pi]$

- $\angle(\boldsymbol{x}, \boldsymbol{y})$ is scale invariant.
  $\angle(\boldsymbol{x}, \boldsymbol{y}) = \angle(\alpha\boldsymbol{x}, \beta\boldsymbol{y})$ for all positive number α, β
- Meaning of $\angle(\boldsymbol{x}, \boldsymbol{y})$
  - If $\angle(\boldsymbol{x}, \boldsymbol{y}) = \frac{\pi}{2}$, then $\boldsymbol{x} \perp \boldsymbol{y}$, $\boldsymbol{x}$ is perpendicular to $\boldsymbol{y}$, or $\boldsymbol{x}$ is orthogonal to $\boldsymbol{y}$
  - Zero vector is orthogonal to any vector
  - If $\angle(\boldsymbol{x}, \boldsymbol{y}) = 0$, then $\boldsymbol{x}, \boldsymbol{y}$ are parallel

- If $\angle(\boldsymbol{x}, \boldsymbol{y}) = \pi$, then $\boldsymbol{x}, \boldsymbol{y}$ are anti-parallel
- If $\angle(\boldsymbol{x}, \boldsymbol{y}) > \pi/2$, then $\boldsymbol{x}, \boldsymbol{y}$ have negative inner product, the angle in-between is obtuse
- If $\angle(\boldsymbol{x}, \boldsymbol{y}) < \pi/2$, then $\boldsymbol{x}, \boldsymbol{y}$ have positive inner product, the angle in-between is acute

3.1.3.14. **Examples**

- $\boldsymbol{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} -0.5 \\ -1 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- Draw $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}$
- $\mathrm{dist}(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|_2 = \sqrt{(2-0)^2 + (3-0)^2} = \sqrt{13}$
- $\mathrm{dist}(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{2}$
- $\mathrm{dist}(\boldsymbol{v}, \boldsymbol{w}) = \|\boldsymbol{v} - \boldsymbol{w}\|_2 = \sqrt{(2+0.5)^2 + (3+1)^2} = \sqrt{22.25}$
- $\mathrm{dist}(\boldsymbol{u}, \boldsymbol{x}) = \mathrm{dist}(\boldsymbol{u}, \boldsymbol{y}) = 1$
- $\angle(\boldsymbol{x}, \boldsymbol{y}) = \cos^{-1} \dfrac{\boldsymbol{x}^\top \boldsymbol{y}}{\|\boldsymbol{x}\| \|\boldsymbol{y}\|} = \cos^{-1} \dfrac{0}{1 \cdot 1} = \dfrac{\pi}{2}$
- $\angle(\boldsymbol{v}, \boldsymbol{x}) = \cos^{-1} \dfrac{\boldsymbol{v}^\top \boldsymbol{x}}{\|\boldsymbol{v}\| \|\boldsymbol{x}\|} = \cos^{-1} \dfrac{2}{\sqrt{13}} = 0.982793723\pi$

3.1.4 Linear Independence

3.1.4.1. Two vectors $\boldsymbol{x}, \boldsymbol{y}$ are called *linearly independent* if they are orthogonal

- $\boldsymbol{x}, \boldsymbol{y}$ have zero inner product $\boldsymbol{x}^\top \boldsymbol{y} = 0$
- The angle between $\boldsymbol{x}, \boldsymbol{y}$ is 90-degree
- $\boldsymbol{x}, \boldsymbol{y}$ is perpendicular to each other
- $\boldsymbol{x}, \boldsymbol{y}$ are "irrelevant" to each other, $\boldsymbol{x}$ cannot be represented as $\boldsymbol{x} = k\boldsymbol{y}$ for some constant k, and same for $\boldsymbol{y}$

3.1.4.2. Three vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are called *linearly independent* if they are mutually orthogonal $\begin{cases} \boldsymbol{x}, \boldsymbol{y} \text{ are linearly independent} \\ \boldsymbol{x}, \boldsymbol{z} \text{ are linearly independent} \\ \boldsymbol{y}, \boldsymbol{z} \text{ are linearly independent} \end{cases}$

3.1.4.3. Two vectors $\boldsymbol{x}, \boldsymbol{y}$ are called *linearly dependent* if they are not orthogonal, and

- $\boldsymbol{x}$ can be represented by $\boldsymbol{y}$ as $\boldsymbol{x} = \alpha \boldsymbol{y}$ for some non-zero scaling factor α, or equivalently
- $\boldsymbol{y}$ can be represented by $\boldsymbol{x}$ as $\boldsymbol{y} = \beta \boldsymbol{y}$ for some non-zero scaling factor β

3.1.4.4. Three vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are called *linearly dependent* if either

- $\boldsymbol{x}$ can be represented by $\boldsymbol{y}, \boldsymbol{z}$ as $\boldsymbol{x} = \alpha \boldsymbol{y} + \beta \boldsymbol{z}$ for some factors α, β
- $\boldsymbol{y}$ can be represented by $\boldsymbol{x}, \boldsymbol{z}$ as $\boldsymbol{y} = \alpha \boldsymbol{x} + \beta \boldsymbol{z}$ for some factors α, β
- $\boldsymbol{z}$ can be represented by $\boldsymbol{x}, \boldsymbol{y}$ as $\boldsymbol{z} = \alpha \boldsymbol{x} + \beta \boldsymbol{y}$ for some factors α, β

3.1.4.5. A collection of n vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ are *linearly dependent* if

$$c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + \cdots + c_n \boldsymbol{v}_n = \boldsymbol{0}$$

holds for some $c_1, c_2, ..., c_n$ that are <u>not all zero</u>

- It can be all $c_1, c_2, ..., c_n$ non-zero
- It can be only one of $c_1, c_2, ..., c_n$ is non-zero and the rest is zero
- It can be two of $c_1, c_2, ..., c_n$ are non-zero and the rest is zero
- It can be some of $c_1, c_2, ..., c_n$ are non-zero and the rest is zero
- It cannot be $c_1 = c_2 = ... = c_n = 0$

In other words,

- Zero is in the non-zero linear combination (3.1.1.18.) of all the n vectors
- One of the vectors in $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ is in a non-zero linear combination (3.1.1.18.) of the rest of the vectors. For example, suppose $c_1 \boldsymbol{x} + c_2 \boldsymbol{y} + c_3 \boldsymbol{z} = \boldsymbol{0}$ and $c_3 \neq 0$,then we have $\boldsymbol{z} = \Big(-\dfrac{c_1}{c_3}\Big)\boldsymbol{x} + \Big(-\dfrac{c_2}{c_3}\Big)\boldsymbol{y}$

3.1.4.6. A collection of n vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ are *linearly independent* if

$$c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + \cdots + c_n \boldsymbol{v}_n = \boldsymbol{0}$$

holds only for $c_1 = c_2 = ... = c_n = 0$

- Zero is in the zero linear combination (3.1.1.18.) of all the n vectors
- None of the vectors in $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ is in a non-zero linear combination (3.1.1.18.) of the rest of the vectors.

3.1.4.7. **A central theme of linear algebra: giving a bunch of (non-zero) vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$, decide whether or not they are linearly independent.** This is not a simple problem.

3.1.4.8. ⊩If a vector $\boldsymbol{x}$ is a linear combination of the linearly independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$, then there is only one way to represent $\boldsymbol{x}$ by $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$. Or in other words, the coefficients $c_1, ..., c_n$ for $\boldsymbol{x} = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n$ is unique.
Proof by contradiction. Suppose there is another way to write $\boldsymbol{x}$ as a combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$, say $\boldsymbol{x} = d_1\boldsymbol{v}_1 + \cdots + d_n\boldsymbol{v}_n$ with $d_i \neq c_i$. Now we have

$$\begin{array}{rcll} \boldsymbol{x} & = & c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n & (1) \\ \boldsymbol{x} & = & d_1\boldsymbol{v}_1 + \cdots + d_n\boldsymbol{v}_n & (2) \\ \boldsymbol{0} & = & (c_1 - d_1)\boldsymbol{v}_1 + \cdots + (c_n - d_n)\boldsymbol{v}_n & (1)-(2) \end{array}$$

Now by the assumption at the very beginning that $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ are linearly independent, hence $c_i - d_i = 0$ for all i. (See (3.1.4.6.) for why.) Now we have $c_i = d_i$ for all i, which contradicts to the assumption that $d_i \neq c_i$. By The Proof of Contradiction (1.1.6.9.), the assumption is false and therefore there is only one way to represent $\boldsymbol{x}$ by $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$. □

3.1.4.9. Any supersets of linearly dependent vectors are linearly dependent.
Any subsets of linearly independent vectors are linearly independent.

3.1.4.10. (How many linearly independent vectors are possible) A linearly independent collection of n-vectors can have at most n elements.

- Any collection of $n + 1$ or more n-vectors is linearly dependent

3.1.4.11. **Basis** The collection of maximal number of linearly independent vectors is called the *basis*.

- If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ is a basis, then any vector $\boldsymbol{b}$ can be written as a a linear combination of them
- We write $\boldsymbol{b} = a_1\boldsymbol{v}_1 + ... + a_n\boldsymbol{v}_n$
- $a_1, ..., a_n$ are the *coefficients* of $\boldsymbol{b}$ in the basis $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$

3.1.4.12. **Trivial basis (Cartesian coordinate system in Euclidean space)** The n standard unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n$ form a basis.

3.1.4.13. **E.g.**

$$\boldsymbol{b} = \begin{bmatrix} 1.2 \\ -3.7 \end{bmatrix} = 1.2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3.7) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1.2\boldsymbol{e}_1 + (-3.7)\boldsymbol{e}_2$$

3.1.4.14. **Kronecker delta** $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

3.1.4.15. **Orthonormal vectors** A collection of (mutually) orthogonal vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ if $\boldsymbol{v}_i^\top \boldsymbol{v}_j = \delta_{ij}$

- ortho means it is ortohgonal
- normal means it is normalized (unit norm)

3.1.4.16. **E.g.**

- $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$ are orthonormal
- $\dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are orthonormal

3.1.4.17. How to find the coefficient of a basis

- Given a basis $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ and a vector $\boldsymbol{b}$
- To find the coefficient of $\boldsymbol{b}$ in the basis $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$, write down $\boldsymbol{b} = a_1\boldsymbol{v}_1 + ... + a_n\boldsymbol{v}_n$, then compute

$$\boldsymbol{b}^\top \boldsymbol{v}_i = a_i \boldsymbol{v}_i^\top \boldsymbol{v}_i \quad \Longrightarrow \quad a_i = \frac{\boldsymbol{b}^\top \boldsymbol{v}_i}{\boldsymbol{v}_i^\top \boldsymbol{v}_i}$$

3.1.4.18. How to verify a basis represents a vector.

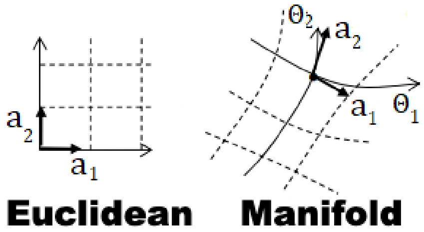
- A vector $\boldsymbol{b}$ represented by a basis $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ has the expression

$$\boldsymbol{b} = (\boldsymbol{v}_1^\top \boldsymbol{b})\boldsymbol{v}_1 + ... + (\boldsymbol{v}_n^\top \boldsymbol{b})\boldsymbol{v}_n$$

This equations is a way to check if $\boldsymbol{b}$ can be represented by the basis $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$

3.1.4.19. **What is the big deal of basis?**

- Basis is the entry point of *abstract vector space* $\longrightarrow$ abstract algebra
- Basis is linked to the Zorn's lemma / Axiom of Choice $\longrightarrow$ set theory, mathematical logic
- Basis is linked to geometry $\longrightarrow$ curvilinear coordinate, differential geometry, manifold



3.1.5 Matrix

3.1.5.1. An example of matrix $\boldsymbol{M} = \begin{bmatrix} 0 & \text{Hi} & a \\ \pi & 2 & 3+2i \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} 0 & \text{Hi} \\ \pi & 2 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} a \\ 3+2i \end{bmatrix}$

- A matrix is an array
- Dimension. $\boldsymbol{M}$ has 2 rows and 3 columns. We call $\boldsymbol{M}$ is 2×3 or 2-by-3
- The top-right corner is the first element, and we move down and move right
- 0 is the $(1,1)$-th element of $\boldsymbol{M}$, or we write $M_{1,1} = 0$
- π is the $(2,1)$-th element of $\boldsymbol{M}$, or we write $M_{2,1} = \pi$
- Hi is the $(1,2)$-th element of $\boldsymbol{M}$, or we write $M_{1,2} =$Hi
- The (i,j)-th element of $\boldsymbol{M}$ is the ith-row, jth-column entry of $\boldsymbol{M}$
- If $\boldsymbol{M}$ is m-by-n, then $i \in \{1,2,...,m\}$ and $j \in \{1,2,...,n\}$
- $[0 \text{ Hi } a]$ is the 1st row of $\boldsymbol{M}$
- $\begin{bmatrix} a \\ 3+2i \end{bmatrix}$ is the 3rd column of $\boldsymbol{M}$, it is also the matrix $\boldsymbol{B}$
- we can also represent $\boldsymbol{M}$ as $[\boldsymbol{A} \;\; \boldsymbol{B}]$, where $\boldsymbol{A}, \boldsymbol{B}$ is submatrices / blocks of $\boldsymbol{M}$

3.1.5.2. **Matrices equivalence.** Given two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.
We said $\boldsymbol{A} = \boldsymbol{B}$ if and only if $\begin{cases} \boldsymbol{A}, \boldsymbol{B} \text{ have the same number of rows and columns} \\ A_{ij} = B_{ij} \forall i, j \end{cases}$

3.1.5.3. We focus on real-valued or complex-valued matrices $\mathbb{R}^{m \times n}$, $\mathbb{C}^{m \times n}$

3.1.5.4. **Special names**

- Tall thin matrix: $m > n$ — it has a technical name called overdetermined
- Short wide matrix: $m < n$ — it has a technical name called underdetermined
- Square matrix: $m = n$
- A 1-by-1 matrix is a scalar
- A $m = 1$ matrix is a row vector
- A $n = 1$ matrix is a column vector

3.1.5.5. **Special matrices**

- **Zero matrix.** A m-by-n matrix that every elements are zero.
- **Diagonal matrix.** A square (n-by-n matrix) that only the diagonal is not zero.
  We write $\text{Diag}(\boldsymbol{u})$ to represent the diagonal
- **Identity matrix.** A diagonal matrix with a all-one diagonal.
- **Triangular matrix.** A matrix with a upper or lower half of the diagonal are all zero.
- **Sparse matrix.** A matrix with many zeros. We write nnz (number of nonzero) to denote the number of non-zero elements in a matrix.

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{a 3-by-3 zero matrix } \mathbf{0}_{3,3}} \qquad \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{a 3-by-3 diagonal matrix}} \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{a 3-by-3 identity matrix}} \qquad \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}}_{\text{a 3-by-3 upper triangular matrix}}$$

3.1.5.6. **Transpose.** $\boldsymbol{A}^\top$ denotes the transpose of $\boldsymbol{A}$, defined by swapping (i,j)-th element with the (j,i)-th element.

$$\boldsymbol{A}=\begin{bmatrix}1&2\\0&4\\0&0\end{bmatrix}\quad \boldsymbol{A}^\top=\begin{bmatrix}1&0&0\\2&4&0\end{bmatrix}\qquad \boldsymbol{x}=\begin{bmatrix}1\\0\\0\\2\end{bmatrix}\quad \boldsymbol{x}^\top=\begin{bmatrix}1&0&0&2\end{bmatrix}\qquad \boldsymbol{M}=\begin{bmatrix}\boldsymbol{P}&\boldsymbol{Q}\\\boldsymbol{R}&\boldsymbol{S}\end{bmatrix}\quad \boldsymbol{M}^\top=\begin{bmatrix}\boldsymbol{P}^\top&\boldsymbol{Q}^\top\\\boldsymbol{R}^\top&\boldsymbol{S}^\top\end{bmatrix}$$

Clearly $(\boldsymbol{A}^\top)^\top=\boldsymbol{A}$

3.1.5.7. **Symmetric matrix.** If $\boldsymbol{A}=\boldsymbol{A}^\top$, we call $\boldsymbol{A}$ a symmetric matrix.

3.1.5.8. **Skew-symmetric matrix.** If $\boldsymbol{A}=-\boldsymbol{A}^\top$, we call $\boldsymbol{A}$ a skew-symmetric matrix.

3.1.5.9. **Matrix addition.** $\boldsymbol{C}=\boldsymbol{A}+\boldsymbol{B}$ is defined as $C_{ij}=A_{ij}+B_{ij}$

3.1.5.10. **Properties of matrix addition.** $(\mathbb{R}^{m\times n},+)$ is an Abelian group.

- Commutativity. $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$.
- Associativity. $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$.
- Additive identity element. $(\boldsymbol{A}+\boldsymbol{0})=\boldsymbol{0}+\boldsymbol{A}=\boldsymbol{A}$.
- Additive inverse. For all $\boldsymbol{A}\in\mathbb{R}^{m\times n}$ there exists $\boldsymbol{B}\in\mathbb{R}^{m\times n}$ such that $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}=\boldsymbol{0}$.
- Transpose distributive over addition. $(\boldsymbol{A}+\boldsymbol{B})^\top=\boldsymbol{A}^\top+\boldsymbol{B}^\top$

3.1.5.11. **Scalar-matrix multiplication.** $\boldsymbol{B}=k\boldsymbol{A}$ is defined as $B_{ij}=kA_{ij}$

3.1.5.12. **Properties of scalar-matrix multiplication.**

- Distributive over transpose. $(k\boldsymbol{A})^\top=k^\top\boldsymbol{A}^\top=k\boldsymbol{A}^\top$
- Distributive over addition $k(\boldsymbol{A}+\boldsymbol{B})=k\boldsymbol{A}+k\boldsymbol{B}$

3.1.5.13. **Matrix Frobenious norm.**
The norm of a matrix is a generalization of the Euclidean norm of a vector.
Given a vector $\boldsymbol{v}$, its Euclidean norm $\|\boldsymbol{v}\|_2=\sqrt{v_1^2+v_2^2+\cdots+v_n^2}$.
Generalizing this, we have that given a matrix $\boldsymbol{X}$, its Frobenious norm is $\|\boldsymbol{X}\|_F=\sqrt{X_{1,1}^2+X_{1,2}^2+\cdots+X_{m,n}^2}$.

$$\|\boldsymbol{X}\|_F\ :=\ \sqrt{\sum_{i=1}^m\sum_{j=1}^n X_{ij}^2}$$

3.1.5.14. ⊩**Properties of Frobenious norm.**

- *Nonnegative* $(\forall\boldsymbol{X}\in\mathbb{R}^{m\times n})\|\boldsymbol{X}\|_F\geq 0$
- *Definiteness* $\|\boldsymbol{X}\|_F=\boldsymbol{0}$ if and only if $\boldsymbol{X}=\boldsymbol{0}$
- *Nonnegative homogeneity* $\|\alpha\boldsymbol{X}\|_F=|\alpha|\cdot\|\boldsymbol{X}\|_F$
- *Sub-additivity / triangle inequality* $(\forall\boldsymbol{X},\boldsymbol{Y})\|\boldsymbol{X}+\boldsymbol{Y}\|_F\geq\|\boldsymbol{X}\|_F+\|\boldsymbol{Y}\|_F$
- *Transpose invariant* $\|\boldsymbol{X}^\top\|_F=\|\boldsymbol{X}\|_F$

3.1.5.15. **Distance between two matrices** $\operatorname{dist}(\boldsymbol{X},\boldsymbol{Y}):=\|\boldsymbol{X}-\boldsymbol{Y}\|_F$

3.1.5.16. **Matrix-vector product** Given $\boldsymbol{A}\in\mathbb{R}^{m\times n}$ and $\boldsymbol{x}\in\mathbb{R}^n$, then $\boldsymbol{A}\boldsymbol{x}$ is defined as

$$\boldsymbol{A}\boldsymbol{x}=x_1\boldsymbol{A}_{:1}+\cdots+x_n\boldsymbol{A}_{:n}\ =\ \text{LinearCombination}\Big(\boldsymbol{A}_{:1},\cdots,\boldsymbol{A}_{:n}\Big)\text{ by coefficients }x_1,\cdots,x_n\quad\text{(Column-thinking)}$$

$$[\boldsymbol{A}\boldsymbol{x}]_i=\langle\boldsymbol{A}_{i:},\boldsymbol{x}\rangle\ =\ \text{inner product between row vector }\boldsymbol{A}_{i:}\text{ and column vector }\boldsymbol{x}\qquad\text{(Row-thinking)}$$

3.1.5.17. **E.g.**

$$\begin{bmatrix}1&5\\2&6\\3&7\\4&8\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}\ =\ 2\begin{bmatrix}1\\2\\3\\4\end{bmatrix}+(-1)\begin{bmatrix}5\\6\\7\\8\end{bmatrix}\ =\ \begin{bmatrix}2\\4\\6\\8\end{bmatrix}+\begin{bmatrix}-5\\-6\\-7\\-8\end{bmatrix}\ =\ \begin{bmatrix}-3\\-2\\-1\\0\end{bmatrix}\qquad\text{(Column-thinking)}$$

$$\begin{bmatrix}1&5\\2&6\\3&7\\4&8\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}1&5\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}\\ \begin{bmatrix}2&6\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}\\ \begin{bmatrix}3&7\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}\\ \begin{bmatrix}4&8\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}\end{bmatrix}=\begin{bmatrix}2-5\\4-6\\6-7\\8-8\end{bmatrix}=\begin{bmatrix}-3\\-2\\-1\\0\end{bmatrix}\qquad\text{(Row-thinking)}$$

3.1.5.18. Special multiplications.
$\mathbf{0}\boldsymbol{x}=\mathbf{0}$ for all $\boldsymbol{x}$
$\boldsymbol{I}\boldsymbol{x}=\boldsymbol{x}$ for all $\boldsymbol{x}$. This is why $\boldsymbol{I}$ is called identity matrix.

3.1.5.19. Special matrix related to identity matrix $\boldsymbol{I}$

- Permutation matrix: $[1,2,3,4]\to[3,2,4,1]$

$$\begin{bmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}c\\b\\d\\a\end{bmatrix}$$

- Multiplying the kth row: multiply the 3rd row

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&m&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}a\\b\\mc\\d\end{bmatrix}$$

- Row-addition: add m times the 2nd row to the 3rd row

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&m&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}a\\b\\mb+c\\d\end{bmatrix}$$

3.1.6 Motivation of matrix

3.1.6.1. **Example of why matrix is useful: Fibonacci's Rabbits**
In the book Liber Abaci (1202) by Leonardo of Pisa (Fibonacci), here is the following
"How many rabbits will be produced n a year, beginning with a single pair, if every month each pair produces a new pair, which becomes productive 2 months after birth."

Month	Number of rabbits
1	1
2	1
3	2
4	3
5	5
6	8
7	13
⋮	⋮

Question: what is the number of rabbits in the 47th month?
Let x_k be the number of rabbit at the kth month.
The equation relating x_k, x_{k-1} and x_{k-2} is

$$x_k=x_{k-1}+x_{k-2}$$

The question is same as: given $x_1=x_2=1$, find x_{47}.

Solution by matrix equation

$$x_k=x_{k-1}+x_{k-2}\quad\Longrightarrow\quad\begin{bmatrix}x_k\\x_{k-1}\end{bmatrix}=\begin{bmatrix}?&?\\?&?\end{bmatrix}\begin{bmatrix}x_{k-1}\\x_{k-2}\end{bmatrix}=\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}x_{k-1}\\x_{k-2}\end{bmatrix}\quad\Longleftrightarrow\quad\boldsymbol{x}_k=\boldsymbol{A}\boldsymbol{x}_{k-1}$$

Now with $\boldsymbol{x}_k=\boldsymbol{A}\boldsymbol{x}_{k-1}$, we can repeat the same expression

$$\boldsymbol{x}_k=\boldsymbol{A}\boldsymbol{x}_{k-1}=\boldsymbol{A}\boldsymbol{A}\boldsymbol{x}_{k-2}=\boldsymbol{A}\boldsymbol{A}\boldsymbol{A}\boldsymbol{x}_{k-3}=\cdots$$

In other words

$$\begin{bmatrix}x_{47}\\x_{46}\end{bmatrix}=\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}x_{46}\\x_{45}\end{bmatrix}=\begin{bmatrix}1&1\\1&0\end{bmatrix}^2\begin{bmatrix}x_{45}\\x_{44}\end{bmatrix}=\begin{bmatrix}1&1\\1&0\end{bmatrix}^3\begin{bmatrix}x_{44}\\x_{43}\end{bmatrix}=\cdots=\begin{bmatrix}1&1\\1&0\end{bmatrix}^{45}\begin{bmatrix}x_2\\x_1\end{bmatrix}=\begin{bmatrix}1&1\\1&0\end{bmatrix}^{45}\begin{bmatrix}1\\1\end{bmatrix}.$$

Perform eigendecomposition (we will talk about it later)

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{bmatrix}\begin{bmatrix}\frac{1+\sqrt{5}}{2} & \\ & \frac{1-\sqrt{5}}{2}\end{bmatrix}\begin{bmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{bmatrix}^\top$$

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^{45} = \begin{bmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{bmatrix}\begin{bmatrix}\frac{1+\sqrt{5}}{2} & \\ & \frac{1-\sqrt{5}}{2}\end{bmatrix}^{45}\begin{bmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{bmatrix}^\top$$

Calculating the product gives us the number of rabbits.

3.1.6.2. Image processing and computer graphics: matrix representation of image

3.1.6.3. Graph theory: matrix representation of connectivity

- counting number of walks of length ℓ is encoded in the matrix power to ℓ of the adjacency matrix

3.1.6.4. Euclidean geometry in $\mathbb{R}^2$

- Scaling a vector (making a vector longer or shorter, but not changing the direction)

$$\boldsymbol{x} \mapsto \begin{bmatrix}|\alpha| & 0\\ 0 & |\alpha|\end{bmatrix}\boldsymbol{x} \;=\; |\alpha|\boldsymbol{I}\boldsymbol{x} \;=\; \begin{bmatrix}|\alpha|x_1\\ |\alpha|x_2\end{bmatrix}$$

  - $|\alpha| > 1$ elongation
  - $|\alpha| = 1$ unchanged
  - $|\alpha| < 1$ compression
- Counterclockwise rotation of a vector (but not changing the length)

$$\boldsymbol{x} \mapsto \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\boldsymbol{x} \;=\; \begin{bmatrix}\cos\theta x_1 - \sin\theta x_2\\ \sin\theta x_1 + \cos\theta x_2\end{bmatrix}$$

- 180-degree rotation

$$\boldsymbol{x} \mapsto -\boldsymbol{x} \quad\Longleftrightarrow\quad \boldsymbol{x} = \alpha\boldsymbol{x} \text{ with } \alpha = -1 \quad\Longleftrightarrow\quad \boldsymbol{x} \mapsto \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\boldsymbol{x} \text{ with } \theta = \pi$$

- Dilation

$$\boldsymbol{x} \mapsto \begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix}\boldsymbol{x}$$

  Difference between scaling and dilation: $d_1 \neq d_2$
- Reflection along the line $L(\theta)$ passing through the origin

$$\boldsymbol{x} \mapsto \begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos\theta\end{bmatrix}\boldsymbol{x}$$

- Projection onto the line $L(\theta)$ passing through the origin

$$\boldsymbol{x} \mapsto \begin{bmatrix}\frac{1+\cos 2\theta}{2} & \frac{\sin 2\theta}{2}\\ \frac{\sin 2\theta}{2} & \frac{1-\cos 2\theta}{2}\end{bmatrix}\boldsymbol{x}$$

3.1.6.5. Markov chain: probability of transition from state i to state j

3.1.6.6. Linear dynamics: first-order Taylor series approximation of an ODE

3.1.6.7. Finite difference and finite element analysis in PDE

3.1.7 Matrix multiplication

3.1.7.1. **Matrix-multiplication between two matrices** Given $\boldsymbol{A} \in \mathbb{R}^{m\times p}$ and $\boldsymbol{B} \in \mathbb{R}^{p\times n}$, then the matrix product between $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted as $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$, is defined as

$$C_{ij} \;=\; \sum_{k=1}^{p} A_{ik}B_{kj}$$

Matrix multiplication as inner products let $\boldsymbol{a}_i^\top$ be the ith row of $\boldsymbol{A}$ and $\boldsymbol{b}_j$ be the jth column of $\boldsymbol{B}$, then

$$\boldsymbol{C} = \boldsymbol{AB} \;=\; \begin{bmatrix} \boldsymbol{a}_1^\top \boldsymbol{b}_1 & \boldsymbol{a}_1^\top \boldsymbol{b}_2 & \cdots & \boldsymbol{a}_1^\top \boldsymbol{b}_n \\ \boldsymbol{a}_2^\top \boldsymbol{b}_1 & \boldsymbol{a}_2^\top \boldsymbol{b}_2 & \cdots & \boldsymbol{a}_2^\top \boldsymbol{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_m^\top \boldsymbol{b}_1 & \boldsymbol{a}_m^\top \boldsymbol{b}_2 & \cdots & \boldsymbol{a}_m^\top \boldsymbol{b}_n \end{bmatrix}$$

3.1.7.2. Example $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 0\cdot 0 & 1\cdot 0 + 0\cdot 1 \\ 2\cdot 3 + 1\cdot 0 & 2\cdot 0 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$

3.1.7.3. $\boldsymbol{IA} = \boldsymbol{A}$ and $\boldsymbol{AI} = \boldsymbol{A}$. Note that the two $\boldsymbol{I}$ are not the same.

3.1.7.4. **Matrix multiplication is not commutative** $\boldsymbol{AB} \neq \boldsymbol{BA}$

- therefore order matters in matrix multiplication

3.1.7.5. **Matrix multiplication is associative** $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$

3.1.7.6. **Matrix multiplication is associative over scalar product** $\gamma(\boldsymbol{AB}) = (\gamma\boldsymbol{A})\boldsymbol{B}$

3.1.7.7. **Matrix multiplication is distributive over addition** $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = (\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C}$ and $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC}+\boldsymbol{BC}$

3.1.7.8. **Matrix multiplication and transpose** $(\boldsymbol{AB})^\top = \boldsymbol{B}^\top\boldsymbol{A}^\top$

3.1.7.9. $\boldsymbol{y}^\top(\boldsymbol{Ax}) = (\boldsymbol{y}^\top\boldsymbol{A})\boldsymbol{x} = (\boldsymbol{A}^\top\boldsymbol{y})^\top\boldsymbol{x}$

3.1.7.10. **Matrix multiplication as matrix-column-vector multiplication**

$$\boldsymbol{AB} = \boldsymbol{A}[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \ldots\ \boldsymbol{b}_n] = [\boldsymbol{Ab}_1\ \boldsymbol{Ab}_2\ \ldots\ \boldsymbol{Ab}_n]$$

This is known as "multiple right-handside"

3.1.7.11. **Matrix multiplication as row-vector-matrix multiplication**

$$\boldsymbol{AB} \;=\; \begin{bmatrix} \boldsymbol{a}_1^\top \\ \boldsymbol{a}_2^\top \\ \vdots \\ \boldsymbol{a}_m^\top \end{bmatrix} \boldsymbol{B} \;=\; \begin{bmatrix} \boldsymbol{a}_1^\top\boldsymbol{B} \\ \boldsymbol{a}_2^\top\boldsymbol{B} \\ \vdots \\ \boldsymbol{a}_m^\top\boldsymbol{B} \end{bmatrix}$$

3.1.7.12. **Gram matrix**

$$\boldsymbol{A}^\top\boldsymbol{A} \;=\; \begin{bmatrix} \boldsymbol{a}_1^\top \boldsymbol{a}_1 & \boldsymbol{a}_1^\top \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_1^\top \boldsymbol{a}_n \\ \boldsymbol{a}_2^\top \boldsymbol{a}_1 & \boldsymbol{a}_2^\top \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_2^\top \boldsymbol{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_m^\top \boldsymbol{a}_1 & \boldsymbol{a}_m^\top \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m^\top \boldsymbol{a}_n \end{bmatrix}$$

- The entries of $\boldsymbol{A}^\top\boldsymbol{A}$ tells the inner products of pairs of columns of $\boldsymbol{A}$
- $\boldsymbol{A}^\top\boldsymbol{A}$ is a symmetric matrix
- This matrix is positive semi-definite (advanced topic, not covered in this course)

3.1.7.13. **Matrix multiplication as outer products**

$$\boldsymbol{AB} \;=\; \boldsymbol{a}_1\boldsymbol{b}_1^\top + \boldsymbol{a}_2\boldsymbol{b}_2^\top + \cdots + \boldsymbol{a}_r\boldsymbol{b}_r^\top$$

3.1.7.14. **Matrix power** For $\boldsymbol{A}$ is a square matrix

- $\boldsymbol{A}^2 = \boldsymbol{AA}$
- $\boldsymbol{A}^k\boldsymbol{A}^\ell = \boldsymbol{A}^{k+\ell}$ for positive integers k, ℓ
- $\boldsymbol{A}^0 = \boldsymbol{I}$ for positive integers k, ℓ
- $\boldsymbol{A}^{\frac{1}{2}}$ is only for positive definite matrix (advanced topic, not covered in this course)
- $\boldsymbol{A}^{-1}$ is only for invertible / non-singular matrix $\boldsymbol{A}$

3.1.7.15. Example. Find the product $\boldsymbol{AB}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

Method 1. Using $C_{ij} = \sum_{k=1}^{p} A_{ik}B_{kj}$

$$\boldsymbol{AB} = \begin{bmatrix} 1(3)+0(-2)+(-2)0 & 1(-1)+0(0)+(-2)2 & 1(0)+0(3)+(-2)(-1) \\ 0(3)+(-3)(-2)+1(0) & 0(-1)+(-3)(0)+1(2) & 0(0)+(-3)(3)+1(-1) \\ 2(3)+1(-2)+0(0) & 2(-1)+1(0)+0(2) & 2(0)+1(3)+0(-1) \end{bmatrix} = \begin{bmatrix} 3 & -5 & 2 \\ 6 & 2 & -10 \\ 4 & -2 & 3 \end{bmatrix}$$

Method 2. Using $\boldsymbol{AB} = [\boldsymbol{Ab}_1 \ \boldsymbol{Ab}_2 \ ... \ \boldsymbol{Ab}_n]$

$$\begin{aligned}
\boldsymbol{AB} &= \left[\begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \right] \\
&= \left[\begin{bmatrix} 1 & 0 \\ 0 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ -3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right] \\
&= \left[\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -9 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 3 & -5 & 2 \\ 6 & 2 & -10 \\ 4 & -2 & 3 \end{bmatrix}
\end{aligned}$$

Method 3. using $\boldsymbol{a}_1^\top \boldsymbol{B}$

$$\begin{aligned}
\boldsymbol{AB} &= \begin{bmatrix} \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -2 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix} \\ \begin{bmatrix} 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -2 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -2 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \\ \begin{bmatrix} -3 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -2 & 0 & 3 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 1(3)+(-2)0 & 1(-1)+(-2)(2) & -2(0)+(-2)(-1) \\ -3(-2)+1(0) & (-3)(0)+1(2) & (-3)(3)+1(-1) \\ 2(3)+1(-2) & 2(-1)+1(0) & 2(0)+1(3) \end{bmatrix} = \begin{bmatrix} 3 & -5 & 2 \\ 6 & 2 & -10 \\ 4 & -2 & 3 \end{bmatrix}
\end{aligned}$$

Method 4. Using outer product $\boldsymbol{AB} = \boldsymbol{a}_1\boldsymbol{b}_1^\top + \cdots + \boldsymbol{a}_r\boldsymbol{b}_r^\top$

$$\begin{aligned}
\boldsymbol{AB} &= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 0 \\ 6 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 6 & 0 & -9 \\ -2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -4 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 2 \\ 6 & 2 & -10 \\ 4 & -2 & 3 \end{bmatrix}
\end{aligned}$$

OK now your turn to show that $\boldsymbol{BA} = \begin{bmatrix} 3 & 3 & -7 \\ 4 & 3 & 4 \\ -2 & -7 & 2 \end{bmatrix}$

3.1.7.16. After learning matrix addition, scalar multiplication, matrix multiplication and inverse, we are now ready to talk about a high-level thing: what is "algebra"?
Algebra A nonempty-set $\mathcal{A}$ of square matrices (i.e., $\mathcal{A}$ denotes a set of square matrices) is called an *algebra* if $\mathcal{A}$ is closed under the operations of matrix addition, scalar multiplication, matrix multiplication.

3.1.7.17. **Def (Algebra)** A set $\mathcal{A}$ is called an algebra if the following hold for all x, y, z in $\mathcal{A}$:

- $(x+y)\cdot z = x\cdot z + y\cdot z$
- $z\cdot(x+y) = z\cdot x + z\cdot y$
- $(ax)\cdot(by) = (ab)(x\cdot y)$

3.1.8 Solving System of Linear Equations

3.1.8.1. A linear equation in unknown $x_1, x_2, ..., x_n$ is an equation in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

- $a_1, a_2, ..., a_n$ and b are constants
- $a_1, a_2, ..., a_n$ are called coefficient
- b is called the constant term
- n is the dimension of the system

3.1.8.2. For small n, usually we do not use $x_1, x_2, ..., x_n$ but x, y, z, w for easier understanding

3.1.8.3. System of linear equations $\begin{cases} a_{11}x + a_{12}y + a_{13}z + a_{14}w = b_1 \\ a_{21}x + a_{22}y + a_{23}z + a_{24}w = b_2 \\ a_{31}x + a_{32}y + a_{33}z + a_{34}w = b_3 \\ a_{41}x + a_{42}y + a_{43}z + a_{44}w = b_4 \end{cases} \iff \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{\boldsymbol{b}}$

- $\boldsymbol{A}$ is a 4-by-4 matrix here called the coefficient matrix
- $\boldsymbol{x}$ is the 4-by-1 column vector that collects all the unknowns x, y, z, w
- $\boldsymbol{b}$ is the 4-by-1 column vector that collects all constant term b_1, b_2, b_3, b_4
- We call $\boldsymbol{Ax} = \boldsymbol{b}$ a system of linear equations, or in short linear system

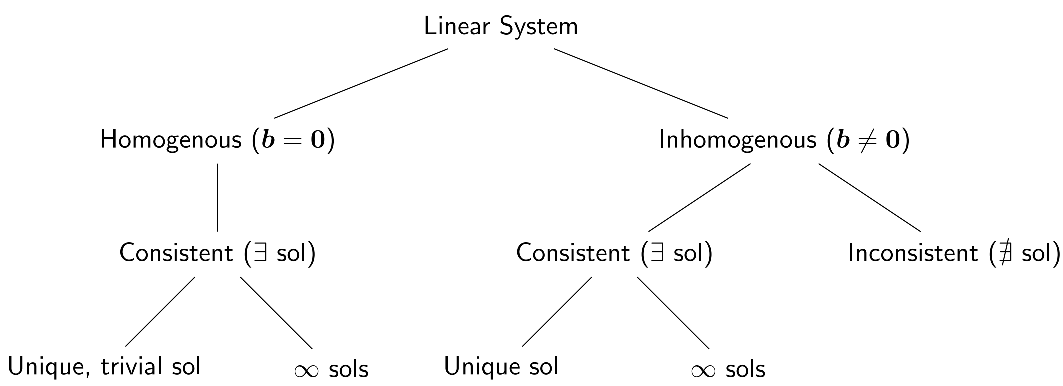
3.1.8.4. A linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is called homogeneous if $\boldsymbol{b} = \boldsymbol{0}$

3.1.8.5. A linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is called inhomogeneous if $\boldsymbol{b} \neq \boldsymbol{0}$

3.1.8.6. A linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is called consistent if solution exists (can be one or more)

3.1.8.7. A linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is called inconsistent if solution does not exist (zero solution)

3.1.8.8. Classification of linear system based on the solution



3.1.8.9. (What if $\boldsymbol{A}$ contains a zero row.) If all $a_i = 0$ for a particular row in $\boldsymbol{A}$, then that linear equation is called degenerate.

- if $b = 0$, that row may be deleted from the system without changing the solution set of the system
- if $b \neq 0$, the system has no solution

3.1.8.10. (What if $\boldsymbol{A}$ contains a zero column.) If all $a_i = 0$ for a particular column in $\boldsymbol{A}$, then the variable corresponding to that column can be removed from the system.

3.1.8.11. How to solve $\boldsymbol{Ax} = \boldsymbol{b}$: we multiply the whole system by a series of matrices $\boldsymbol{M}_1, \boldsymbol{M}_2, ...$ such that we obtain a upper-triangular system $\boldsymbol{Ux} = \boldsymbol{z}$

$$\underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}}_{\boldsymbol{U}} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}}_{\boldsymbol{z}} \iff \begin{cases} x_4 = \dfrac{z_4}{u_{44}} \\ x_3 = \dfrac{z_3 - u_{34}x_4}{u_{33}} \\ x_2 = \dfrac{z_2 - u_{23}x_3 - u_{24}x_4}{u_{22}} \\ x_1 = \dfrac{z_1 - u_{12}x_2 - u_{13}x_3 - u_{14}x_4}{u_{11}} \end{cases}$$

3.1.8.12. How do we get $\boldsymbol{Ux} = \boldsymbol{z}$ from $\boldsymbol{Ax} = \boldsymbol{b}$

$$\begin{array}{rrcll}
 & \boldsymbol{Ax} & = & \boldsymbol{b} & \text{the original } \boldsymbol{Ax}=\boldsymbol{b} \\
\iff & \boldsymbol{M}_1\boldsymbol{Ax} & = & \boldsymbol{M}_1\boldsymbol{b} & \text{left-multiplication by a matrix } \boldsymbol{M}_1 \\
\iff & \boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{Ax} & = & \boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{b} & \text{left-multiplication by a matrix } \boldsymbol{M}_2 \\
 & & \vdots & & \\
\iff & \boldsymbol{M}_n\cdots\boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{Ax} & = & \boldsymbol{M}_n\cdots\boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{b} & \text{left-multiplication by a matrix } \boldsymbol{M}_2 \\
\iff & \boldsymbol{Ux} & = & \boldsymbol{z} & \boldsymbol{U} := \boldsymbol{M}_n\cdots\boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{A},\quad \boldsymbol{z} := \boldsymbol{M}_n\cdots\boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{b}
\end{array}$$

What we need to do: find those $\boldsymbol{M}_1, \boldsymbol{M}_2, \cdots, \boldsymbol{M}_n$

3.1.8.13. ⊩**LU-factorization** If $\boldsymbol{M}_j$ are a bunch of lower-trangular matrix with non-zero diagonal, then $\boldsymbol{M}_j^{-1}$ exists

$$\begin{array}{rrcl}
 & \boldsymbol{M}_n\cdots\boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{Ax} & = & \boldsymbol{M}_n\cdots\boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{b} \\
\iff & (\boldsymbol{M}_n\cdots\boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{A})\boldsymbol{x} & = & (\boldsymbol{M}_n\cdots\boldsymbol{M}_2\boldsymbol{M}_1)\boldsymbol{b} \\
\iff & (\boldsymbol{M}_n\cdots\boldsymbol{M}_2\boldsymbol{M}_1)^{-1}(\boldsymbol{M}_n\cdots\boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{A})\boldsymbol{x} & = & \boldsymbol{b} \\
\iff & \boldsymbol{LUx} & = & \boldsymbol{b}
\end{array}$$

$\boldsymbol{L}$ is lower-triangular, $\boldsymbol{U}$ is upper-triangular

3.1.8.14. **E.g.** Solve $\begin{cases} x+2y+2z=3 \\ 4x+4y+2z=6 \\ 4x+6y+4z=10 \end{cases}$

First we write down the matrix-vector form

$$\begin{cases} x+2y+2z=3 \\ 4x+4y+2z=6 \\ 4x+6y+4z=10 \end{cases} \iff \begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 2 \\ 4 & 6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \iff \boldsymbol{Ax} = \boldsymbol{b}$$

Now to find $\boldsymbol{M}_1$, note that we want

- to add $(-4)\times$(1st-row) to the (2nd-row) to removed the "4"
- to add $(-4)\times$(1st-row) to the (3rd-row) to removed the "4"

$$\text{left-multiplication with } \begin{bmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{add } (-4)\times\text{(1st-row) to the (2nd-row) to removed the "4"}$$

$$\text{left-multiplication with } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} = \text{add } (-4)\times\text{(1st-row) to the (3rd-row) to removed the "4"}$$

$$\text{left-multiplication with the identity matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{keep all rows unchanged } = \text{do nothing}$$

Therefore

$$\text{left-multiplication with } \left(\begin{bmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \begin{array}{c} \text{add } (-4)\times\text{(1st-row) to the (2nd-row)} \\ \text{add } (-4)\times\text{(1st-row) to the (3rd-row)} \\ \text{keep all rows unchanged for the addition} \end{array}$$

We now have

$$\begin{array}{rrcll}
 & \begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 2 \\ 4 & 6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} & = & \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} & \\
\iff & \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 2 \\ 4 & 6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} & = & \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} & \begin{array}{c} \text{add -4(1st-row) to 2nd-row} \\ \text{add -4(1st-row) to 3rd-row} \end{array} \\
\iff & \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -6 \\ 0 & -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} & = & \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} &
\end{array}$$

Now for $\boldsymbol{M}_2$, in the same spirit

$$\iff \begin{bmatrix}1 & & \\ & 1 & \\ & -0.5 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 0 & -4 & -6\\ 0 & -2 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1 & & \\ & 1 & \\ & -0.5 & 1\end{bmatrix}\begin{bmatrix}3\\ -6\\ -2\end{bmatrix} \quad \text{add -2(2nd-row) to 3rd-row}$$

$$\iff \begin{bmatrix}1 & 2 & 2\\ & -4 & -6\\ & & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}3\\ -6\\ 1\end{bmatrix}$$

Backward substitution gives $z=-1$, $y=3$ and $x=-1$

$$\begin{bmatrix}1 & 2 & 2\\ 4 & 4 & 2\\ 4 & 6 & 4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}3\\ 6\\ 10\end{bmatrix} \iff \begin{bmatrix}1 & & \\ & 1 & \\ & -0.5 & 1\end{bmatrix}\begin{bmatrix}1 & & \\ -4 & 1 & \\ -4 & & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 4 & 4 & 2\\ 4 & 6 & 4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1 & & \\ & 1 & \\ & -0.5 & 1\end{bmatrix}\begin{bmatrix}1 & & \\ -4 & 1 & \\ -4 & & 1\end{bmatrix}\begin{bmatrix}3\\ 6\\ 10\end{bmatrix}$$

3.1.8.15. Elementary operations: the matrices $\boldsymbol{M}_1, \boldsymbol{M}_2$ are called elementary operations. There are three type of elementary operations

E1: swap equation-i and equation-j

E2: replace equation-i by k times of equation-i, where $k \neq 0$

E3: replace equation-j by k times of equation-i plus equation-j, where $k \neq 0$

3.1.8.16. **Row echelon form** The idea to solve linear system is to turn $\boldsymbol{A}$ into an upper triangular matrix called row echelon form.

$$\begin{bmatrix}1 & 2 & 2\\ 4 & 4 & 2\\ 4 & 6 & 4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}3\\ 6\\ 10\end{bmatrix} \iff \begin{bmatrix}1 & 2 & 2\\ & -4 & -6\\ & & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}3\\ -6\\ 1\end{bmatrix}$$

- $\begin{bmatrix}1 & 2 & 2\\ & -4 & -6\\ & & -1\end{bmatrix}$ is an upper triangular matrix called row echelon form
- the first nonzero entry of each row is called a pivot
- how to turn a matrix into row echelon form: Gaussian elimination

3.1.8.17. **Solving linear equations by determinant formula**.

- The general approach of solving linear equations is to use Gaussian elimination to turn the system matrix $\boldsymbol{A}$ to row echelon form.
  (Echelon comes from the French word échelon, meaning "ladder")
- Sometimes it is tedious to do the Gaussian elimination, so we look for formula
- Here is a formula based on determinant
- This part should be viewed after you learn what is a determinant.

Consider $\begin{cases}a_{11}x + a_{12}y = b_1\\ a_{21}x + a_{22}y = b_2\end{cases}$, then $x = \dfrac{\det\begin{bmatrix}b_1 & a_{12}\\ b_2 & a_{22}\end{bmatrix}}{\det\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}}$ and $y = \dfrac{\det\begin{bmatrix}a_{11} & b_1\\ a_{21} & b_2\end{bmatrix}}{\det\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}}$

3.1.8.18. **E.g.** Solve $\begin{cases}4x - 3y = 15\\ 2x + 5y = 1\end{cases}$ $\quad x = \dfrac{\det\begin{bmatrix}15 & -3\\ 1 & 5\end{bmatrix}}{\det\begin{bmatrix}4 & -3\\ 2 & 5\end{bmatrix}} = 3$ and $y = \dfrac{\det\begin{bmatrix}4 & 15\\ 2 & 1\end{bmatrix}}{\det\begin{bmatrix}4 & -3\\ 2 & 5\end{bmatrix}} = -1$

3.1.8.19. A homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ with more unknowns than equations has a nonzero solution.

3.1.8.20. **E.g.** $\begin{cases}x + y - z = 0\\ 2x - 3y + z = 0\\ x - 4y + 2z = 0\end{cases}$ after reducing the system to echelon form gives $\begin{cases}x + y - z = 0\\ 5y + 3z = 0\end{cases}$, we have more unknowns than equations. In this case one variable (e.g. z) is a free variable, and x, y can be expressed as a function of z, we have

$$x = \frac{8}{5}z, \quad y = -\frac{3}{5}z, \quad z \in \mathbb{R}$$

- Since $\boldsymbol{z} \in \mathbb{R}$, there are uncountably-infinite many solution for the linear system
- If you forgot what is countable infinity and uncountable infinity, go back to Section 1.2.5

3.1.9 Matrix inverse

3.1.9.1. **Def (Left-inverse)** Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, a matrix $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ is called the *left-inverse* of $\boldsymbol{A}$ if the condition $\boldsymbol{BA} = \boldsymbol{I}_n$ holds

3.1.9.2. Example (left-inverse is not unique)

$$\boldsymbol{A} = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix},\ \boldsymbol{B}_1 = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix},\ \boldsymbol{B}_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix},\quad \boldsymbol{B}_1\boldsymbol{A} = \boldsymbol{B}_2\boldsymbol{A} = \boldsymbol{I}_2.$$

This example showed that inverse of a matrix is different from inverse of a number

- (non-uniqueness) there can be more than one left-inverse of a given matrix
- Fact: if a matrix has non-unique left-inverse, then it has infinitely many left-inverse

3.1.9.3. Example (left-inverse may not exist)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \end{bmatrix} \text{ has no left-inverse}$$

3.1.9.4. Example. Find the left-inverse of $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \\ 0 & 0 \end{bmatrix}$.

- By definition, the left-inverse $\boldsymbol{B}$ of $\boldsymbol{A}$ is $\boldsymbol{BA} = \boldsymbol{I}_2$, hence

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- By the 3.1.7.10., we have

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \text{(logic and)} \quad \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Change name

$$\begin{bmatrix} x & y & z \\ r & s & t \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \wedge \quad \begin{bmatrix} x & y & z \\ r & s & t \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- That is, we have to solve a system of linear equations (6 unknowns, 4 equations)

$$\begin{cases} 0x & +1y & +0z & = 1 \\ 0r & +1s & +0t & = 0 \\ 1x & -3y & +0z & = 0 \\ 1r & -3s & +0t & = 1 \end{cases} \iff \begin{cases} & y & = 1 \\ & s & = 0 \\ x & -3y & = 0 \\ r & -3s & = 1 \end{cases} \iff \begin{cases} & y & = 1 \\ & s & = 0 \\ x & -3 & = 0 \\ r & & = 1 \end{cases} \iff \begin{cases} x = 3 \\ y = 1 \\ r = 1 \\ s = 0 \end{cases}, z \in \mathbb{R}, t \in \mathbb{R}$$

- The left-inverse of $\boldsymbol{A}$ is

$$\boldsymbol{B} = \begin{bmatrix} 3 & 1 & z \\ 1 & 0 & t \end{bmatrix}$$

3.1.9.5. Exercise: show that $\boldsymbol{A} = [1 \;\; 2]$ has no left-inverse.

3.1.9.6. Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$. If $\boldsymbol{A}^\top$ is its left-inverse, then $\boldsymbol{A}$ is orthogonal $\iff$ $\boldsymbol{A}$ has orthonormal columns.

3.1.9.7. If $\boldsymbol{A}$ has left-inverse then the columns of $\boldsymbol{A}$ are linearly independent

- Suppose $\boldsymbol{Ax} = \boldsymbol{0}$
- Seeing matrix-vector multiplication as taking linear combination of columns of $\boldsymbol{A}$ by coefficients in $\boldsymbol{x}$, then $\boldsymbol{Ax} = \boldsymbol{0}$ is saying "what $\boldsymbol{x}$ will give linear combination of columns of $\boldsymbol{A}$ to generate zero"
- Since $\boldsymbol{A}$ has left-inverse $\boldsymbol{B}$, then

$$\begin{aligned} \boldsymbol{Ax} &= \boldsymbol{0} && \text{what we suppose in the beginning} \\ \boldsymbol{BAx} &= \boldsymbol{B0} && \text{multiply the whole expression by } \boldsymbol{B} \\ \boldsymbol{BA} &= \boldsymbol{0} && \text{any matrix multiply zero vector gives a zero vector} \\ \boldsymbol{Ix} &= \boldsymbol{0} && \boldsymbol{BA} = \boldsymbol{I}\text{because } \boldsymbol{B} \text{ is the left-inverse of } \boldsymbol{A} \\ \boldsymbol{x} &= \boldsymbol{0} \end{aligned}$$

So the only possible vector $\boldsymbol{x}$ that $\boldsymbol{Ax} = \boldsymbol{0}$ is zero vector. This means the column of $\boldsymbol{A}$ are linearly independent by definition 3.1.4.6..

3.1.9.8. The last point is $\{\boldsymbol{A}$ has left-inverse $\} \implies \{$columns of $\boldsymbol{A}$ are linearly independent$\}$. The converse (1.1.3.27.) is

$$\{\text{columns of } \boldsymbol{A} \text{ are linearly independent}\} \implies \{\boldsymbol{A} \text{ has left-inverse}\}$$

which is also true, so we have

$$\{\text{columns independence}\} \iff \{\text{left-invertibility}\}$$

3.1.9.9. Only tall or square matrices can be left-invertible

3.1.9.10. We can solve system of linear equations by left-inverse. Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution and we want to find the solution $\boldsymbol{x}$. Then if $\boldsymbol{A}$ has a left-inverse $\boldsymbol{C}$:

$$\boldsymbol{Ax} = \boldsymbol{b} \iff \boldsymbol{CAx} = \boldsymbol{Cb}$$

3.1.9.11. Example (Using left-inverse to solve system of linear equation). Given $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. We know from 3.1.9.4. that its left-inverse $\boldsymbol{B} = \begin{bmatrix} 3 & 1 & z \\ 1 & 0 & t \end{bmatrix}$, solve $\boldsymbol{Ax} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- Since $\boldsymbol{B} = \begin{bmatrix} 3 & 1 & z \\ 1 & 0 & t \end{bmatrix}$ for any $z \in \mathbb{R}, t \in \mathbb{R}$ is a left-inverse of $\boldsymbol{A}$, when we compute with left-inverse we want to set $z = t = 0$ so that we can be as lazy as possible, so we pick $\boldsymbol{B} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
- $\boldsymbol{BAx} = \boldsymbol{Bb}$ gives $\boldsymbol{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.
- Note that $\boldsymbol{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ solves $\boldsymbol{BAx} = \boldsymbol{Bb}$, not $\boldsymbol{Ax} = \boldsymbol{b}$. If you put $\boldsymbol{x}$ into $\boldsymbol{Ax}$ it gives $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \boldsymbol{b}$, so we can conclude that $\boldsymbol{Ax} = \boldsymbol{b}$ has no solution for $\boldsymbol{x}$. That is, it is impossible to find a $\boldsymbol{x}$ such that $\boldsymbol{Ax} = \boldsymbol{b}$.
- In fact you can see why $\boldsymbol{Ax} = \boldsymbol{b}$ has no solution: the 3rd row of $\boldsymbol{A}$ are all zeros, which is impossible to give any non-zero value in the 3rd dimension in the vector $\boldsymbol{b}$

3.1.9.12. Example (Using left-inverse to solve system of linear equation). Given $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. We know from 3.1.9.4. that its left-inverse $\boldsymbol{B} = \begin{bmatrix} 3 & 1 & z \\ 1 & 0 & t \end{bmatrix}$, solve $\boldsymbol{Ax} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

- Since $\boldsymbol{B} = \begin{bmatrix} 3 & 1 & z \\ 1 & 0 & t \end{bmatrix}$ for any $z \in \mathbb{R}, t \in \mathbb{R}$ is a left-inverse of $\boldsymbol{A}$, when we compute with left-inverse we want to set $z = t = 0$ so that we can be as lazy as possible, so we pick $\boldsymbol{B} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
- $\boldsymbol{BAx} = \boldsymbol{Bb}$ gives $\boldsymbol{x} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.
- Note that $\boldsymbol{x} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ solves both $\boldsymbol{BAx} = \boldsymbol{Bb}$ and $\boldsymbol{Ax} = \boldsymbol{b}$.

3.1.9.13. The two examples above showed that

- how to use left-inverse to determine the existence of solution in a system of linear equations
- how to use left-inverse to solve a system of linear equations

3.1.9.14. **Def (Right-inverse)** Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, a matrix $\boldsymbol{B} \in \mathbb{R}^{n \times m}$ is called the *right-inverse* of $\boldsymbol{A}$ if the condition $\boldsymbol{AB} = \boldsymbol{I}_m$ holds

3.1.9.15. Based on transpose, if $\boldsymbol{B}$ is the right-inverse of $\boldsymbol{A}$, then $\boldsymbol{B}^\top$ is the left-inverse of $\boldsymbol{A}^\top$

- so we only need to study left-inverse and then carry everything from left to the right-version by the operation transpose

3.1.9.16. A matrix is right-invertible if and only if its rows are linearly independent

3.1.9.17. Only square or wide matrices can be right-invertible

3.1.9.18. Given $\boldsymbol{A}, \boldsymbol{b}$ and we want to solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Suppose $\boldsymbol{A}$ is right-invertible with right-inverse $\boldsymbol{B}$, then $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{b}$.

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{B}\boldsymbol{b} = \boldsymbol{I}\boldsymbol{b} = \boldsymbol{b}$$

3.1.9.19. **Def (Inverse)** If a matrix $\boldsymbol{A}$ has both left- and right- inverse, then we call $\boldsymbol{A}$ invertible / non-singular.

3.1.9.20. If $\boldsymbol{A}$ is invertible, then its left-inverse and right-inverse are equal.

- Suppose that $\boldsymbol{B}_{\text{left}}\boldsymbol{A} = \boldsymbol{I}$ and $\boldsymbol{A}\boldsymbol{B}_{\text{right}} = \boldsymbol{I}$
- We have $\boldsymbol{B}_{\text{right}} = \boldsymbol{I}\boldsymbol{B}_{\text{right}} = \boldsymbol{B}_{\text{left}}\boldsymbol{A}\boldsymbol{B}_{\text{right}} = \boldsymbol{B}_{\text{left}}\boldsymbol{I} = \boldsymbol{B}_{\text{left}}$

3.1.9.21. A matrix that is not invertible is called singular

3.1.9.22. We write $\boldsymbol{A}^{-1}$ to denote the inverse of an non-singular matrix $\boldsymbol{A}$. We have

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}, \quad (\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$$

3.1.9.23. If $\boldsymbol{A}$ has an inverse, then we can solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ as $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$ (here $\boldsymbol{A}^{-1}$ is the right-inverse of $\boldsymbol{A}$)

3.1.9.24. Example. $\boldsymbol{A} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$ and $\boldsymbol{A}^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$. Then we can solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ by $\boldsymbol{x} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}\boldsymbol{b}$

3.1.9.25. For square matrix, the following are equivalent

- left-invertibility
- right-invertibility
- invertibility
- columns are linearly independent
- rows are linearly independent

3.1.9.26. Example. Find the inverse of $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix}$.

Let $\boldsymbol{B}$ be the inverse of $\boldsymbol{A}$, where $\boldsymbol{B}\boldsymbol{A} = \boldsymbol{I}$. We find $\boldsymbol{B}$ using products of elementary operations $\boldsymbol{B} = \boldsymbol{E}_k ... \boldsymbol{E}_2\boldsymbol{E}_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix} \quad \text{start with } \boldsymbol{A} = \boldsymbol{A}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{left multiply with } \boldsymbol{E}_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 5 \end{bmatrix} \quad \text{left multiply with } \boldsymbol{E}_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{left multiply with } \boldsymbol{E}_3$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{left multiply with } \boldsymbol{E}_4$$

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{left multiply with } \boldsymbol{E}_5$$

Thus

$$\underbrace{\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}}_{=\boldsymbol{B}}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix} = \boldsymbol{I}$$

Therefore

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -26 & -7 & 12 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{bmatrix}$$

3.1.9.27. Example of invertible matrix

- $\boldsymbol{I}^{-1} = \boldsymbol{I}$

- diagonal matrix
- Formula for 2-by-2 matrix $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\boldsymbol{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & b \end{bmatrix}$ if $ad - bc \neq 0$.
  (Actually $ad - bc$ is the determinant of $\boldsymbol{A}$)
- $(\boldsymbol{A}^{-1})^\top = (\boldsymbol{A}^\top)^{-1}$
- $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ if both $\boldsymbol{A}, \boldsymbol{B}$ are invertible

3.1.9.28. Matrix to negative power $\boldsymbol{A}^{-k} = (\boldsymbol{A}^{-1})^k$

3.1.9.29. **Pseudo-inverse (Moore-Penrose inverse)** for non-square $\boldsymbol{A}$

- "A matrix has inverse" is only for square matrix
- We want something similar for rectangular matrix
- pseudo-inverse is defined for any matrix
  - if $\boldsymbol{A}$ has linearly independent columns: $\boldsymbol{A}^\dagger := (\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top$
  - if $\boldsymbol{A}$ has linearly independent rows: $\boldsymbol{A}^\dagger := \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{A}^\top)^{-1}$
- pseudo-inverse for square-invertible matrix reduces to $\boldsymbol{A}^{-1}$

3.1.9.30. **Minor, cofactor and adjoint**

- The minor at the (i, j) position of a matrix $\boldsymbol{A}$ is the determinant of $\boldsymbol{A}$ without the ith row and jth column
- We write minor at the (i, j) position as M_{ij}
- Cofactor, denoted as C_{ij}, is defined as $(-1)^{i+j}M_{ij}$
- The adjoint matrix of $\boldsymbol{A}$, denoted as adj$\boldsymbol{A}$, is the transpose of the cofactor matrix
- We have $\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}}\text{adj}\boldsymbol{A}$ or $\boldsymbol{A}\,\text{adj}\boldsymbol{A} = \det(\boldsymbol{A})\boldsymbol{I}$
- At this stage you probably is thinking "what the heck are these things", okay there are some deep reasons (related to commutative ring in abstract algebra)

3.1.9.31. **E.g.** $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then adj$\boldsymbol{A} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Together with $\det \boldsymbol{A}$ we have $\boldsymbol{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3.1.9.32. **E.g.** $\boldsymbol{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, then

$$\boldsymbol{C} = \begin{bmatrix} +\det\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} & -\det\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} & +\det\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \\ -\det\begin{bmatrix} a_2 & a_3 \\ c_2 & c_3 \end{bmatrix} & +\det\begin{bmatrix} a_1 & a_3 \\ c_1 & c_3 \end{bmatrix} & -\det\begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix} \\ +\det\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} & -\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} & +\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \end{bmatrix}, \qquad \text{adj}\boldsymbol{A} = \boldsymbol{C}^\top$$

Together with $\det \boldsymbol{A}$ we can compute $\boldsymbol{A}^{-1}$ for any 3-by-3 matrix.

3.1.9.33. Which way of computing inverse is the fastest? All of them are equally slow.

3.1.10 Eigenvalues and eigenvectors

3.1.10.1. Eigenvalues and eigenvectors belong to a mathematical area called spectral theory. The Eigenvalues of a matrix is called the spectrum of the matrix.

3.1.10.2. **Determinant of a matrix** Given a square matrix $\boldsymbol{M}$, its determinant is denoted as $\det \boldsymbol{M}$.

- determinant is a function that maps a matrix into a scalar
- some people write it as $|\boldsymbol{M}|$, which is possibly confusing.
- determinant was first introduced when people study how to solve system of linear equations

3.1.10.3. How to compute determinant

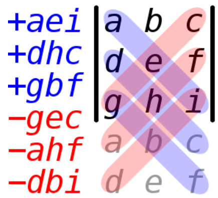
- Formula of Rule of Sarrus
- Leibniz formula
- Laplace's cofactor expansion

3.1.10.4. **Rule of Sarrus formula of the determinant of 1-by-1 matrix** $\det\begin{pmatrix} a \end{pmatrix} = a$

3.1.10.5. **Rule of Sarrus formula of the determinant of 2-by-2 matrix** $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

3.1.10.6. **E.g.** $\det\begin{pmatrix} 5 & 3 \\ 4 & 6 \end{pmatrix} = 5(6) - 3(4) = 30 - 12 = 18$

3.1.10.7. **Rule of Sarrus formula of the determinant of 3-by-3 matrix**



$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + dhc + gbf - gec - ahf - dbi.$$

3.1.10.8. **E.g.**

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - 6 = -2, \quad \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1, \quad \det\begin{bmatrix} 1+x & 0 \\ 0 & 1-x \end{bmatrix} = \det\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} = 1 - x^2$$

3.1.10.9. **E.g.** $\det\begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{bmatrix} = 2(5)(4) + (-2) + 0 - 5 - (-3)(-2)(2) - 0 = 21$

3.1.10.10. **E.g.** $\det\begin{bmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{bmatrix} = 81$

3.1.10.11. The rule of Sarrus does not work in 4-by-4 matrix. The rule of Sarrus is just "an experimental formula", not a mathematical theorem. The rule of Sarrus formula is just a special case of Leibniz formula, which uses the Levi-Civita symbol (something crazy so out of scope in this module). The Levi-Civita symbol is the reason of why there are those $+1, -1$ in the formula.

3.1.10.12. **Recursive definition of determinant: Laplace's cofactor expansion**

- Suppose we want to find $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- We take a row, say the first row (a, b), then we do the following

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \times \det\begin{bmatrix} d \end{bmatrix} - b \times \det\begin{bmatrix} c \end{bmatrix}$$

- Since determinant of a scalar is the scalar itself, we have

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- Let $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if we remove the row and colums of $\boldsymbol{A}$ that contains a, we get a submatrix $\begin{bmatrix} d \end{bmatrix}$
- Later we will know that this submatrix is called the *minor* of a. The *cofactor* is a *signed* minor, with the sign determined by $(-1)^{ij}$

3.1.10.13. Sign in the Laplace's cofactor expansion

$$\begin{bmatrix} + \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}, \dots$$

3.1.10.14. **E.g.** Find $\det \boldsymbol{A}$ where $\boldsymbol{A} = \begin{bmatrix} -4 & 3 & 3 \\ 8 & 7 & 3 \\ 4 & 3 & 3 \end{bmatrix}$

$$\begin{aligned} \det\begin{bmatrix} \color{red}{-4} & \color{red}{3} & \color{red}{3} \\ 8 & 7 & 3 \\ 4 & 3 & 3 \end{bmatrix} = \color{red}{(-4)}\det\begin{bmatrix} 7 & 3 \\ 3 & 3 \end{bmatrix} - \color{red}{3}\det\begin{bmatrix} 8 & 3 \\ 4 & 3 \end{bmatrix} + \color{red}{3}\det\begin{bmatrix} 8 & 7 \\ 4 & 3 \end{bmatrix} &= -4(21-9) - 3(24-12) + 3(24-28) \\ &= -4(12) - 3(12) + 3(-4) = -96 \end{aligned}$$

$$\det\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix} = (-4)\det\begin{bmatrix}7 & 3\\ 3 & 3\end{bmatrix} - 8\det\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix} + 4\det\begin{bmatrix}3 & 3\\ 7 & 3\end{bmatrix} \begin{aligned} &= -4(21-9) - 8(9-9) + 4(9-21)\\ &= -4(12) - 8(0) + 4(-12) \;=\; -96\end{aligned}$$

$$\det\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix} = -8\det\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix} + 7\det\begin{bmatrix}-4 & 3\\ 4 & 3\end{bmatrix} - 3\det\begin{bmatrix}-4 & 3\\ 4 & 3\end{bmatrix} \begin{aligned} &= -8(9-9) + 7(-12-12) - 3(-12-12)\\ &= -8(0) + 7(-24) - 3(-24) \;=\; -96\end{aligned}$$

3.1.10.15. **Determinant of product** For two square matrices $\boldsymbol{A}, \boldsymbol{B}$ we have $\det(\boldsymbol{AB}) = (\det \boldsymbol{A})(\det \boldsymbol{B})$. Giving N square matrices $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_N$, we have $\det(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_N) = (\det \boldsymbol{A}_1)\cdots(\det \boldsymbol{A}_N)$

3.1.10.16. In general, $\det(\boldsymbol{A}+\boldsymbol{B}) \neq \det \boldsymbol{A} + \det \boldsymbol{B}$

3.1.10.17. **Properties of determinant**

- Interchanging two rows negates the determinant.

  E.g. If we swap the first two rows of $\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix}$, then $\det\begin{bmatrix}8 & 7 & 3\\ -4 & 3 & 3\\ 4 & 3 & 3\end{bmatrix} = -\det\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix}$.

$$\begin{aligned}\det\begin{bmatrix}8 & 7 & 3\\ -4 & 3 & 3\\ 4 & 3 & 3\end{bmatrix} &= \det\left(\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix}\right)\\ &= \det\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\det\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix} \quad \text{based on (3.1.10.15.)}\\ &= -\det\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix}\end{aligned}$$

- Scaling a row by a constant multiplies the determinant by that constant.

  E.g. If we scale the first row of $\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix}$, then $\det\begin{bmatrix}-8 & 6 & 6\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix} = 2\det\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix}$.

$$\begin{aligned}\det\begin{bmatrix}-8 & 6 & 6\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix} &= \det\left(\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix}\right)\\ &= \det\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\det\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix} \quad \text{based on (3.1.10.15.)}\\ &= 2\det\begin{bmatrix}-4 & 3 & 3\\ 8 & 7 & 3\\ 4 & 3 & 3\end{bmatrix}\end{aligned}$$

  - This lead to the following: if there is a zero row in $\boldsymbol{A}$, then $\det \boldsymbol{A} = 0$

- Adding a scalar multiple of one row to another does not change the determinant.

E.g. For $\begin{bmatrix}-4&3&3\\8&7&3\\4&3&3\end{bmatrix}$, add 2*row1 to row2, then $\det\begin{bmatrix}-4&3&3\\0&13&12\\4&3&3\end{bmatrix}=\det\begin{bmatrix}-4&3&3\\8&7&3\\4&3&3\end{bmatrix}$

$$\begin{aligned}\det\begin{bmatrix}-4&3&3\\0&13&12\\4&3&3\end{bmatrix} &= \det\left(\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-4&3&3\\8&7&3\\4&3&3\end{bmatrix}\right)\\ &= \det\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\det\begin{bmatrix}-4&3&3\\8&7&3\\4&3&3\end{bmatrix} \quad \text{based on (3.1.10.15.)}\\ &= \det\begin{bmatrix}-4&3&3\\8&7&3\\4&3&3\end{bmatrix}\end{aligned}$$

3.1.10.18. Using properties of determinant to perform Laplace's cofactor expansion

$$\begin{aligned}\det\begin{bmatrix}-4&3&3\\8&7&3\\4&3&3\end{bmatrix} &= \det\begin{bmatrix}-4&3&3\\0&13&9\\0&6&6\end{bmatrix} && \text{2 row1 + row2, row1 + row3}\\ &= -4\det\begin{bmatrix}13&9\\6&6\end{bmatrix} && \text{cofactor expansion along first column}\\ &= -4(13(6)-9(6)) = -4(24) = -96\end{aligned}$$

3.1.10.19. **Characteristic polynomial** Given a square matrix $\boldsymbol{A}$, the expression $\det(\lambda\boldsymbol{I}-\boldsymbol{A})$ is call the characteristic polynomial of $\boldsymbol{A}$.

- The name suggests it is telling the "characteristic" of $\boldsymbol{A}$, we will know what are those "characteristic" later

3.1.10.20. **E.g.** $\boldsymbol{A}=\begin{bmatrix}1&3\\4&5\end{bmatrix}$, then $\det(\lambda\boldsymbol{I}-\boldsymbol{A})=(\lambda-1)(\lambda-5)-12=\lambda^2-6\lambda-7$

3.1.10.21. **Trace** Given a square matrix $\boldsymbol{A}$, the trace of $\boldsymbol{A}$, denoted as $\mathrm{Tr}(\boldsymbol{A})$, is the sum of the main diagonal of $\boldsymbol{A}$

3.1.10.22. **E.g.** $\boldsymbol{A}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, then $\det(\lambda\boldsymbol{I}-\boldsymbol{A})=\lambda^2-(a+d)\lambda-\det\boldsymbol{A}=\lambda^2-(\mathrm{Tr}\boldsymbol{A})\lambda-\det\boldsymbol{A}$

3.1.10.23. **Eigenvalues** The root of the characteristic polynomial is the eigenvalue of the matrix.

3.1.10.24. **E.g.** The eigenvalues of $\boldsymbol{A}=\begin{bmatrix}4&2\\3&-1\end{bmatrix}$ are 5 and -2

$$\begin{aligned}\det(\lambda\boldsymbol{I}-\boldsymbol{A})=0 \iff \det\left(\begin{bmatrix}\lambda&\\&\lambda\end{bmatrix}-\begin{bmatrix}4&2\\3&-1\end{bmatrix}\right)=0 &\iff \det\begin{bmatrix}\lambda-4&-2\\-3&\lambda+1\end{bmatrix}=0\\ &\iff (\lambda-4)(\lambda+1)-6=0\\ &\iff \lambda^2-4\lambda+\lambda-4-6=0\\ &\iff \lambda^2-3\lambda-10=0\\ &\iff (\lambda-5)(\lambda+2)=0\\ &\iff \lambda=5,-2\end{aligned}$$

3.1.10.25. **Eigenvectors** A vector $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalues λ if $\boldsymbol{Ax}=\lambda\boldsymbol{x}$

- $\boldsymbol{Ax}=\lambda\boldsymbol{x}$ means that "multiplying $\boldsymbol{A}$ on $\boldsymbol{x}$ is just the same as scaling $\boldsymbol{x}$ with λ"

3.1.10.26. **E.g.** The eigenvalues of $\boldsymbol{A}=\begin{bmatrix}4&2\\3&-1\end{bmatrix}$ are $\lambda_1=5,\lambda_2=-2$, find the eigenvector

$$\begin{aligned}\boldsymbol{Av}=\lambda_1\boldsymbol{v} \iff \boldsymbol{Av}=\lambda_1\boldsymbol{Iv} \iff (\boldsymbol{A}-\lambda_1\boldsymbol{I})\boldsymbol{v}=\boldsymbol{0} &\iff \begin{bmatrix}-1&2\\3&-6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\\ &\iff -x+2y=0\end{aligned}$$

So there are infinitely many possible (x,y) to solve the linear system, we pick one particular one, say $\boldsymbol{v}_1=\begin{bmatrix}2\\1\end{bmatrix}$

Similarly, for $\lambda_2 = -2$

$$\boldsymbol{A}\boldsymbol{v} = \lambda_2 \boldsymbol{v} \iff (\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0} \iff \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff 3x + y = 0$$

So there are infinitely many possible (x, y) to solve the linear system, we pick one particular one, say $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

3.1.10.27. **How to find the eigenvalues and eigenvectors, summary**

- Computing the characteristic polynomial of the matrix
- Find the roots of the characteristic polynomial, the roots are the eigenvalues
- Solve $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$ to find the eigenvectors

3.1.10.28. **Eigenbasis** The matrix whose columns are the eigenvectors is called eigenbasis.

- ⊩This matrix is always invertiable.

3.1.10.29. ⊩**Eigendecomposition** Let $\boldsymbol{V}$ be the matrix whose columns are the eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ...$, and then let $\boldsymbol{\Lambda}$ be the diagonal matrix whose diagonal entries are the eigenvalues $\lambda_1, \lambda_2, ...$, then we have

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}.$$

This is known as the eigendecomposition theorem.

3.1.10.30. **E.g.** For $\boldsymbol{A} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$, we have $\boldsymbol{V} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ thus $\boldsymbol{V}^{-1} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$ and $\boldsymbol{\Lambda} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$, we have

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

3.1.10.31. Why do we care about eigenvalue and eigenvectors?

- We have $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$
- Suppose we want to find $\boldsymbol{A}^{2024}$. You can find this by multiplying $\boldsymbol{A}$ 2024 times, or, you can use eigendecomposition
- $\boldsymbol{A}^{2024} = \underbrace{(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1})(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1})\cdots(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1})}_{\text{2024 times}} = \boldsymbol{V}\underbrace{\boldsymbol{\Lambda}\boldsymbol{\Lambda}\cdots\boldsymbol{\Lambda}}_{\text{2024 times}}\boldsymbol{V}^{-1} = \boldsymbol{V}\boldsymbol{\Lambda}^{2024}\boldsymbol{V}^{-1}$. Hence

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}^{2024} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}^{2024} \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5^{2024} & 0 \\ 0 & (-2)^{2024} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

In other words, one application of eigendecomposition is to compute matrix power

3.1.10.32. Here are two nontrivial facts

- determinant is the product of eigenvalues
- trace is the sum of eigenvalues

3.1.10.33. **E.g.** $\boldsymbol{A} = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$. We have $\det \boldsymbol{A} = 16$ and $\mathrm{Tr}\boldsymbol{A} = 8$. That means if we know λ_1, we have $\lambda_2 = \mathrm{Tr}\boldsymbol{A} - \lambda_1 = (\det \boldsymbol{A})/\lambda_1$. Calculate the characteristic equation gives $(\lambda - 4)^2$, so $4, 4$ are the eigenvalues, and they satisfy $\lambda_2 = \mathrm{Tr}\boldsymbol{A} - \lambda_1 = (\det \boldsymbol{A})/\lambda_1$

3.1.10.34. ⊩**Cayley-Hamilton** "All square matrix satisfies its own characteristic equation."

Given a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. The characteristic polynomial of the eigenvalues of $\boldsymbol{A}$ is

$$\begin{aligned} p_{\boldsymbol{A}}(\lambda) &= \det(\boldsymbol{A} - \lambda \boldsymbol{I}_n) \\ &= \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0. \end{aligned}$$

Cayley-Hamilton Theorem says that, if we put the matrix $\boldsymbol{A}$ into the characteristic polynomial, we get zero:

$$\begin{aligned} p_{\boldsymbol{A}}(\boldsymbol{A}) &= \boldsymbol{A}^n + a_{n-1}\boldsymbol{A}^{n-1} + ... + a_1\boldsymbol{A} + a_0\boldsymbol{I}. \\ &= \boldsymbol{0}_n \end{aligned}$$

Proof. Out of scope for a year 1 course.

3.1.10.35. **Cayley-Hamilton Theorem for 2-by-2**

Given a matrix $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the characteristic polynomial of $\boldsymbol{A}$'s eigenvalue is

$$\begin{aligned} p_{\boldsymbol{A}}(\lambda) &= \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 + \underbrace{(a+d)}_{\mathrm{Tr}\boldsymbol{A}}\lambda + \underbrace{ad-bc}_{\det \boldsymbol{A}} \end{aligned}$$

By Cayley-Hamilton Theorem,

$$\boldsymbol{A}^2 - \mathrm{Tr}(\boldsymbol{A})\boldsymbol{A} + \det(\boldsymbol{A})\boldsymbol{I} = \boldsymbol{0}_2. \qquad \text{(Cayley-Hamilton Theorem 2-by-2)}$$

3.1.10.36. **E.g.** $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$. $\mathrm{Tr}\boldsymbol{A} = 6$, $\det \boldsymbol{A} = 5$. Then $\boldsymbol{A}^2 - 6\boldsymbol{A} + 5\boldsymbol{I} = \begin{bmatrix} 7 & 6 \\ 18 & 19 \end{bmatrix} - \begin{bmatrix} 12 & 6 \\ 18 & 24 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3.1.10.37. **Application of Cayley-Hamilton Theorem.** Any polynomial in $\boldsymbol{A}$ can be expressed as linear combination of $\{\boldsymbol{I}, \boldsymbol{A}\}$.

E.g. $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. Then by (Cayley-Hamilton Theorem 2-by-2) we have $\boldsymbol{A}^2 = 2\boldsymbol{A} + \boldsymbol{I}$

$$\begin{aligned} \boldsymbol{A}^2 &= 2\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \end{aligned}$$

How do we find $\boldsymbol{A}^5$? You can do eigen-decompisition $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ and then $\boldsymbol{A}^5 = \boldsymbol{V}\boldsymbol{\Lambda}^5\boldsymbol{V}^{-1}$. But let's see how do we make use of Cayley-Hamliton theorem.

$$\begin{array}{rcl|l} \boldsymbol{A}^5 &=& (\boldsymbol{A}^2)^2\boldsymbol{A} & \square^5 = \square^{4+1} = \square^4\square = (\square^2)^2\square \\ &=& (2\boldsymbol{A}+\boldsymbol{I})^2\boldsymbol{A} & \boldsymbol{A}^2 = 2\boldsymbol{A}+\boldsymbol{I} \\ &=& 4\boldsymbol{A}^3 + 4\boldsymbol{A}^2 + \boldsymbol{A} & \text{expand the bracket} \\ &=& 4\boldsymbol{A}^2\boldsymbol{A} + 4\boldsymbol{A}^2 + \boldsymbol{A} & \text{create the term } \boldsymbol{A}^2 \text{ so that we can use } \boldsymbol{A}^2 = 2\boldsymbol{A}+\boldsymbol{I} \\ &=& 4(2\boldsymbol{A}+\boldsymbol{I})\boldsymbol{A} + 4(2\boldsymbol{A}+\boldsymbol{I}) + \boldsymbol{A} & \boldsymbol{A}^2 = 2\boldsymbol{A}+\boldsymbol{I} \\ &=& 8\boldsymbol{A}^2 + 4\boldsymbol{A} + 8\boldsymbol{A} + 4\boldsymbol{I} + \boldsymbol{A} & \\ &=& 8\boldsymbol{A}^2 + 13\boldsymbol{A} + 4\boldsymbol{I} & \\ &=& 8(2\boldsymbol{A}+\boldsymbol{I}) + 13\boldsymbol{A} + 4\boldsymbol{I} & \boldsymbol{A}^2 = 2\boldsymbol{A}+\boldsymbol{I} \\ &=& 29\boldsymbol{A} + 12\boldsymbol{I} & \\ &=& 29\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + 12\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} & \\ &=& \begin{bmatrix} 41 & 58 \\ 29 & 41 \end{bmatrix} & \end{array}$$

3.1.10.38. **Practice** $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, use Cayley-Hamilton Theorem to show that $\boldsymbol{A}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

3.1.10.39. **Example Cayley-Hamilton Theorem for 3-by-3**.

Suppose $\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & 1 \\ 2 & 5 & 4 \end{bmatrix}$, find $\boldsymbol{A}^{-1}$.

Here is the Cayley-Hamilton Theorem. Let $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ and $\boldsymbol{M}_{ij}$ the minor of $\boldsymbol{A}$ wrt. $(i,j)$the element. Then we have

$$\boldsymbol{A}^3 - \mathrm{Tr}(\boldsymbol{A})\boldsymbol{A}^2 + \Big(\det \boldsymbol{M}_{1,1} + \det \boldsymbol{M}_{2,2} + \det \boldsymbol{M}_{3,3}\Big)\boldsymbol{A} - \det(\boldsymbol{A})\boldsymbol{I} = \boldsymbol{0}. \qquad \text{(Cayley-Hamilton Theorem, 3-by-3)}$$

Now from $\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & 1 \\ 2 & 5 & 4 \end{bmatrix}$ we have

$$\boldsymbol{A}^3 - 10\boldsymbol{A}^2 + 20\boldsymbol{A} - 35\boldsymbol{I} = \boldsymbol{0}$$

Multiply the whole equation by $\boldsymbol{A}^{-1}$ gives

$$\boldsymbol{A}^2 - 10\boldsymbol{A} + 20\boldsymbol{I} - 35\boldsymbol{A}^{-1} = \boldsymbol{0}$$

Hence we have
$$A^{-1} = \frac{1}{35}A^2 - \frac{10}{35}A + \frac{20}{35}I$$

We do need to find A^2, which is $\begin{bmatrix}17 & 16 & 15\\ 26 & 18 & 15\\ 34 & 37 & 25\end{bmatrix}$, hence

$$A^{-1} = \frac{1}{35}\begin{bmatrix}17 & 16 & 15\\ 26 & 18 & 15\\ 34 & 37 & 25\end{bmatrix} - \frac{10}{35}\begin{bmatrix}3 & 1 & 2\\ 4 & 3 & 1\\ 2 & 5 & 4\end{bmatrix} + \frac{20}{35}\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix} = \frac{1}{35}\begin{bmatrix}7 & 6 & -5\\ -14 & 8 & 5\\ 14 & 13 & 5\end{bmatrix}$$

3.1.11 Geometry of linear algebra

3.1.11.1. Given a vector $u \in \mathbb{R}^n$ that is nonzero, the direction of u, called the unit vector, is $\hat{u} := \dfrac{u}{\|u\|}$

3.1.11.2. Given two vectors $u, v \in \mathbb{R}^n$, the angle between them is $\angle(u, v) := \cos^{-1}\dfrac{u^\top v}{\|u\|\|v\|}$

3.1.11.3. Given two vectors $u, v \in \mathbb{R}^n$, the Euclidean distance between them is $\mathrm{dist}(u, v) := \|u - v\|$

3.1.11.4. Given two vectors $u, v \in \mathbb{R}^n$, the Euclidean projection of u onto the direction of v is $\mathrm{proj}_{\hat{v}}(u) = \dfrac{u^\top v}{\|v\|^2}v$

How to understand this: $\dfrac{u^\top v}{\|v\|^2}v = \left(u^\top \dfrac{v}{\|v\|}\right)\dfrac{v}{\|v\|} = (u^\top \hat{v})\hat{v}$

- $\hat{v}$ is the direction of v, so it is a unit vector with unit norm
- $u^\top \hat{v}$ is the amount of u projected along the direction of $\hat{v}$

3.1.11.5. Line in $\mathbb{R}^n$. The line L passing through the point $p = (p_1, p_2, ..., p_n)$ in the direction $v \neq \mathbf{0}$ is the set $L := \Big\{p + tv \ : \ t \in \mathbb{R}\Big\}$

3.1.11.6. Inner product is describing a plane. Given a vector $n \in \mathbb{R}^n$ and a constant $c \in \mathbb{R}$, the equation $n^\top x = c$ is describing a hyperplane

- the plane has an orientation (angle), specified by its normal vector n
- the plane has an offset, specified by its constant c

3.1.11.7. In $\mathbb{R}^2$, two linear equations $\begin{cases} a_{11}x + a_{12}y = b_1\\ a_{21}x + a_{22}y = b_2\end{cases}$ are describing two straight lines.

- $\det\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \neq 0 \iff$ the two lines intersect $\iff$ a solution exists for the linear system
- $\det\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} = 0 \iff$ the two lines are parallel $\iff$ a solution does not exist for the linear system

3.1.11.8. In $\mathbb{R}^3$, three linear equations $\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1\\ a_{21}x + a_{22}y + a_{23}z = b_2\\ a_{31}x + a_{32}y + a_{33}z = b_3\end{cases}$ are describing three planes.

- $\det A \neq 0 \iff$ the three planes intersect $\iff$ a solution exists for the linear system
- $\det A = 0 \iff$ the three planes are parallel $\iff$ a solution does not exist for the linear system

3.1.11.9. In $\mathbb{R}^3$, three linear equations describing three planes have the following 8 possibilities

case 1. Single point, unique solution. The three planes intersect at one point.

case 2a. Infinite many solution, the three planes intersect along a line.

case 2b. Infinite many solution, two planes are coincident, the third plane cuts through the two planes.

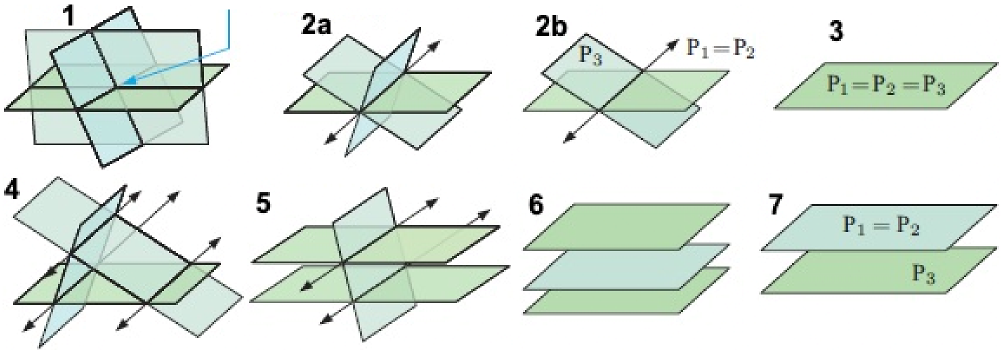
case 3. Infinite many solution, the three planes are identical that coincide with each other

case 4. no solution, the three planes form a triangular prism

case 5. no solution, two non-coincident parallel planes each intersect a third plane.

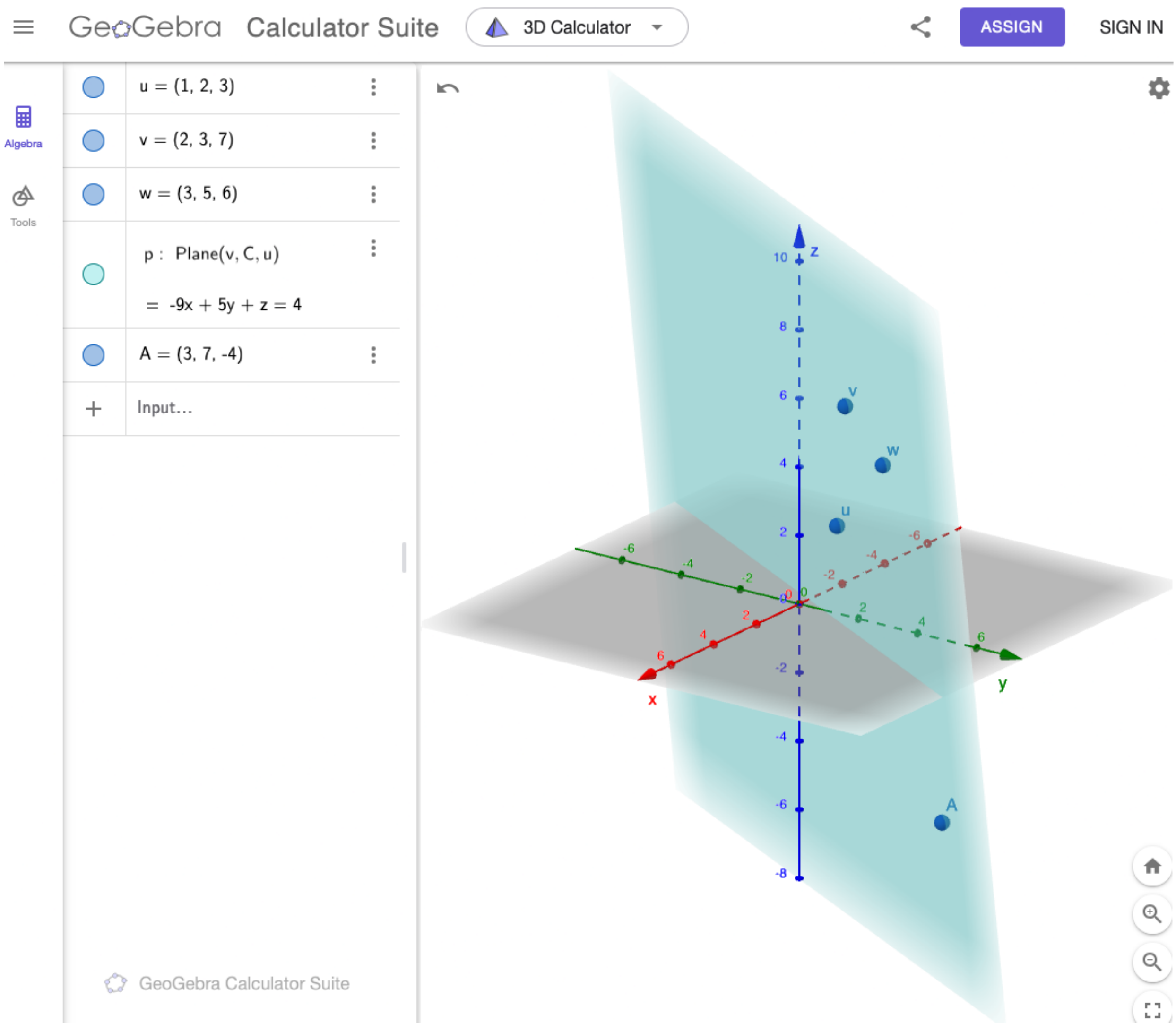
case 6. no solution, the three planes are parallel and non-coincident

case 7. no solution, two planes are coincident and parallel to the third plane



3.1.11.10. **Visualization tool: GeoGebra**

- `https://www.geogebra.org/calculator`
- Click 3D Calculator
- Click Tools to make a 3D plane



3.2 Abstract linear algebra

3.2.1 Vector space

3.2.1.1. **What is linear algebra anyway?**
Linear algebra is not about matrix multiplication, determinant, inverse. The core of linear algebra is vector space.

3.2.1.2. **What is a vector anyway?**
In 3.1.1.2. we said it is wrong to understand vector as a list of number. We now explain it why.

- When you writ a vector as a tuple, you're implicitly assuming a thing: the basis. You are assuming the vector is written as the expansion of a particular basis.

- There are infinitely many basis
- Note that this basis is very often not even specified.
- Not specifying the basis is often ok, because the "normal vector operations" don't in fact depend on the basis, i.e. if you transformed all you vectors to be written out in some other basis, you would have all different numbers but the same calculations would still yield correct results.
- In other words, the nice properties of linear algebra is actually this fact that "you don't need to specify a basis", because the "symmetry" of the linear algebra allows you to be "careless about the specification of basis"
- A vector is not a tuple of individual components. A vector is an element of some vector space.
- This is the same for matrices. "Matrix" and "linear mapping" are not synonyms, they are in fact not the same: matrices refer to a particular basis while linear mappings need no such thing.

3.2.1.3. **Def (Abstract vector space)** A *vector space* is a set V along with an addition $+$ on V and a scalar multiplication on V such that the following hold

- $\boldsymbol{u}+\boldsymbol{w}=\boldsymbol{w}+\boldsymbol{u}$ for all $\boldsymbol{u},\boldsymbol{w}\in V$ — commutativity
- $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$ and $(ab)\boldsymbol{u}=a(b\boldsymbol{u})$ for all $\boldsymbol{u},\boldsymbol{w}\in V$ and all $a,b\in\mathbb{R}$ — associativity
- there exists $\mathbf{0}\in V$ such that $\boldsymbol{u}+\mathbf{0}=\boldsymbol{u}$ for all $\boldsymbol{u}\in V$ — additive identity
- for all $\boldsymbol{u}\in V$, there exists $\boldsymbol{w}\in V$ such that $\boldsymbol{u}+\boldsymbol{w}=\mathbf{0}$ — additive inverse
- $1\boldsymbol{u}=\boldsymbol{u}$ for all $\boldsymbol{u}\in V$ — multiplicative identity
- $a(\boldsymbol{u}+\boldsymbol{w})=a\boldsymbol{u}+a\boldsymbol{w}$ and $(a+b)\boldsymbol{u}=a\boldsymbol{u}+b\boldsymbol{u}$ and $(ab)\boldsymbol{u}=a(b\boldsymbol{u})$ for all and $a,b\in\mathbb{R}$ all $\boldsymbol{u},\boldsymbol{w}\in V$ — distributive

The first 4 is saying V is a Abelian group (commutative group) under addition.

- addition requires no parenthesis and is order independent
- zero vector is unique, the negative is also unique
- cancellation law: if $\boldsymbol{u}+\boldsymbol{w}=\boldsymbol{v}+\boldsymbol{w}$ then $\boldsymbol{u}=\boldsymbol{v}$

3.2.1.4. **E.g.** $\mathbb{R}^n$ is the standard n-dimensional space over real numbers: the space consisting of n-tuples of real numbers

3.2.1.5. **E.g.** The set $\mathbb{R}^3$ with the standard basis $\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3$, vector addition and scalar multiplication, forms a vector space.

3.2.1.6. **E.g.** (Fourier series). The set

$$1(x),\sqrt{2}\sin(x),\sqrt{2}\sin(2x),\sqrt{2}\sin(3x),...,\sqrt{2}\cos(x),\sqrt{2}\cos(2x),\sqrt{2}\cos(3x),...,$$

is a vector space (for function).

3.2.1.7. **E.g.** All polynomials up to degree n form a vector space.

3.2.1.8. **E.g.** All n-by-m real matrices form a vector space.

3.2.1.9. **E.g.** All functions $f:\mathbb{R}\to\mathbb{R}$ form a vector space.

3.2.1.10. **Def (subspace)** A non-empty subset U of a vector space V is called the *subspace* of V

3.2.1.11. **Def (generator and span)** Given a vector space V, a set of vectors $\{\boldsymbol{v}_1,...,\boldsymbol{v}_r\}$ is called a *generator* of V if all vectors in V can be expressed as a linear combination (3.1.1.18.) of $\{\boldsymbol{v}_1,...,\boldsymbol{v}_r\}$. We write $V=\mathrm{Span}\{\boldsymbol{v}_1,...,\boldsymbol{v}_r\}$, meaning "the vectors $\{\boldsymbol{v}_1,...,\boldsymbol{v}_r\}$ *span* (generate) the space V"

- Generator vs basis
  - both are name of a set of vectors
  - basis: linear independence
  - generator: linear combination
  - it is possible to have a generating set that is not linearly independent

3.2.1.12. **E.g.** of linearly dependent set that is a generating set. In $\mathbb{R}^2$, the set $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis since the vectors in the set is linearly independent. The basis is a generating set, it generate all vectors in $\mathbb{R}^2$. That is, for all $\boldsymbol{x}\in\mathbb{R}^n$, we have some $\alpha,\beta\in\mathbb{R}$ that $\boldsymbol{x}=\alpha\boldsymbol{e}_1+\beta\boldsymbol{e}_2$.

For the set $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is not a basis because the second vector is linear combination of the other vectors. However this set is a generator of $\mathbb{R}^2$, for all $\boldsymbol{x}\in\mathbb{R}^n$, we have some $\alpha,\beta,\gamma\in\mathbb{R}$ that $\boldsymbol{x}=\alpha\boldsymbol{e}_1+\beta\boldsymbol{e}_2+\gamma\begin{bmatrix}1\\1\end{bmatrix}$. In fact, since we know that $\begin{bmatrix}1\\1\end{bmatrix}=\boldsymbol{e}_1+\boldsymbol{e}_2$, we get $\boldsymbol{x}=\alpha\boldsymbol{e}_1+\beta\boldsymbol{e}_2+\gamma\begin{bmatrix}1\\1\end{bmatrix}=(\alpha+\gamma)\boldsymbol{e}_1+(\beta+\gamma)\boldsymbol{e}_2$.

3.2.1.13. Given a set of vectors $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_r\}$ that is possibly not a generating set, the subspace generated by these vectors is denoted as $\mathrm{Span}\{\boldsymbol{v}_1, ..., \boldsymbol{v}_r\}$

3.2.1.14. **E.g.** of span. Consider $\mathbb{R}^2$, the set $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\}$ can only span the x-axis.

3.2.1.15. **Extra (Hilbert space)** First recall vector space. In 2-dimensional vector space, given two vectors $\boldsymbol{x} = [x_1, x_2]$, $\boldsymbol{y} = [y_1, y_2]$. We have the inner product $\boldsymbol{x}^\top \boldsymbol{y} = x_1y_1 + x_2y_2 = \displaystyle\sum_{i=1}^{2} x_iy_i$ and the norm $\|\boldsymbol{x}\|_2 = \sqrt{\boldsymbol{x}^\top \boldsymbol{x}} = \sqrt{\displaystyle\sum_{i=1}^{2} x_i^2}$.
Now instead of 2-dimensional vector, we have ∞-dimensional vector: $\boldsymbol{x} = [x_1, x_2, ...]$, $\boldsymbol{y} = [y_1, y_2, ...]$. We consider inner product $\boldsymbol{x}^\top \boldsymbol{y} = \displaystyle\sum_{i=1}^{\infty} x_iy_i$ and $\|\boldsymbol{x}\|_2 = \sqrt{\displaystyle\sum_{i=1}^{\infty} x_i^2}$. We further want the norm to be finite so $\|\boldsymbol{x}\|_2 = \sqrt{\displaystyle\sum_{i=1}^{\infty} x_i^2} < \infty$.
Then infinite-dimensional vector space satisfying these conditions is called a *Hilbert space*. In other words, Hilbert space is a generalization of vector space to infinite dimension (in which vector is now called *function*).

3.2.1.16. **E.g. (Deciding whether a vector is in the linear combination of others)**
Given $\boldsymbol{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix}2\\3\\7\end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix}3\\5\\6\end{bmatrix}$. Is the vector $\boldsymbol{x} = \begin{bmatrix}3\\7\\-4\end{bmatrix}$ in the span of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$?

$$\begin{aligned}
\boldsymbol{x} \in \text{span}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \quad &\iff \quad \boldsymbol{x} \text{ is a linear combination of } \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \\
&\iff \quad \boldsymbol{x} = \alpha\boldsymbol{u} + \beta\boldsymbol{v} + \gamma\boldsymbol{w} \text{ for some } \alpha, \beta, \gamma \in \mathbb{R} \\
&\iff \quad \begin{bmatrix}3\\7\\-4\end{bmatrix} = \alpha\begin{bmatrix}1\\2\\3\end{bmatrix} + \beta\begin{bmatrix}2\\3\\7\end{bmatrix} + \gamma\begin{bmatrix}3\\5\\6\end{bmatrix} \\
&\iff \quad \begin{bmatrix}1&2&3\\2&3&5\\3&7&6\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} = \begin{bmatrix}3\\7\\-4\end{bmatrix}
\end{aligned}$$

Solving the linear system gives $\alpha = 2, \beta = -4, \gamma = 3$, so $\boldsymbol{x} \in \text{span}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$

Use the 3D calculator in `https://www.geogebra.org/calculator` to visualize.

The picture is in 3.1.11.10.

3.2.1.17. The meaning of spanning set / generator

- The generators $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_r\}$ tell "what can you possibility get"
- Everything within span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_r\}$ can be written as a particular linear combination of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_r\}$
- Everything outside span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_r\}$ can not be written as linear combination of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_r\}$
- If span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_r\}$ is not able to "capture" everything, then we have a *subspace*

3.2.1.18. **Subspace** Let V be a vector space. If W is a subset of V, then W is a subspace of V if W itself is also a vector space (following the same addition and multiplication)

3.2.1.19. **E.g.** In $\mathbb{R}^3$, the three vectors $\begin{bmatrix}0.2\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}1/\sqrt{2}\\0.2\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$ is generating the xy-plane, and the xy-plane is only a subspace of the whole $\mathbb{R}^3$

3.2.1.20. **Column space** The expression $\boldsymbol{Ax}$ is known as the column space of $\boldsymbol{A}$, written as Col($\boldsymbol{A}$) or ColSpan($\boldsymbol{A}$)

$$\text{Col}(\boldsymbol{A}) := \Big\{\boldsymbol{Ax} \;:\; \boldsymbol{x} \in \mathbb{R}^n\Big\}$$

How to see this: based on the column-thinking of matrix-vector product (3.1.5.16.), $\boldsymbol{Ax}$ is just all the possible linear combination of the columns in $\boldsymbol{A}$

3.2.1.21. For a matrix, column space = image = range

3.2.1.22. **Basis** Let V be a vector space and $\{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_r\}$ be a set of vectors. If every $\boldsymbol{v} \in \boldsymbol{V}$ can be written uniquely as a linear combination of $\{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_r\}$, then we call $\{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_r\}$ a basis of V

- the word "uniquely" implies $\{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_r\}$ are linearly independent
- in other words, a basis is a linearly independent spanning set.

3.2.1.23. $\Vdash$**Existence of basis of vector space** All vector space has a basis.
Proof. Too deep for a year one course. The proof involves the use of Zorn's lemma (equivalently, Axiom of Choice).

3.2.1.24. **Coordinate** In the example 3.2.1.16., the α, β, γ are called the *coordinate* of $\boldsymbol{x}$ in span$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$

- If we use the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ then we have the coordinate of $\boldsymbol{x}$ as $(3, 7, -4)$.

- If we use the basis $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, then we have the coordinate of $\boldsymbol{x}$ as $(2, -4, 3)$.

- Since we are working on the same $\boldsymbol{x}$, having two coordinate system means that there is a transformation (a linear function) that connects the two basis (and in fact such a transformation is a matrix)

3.2.1.25. **Dimension** Let V be a vector space with a basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_r\}$. The number of vectors in $\{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_r\}$ is the dimension of the vector space.

3.2.1.26. **E.g.** $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$, this basis is the "simplest" one and is known as the standard basis, canonical basis. $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right\}$ is another basis of $\mathbb{R}^2$. Both bases have two vectors, so the dimension of $\mathbb{R}^2$ is 2.

3.2.1.27. **E.g.** The matrices $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^{2\times 3}$. The dimension of $\mathbb{R}^{2\times 3}$ is 6 because the spanning set has 6 elements.

3.2.1.28. **E.g.** (Integers form a vector space) For the set of natural number less than or equal to 12 $\{n \in \mathbb{N} \ : \ n \leq 12\}$, we have the following prime numbers

$$2, 3, 5, 7, 11$$

Every integer $n \in \mathbb{N}$ below 12 can be written as a prime factorization

$$\begin{aligned} 1 &= 2^0 \cdot 3^0 \cdot 5^0 \cdot 7^0 \cdot 11^0 \\ 2 &= 2^1 \cdot 3^0 \cdot 5^0 \cdot 7^0 \cdot 11^0 \\ 3 &= 2^0 \cdot 3^1 \cdot 5^0 \cdot 7^0 \cdot 11^0 \\ 4 &= 2^2 \cdot 3^0 \cdot 5^0 \cdot 7^0 \cdot 11^0 \end{aligned}$$

Therefore $\{2, 3, 5, 7, 11\}$ span all the natural number below 12 and is a basis. The dimension of the natural number below 12 is the number of element in the spanning set, so the dimension of $\{n \in \mathbb{N} \ : \ n \leq 12\}$ is 5.

3.2.1.29. **E.g.** Chemical spectra form a vector space.

3.2.1.30. **Rank of a matrix** The rank of a matrix $\boldsymbol{A}$, written $\operatorname{rank}(\boldsymbol{A})$, is the maximum number of linearly independent columns of $\boldsymbol{A}$. Equivalently, the dimension of the column space of $\boldsymbol{A}$

3.2.1.31. ⊩**Invertible matrix theorem.** If $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, then following are all equivalent (either all true or all false for $\boldsymbol{A}$)

$$\begin{array}{lll}
 & 1 & \boldsymbol{A} \text{ is non-singular } \iff \boldsymbol{A}^{-1} \text{ exists} \\
\iff & 2a & \boldsymbol{A} \text{ has left inverse under matrix multiplication } \iff \text{ there exists } \boldsymbol{B} \text{ such that } \boldsymbol{BA} = \boldsymbol{I} \\
\iff & 2b & \boldsymbol{A} \text{ has right inverse under matrix multiplication } \iff \text{ there exists } \boldsymbol{C} \text{ such that } \boldsymbol{AC} = \boldsymbol{I} \\
\iff & 2c & \boldsymbol{A} \text{ has inverse under matrix multiplication } \iff \text{ there exists } \boldsymbol{D} \text{ such that } \boldsymbol{DA} = \boldsymbol{AD} = \boldsymbol{I} \\
\iff & 3a & \text{the reduced row-echelon form of } \boldsymbol{A} \text{ is } \boldsymbol{I}_n \\
\iff & 3b & \text{the reduced column-echelon form of } \boldsymbol{A} \text{ is } \boldsymbol{I}_n \\
\iff & 3c & \boldsymbol{A} \text{ has } n \text{ pivots} \\
\iff & 3d & \boldsymbol{A} \text{ can be expressed as a product of elementary matrices } \iff \boldsymbol{A} = \boldsymbol{E}_1\boldsymbol{E}_2\cdots \\
\iff & 4a & \boldsymbol{A} : \boldsymbol{x} \mapsto \boldsymbol{Ax} \text{ is surjective } \iff \text{ there exists at least one } \boldsymbol{x} \text{ that solves } \boldsymbol{Ax} = \boldsymbol{b} \\
\iff & 4b & \boldsymbol{A} : \boldsymbol{x} \mapsto \boldsymbol{Ax} \text{ is injective } \iff \text{ there exists at most one } \boldsymbol{x} \text{ that solves } \boldsymbol{Ax} = \boldsymbol{b} \\
\iff & 4c & \boldsymbol{A} : \boldsymbol{x} \mapsto \boldsymbol{Ax} \text{ is bijective } \iff \text{ there exists exactly one } \boldsymbol{x} \text{ that solves } \boldsymbol{Ax} = \boldsymbol{b} \\
\iff & 4d & \boldsymbol{Ax} = \boldsymbol{b} \text{ is consistent for all } \boldsymbol{b} \in \mathbb{R}^n \\
\iff & 5a & \text{columns of } \boldsymbol{A} \text{ are linearly independent} \\
\iff & 5b & \text{rows of } \boldsymbol{A} \text{ are linearly independent} \\
\iff & 5c & \text{columns of } \boldsymbol{A} \text{ span } \mathbb{R}^n \\
\iff & 5d & \text{rows of } \boldsymbol{A} \text{ span } \mathbb{R}^n \\
\iff & 5e & \text{columns of } \boldsymbol{A} \text{ form a basis for } \mathbb{R}^n \\
\iff & 5f & \text{rows of } \boldsymbol{A} \text{ form a basis for } \mathbb{R}^n \\
\iff & 6a & \boldsymbol{A} \text{ has rank } n \\
\iff & 6b & \boldsymbol{A} \text{ has nullity } 0 \\
\iff & 6c & \boldsymbol{Ax} = \boldsymbol{0} \text{ has only the trivial solution } \boldsymbol{x} = \boldsymbol{0} \quad \iff \quad \ker\boldsymbol{A} = \{\boldsymbol{0}_n\} \\
\iff & 6d & \text{the orthogonal complement of the null space of } \boldsymbol{A} \text{ is } \mathbb{R}^n \\
\iff & 6e & \text{the orthogonal complement of the row space of } \boldsymbol{A} \text{ is } \boldsymbol{0}_n \\
\iff & 6f & \text{the range of } \boldsymbol{A} \text{ is } \mathbb{R}^n \\
\iff & 7a & \det(\boldsymbol{A}) \neq 0 \\
\iff & 7b & \text{the eigenvalues of } \boldsymbol{A} \text{ contains no zero } \iff 0 \text{ is not an eigenvalue of } \boldsymbol{A} \\
\iff & 8a & \boldsymbol{A}^\top \text{ is invertible} \\
\iff & 8b & \boldsymbol{A}^\top\boldsymbol{A} \text{ is invertible} \\
\iff & 8c & \boldsymbol{AA}^\top \text{ is invertible}
\end{array}$$

3.2.1.32. Set addition, revision. Let U, W be subsets of a vector space V. Then

- $U + W = \{\boldsymbol{u} + \boldsymbol{w} \ : \ \boldsymbol{u} \in U, \boldsymbol{w} \in W\}$
- $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$

3.2.1.33. **Direct sum** A vector space V is the direct sum of its subspaces U and W, written as $V = U \oplus W$, if all $\boldsymbol{v} \in V$ can be written uniquely as $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ for $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$

3.2.1.34. **E.g.** For vector space $V = \mathbb{R}^3$, consider $U = \{(a, b, 0) : a, b \in \mathbb{R}\}$, $W = \{(0, c, d) : c, d \in \mathbb{R}\}$. Then $V = U + W$ because every vector in V is a sum of vector in U and a vector of W. However $V \neq U \oplus W$ because there are vectors in V that can have more than one sum. E.g., $(1, 2, 1) = (1, 1, 0) + (0, 1, 1) = (1, -1, 0) + (0, 3, 1) = (1, 0, 0) + (0, 2, 1) = \ldots$

3.2.1.35. **E.g.** xyz space in $\mathbb{R}^3$ = x-line $\oplus$ y-line $\oplus$ z-line

3.2.1.36. **E.g.** Cylinder $\{(x, y, z) \ : \ x^2 + y^2 = 1, z \in \mathbb{R}\}$ = xy-circle $\oplus$ z-line

3.2.1.37. **E.g.** The matrices $U = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right\}$, $V = \left\{\begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}\right\}$ in $\mathbb{R}^{2\times 2}$. Then $U + W = \left\{\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\right\}$. Note that we do not write $U + W = \left\{\begin{bmatrix} a + c & b \\ c & 0 \end{bmatrix}\right\}$ because the meaning of a, b, c here are just dummy variables. We also have $U \cap W = \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right\}$. We have $\dim U = 2, \dim W = 2, \dim(U \cap W) = 1$ and therefore $\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 2 + 2 - 1 = 3$

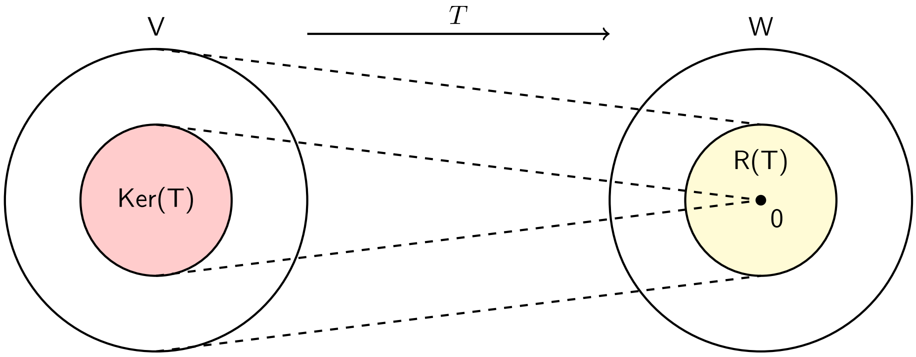
3.2.1.38. ⊩If $V = W_1 + W_2 + \cdots + W_r$, then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$ if and only if $\dim V = \sum_k \dim W_k$

- ($\implies$) If $\dim V = \sum_k \dim W_k$, then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$.
- ($\impliedby$) If $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$, then $\dim V = \sum_k \dim W_k$.

3.2.1.39. ⊩If $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$, then suppose B_k is a basis for W_k, then $\bigcup_k S_k$ is a basis of V

3.2.1.40. **Kernel** Suppose $\boldsymbol{T} : V \to \boldsymbol{W}$. The set of vector $\boldsymbol{x} \in V$ such that $\boldsymbol{Tx} = \boldsymbol{0} \in W$ is called the kernel of $\boldsymbol{T}$

- we write Ker$\boldsymbol{T}$
- another name is null space, Null$\boldsymbol{T}$
- Note that kernel is a subset of the domain, not on the range
- Since $\boldsymbol{x} \in$Ker$\boldsymbol{T}$ is mapped to $\mathbf{0} \in W$, so we have the following figure



Recall R is the range of $\boldsymbol{T}$, also called the image, written $\mathrm{Im}\boldsymbol{T}$ The set W is the codomain of $\boldsymbol{T}$.

3.2.1.41. Finding eigenvector $\boldsymbol{Ax} = \lambda \boldsymbol{x} \iff (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \mathbf{0} \iff \boldsymbol{x} \in$Ker$(\boldsymbol{A} - \lambda \boldsymbol{I})$

3.2.1.42. $\boldsymbol{AB} = \mathbf{0} \iff \mathrm{Im}(\boldsymbol{B}) \subset \mathrm{Ker}(\boldsymbol{A})$

($\implies$) Suppose $\boldsymbol{AB} = \mathbf{0}$. Let $\boldsymbol{x} \in \mathrm{Im}(\boldsymbol{B})$. Then $\boldsymbol{x} = \boldsymbol{By}$ for some $\boldsymbol{y}$. So

$$\boldsymbol{Ax} = \boldsymbol{A}(\boldsymbol{By}) = (\boldsymbol{AB})\boldsymbol{y} = \mathbf{0}\boldsymbol{y} = \mathbf{0}$$

($\impliedby$) Suppose $\mathrm{Im}(\boldsymbol{B}) \subset \mathrm{Ker}(\boldsymbol{A})$. Then for all $\boldsymbol{x}$, $\boldsymbol{Bx} \in \mathrm{Im}(\boldsymbol{B})$ and thus $\boldsymbol{Bx} \in \mathrm{Ker}(\boldsymbol{A})$, i.e, $\boldsymbol{ABx} = \mathbf{0}$. Since $\boldsymbol{x}$ is any vector, thus $\boldsymbol{AB} = \mathbf{0}$.

3.2.1.43. Suppose S is the set of solution $\boldsymbol{x}$ that satisfies $\boldsymbol{Ax} = \boldsymbol{b}$. Let $W = \mathrm{Ker}\boldsymbol{A}$. Then any vector in the set $S + W$ also solves $\boldsymbol{Ax} = \boldsymbol{b}$. In fact, the general solution set of any linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has the following form

$$\Big\{\boldsymbol{x} + \boldsymbol{y} \ : \ \boldsymbol{Ax} = \boldsymbol{b}, \ \boldsymbol{y} \in \mathrm{Ker}\boldsymbol{A}\Big\}$$

3.2.1.44. For a matrix $\boldsymbol{A}$, the matrix multiplication $\boldsymbol{AX}$ for any matrix $\boldsymbol{X}$ has the general form $\boldsymbol{A}(\boldsymbol{X} + \boldsymbol{Y})$ where $\boldsymbol{Y} \in \mathrm{Ker}\boldsymbol{A}$.

3.2.1.45. ⊩**Fundamental Theorem of Linear Algebra** Let $T : V \to W$ be a linear transformation between two vector spaces. Then

$$\dim(\mathrm{Im}T) + \dim(\mathrm{Ker}T) = \dim(\mathrm{dom}T)$$

or equivalently

$$\mathrm{rank}(T) + \dim(\mathrm{Ker}T) = \dim V$$

This is the cover image.

3.3 Algorithms in linear algebra

3.3.0.1. Given a set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$, how to find the number of independent vectors?

Algorithm 1: Find the number of independent vectors

1 Form a matrix $\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n]$ by placing the vectors as columns
2 Find the rank of $\boldsymbol{V}$

3.3.0.2. Given a set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$, how to find a maximal linearly independent subset?

Algorithm 2: Find a maximal linearly independent subset

1 Form a matrix $\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n]$ by placing the vectors as columns
2 Use Gaussian elimination to reduce $\boldsymbol{V}$ to row-echelon form $\boldsymbol{W}$
3 Identify the columns index j of $\boldsymbol{W}$ that contain the leading 1s (the pivots)
4 The columns $\boldsymbol{V}_{:j}$ for j identified in step 3 form a maximal linearly independent set of the vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$

3.3.0.3. Given a set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$, how to find a maximal linearly independent subset?

3.3.0.4. **Power iteration** Given a matrix $\boldsymbol{A}$, how to we compute its eigenvector and eigenvalue?

Algorithm 3: Find a maximal linearly independent subset

Let set A represents $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ and set $B = \varnothing$
Remove from A any repetitions and all zero vectors
while $A \neq \varnothing$ **do**
  Pick a vector $\boldsymbol{v}$ from A
  if $\boldsymbol{v} \in$ *span*B **then**
    $A \leftarrow A \setminus \{\boldsymbol{v}\}$
  else
    $A \leftarrow A \setminus \{\boldsymbol{v}\}$
    $B \leftarrow B \cup \{\boldsymbol{v}\}$
  B contains a maximal linearly independent subset of A

Algorithm 4: Power iteration

for $k = 1, 2, 3, ...$ **do**
  $\boldsymbol{x}_{k+1} = \dfrac{\boldsymbol{A}\boldsymbol{x}_k}{\|\boldsymbol{A}\boldsymbol{x}_k\|_2}$

3.3.0.5. $\Vdash$If $\boldsymbol{A}$ has eigenvector $\boldsymbol{v}_1$ with the largest eigenvalue $\lambda_1 \in \mathbb{R}$, then for $\boldsymbol{x}_k$, $k \in \mathbb{N}$ produced by Algorithm 4, we have $\lim\limits_{k\to\infty} \boldsymbol{x}_k = \boldsymbol{v}_1$.

Proof First we show that $\boldsymbol{x}_k = \dfrac{\boldsymbol{A}^k\boldsymbol{x}_0}{\|\boldsymbol{A}^k\boldsymbol{x}_0\|_2}$

$$
\begin{aligned}
\boldsymbol{x}_k &= \frac{\boldsymbol{A}\boldsymbol{x}_{k-1}}{\|\boldsymbol{A}\boldsymbol{x}_{k-1}\|_2} && \text{by definition of the algorithm on } k \\
&= \frac{\boldsymbol{A}\dfrac{\boldsymbol{A}\boldsymbol{x}_{k-2}}{\|\boldsymbol{A}\boldsymbol{x}_{k-2}\|_2}}{\left\|\boldsymbol{A}\dfrac{\boldsymbol{A}\boldsymbol{x}_{k-2}}{\|\boldsymbol{A}\boldsymbol{x}_{k-2}\|_2}\right\|_2} && \text{by definition of the algorithm on } k-1 \\
&= \frac{\boldsymbol{A}\dfrac{\boldsymbol{A}\boldsymbol{x}_{k-2}}{\|\boldsymbol{A}\boldsymbol{x}_{k-2}\|_2}}{\dfrac{1}{\|\boldsymbol{A}\boldsymbol{x}_{k-2}\|_2}\|\boldsymbol{A}\boldsymbol{A}\boldsymbol{x}_{k-2}\|_2} && \|c\boldsymbol{x}\|_2 = c\|\boldsymbol{x}\|_2 \\
&= \frac{\boldsymbol{A}^2\boldsymbol{x}_{k-2}}{\|\boldsymbol{A}^2\boldsymbol{x}_{k-2}\|_2} && \boldsymbol{A}\boldsymbol{A} = \boldsymbol{A}^2 \text{ and cancel the denominator} \\
&= \frac{\boldsymbol{A}^3\boldsymbol{x}_{k-3}}{\|\boldsymbol{A}^3\boldsymbol{x}_{k-3}\|_2} = \frac{\boldsymbol{A}^4\boldsymbol{x}_{k-4}}{\|\boldsymbol{A}^4\boldsymbol{x}_{k-4}\|_2} = \cdots = \frac{\boldsymbol{A}^k\boldsymbol{x}_0}{\|\boldsymbol{A}^k\boldsymbol{x}_0\|_2} && \text{apply the same logic}
\end{aligned}
$$

Now by assumption that $\boldsymbol{A}$ has eigendecomposition, we have $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$

$$
\begin{aligned}
\boldsymbol{x}_k = \frac{\boldsymbol{A}^k\boldsymbol{x}_0}{\|\boldsymbol{A}^k\boldsymbol{x}_0\|_2} &= \frac{(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1})^k\boldsymbol{x}_0}{\|(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1})^k\boldsymbol{x}_0\|_2} && \text{by } \boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1} \\
&= \frac{\boldsymbol{V}\boldsymbol{\Lambda}^k\boldsymbol{V}^{-1}\boldsymbol{x}_0}{\|\boldsymbol{V}\boldsymbol{\Lambda}^k\boldsymbol{V}^{-1}\boldsymbol{x}_0\|_2} && \text{by property of eigen-matrices} \\
&= \frac{\boldsymbol{V}\boldsymbol{\Lambda}^k\boldsymbol{V}^{-1}(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n)}{\|\boldsymbol{V}\boldsymbol{\Lambda}^k\boldsymbol{V}^{-1}(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n)\|_2} && \text{eigenvectors form a basis, express } \boldsymbol{x} \text{ in such basis} \\
&= \frac{\boldsymbol{V}\boldsymbol{\Lambda}^k(c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)}{\|\boldsymbol{V}\boldsymbol{\Lambda}^k(c_1\boldsymbol{v}_1 + c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)\|_2} && \text{left-multiply } \boldsymbol{V}^{-1} \text{ map back to standard basis} \\
&= \frac{\boldsymbol{V}\boldsymbol{\Lambda}^k c_1\boldsymbol{e}_1 + \boldsymbol{V}\boldsymbol{\Lambda}^k(c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)}{\|\boldsymbol{V}\boldsymbol{\Lambda}^k c_1\boldsymbol{e}_1 + \boldsymbol{V}\boldsymbol{\Lambda}^k(c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)\|_2} && \text{distribute the multiplication} \\
&= \frac{c_1}{|c_1|}\frac{\boldsymbol{V}\boldsymbol{\Lambda}^k\boldsymbol{e}_1 + \frac{1}{c_1}\boldsymbol{V}\boldsymbol{\Lambda}^k(c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)}{\left\|\boldsymbol{V}\boldsymbol{\Lambda}^k\boldsymbol{e}_1 + \frac{1}{c_1}\boldsymbol{V}\boldsymbol{\Lambda}^k(c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)\right\|_2} && \text{factor out } c_1 \text{ and using } \|c\boldsymbol{x}\| = |c| \cdot \|\boldsymbol{x}\|
\end{aligned}
$$

Now we focus on the term $\boldsymbol{V}\boldsymbol{\Lambda}^k\boldsymbol{e}_1$

$$
\begin{aligned}
\boldsymbol{V}\boldsymbol{\Lambda}^k\boldsymbol{e}_1 &= \begin{bmatrix} | & | & \cdots & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \\ | & | & \cdots & | \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}^k \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\boldsymbol{e}_1} \\
&= \begin{bmatrix} | & | & \cdots & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \\ | & | & \cdots & | \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{power of diagonal matrix} \\
&= \begin{bmatrix} | & | & \cdots & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \\ | & | & \cdots & | \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1^k \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_1^k \boldsymbol{v}_1
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\boldsymbol{x}_k &= \frac{c_1}{|c_1|} \frac{\lambda_1^k\boldsymbol{v}_1 + \frac{1}{c_1}\boldsymbol{V}\boldsymbol{\Lambda}^k(c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)}{\left\|\lambda_1^k\boldsymbol{v}_1 + \frac{1}{c_1}\boldsymbol{V}\boldsymbol{\Lambda}^k(c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)\right\|_2} \\
&= \left(\frac{\lambda_1}{|\lambda_1|}\right)^k \frac{c_1}{|c_1|} \frac{\boldsymbol{v}_1 + \frac{1}{c_1}\boldsymbol{V}\left(\frac{1}{\lambda_1}\boldsymbol{\Lambda}\right)^k(c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)}{\left\|\boldsymbol{v}_1 + \frac{1}{c_1}\boldsymbol{V}\left(\frac{1}{\lambda_1}\boldsymbol{\Lambda}\right)^k(c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)\right\|_2} \qquad \text{factor out } \lambda_1^k \text{ and using } \|c\boldsymbol{x}\| = |c| \cdot \|\boldsymbol{x}\|
\end{aligned}
$$

Now consider the term $\left(\frac{1}{\lambda_1}\boldsymbol{\Lambda}\right)^k$

$$
\left(\frac{1}{\lambda_1}\boldsymbol{\Lambda}\right)^k = \left(\frac{1}{\lambda_1}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}\right)^k = \begin{bmatrix} 1 & & & \\ & \frac{\lambda_2}{\lambda_1} & & \\ & & \ddots & \\ & & & \frac{\lambda_n}{\lambda_1} \end{bmatrix}^k = \begin{bmatrix} 1 & & & \\ & \left(\frac{\lambda_2}{\lambda_1}\right)^k & & \\ & & \ddots & \\ & & & \left(\frac{\lambda_n}{\lambda_1}\right)^k \end{bmatrix}
$$

Since λ_1 has the largest magnitude,

$$
\lim_{k\to\infty}\left(\frac{1}{\lambda_1}\boldsymbol{\Lambda}\right)^k = \begin{bmatrix} 1 & & \\ & \lim_{k\to\infty}\left(\frac{\lambda_2}{\lambda_1}\right)^k & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{bmatrix}
$$

Now

$$
\lim_{k\to\infty}\left(\frac{1}{\lambda_1}\boldsymbol{\Lambda}\right)^k(c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n) = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \left(c_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

Now we see that $\lim_{k\to\infty} \frac{1}{c_1}\boldsymbol{V}\left(\frac{1}{\lambda_1}\boldsymbol{\Lambda}\right)^k (c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n) = \boldsymbol{0}$, therefore Now we have

$$
\begin{aligned}
\lim_{k\to\infty} \boldsymbol{x}_k &= \lim_{k\to\infty} \left(\frac{\lambda_1}{|\lambda_1|}\right)^k \frac{c_1}{|c_1|} \frac{\boldsymbol{v}_1 + \frac{1}{c_1}\boldsymbol{V}\left(\frac{1}{\lambda_1}\boldsymbol{\Lambda}\right)^k (c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)}{\left\|\boldsymbol{v}_1 + \frac{1}{c_1}\boldsymbol{V}\left(\frac{1}{\lambda_1}\boldsymbol{\Lambda}\right)^k (c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)\right\|_2} \\
&= \frac{c_1}{|c_1|} \left(\lim_{k\to\infty} \left(\frac{\lambda_1}{|\lambda_1|}\right)^k\right) \left(\lim_{k\to\infty} \frac{\boldsymbol{v}_1 + \frac{1}{c_1}\boldsymbol{V}\left(\frac{1}{\lambda_1}\boldsymbol{\Lambda}\right)^k (c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)}{\left\|\boldsymbol{v}_1 + \frac{1}{c_1}\boldsymbol{V}\left(\frac{1}{\lambda_1}\boldsymbol{\Lambda}\right)^k (c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n)\right\|_2}\right) \\
&= \frac{c_1}{|c_1|} \left(\lim_{k\to\infty} \left(\frac{\lambda_1}{|\lambda_1|}\right)^k\right) \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|_2}
\end{aligned}
$$

If $\lambda_1 > 0$ then $|\lambda_1| = \lambda_1$ and then

$$
\lim_{k\to\infty} \boldsymbol{x}_k = \frac{c_1}{|c_1|} \left(\lim_{k\to\infty} (1)^k\right) \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|_2} = \frac{c_1}{|c_1|} \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|_2} = \frac{c_1}{|c_1|\|\boldsymbol{v}_1\|_2}\boldsymbol{v}_1
$$

Since eigenvector is about direction and not about norm, so we can ignore the scaling factor, and therefore

$$
\lim_{k\to\infty} \boldsymbol{x}_k = \boldsymbol{v}_1.
$$

END