Complex Numbers: The Basics

Mathematicians say that $\sqrt{-1} = i$. Using surds skills, we may therefore say e.g. $\sqrt{-8} = \sqrt{8}\sqrt{-1} = 2\sqrt{2}i$. Since these are not "real", physical quantities, any multiple of i is called an **imaginary number**.

The set of **complex numbers** (notated $\mathbb{C}$) is the set of numbers which may be written in the form $a + bi$. In other words, they may consist of a real and/or imaginary part. Formally, $\mathbb{C}$ contains $\mathbb{R}$ since the imaginary part of a complex number may be zero. This is a natural extension of the idea that $\mathbb{R}$ contains $\mathbb{Z}$.

Manipulating complex numbers works in an identical fashion to manipulating numbers involving real surds: gather like terms, expand brackets, use "conjugates" to simplify fractions with awkward denominators etc. It can be helpful to remember the following:

- $i^2 = -1$
- $i^3 = -i$
- $i^4 = 1$

Subsequent powers of i carry on round the loop $i \to -1 \to -i \to 1$

We often make use of a complex number's **complex conjugate**. The complex conjugate of $a + bi$ is $a - bi$.

Adding and subtracting complex numbers

- Gather real and imaginary terms

e.g. $x = 2 + 3i$ and $y = 5 - i$. Find $x - y$.

$$(2 + 3i) - (5 - i) = (2 - 5) + (3 + 1)i = -3 + 4i$$

Multiplying complex numbers

- Expand brackets like algebra, remembering that $i^2 = -1$ etc
- Gather real and imaginary terms

e.g. $x = 2 + 3i$ and $y = 5 - i$. Find xy.

$$(2 + 3i)(5 - i) = 10 - 2i + 15i - 3i^2 = 10 - 2i + 15i + 3 = 13 + 13i$$

Dividing complex numbers

- Write the division as a fraction
- Multiply top and bottom by the complex conjugate of the denominator.
  We are "realizing" the denominator in the same way as you might "rationalize" with surds
- Split into real and imaginary parts if required

e.g. $x = 2 + 3i$ and $y = 5 - i$. Find $x \div y$

$$\frac{2+3i}{5-i} = \frac{(2+3i)(5+i)}{(5-i)(5+i)} = \frac{7+17i}{26} = \frac{7}{26} + \frac{17}{26}i$$