Proof by Induction: Matrices

Proof by Induction of a formula for the n[th] power of a matrix

The good news is that matrices proofs are almost identical in structure to series proofs and the algebra is normally easier.

With series proofs, the trick we used was to split the sum to $k+1$ into the sum to k plus the $k+1^{\text{th}}$ term. Our trick now is to split the $k+1^{\text{th}}$ power of matrix M into the k^{th} power of M multiplied by M once more.

e.g. $M = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$. Prove that $M^n = \begin{pmatrix} 2n+1 & -n \\ 4n & 1-2n \end{pmatrix}$ for all positive integers n.

Base case If $n = 1$	LHS = $\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}^1 = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$ RHS = $\begin{pmatrix} 2+1 & -1 \\ 4 & 1-2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$	These are equal so the statement is true for $n = 1$
Induction step Assume true for $n = k$ Then for $n = k+1$	$\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}^k = \begin{pmatrix} 2k+1 & -k \\ 4k & 1-2k \end{pmatrix}$ LHS $= \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}^{\mathrm{k+1}}$ $= \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}^{\mathrm{k}} \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$ $= \begin{pmatrix} 2k+1 & -k \\ 4k & 1-2k \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$ $= \begin{pmatrix} 2k+3 & -k-1 \\ 4k+4 & -2k-1 \end{pmatrix}$ $= \begin{pmatrix} 2[k+1]+1 & -[k+1] \\ 4[k+1] & 1-2[k+1] \end{pmatrix}$ = RHS	If the statement is true for $n = k$… … … … … … … … … … … … …then it is also true for $n = k+1$
Conclusion	• True for $n = 1$ • If true for $n = k$ then true for $n = k+1$ • Hence true for all positive integers n	