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Edexcel AS Further Mathematics: Vectors

Section 4: Finding distances

Notes and Examples

These notes contain the following subsections:

The distance of a point from a plane

The shortest distance of a point A to a plane is the distance AP where AP is a line perpendicular to the plane and P is a point on the plane.

You can find this distance by thinking about the line $\mathbf{r} = \mathbf{p} + \lambda\mathbf{n}$, which goes through the point P and is perpendicular to the plane. You can find the intersection of the line and the plane, and then find the distance from the point P to the intersection point.

The following example shows this process being applied.

Example 1

A plane has equation $2x + y - 3z = 4$.

(a) Find the point M on the plane which is closest to the point P $(2, -2, 4)$.

(b) Hence find the distance of P from the plane.

Solution

(a)

The normal to the plane is $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, so write down the equation of a line through P perpendicular to the plane.

This line will intersect the plane at the point M.

$$\mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2+2\lambda \\ -2+\lambda \\ 4-3\lambda \end{pmatrix}.$$

Substitute $x = 2+2\lambda$, $y = -2+\lambda$ and $z = 4-3\lambda$ into the equation of the plane and solve to find λ

$$2(2+2\lambda) + (-2+\lambda) - 3(4-3\lambda) = 4$$
$$14\lambda = 14$$
$$\lambda = 1$$

Use this value of λ to find the coordinates of M

$$\mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

M has coordinates $(4, -1, 1)$

(b) $\overrightarrow{\mathbf{PM}} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

$$\left|\overrightarrow{\mathbf{PM}}\right| = \sqrt{4+1+9} = \sqrt{14}$$

So the distance of the point from the plane is $\sqrt{14}$.

The process shown in Example 1 can be generalised to give a formula for the distance of a point from a plane.

The distance of a point (α, β, γ) from a plane $n_1x + n_2y + n_3z + d = 0$ is given by the formula

$$\left| \frac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{n_1^2 + n_2^2 + n_3^2}} \right|$$

The next example shows this formula being applied to find the same distance as in Example 1.

Example 2

Find the distance of the point P $(2,-2,4)$ from the plane $2x+y-3z=4$.

Solution

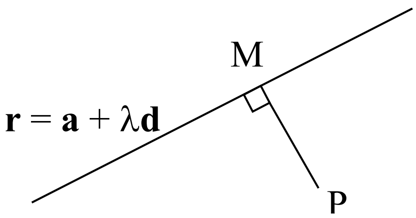
The equation of the plane can be written as $2x+y-3z-4=0$, so $a=2$, $b=1$, $c=-3$ and $d=-4$.

Distance of point from plane $=\dfrac{\left|(2\times 2)+(1\times -2)+(-3\times 4)-4\right|}{\sqrt{2^2+1^2+3^2}}=\dfrac{14}{\sqrt{14}}=\sqrt{14}$

Notice that in Example 1, you were required to find the coordinates of the point on the plane that is closest to point P. So you needed to work from first principles to find this point, and then the distance followed from that working, and so the formula was not needed. Make sure you understand the principles involved.

The distance of a point from a line

The distance of a point from a line is the perpendicular distance from the point to the line, which is the shortest possible distance.



To find the distance of a point P from a line with equation $\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}$, you need to find a point M which is on this line, and for which the vector $\overrightarrow{\mathbf{PM}}$ is perpendicular to $\mathbf{d}$ (i.e. the point M is the foot of the perpendicular from the point to the line).

The next example shows how this can be done.

Example 3

Find the foot of the perpendicular from the point P $(2,1,-2)$ to the line $\mathbf{r}=\begin{pmatrix}1\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$.

Hence find the shortest distance between the point P and the line.

Solution

Let M be the foot of the perpendicular from P to the line.

M is on the line, so its position vector must satisfy $\mathbf{r}=\begin{pmatrix}1\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$.

The position vector of M is of the form $\overrightarrow{\mathbf{OM}}=\begin{pmatrix}1+2\lambda\\-1-\lambda\\2+3\lambda\end{pmatrix}$.

So $\overrightarrow{\mathbf{PM}}=\overrightarrow{\mathbf{OM}}-\overrightarrow{\mathbf{OP}}=\begin{pmatrix}1+2\lambda\\-1-\lambda\\2+3\lambda\end{pmatrix}-\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}-1+2\lambda\\-2-\lambda\\4+3\lambda\end{pmatrix}$

The vector $\overrightarrow{\mathbf{PM}}$ is perpendicular to the line with direction vector $\begin{pmatrix}2\\-1\\3\end{pmatrix}$

$$\begin{pmatrix}-1+2\lambda\\-2-\lambda\\4+3\lambda\end{pmatrix}.\begin{pmatrix}2\\-1\\3\end{pmatrix}=0$$

$$2(-1+2\lambda)-(-2-\lambda)+3(4+3\lambda)=0$$

$$14\lambda=-12$$

$$\lambda=-\frac{6}{7}$$

So $\overrightarrow{\mathbf{OM}}=\begin{pmatrix}1+2(-\frac{6}{7})\\-1-(-\frac{6}{7})\\2+3(-\frac{6}{7})\end{pmatrix}=\begin{pmatrix}-\frac{5}{7}\\-\frac{1}{7}\\-\frac{4}{7}\end{pmatrix}$

So the coordinates of the foot of the perpendicular are $\left(-\frac{5}{7},-\frac{1}{7},-\frac{4}{7}\right)$.

$$\overrightarrow{\mathbf{PM}}=\begin{pmatrix}-1-\frac{12}{7}\\-2+\frac{6}{7}\\4-\frac{18}{7}\end{pmatrix}=\frac{1}{7}\begin{pmatrix}-19\\-8\\10\end{pmatrix}$$

Distance of point from line $=\left|\overrightarrow{\mathbf{PM}}\right|=\frac{1}{7}\sqrt{19^2+8^2+10^2}=\frac{1}{7}\sqrt{525}=\frac{5}{7}\sqrt{21}$

The distance between two parallel lines

Two lines in 3D space which do not meet are either parallel or skew. You will see how to find the distance between skew lines in Example 13. However, finding the distance between parallel lines is equivalent to finding the shortest distance between any point on one line and the other line, so you can use the same method as in Example 11.

Example 4

Show that the lines $\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$ and $\frac{x+1}{-2}=y-3=\frac{z+2}{-1}$ are parallel and find the distance between them.

Solution

The second line can be written as $\mathbf{r}=\begin{pmatrix}-1\\3\\-2\end{pmatrix}+\mu\begin{pmatrix}-2\\1\\-1\end{pmatrix}$.

Since $\begin{pmatrix}-2\\1\\-1\end{pmatrix}=-1\begin{pmatrix}2\\-1\\1\end{pmatrix}$, the direction vectors are parallel and so the lines are parallel.

Let P be the point $(1,2,0)$ which is on the first line.

Let M be the point on the second line which is closest to P.

So $\overrightarrow{\mathbf{OM}}=\begin{pmatrix}-1-2\mu\\3+\mu\\-2-\mu\end{pmatrix}$

and therefore $\overrightarrow{\mathbf{PM}}=\overrightarrow{\mathbf{OM}}-\overrightarrow{\mathbf{OP}}=\begin{pmatrix}-1-2\mu\\3+\mu\\-2-\mu\end{pmatrix}-\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}-2-2\mu\\1+\mu\\-2-\mu\end{pmatrix}$

The vector $\overrightarrow{\mathbf{PM}}$ is perpendicular to the direction vector of the second line

$$\overrightarrow{\mathbf{PM}}.\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}-2-2\mu\\1+\mu\\-2-\mu\end{pmatrix}.\begin{pmatrix}2\\-1\\1\end{pmatrix}=0$$

$$-4-4\mu-1-\mu-2-\mu=0$$

$$6\mu=-7$$

$$\mu=-\tfrac{7}{6}$$

$$\overrightarrow{\mathbf{PM}}=\begin{pmatrix}-2+\frac{14}{6}\\1-\frac{7}{6}\\-2+\frac{7}{6}\end{pmatrix}=\begin{pmatrix}\frac{2}{6}\\-\frac{1}{6}\\-\frac{5}{6}\end{pmatrix}$$

So $\left|\overrightarrow{\mathbf{PM}}\right|=\tfrac{1}{6}\sqrt{2^2+1^2+5^2}=\tfrac{1}{6}\sqrt{30}$

The distance between two skew lines

To find the distance between two skew lines, you need to find point P on one line and point Q on the other such that PQ is perpendicular to both lines. Example 13 shows how this is done.

Example 5

Find the shortest distance between the two skew lines

$$\frac{x-1}{2}=\frac{y+1}{1}=\frac{z+2}{1} \text{ and } \frac{x-3}{1}=\frac{y+2}{-2}=\frac{z-7}{3}$$

Solution

The lines can be written as $\mathbf{r}=\begin{pmatrix}1\\-1\\-2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\1\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}3\\-2\\7\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\3\end{pmatrix}$

Let P be a point on the first line, so $\overrightarrow{\mathbf{OP}}=\begin{pmatrix}1+2\lambda\\-1+\lambda\\-2+\lambda\end{pmatrix}$.

Let Q be a point on the second line, so $\overrightarrow{\mathbf{OQ}} = \begin{pmatrix} 3+\mu \\ -2-2\mu \\ 7+3\mu \end{pmatrix}$

Then $\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{OQ}} - \overrightarrow{\mathbf{OP}} = \begin{pmatrix} 3+\mu \\ -2-2\mu \\ 7+3\mu \end{pmatrix} - \begin{pmatrix} 1+2\lambda \\ -1+\lambda \\ -2+\lambda \end{pmatrix} = \begin{pmatrix} 2+\mu-2\lambda \\ -1-2\mu-\lambda \\ 9+3\mu-\lambda \end{pmatrix}$

$\overrightarrow{\mathbf{PQ}}$ is perpendicular to the first line

$$\begin{pmatrix} 2+\mu-2\lambda \\ -1-2\mu-\lambda \\ 9+3\mu-\lambda \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$4+2\mu-4\lambda-1-2\mu-\lambda+9+3\mu-\lambda=0$$

$$-6\lambda+12+3\mu=0$$

$$2\lambda-\mu=4$$

$\overrightarrow{\mathbf{PQ}}$ is perpendicular to the second line

$$\begin{pmatrix} 2+\mu-2\lambda \\ -1-2\mu-\lambda \\ 9+3\mu-\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 0$$

$$2+\mu-2\lambda+2+4\mu+2\lambda+27+9\mu-3\lambda=0$$

$$-3\lambda+31+14\mu=0$$

$$3\lambda-14\mu=31$$

You can use a calculator to solve these equations simultaneously

$\lambda=1, \mu=-2$.

So $\overrightarrow{\mathbf{PQ}} = \begin{pmatrix} 2-2-2 \\ -1+4-1 \\ 9-6-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$

$$\left|\overrightarrow{\mathbf{PQ}}\right| = \sqrt{2^2+2^2+2^2} = 2\sqrt{3}$$

So the distance between the two lines is $2\sqrt{3}$.