Complex Numbers: Higher Order Polynomials

The methods and ideas of quadratic equations follow through naturally into cubics, quartics and so on.

- If a polynomial has real coefficients than solutions are either **real or in complex conjugate pairs**
- A polynomial of degree n is considered to have exactly n solutions and n factors, though some may be repeated real roots and, equivalently, powers of factors

Recall that knowing a solution $z = a + bi$ led to the quadratic equation $z^2 - 2az + a^2 + b^2 = 0$
In higher order polynomials, knowing a solution $z = a + bi$ leads to the **factor** $(z^2 - 2az + a^2 + b^2)$
We often need to use polynomial division or some other method of factorising to find all solutions.

The factor theorem is used frequently:

- $f(a) = 0 \Leftrightarrow (z - a)$ is a factor of $f(z)$
- If we have a non-integer root: $f\left(\frac{a}{b}\right) = 0 \Leftrightarrow (bz - a)$ is a factor of $f(z)$

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e.g. $z^3 + az^2 + bz + c$ is known to have solutions $z = 2 + 3i$ and $z = -2$.
Find the values of a, b, c

Since $z = 2 + 3i$ is a solution, $z^* = 2 - 3i$ is also a solution and $(z^2 - 4z + 13)$ is a factor
Since $z = -2$ is a solution, $(z + 2)$ is a factor
The cubic equation is therefore $(z^2 - 4z + 13)(z + 2) = 0$
This multiplies out to $z^3 - 2z^2 + 5z + 26 = 0$ i.e. $a = -2, b = 5, c = 26$

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e.g. $f(z) = z^3 + 3z^2 + 7z - 75$

a) Use the factor theorem to show that $(z - 3)$ is a factor of $f(z)$
b) Hence solve $f(z)$ completely

a) $f(3) = 27 + 27 + 21 - 75 = 0$
$z = 3$ is a solution to $f(z) = 0 \Leftrightarrow (z - 3)$ is a factor of $f(z)$

b) Using polynomial division:

$$
\begin{array}{r|l}
 & z^2 + 6z + 25 \\
\hline
z - 3 & z^3 + 3z^2 + 7z - 75 \\
 & -(z^3 - 3z^2) \\
\hline
 & 6z^2 + 7z \\
 & -(6z^2 - 18z) \\
\hline
 & 25z - 75 \\
 & -(25z - 75) \\
\hline
 & 0
\end{array}
$$

So $f(z) = (z - 3)(z^2 + 6z + 25)$
Finding solutions for the quadratic factor:
$z^2 + 6z + 25 = 0$
$(z + 3)^2 + 16 = 0$
$z = -3 \pm 4i$

So the complete solution set is $z = 3, -3 + 4i, -3 - 4i$