Complex Numbers: Quadratic Roots

Forwards – solutions from an equation

To solve a quadratic equation with complex roots, you may use your old surds skills and your new knowledge that $i = \sqrt{-1}$ along with the usual methods of solving:

- completing the square
- quadratic formula

e.g. Solve $z^2 + 2z + 5 = 0$ in exact form

$$(z+1)^2 + 4 = 0$$
$$(z+1)^2 = -4$$
$$z + 1 = \pm\sqrt{-4} = \pm 2i$$
$$z = -1 \pm 2i$$

There is nothing magical about the letter z. The letter x would do just as well but it is convention to use z when complex numbers are involved.

e.g. Solve $2z^2 - 3z + 5 = 0$ in exact form

$$a = 2, b = -3, c = 5$$
$$z = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(5)}}{2(2)}$$
$$z = \frac{3}{4} \pm \frac{\sqrt{31}}{4} i$$

Backwards – equation from solution(s)

When a quadratic equation has real coefficients, then it has either:

- **two real roots**
- **one repeated real root** (i.e. a squared factor) or
- **two complex conjugate roots** (i.e. $z = a + bi$ and $z^* = a - bi$)

This last fact helps us to recover the original equation if we have one complex root.
The long way around this is to factorize and expand e.g. if $3 + 5i$ and $3 - 5i$ are the roots then we may factorize as $\big(z - (3+5i)\big)\big(z - (3-5i)\big) = 0$ before expanding and simplifying.

We may shortcut the process by noticing that when the roots are α, β then $(z-\alpha)(z-\beta) = 0$ expands to $z^2 - (\alpha + \beta)z + \alpha\beta = 0$ i.e. the z coefficient is minus the sum of the roots and the constant term is the product (more on this in the "roots of polynomials" chapter). Since our two roots are conjugates, this simplifies nicely since $(a+bi) + (a-bi) = 2a$ and $(a+bi)(a-bi) = a^2 + b^2$

If the roots of a quadratic with real coefficients are $a \pm bi$ then (assuming $1z^2$) the equation is:

$$z^2 - 2az + a^2 + b^2 = 0$$

e.g. $z = 2 + 3i$ is a root of $x^2 + px + q = 0$, where $p, q \in \mathbb{R}$.
State the other root and then find the values of p, q

Since $2 + 3i$ is a root, the other root is $2 - 3i$
The equation is $z^2 - 4z + 13 = 0$ i.e. $p = -4$ and $q = 13$