Developed by MEI Mathematics Education Innovation

integral

Edexcel AS Further Mathematics: Vectors

Section 2: The vector equation of a line

Notes and Examples

These notes contain the following subsections:

The vector equation of a line in two dimensions

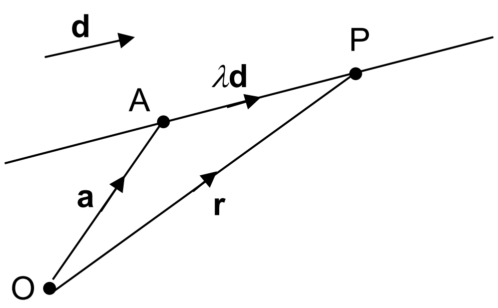
To find the cartesian equation of a line you need to know two things:

- the coordinates of one point on the line
- the gradient of the line (or the coordinates of a second point)

Similarly, to find the vector equation of a line you need to know two things:

- the position vector of one point on the line
- the direction of the line (or the position vector of a second point)

The vector equation of a line is a bit different from the cartesian equation. A cartesian equation $y = mx + c$ gives a relationship between the coordinates of points on the line. A vector equation, however, gives a general position vector for points on the line.



The diagram shows a general point P on a line which goes through the point A and is parallel to the vector **d**. To get to point P from the origin O, you can first move to point A and then move in the direction of vector **d**. So the position vector of P is **a** + some multiple of **d**. You can write this as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, where λ is a parameter. Each different value of λ corresponds to a different point on the line.

So the general vector equation of a line AB is given by:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

where **a** is the position vector of a point on the line (this is like the $+c$ part of a cartesian equation of a line), and **d** is the direction vector of the line (this is like the gradient of the line – the 'm' part of a cartesian equation).

Example 1

Find a vector equation of the line through the point A $(-4,-1)$ parallel to $4\mathbf{i}+2\mathbf{j}$.

Solution

$\mathbf{a}=\begin{pmatrix}-4\\-1\end{pmatrix}$ and the direction vector of the line is $\mathbf{d}=\begin{pmatrix}4\\2\end{pmatrix}$

$$\begin{aligned}\mathbf{r}&=\mathbf{a}+\lambda\mathbf{d}\\&=\begin{pmatrix}-4\\-1\end{pmatrix}+\lambda\begin{pmatrix}4\\2\end{pmatrix}\end{aligned}$$

Since the vector $\begin{pmatrix}4\\2\end{pmatrix}$ gives the direction of the line, you can simplify it to $\begin{pmatrix}2\\1\end{pmatrix}$ as this vector has the same direction. It is usually easier to work with the simplest possible direction vector.

A vector equation for the line is $\mathbf{r}=\begin{pmatrix}-4\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\end{pmatrix}$

If A and B are the points with position vectors **a** and **b**, you can write the vector equation of a line as:

$$\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$$

Example 2

Find the vector equation of the line joining the points A $(5,-2)$ and B $(1,3)$.

Solution

$\mathbf{a}=\begin{pmatrix}5\\-2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}1\\3\end{pmatrix}$

$$\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$$

So $\mathbf{r} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \lambda\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \end{pmatrix}\right)$

$\Rightarrow \mathbf{r} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -4 \\ 5 \end{pmatrix}$

Notice that there are an infinite number of possible ways to write the equation of the line. As well as different multiples of the direction vector, such as $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$, you could use any point on the line for the first vector. If you used B instead of A you could have $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -4 \\ 5 \end{pmatrix}$. You could work out any other point on the line and use that, but of course it is much easier to use one of the ones that you are given.

The vector equation of a line in three dimensions

The vector equation of a line in three dimensions has the same form as in two dimensions

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

Example 3

The points A and B have coordinates $(-1, 2, -3)$ and $(0, -2, 1)$ respectively.

Find a vector equation for the line AB.

Solution

$\overrightarrow{\mathrm{OA}} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{\mathrm{OB}} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

$\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{OB}} - \overrightarrow{\mathrm{OA}}$

$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix}$

A vector equation of the line is $\mathbf{r} = \overrightarrow{\mathrm{OA}} + \lambda\overrightarrow{\mathrm{AB}}$

So: $\mathbf{r}=\begin{pmatrix}-1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\-4\\4\end{pmatrix}$

Cartesian equation of a line in three dimensions

The vector equation of a line $\mathbf{r}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ can be written as $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$.

This can be expressed as three equations: $x=a_1+\lambda u_1$

$$y=a_2+\lambda u_2$$

$$z=a_3+\lambda u_3$$

which can each be rearranged to give: $\lambda=\dfrac{x-a_1}{u_1}$

$$\lambda=\frac{y-a_2}{u_2}$$

$$\lambda=\frac{z-a_3}{u_3}$$

Equating the three expressions for λ gives: $\dfrac{x-a_1}{u_1}=\dfrac{y-a_2}{u_2}=\dfrac{z-a_3}{u_3}$.

This is the equation of a line in Cartesian form.

It may feel strange that there are two equals signs in the equation. It really means that the line consists of points (x,y,z) for which **both** $\dfrac{x-a_1}{u_1}=\dfrac{y-a_2}{u_2}$ **and** $\dfrac{y-a_2}{u_2}=\dfrac{z-a_3}{u_3}$ are true.

Example 4

Write down the cartesian equation of the line $\mathbf{r}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\-5\\1\end{pmatrix}$

Solution

Substitute into the general form $\dfrac{x-a_1}{u_1}=\dfrac{y-a_2}{u_2}=\dfrac{z-a_3}{u_3}$

$$\frac{x-(-2)}{3}=\frac{y-1}{-5}=\frac{z-3}{1}$$

This could be written as $\dfrac{x+2}{3}=\dfrac{1-y}{5}=z-3$

Example 5

Write down the vector equation of the line $\dfrac{3-x}{2}=y-4=\dfrac{z}{5}$

Solution

Write the equation in the same form as $\dfrac{x-a_1}{u_1}=\dfrac{y-a_2}{u_2}=\dfrac{z-a_3}{u_3}$

$$\frac{3-x}{2}=y-4=\frac{z}{5}\Rightarrow\frac{x-3}{-2}=\frac{y-4}{1}=\frac{z-0}{5}$$

The numbers in the numerator give the components of vector $\mathbf{a}$, and the numbers in the denominators give the components of the direction vector $\mathbf{d}$.

So the vector equation is:

$$\mathbf{r}=\begin{pmatrix}3\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\5\end{pmatrix}$$

Special cases of a cartesian equation of a line in 3-D

When the direction vector of a line in three dimensions contains one zero, this means that the line is running parallel to either the xy plane, xz plane or the yz plane.

In this case, the Cartesian equation is written slightly differently, as is shown in the next example.

Example 6

Write down the cartesian equation of the line $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$

In the line the x coordinate is always 4 as the line isn't 'moving' in the x direction

Solution

Substitute into the general form $\frac{x-a_1}{u_1} = \frac{y-a_2}{u_2} = \frac{z-a_3}{u_3}$

$$\frac{x-4}{0} = \frac{y-(-1)}{-2} = \frac{z-2}{3}$$

This means that you would be dividing by 0 which is undefined, so the equation is written as shown below.

$x = 4$ and $\frac{y+1}{-2} = \frac{z-2}{3}$

The next example shows you how to treat two zeros in the direction vector.

Example 7

Write down the cartesian equation of the line $\mathbf{r} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$

In this line the x coordinate is always -3 and the z coordinate is always 1 as the line isn't 'moving' in the x or z direction

Solution

Substitute into the general form $\frac{x-a_1}{u_1} = \frac{y-a_2}{u_2} = \frac{z-a_3}{u_3}$

$$\frac{x-(-3)}{0}=\frac{y-2}{4}=\frac{z-1}{0}$$

This means that the y coordinate can take any value, so the equation is written as shown below.

$x=-3$ and $z=1$

Notice that in all cases, the Cartesian equation of a line in three dimensions involves two equations. In the standard format, you have $\frac{x-a_1}{u_1}=\frac{y-a_2}{u_2}$ and $\frac{y-a_2}{u_2}=\frac{z-a_3}{u_3}$. If there are zeros in the direction vectors, one or both of these equations takes the form $x=k$ or similar.

The angle between two lines

To find the angle between two lines simply find the angle between their two direction vectors, using the scalar product.

Note that two lines in three dimensions may not intersect but you can still work out the angle between them in the same way.

Example 8

Find the angle between the lines $\mathbf{r}=\begin{pmatrix}-3\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\1\\-2\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}1\\-5\\-2\end{pmatrix}+\mu\begin{pmatrix}1\\0\\-3\end{pmatrix}$

Solution

Find the angle between the two direction vectors $\begin{pmatrix}3\\1\\-2\end{pmatrix}$ and $\begin{pmatrix}1\\0\\-3\end{pmatrix}$ using the formula $\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$$\begin{pmatrix}3\\1\\-2\end{pmatrix}.\begin{pmatrix}1\\0\\-3\end{pmatrix}=(3\times1)+(1\times0)+(-2\times-3)=3+0+6=9$$

$$\left|\begin{pmatrix}3\\1\\-2\end{pmatrix}\right| = \sqrt{3^2+1^2+(-2)^2} = \sqrt{9+1+4} = \sqrt{14}$$

$$\left|\begin{pmatrix}1\\0\\-3\end{pmatrix}\right| = \sqrt{1^2+0^2+(-3)^2} = \sqrt{1+0+9} = \sqrt{10}$$

So $\cos\theta = \dfrac{9}{\sqrt{14}\sqrt{10}}$

$\Rightarrow \theta = 40.5°\ (3\text{ s.f.})$

Finding the intersection of two lines in two dimensions

If two lines intersect, then there is a point which lies on both lines. This means that the position vector of the intersection point satisfies both equations.

Example 9

Find the position vector of the point where the lines:

$$\mathbf{r} = \begin{pmatrix}5\\0\end{pmatrix} + \lambda\begin{pmatrix}4\\1\end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix}-2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\3\end{pmatrix} \text{ intersect.}$$

Solution

Remember that the equation of a line gives a generalised position vector for points on the line, with different values of the parameter corresponding to different points on the line. To find the intersection point, you need to find values of λ and μ which give the same position vector $\mathbf{r}$ for both lines, so you can equate the two equations to each other. Notice that it is important to use different parameters for the two equations (in this case λ and μ are used), as they won't necessarily have the same value at the intersection point.

$$\begin{pmatrix}5\\0\end{pmatrix} + \lambda\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\3\end{pmatrix}$$

This vector equation must be true for each of the components. So you can read across to write down two equations. The first equation below is for the **i** component and the second is for the **j** component.

$$5 + 4\lambda = -2 + \mu$$

$$\lambda = 1 + 3\mu$$

You now have two equations and two unknowns. Solve them simultaneously to find λ and μ. You can use a calculator to do this, or you can do it algebraically.

$$\lambda = -2, \mu = -1$$

You can now use one of the original vector equations, with the appropriate value of λ or μ to find the position vector of the intersection point. Either will give you the same answer.

$$\mathbf{r} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

So the lines intersect at the point with position vector $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$.

The intersection of two lines in three dimensions

In two dimensions, two lines which are not parallel will always intersect. This is not the case in three dimensions.

Two lines in three dimensions which do not intersect and are not parallel are called **skew** lines.

Example 10

Lines l_1 and l_2 have vector equations $\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ respectively. Do these lines intersect?

Solution

First check whether the lines are parallel. If they are, their direction vectors will be multiples of one another.

The lines are not parallel as $\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ are not multiples of one another.

As they are not parallel, you must establish whether or not they are skew.

If the lines intersect, $\begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

From the vector equation above, write down equations for each of the three components.

$$\begin{aligned} 3\lambda &= 2 + \mu && (1) \\ -3 + 2\lambda &= 5 - \mu && (2) \\ 4 + 3\lambda &= -1 + 2\mu && (3) \end{aligned}$$

If the lines intersect then there will exist values of λ and μ which will satisfy **all three** of these equations simultaneously. Pick any pair of equations and solve them simultaneously – either using a calculator or algebraically.

Solving equations (1) and (2) gives $\lambda = 2$ and $\mu = 4$

If the lines intersection, these values of λ and μ must also satisfy equation (3).

Substituting $\lambda = 2$ and $\mu = 4$ into $4 + 3\lambda = -1 + 2\mu$ gives left hand side 10 and right hand side 7, so equation (1) is not satisfied.

So the equations cannot be solved simultaneously, and the lines must be skew.

In the example above, if the lines did meet, the point of intersection could be found by substituting one or other of the parameters into the appropriate vector equation.

Note that as in the example in two dimensions, different symbols must be used to denote the parameter in the two lines. This must be done, as if the same symbol were used it would imply that the parameters in each line always have equal values, which is certainly not true.