Proof by Induction: Series

Types of Proof

There are numerous ways to prove a mathematical or logical statement. You have previously met:

- Deductive (direct) proof – combine known facts and theorems and reason on to new ones
- Proof by exhaustion – split the theory up into individual cases and test them all
- Proof by contradiction – Assume the logical opposite and show that it leads to contradiction

There are further types of proof that you will most likely not meet until degree level (e.g. contrapositive proof, proof by construction…) but CP1 features "proof by induction" which is suitable when we are trying to prove that some mathematical statement is true for all positive integers. e.g. $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ is the statement that for all positive integers n, the sum of the integers from 1 to n is given by $\frac{1}{2}n(n+1)$.

The Idea of Proof by Induction

The analogy that is always used for proof by induction is toppling a line of dominoes. The two things you need to know to be sure that all the dominos will eventually be knocked over are:

- Will the first domino be knocked over?
- Once any given domino is knocked over, is it close enough to knock over the next domino?

If these two conditions are met, then you can be sure that they will all, eventually, be toppled – however long the line of dominoes.

The mathsy equivalent conditions are:

- Is the statement true when $n = 1$?
- If we assume the statement is true for some general value of n, say $n = k$, then can we show that it would also be true for the next value of n i.e. $n = k + 1$?

If these two conditions can be shown then:

… since it's true for $n = 1$ it must also be true for $n = 2$ (the next value) and then…
… since it's true for $n = 2$ it must also be true for $n = 3$ (the next value) and then…
… since it's true for $n = 3$ it must also be true for $n = 4$ (the next value) and then…
… by the same pattern, we can reason that it must eventually be true for all positive integer values of n

The Structure

(Almost) always, the structure of a proof by induction should be exactly the same:

1. **Show** explicitly that the statement is true when $\boldsymbol{n = 1}$
2. **Assume** that the statement is true for some value $\boldsymbol{n = k}$
3. Show that **if** that assumption is correct **then** the statement is also true for the next value $\boldsymbol{n = k + 1}$
4. Conclude with a well-rehearsed bit of **waffle** that is tedious but nevertheless important

Proof by Induction of a Series Statement

The trick with a series induction is to notice that:

The sum of the first $k+1$ terms can be written as the sum of the first k terms plus the $k+1^{\text{th}}$ term

Let's see this in action:

e.g. Prove by induction that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ for all positive integers n

Base case If $n = 1$	LHS = $\sum_{r=1}^{1} r = 1$ RHS = $\frac{1}{2} \times 1 \times 2 = 1$	These are equal so the statement is true for $n = 1$
Induction step Assume true for $n = k$ Then for $n = k + 1$	i.e. $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$ LHS $= \sum_{r=1}^{k+1} r$ $= 1 + 2 + \cdots + \mathrm{k} + (\mathrm{k} + 1)$ $= \sum_{i=1}^{k} r + (k+1)$ $= \frac{1}{2}k(k+1) + (k+1)$ $= \frac{1}{2}(k+1)(k+2)$ $= \frac{1}{2}([k+1])([k+1]+1)$ = RHS	If the statement is true for $n = k$… … … … … … … … … … … … … … …then it is also true for $n = k + 1$
Conclusion	• True for $n = 1$ • If true for $n = k$ then true for $n = k + 1$ • Hence true for all positive integers n	

Another e.g. Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{(n+1)}$ for all positive integers n

Base case If $n = 1$	LHS = $\sum_{r=1}^{1} \frac{1}{r(r+1)} = \frac{1}{1\times2} = \frac{1}{2}$ RHS = $\frac{1}{1+1} = \frac{1}{2}$	These are equal so the statement is true for $n = 1$
Induction step Assume true for $n = k$ Then for $n = k + 1$	i.e. $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$ LHS $= \sum_{r=1}^{k+1} \frac{1}{r(r+1)}$ $= \frac{1}{(1)(2)} + \cdots + \frac{1}{\mathrm{k}(\mathrm{k}+1)} + \frac{1}{(\mathrm{k}+1)(\mathrm{k}+2)}$ $= \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$ $= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ $= \frac{k(k+2)+1}{(k+1)(k+2)}$ $= \frac{k^2+2k+1}{(k+1)(k+2)}$ $= \frac{(k+1)^2}{(k+1)(k+2)}$ $= \frac{k+1}{k+2}$ $= \frac{[k+1]}{[k+1]+1}$ = RHS	If the statement is true for $n = k$… …then it is also true for $n = k + 1$
Conclusion	• True for $n = 1$ • If true for $n = k$ then true for $n = k + 1$ • Hence true for all positive integers n	