Edexcel AS Further Mathematics: Vectors

Section 3: The equation of a plane

Notes and Examples

These notes contain the following subsections:

The vector equation of a plane

To find the equation of a plane, you need to know two things about it: its orientation in space and its position.

The orientation of the plane can be defined by a vector which is perpendicular to the plane. This is called the normal vector to the plane and is usually denoted by $\mathbf{n}$.

If you know a point in the plane, say with position vector $\mathbf{a}$, and $\mathbf{r}$ is any general point in the plane, then $\mathbf{r}-\mathbf{a}$ is a vector within the plane.

So $\mathbf{r}-\mathbf{a}$ must be perpendicular to the normal vector $\mathbf{n}$.

Therefore $(\mathbf{r}-\mathbf{a}).\mathbf{n}=0$

This can be rewritten as $\mathbf{r}.\mathbf{n}=\mathbf{a}.\mathbf{n}$.

The Cartesian equation of a plane

The vector equation of a plane given above can be written as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}=d$, where

$d=\mathbf{a}\cdot\mathbf{n}$ (remember that the result of a scalar product is just a number).

So this can be written in the form:

$$n_1x+n_2y+n_3z=d$$

Example 1

(a) Find the Cartesian equation of the plane through the point $(4,5,-2)$, given that the vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ is perpendicular to the plane.

(b) Verify that the point $(2,4,-1)$ also lies on the plane.

Solution

(a)

> The normal vector gives the coefficients for x, y and z in the equation of the plane

The equation of the plane is $2x - y + 3z = d$

> The point $(4,5,-2)$ lies on the plane so it must satisfy the equation. Substitute these coordinates into the equation.

$$2 \times 4 - 5 + 3 \times -2 = d$$
$$8 - 5 - 6 = d$$
$$-3 = d$$

The equation of the plane is $2x - y + 3z + 3 = 0$

> All planes with these coefficients of x, y and z will be parallel. The value of d locates the plane.

(b)

> You need to verify that $2x - y + 3z + 3 = 0$ when $x = 2$, $y = 4$ and $z = -1$

$(2 \times 2) - 4 + (3 \times -1) + 3 = 4 - 4 - 3 + 3 = 0$ as required

So the point $(2,4,-1)$ lies on the plane.

Another vector form for the equation of a plane

Another way of expressing the equation of a plane is to think about two non-parallel vectors, say $\mathbf{b}$ and $\mathbf{c}$, that lie within the plane. If you also know one point in the plane, say with position vector $\mathbf{a}$, you can express any point in the plane in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. (The position vector $\mathbf{a}$ gets you to the plane, and then you can get to any point on the plane by a combination of $\mathbf{b}$ and $\mathbf{c}$).

This means that if you know three points in the plane, you can find two vectors that lie within the plane and hence the equation of the plane.

Example 2

Find the equation of the plane containing the points P $(2, 3, -1)$, Q $(1, 0, 4)$ and R $(-1, 1, 1)$.

Solution

A vector in the plane is $\overrightarrow{\mathbf{PQ}} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$.

Another vector in the plane is $\overrightarrow{\mathbf{PR}} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$

Use $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ with $\overrightarrow{\mathbf{OP}}$ as $\mathbf{a}$, $\overrightarrow{\mathbf{PQ}}$ as $\mathbf{b}$ and $\overrightarrow{\mathbf{PR}}$ as $\mathbf{c}$

So an equation of the plane is $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$

Notice that there are many different possibilities for this equation – you could have chosen to use a different pair of vectors, such as $\overrightarrow{\mathbf{QR}}$ and $\overrightarrow{\mathbf{RP}}$. You could also have chosen any of the three points as the first vector in the equation.

You can convert this form into the Cartesian form with a bit of algebra. This is shown in the next example.

Example 3

Find the equation of the plane $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$ in Cartesian form.

Solution

A general point on the plane can be written as $\mathbf{r} = \begin{pmatrix} 2-\lambda-3\mu \\ 3-3\lambda-2\mu \\ -1+5\lambda+2\mu \end{pmatrix}$

So (1) $x = 2-\lambda-3\mu$

(2) $y = 3-3\lambda-2\mu$

(3) $z = -1+5\lambda+2\mu$

(2) + (3) $\Rightarrow y+z = 2+2\lambda \quad \Rightarrow \lambda = \frac{1}{2}(y+z-2)$

Substitute into equation 2

$$y = 3-\tfrac{3}{2}(y+z-2)-2\mu$$
$$2y = 6-3y-3z+6-4\mu$$
$$\mu = \tfrac{1}{4}(12-5y-3z)$$

Substitute both expressions into equation 1

$$x = 2-\tfrac{1}{2}(y+z-2)-\tfrac{3}{4}(12-5y-3z)$$
$$4x = 8-2y-2z+4-36+15y+9z$$
$$4x-13y-7z+24 = 0$$

Notice that since the plane in Example 3 is the same as the one in Example 2, you can check the answer here by substituting the coordinates of the points P, Q and R from Example 2, and showing that they fit the Cartesian equation found in Example 3.

Finding the angle between two planes

The angle between two planes is the same as the angle between the normal vectors for the two planes.

Example 4

Find the angle between the planes $2x+3y-5z=3$ and $4x-3z=2$

Solution

The normal vectors to the planes are $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$.

Use $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ to find the angle between the two normal vectors

$$\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}.\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} = 8+0+15 = 23$$

$$\left|\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}\right| = \sqrt{2^2+3^2+(-5)^2} = \sqrt{38}$$

$$\left|\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}\right| = \sqrt{4^2+0^2+(-3)^2} = \sqrt{25} = 5$$

$$\cos\theta = \frac{23}{5\sqrt{38}}$$

$$\theta = 41.7° \text{ (3 s.f.)}$$

Finding the intersection of a line and a plane

The following example shows you how to find the intersection of a line and a plane. Remember that you are trying to find values for x, y and z that satisfy the equation of the plane, and also satisfy the equation of the line for a particular value of the parameter λ.

Example 5

Find the point of intersection of the line $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ and the plane $2x-3y+z=6$

Solution

Write the equation of the line using x, y and z

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

Use this to write down three equations by reading across

$$x = 2 - \lambda$$
$$y = -3 + 2\lambda$$
$$z = 4 - 3\lambda$$

Substitute these into the equation of the plane $2x - 3y + z = 6$

$$2(2-\lambda) - 3(-3+2\lambda) + (4-3\lambda) = 6$$

Solve this equation to find a value for λ.

$$4 - 2\lambda + 9 - 6\lambda + 4 - 3\lambda = 6$$
$$\Rightarrow 11\lambda = 11$$
$$\Rightarrow \lambda = 1$$

Now substitute $\lambda = 1$ into the equation of the line to find the position vector of the point of intersection.

$$\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

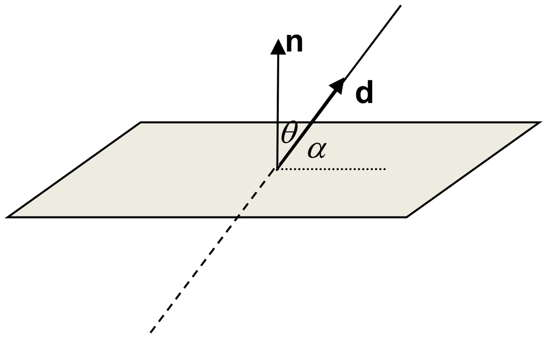
So the coordinates of the point of intersection are $(1, 1, 1)$

Check this point lies on the plane $2x - 3y + z = 6$:

$2 \times 1 - 3 \times (-1) + 1 = 2 + 3 + 1 = 6$ as required..

Finding the angle between a line and a plane

The angle between a line and a plane can be found using the direction vector of the line and the normal vector to the plane.



In the diagram above, the angle between the normal vector $\mathbf{n}$ and the direction vector $\mathbf{d}$ is the angle marked θ. The angle between the line and the plane is marked as α. You can see that to find α you need to find $90° - \theta$.

Remember that as when finding the angle between lines, you may end up with an obtuse angle instead of an acute one for the angle between the direction vector and the normal vector. If this is the case you need to subtract from $180°$ to get the acute angle between the normal vector and the direction vector, and then subtract the result from $90°$ to get the angle between the line and the plane.

Example 6

Find the angle between the line $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ and the plane $2x - 3y + z = 6$

Solution

The direction vector of the line is $\mathbf{d} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$.

The normal vector to the plane is $\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

Find the angle between these two vectors using $\cos\theta = \frac{\mathbf{n}.\mathbf{d}}{|\mathbf{n}||\mathbf{d}|}$

$$\mathbf{n}.\mathbf{d} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} . \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = -2-6-3 = -11$$

$$|\mathbf{d}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$|\mathbf{n}| = \sqrt{2^2+3^2+1^2} = \sqrt{14}$$

$$\cos\theta = \frac{\mathbf{n}.\mathbf{d}}{|\mathbf{n}||\mathbf{d}|} = \frac{-11}{\sqrt{14}\sqrt{14}}$$

$$\theta = 141.8°$$

The acute angle between the vectors is $38.2°$

The angle between the line and the plane is $90° - 38.2° = 51.8°$