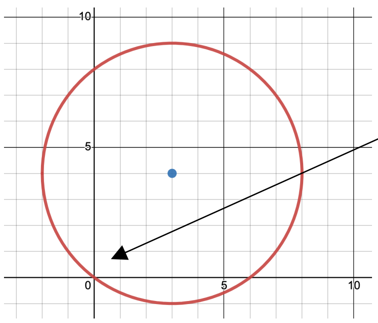
Loci on the Complex Plane

For any two points z_1 and z_2, $|z_1 - z_2|$ is the distance between those points on the complex plane.

$|z - z_1| = k$ is a circle centre z_1 radius k
It is the locus of all points which are distance k away from point z_1

e.g. $|z - 3 - 4i| = 5$ could be rewritten as $|z - (3 + 4i)| = 5$
It is a circle, centre $(3 + 4i)$ radius 5.



Look out for key points e.g. this circle goes through the origin

Split into real and imaginary parts

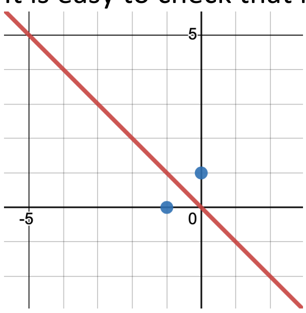
Algebraic proof:
Suppose that point $z = x + iy$

$$|z - 3 - 4i| = 5$$
$$|x + iy - 3 - 4i| = 5$$
$$|(x - 3) + i(y - 4)| = 5$$
$$(x - 3)^2 + (y - 4)^2 = 25$$ i.e. circle centre $(3, -4)$ radius 5

Since $|a + bi|^2 = a^2 + b^2$

$|z - z_1| = |z - z_2|$ is the perpendicular bisector of points z_1 and z_2
It is the locus of points for which the distance from z_1 equals the distance from z_2

e.g. $|z - i| = |z + 1|$ could be rewritten as $|z - i| = |z - (-1)|$
It is the perpendicular bisector of points i and -1
The gradient between i and -1 is 1, so the perpendicular bisector has gradient -1
It is easy to check that it has equation $y = -x$



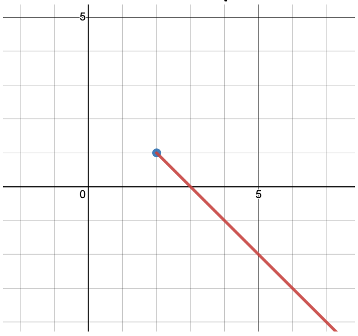
Algebraic proof:
Suppose that point $z = x + iy$

$$|z - i| = |z + 1|$$
$$|x + iy - i| = |x + iy + 1|$$
$$|x + i(y - 1)| = |(x + 1) + iy|$$
$$x^2 + (y - 1)^2 = (x + 1)^2 + y^2$$
$$x^2 + y^2 - 2y + 1 = x^2 + 2x + 1 + y^2$$
$$-2y + 1 = 2x + 1$$
$$y = -x$$

Split into real and imaginary parts and use $|a + bi|^2 = a^2 + b^2$ again

$\arg(z - z_1) = \theta$ is a half line, from point z_1 (though not including z_1) at angle θ
It is the locus of points for which the vector between z_1 and z has angle θ

e.g. $\arg(z - 2 - i) = -\frac{\pi}{4}$ could be rewritten as $\arg\big(z - (2 + i)\big) = -\frac{\pi}{4}$
It is a half-line from the point $2 + i$ at angle $-\frac{\pi}{4}$
It coincides with part of the line $y = -x + 3$



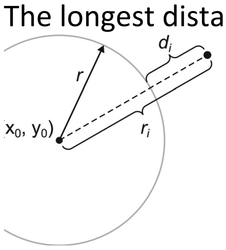
Algebraic proof:
Suppose that point $z = x + iy$

$$\arg(z - 2 - i) = -\frac{\pi}{4}$$
$$\arg(x + iy - 2 - i) = -\frac{\pi}{4}$$
$$\arg\big((x - 2) + i(y - 1)\big) = -\frac{\pi}{4}$$
$$\frac{y-1}{x-2} = \tan\left(-\frac{\pi}{4}\right)$$
$$\frac{y-1}{x-2} = -1$$
$$y - 1 = 2 - x$$
$$y = -x + 3$$

This jump essentially uses the idea that $\tan(\text{an angle}) = \frac{opp}{adj}$ which on the complex plane means $\tan\theta = \frac{Im(z)}{Re(z)}$

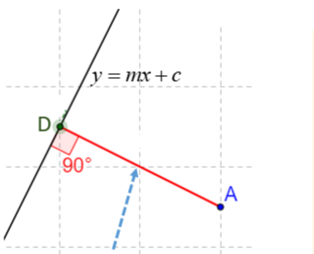
A Few Useful Facts for Problem-Solving

- The shortest distance from a point to a circle = the distance to the centre minus the radius
  The longest distance from a point to a circle = the distance to the centre plus the radius



$d_i = r_i - r$ in the diagram on the left

- The shortest distance from a point to a straight line is the perpendicular distance



The shortest distance between a point and a line is a perpendicular line segment.

- To find where two complex loci meet, you may convert to Cartesian equations and solve simultaneously. Sometimes it is easier to use geometric facts. For instance, if one of the loci is a half-line at angle $\frac{\pi}{4}$ (45°) there is a strong chance that you will be able to form an isosceles triangle