Section 2: Matrices and transformations

Notes and Examples

These notes contain subsections on

Transformations in two dimensions

Although matrices have many applications, their use to describe transformations is particularly useful.

A linear transformation is a transformation in which the image (x', y') of a point (x, y) can be written as

$$x' = ax + by$$
$$y' = cx + dy$$

This can be written in the form of a matrix equation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix form makes it easy to find the image of a point using matrix multiplication.

Example 1

Find the image of the shape A (3, 1), B (-2, 4) and C (5, -1) under the transformation represented by the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$.

Solution

The points A, B and C can be written as the single matrix $\begin{pmatrix} 3 & -2 & 5 \\ 1 & 4 & -1 \end{pmatrix}$.

Multiplying by the transformation matrix: $\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 & -2 & 5 \\ 1 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 8 & 7 \\ -3 & 2 & -5 \end{pmatrix}$

The image of A is (9, -3), the image of B is (8, 2) and the image of C is (7, -5).

By the time you have worked through this section, you should be familiar with the matrices for simple transformations such as reflections in the x axis, the y axis and the lines $y = \pm x$, rotations, enlargements and stretches.

An easy way to find the matrices representing simple transformations is to think about the images of the points (1, 0) and (0, 1).

Under the transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

- The image of the point (1, 0) is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$, which is the first column of the matrix.
- The image of the point (0, 1) is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$, which is the second column of the matrix.

Stretches and enlargements

A stretch parallel to the x-axis with scale factor k maps the point (1, 0) to the point (k, 0), but leaves the point (0, 1) unchanged.

So the matrix representing this transformation is $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$.

A stretch parallel to the y-axis with scale factor k leaves the point (1, 0) unchanged by maps the point (0, 1) to the point (0, k).

So the matrix representing this transformation is $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$.

An enlargement, centre the origin, with scale factor k maps the point (1, 0) to the point (k, 0) and the point (0, 1) to the point (0, k).

So the matrix representing this transformation is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

Rotations about the origin

A rotation through 90° anticlockwise about the origin maps the point (1, 0) to the point (0, 1) and the point (0, 1) to the point (-1, 0).

So the matrix representing this transformation is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

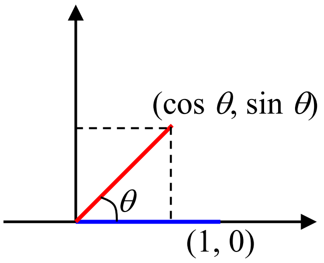
A rotation through 180° about the origin maps the point (1, 0) to the point (-1, 0) and the point (0, 1) to the point (0, -1).

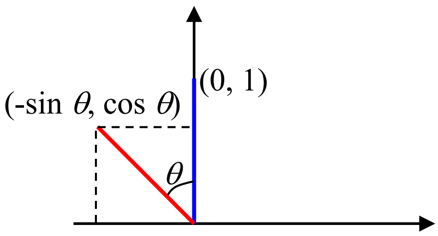
So the matrix representing this transformation is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

A rotation through 90° clockwise about the origin maps the point (1, 0) to the point (0, -1) and the point (0, 1) to the point (1, 0).

So the matrix representing this transformation is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The diagrams above show that an anticlockwise rotation through an angle θ maps the point (1, 0) to the point (cos θ, sin θ) and the point (0, 1) to the point (-sin θ, cos θ).





So the matrix which represents a anticlockwise rotation through any angle θ is given by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

To deal with rotations through angles greater than 90°, you need to know a little about the sines and cosines of such angles. You also need to recognise the exact values of sin θ, cos θ and tan θ for common values of θ such as 30°, 45° and 60°.

To recognise a rotation matrix, note that the elements on the leading diagonal are the same, with the same sign (either both positive or both negative), and the elements on the other diagonal are the same but with opposite signs (either could be the negative one, depending on which quadrant the angle is in).

Example 2

Find the matrix which describes

(i) Anticlockwise rotation through 140° about the origin

(ii) Clockwise rotation through 62° about the origin

Solution

(i) The matrix for anticlockwise rotation through angle θ about the origin is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

In this case θ = 140°, so the matrix is
$$\begin{pmatrix} \cos 140^\circ & -\sin 140^\circ \\ \sin 140^\circ & \cos 140^\circ \end{pmatrix}$$
$$= \begin{pmatrix} -0.766 & -0.642 \\ 0.642 & -0.766 \end{pmatrix}$$

(ii) A clockwise rotation involves a negative angle, so in this case θ = -62°.

The matrix is $\begin{pmatrix} \cos(-62^\circ) & -\sin(-62^\circ) \\ \sin(-62^\circ) & \cos(-62^\circ) \end{pmatrix} = \begin{pmatrix} 0.469 & 0.883 \\ -0.883 & 0.469 \end{pmatrix}$

Alternatively you can use the matrix for clockwise rotation through angle θ about the origin, $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, with $\theta = 62°$. This will give the same result.

Example 3

Each of the following matrices represents a rotation about the origin. Find the angle and direction of rotation in each case.

(i) $\begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$ (ii) $\begin{pmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{pmatrix}$

(iii) $\begin{pmatrix} -0.966 & -0.259 \\ 0.259 & -0.966 \end{pmatrix}$ (iv) $\begin{pmatrix} 0.766 & 0.643 \\ -0.643 & 0.766 \end{pmatrix}$

Solution

In each case, compare the given matrix with the matrix for anticlockwise rotation through angle θ, $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(i) In this case $\cos\theta = \frac{1}{2}\sqrt{3}, \sin\theta = \frac{1}{2}$.
Both are positive, so the angle is in the first quadrant.
$\arcsin\left(\frac{1}{2}\right) = 30°$.
The matrix represents an anticlockwise rotation of 30° about the origin.

(ii) In this case $\cos\theta = -0.6$, $\sin\theta = -0.8$.
Both are negative, so the angle is in the third quadrant.
Using a calculator, arccos 0.6 = 53.1°.
The corresponding angle in the third quadrant is 180° + 53.1° = 233.1°.
The matrix represents an anticlockwise rotation of 233.1° about the origin.
(OR a clockwise rotation of 126.9°).

(iii) In this case $\cos\theta = -0.966$, $\sin\theta = 0.259$.
$\cos\theta$ is negative and $\sin\theta$ is positive, so the angle is in the second quadrant.
Using a calculator, arcsin 0.259 = 15°.
The corresponding angle in the second quadrant is 165°.
The matrix represents an anticlockwise rotation of 165° about the origin.

(iv) In this case $\cos\theta = 0.766$, $\sin\theta = -0.463$.
$\cos\theta$ is positive and $\sin\theta$ is negative, so the angle is in the fourth quadrant.
Using a calculator, arcos 0.766 = 40°.
The corresponding angle in the fourth quadrant is 320° or –40°.
The matrix represents a clockwise rotation of 40° about the origin.
(OR an anticlockwise rotation of 320°).

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Reflections

Reflection in the x-axis leaves the point (1, 0) unchanged but maps the point (0, 1) to the point (0, -1).

So the matrix representing this transformation is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Reflection in the y-axis maps the point (1, 0) to the point (-1, 0) but leaves the point (0, 1) unchanged.

So the matrix representing this transformation is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Reflection in the line $y = x$ maps the point (1, 0) to the point (0, 1) and maps the point (0, 1) to the point (1, 0).

So the matrix representing this transformation is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Reflection in the line $y = -x$ maps the point (1, 0) to the point (0, -1) and maps the point (0, 1) to the point (-1, 0).

So the matrix representing this transformation is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Transformations in three dimensions

The transformation matrices you have met so far are all 2×2 matrices. They represent mappings from two-dimensional space to two-dimensional space. You can also use 3×3 transformation matrices which represent mappings from three-dimensional space to three-dimensional space.

For a transformation in three dimensions, represented by a 3 × 3 matrix, the columns of the matrix represent the images of the point (1, 0, 0), (0, 1, 0) and (0, 0, 1) respectively.

In most of the simple transformations in three dimensions that you will meet, you will see that at least one of the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) maps to itself. One way to identify the transformation is to ignore the row and column for this point, and look at the remaining 2 × 2 matrix. Identify the transformation, and then express it in terms of a three dimensional transformation.

The list below explains how to recognise each of the different types of three-dimensional transformation that you might meet.

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Reflections

- Under the matrix $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the points (0, 1, 0) and (0, 0, 1) are mapped to themselves, and the point (1, 0, 0) is mapped to the point (-1, 0, 0). So this represents a reflection in the plane $x = 0$.
  Reflections in the planes $y = 0$ and $z = 0$ are similar.
- Under the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, the point (1, 0, 0) is mapped to itself. If you ignore the first row and column, you have the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In two dimensions, this would be a reflection in the line $y = x$, but in this case it involves y and z, so this is a reflection in the plane $y = z$ (note that it is a plane rather than a line). Reflections in the planes $x = y$ and $x = z$ are similar.

Rotations

- Under the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, the point (1, 0, 0) is mapped to itself. If you ignore the first row and column, you have the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.In two dimensions, this represents a rotation of 90° anticlockwise about the origin. Since x-coordinates are unchanged, this means that the three-dimensions transformation is a rotation of 90° about the x-axis. (Note that it is not necessary in three dimensions to state a direction, since it is difficult to define "clockwise" and "anticlockwise" senses when working in three dimensions). Rotations about the y-axis and the z-axis are similar.

Note

It is also possible to have transformation matrices which are not square. For example, when you draw the plan of a building to a scale of 1:100, you start with an object in three dimensional space and end up with a two-dimensional drawing. You could write this as the matrix $\begin{pmatrix} \frac{1}{100} & 0 & 0 \\ 0 & \frac{1}{100} & 0 \end{pmatrix}$.

So a matrix with m rows and n columns is used to describe a transformation from n dimensional space to m dimensional space.

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Composition of transformations

The Explore resource ***Successive transformations*** looks at carrying out one transformation followed by another.

In both two dimensions and three dimensions, the matrix representing a combination of transformations can be found by multiplying the matrices for the individual transformations.

The important thing is to get the order of multiplication right.
The matrix **AB** represents first applying the transformation represented by matrix **B**, and then applying the transformation represented by matrix **A**.