Section 3: Matrices and simultaneous equations

Notes and Examples

These notes contain subsections on
- **Solving simultaneous equations using matrices**
- **The arrangements of three planes in 3D space**

Solving simultaneous equations using matrices

You already know how to solve a pair of simultaneous linear equations using the elimination or substitution methods. In this section you see that such equations can also be solved by writing them as a single matrix equation and solving this by using the inverse matrix.

You may wonder why you should use this method when the familiar elimination method may seem easier to you. The truth is that in the case of two simultaneous equations in two unknowns, it probably is. However, if you have to deal with a larger number of equations, the matrix method becomes more efficient than using algebraic methods. The method is the same for any number of equations:

- write the equations as a matrix equation: $\mathbf{Mp} = \mathbf{p}'$
- pre-multiply by the inverse matrix $\mathbf{M}^{-1}$: $\mathbf{p} = \mathbf{M}^{-1}\mathbf{p}'$

The only difficulty is finding the inverse matrix. If you have n simultaneous equations in n unknowns, you need to be able to find the inverse of an $n \times n$ matrix. However, computers and some calculators can do this for you.

The example below shows how the matrix method can be used to solve two equations in two unknowns. Although you would probably not use this method for two equations, it illustrates the method, before it is extended to three equations.

Example 1
Solve the linear simultaneous equations

$$3x + 2y = 9$$
$$4x + 5y = 5$$

using a matrix method.

Solution

Write the equations in matrix form

$$\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \end{pmatrix}$$

Find the inverse matrix

$$\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = (3\times 5) - (2\times 4) = 15 - 8 = 7$$

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The inverse of the matrix $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$ is $\frac{1}{7}\begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix}$.

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{7}\begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 9 \\ 5 \end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix} 35 \\ -21 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ -3 \end{pmatrix} \end{aligned}$$

Pre-multiply both sides of the matrix equation by the inverse matrix. On the left hand side this “undoes” the effect of the original matrix.

The solution is $x = 5$, $y = -3$.

Example 1 resulted in a unique solution for the system of equations. This is not always the case.

If the determinant of a matrix is zero, the matrix does not have an inverse. In a case like this you cannot solve a system of linear equations using the inverse matrix method.

There are three possible outcomes when you try to solve a system of linear equations.

- There may be a unique solution, as in Example 1 above.
- There may be no solution.
- There may be an infinite number of solutions.

Example 2 looks at these three possibilities for a system of two equations in two unknowns.

Example 2

Solve, if possible, each of the following systems of equations:

(i) $3x+2y=5$, $2x-y=8$ (ii) $4x-3y=2$, $8x-6y=4$ (iii) $3x-y=3$, $-6x+2y=1$

Solution

(i) $\begin{aligned} 3x+2y&=5 \\ 2x-y&=8 \end{aligned} \Rightarrow \begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$

$$\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = (3\times -1)-(2\times 2) = -3-4 = -7$$

The inverse of $\begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix}$ is $-\frac{1}{7}\begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} -21 \\ 14 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

The solution is $x = 3$, $y = -2$.

(ii) $\begin{matrix} 4x-3y=2 \\ 8x-6y=4 \end{matrix} \Rightarrow \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$\begin{vmatrix} 4 & -3 \\ 8 & -6 \end{vmatrix} = (4\times -6)-(-3\times 8) = -24+24=0$

The matrix has no inverse, so there is no unique solution.
The second equation is twice the first equation, so there are infinitely many solutions.
These can be expressed in terms of a parameter:
The solutions are: $x=\lambda, y=\frac{4}{3}\lambda-\frac{2}{3}$

(iii) $\begin{matrix} 3x-y=3 \\ -6x+2y=1 \end{matrix} \Rightarrow \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\begin{vmatrix} 3 & -1 \\ -6 & 2 \end{vmatrix} = (3\times 2)-(-1\times -6) = 6-6=0$

The matrix has no inverse, so there is no unique solution.
The equations are inconsistent since the second equation can be written as $3x-y=-\frac{1}{2}$.
There are no solutions.

Example 2 illustrates the three types of solution:

- a unique solution
- infinitely many solutions (which may be expressed in terms of a parameter)
- no solutions

It is helpful to think of the geometrical interpretation of these three situations. An equation of the form $ax+by=c$ represents a straight line in two dimensions. So a pair of linear simultaneous equations in two unknowns represent two straight lines.

- If the two lines cross at a single point, there is a unique solution. The equations are consistent.
- If the two lines are the same line, all points on the line are solutions, so there are infinitely many solutions. The equations are consistent.
- If the two lines are parallel, there are no common points and so no solutions. The equations are inconsistent.

In Example 2 parts (ii) and (iii), you would probably spot at once that the equations were the same in the case of (ii) and inconsistent in the case of (iii), so it would not be necessary to show that the determinant of the matrix is zero. However, when you deal with three or more equations, it is not always obvious when the set of equations is inconsistent or will result in infinitely many solutions.

Example 3 shows how the matrix method can be extended to three linear simultaneous equations.

Example 3
Solve the linear simultaneous equations

$$2x + 3y - z = 5$$
$$x - 4y + 2z = 3$$
$$3x - 2y + 4z = 3$$

using a matrix method.

Solution

Write the equations in matrix form

$$\begin{pmatrix} 2 & 3 & -1 \\ 1 & -4 & 2 \\ 3 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$$

Use a calculator to find the inverse matrix

$$\text{Inverse matrix} = -\frac{1}{28}\begin{pmatrix} -12 & -10 & 2 \\ 2 & 11 & -5 \\ 10 & 13 & -11 \end{pmatrix}$$

Pre-multiply by the inverse matrix

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{28}\begin{pmatrix} -12 & -10 & 2 \\ 2 & 11 & -5 \\ 10 & 13 & -11 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$$

$$= -\frac{1}{28}\begin{pmatrix} -84 \\ 28 \\ 56 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

The solution is $x = 3$, $y = -1$, $z = -2$.

The arrangements of three planes in 3D space

As for two equations, the matrix method fails if the inverse of the matrix does not exist, i.e. if its determinant is zero. If the inverse matrix does exist, the matrix method gives a unique solution to the equations. Otherwise, there may be no solutions or an infinite number of solutions.

An equation of the form $ax + by + cz = d$ represents a plane, so a set of three linear simultaneous equations in three unknowns represents three planes.

Assuming that none of the planes are the same, there are five different possible configurations of three planes.

No solutions

- Three parallel planes
- Two planes are parallel and the third is not
- Triangular prism

Infinitely many solutions
- Sheaf of planes

Unique solution
- Planes intersect at a single point.

It is usually easy to spot cases where planes are parallel, since if two planes are parallel, they can be written so that the coefficients of x, y and z are the same for both equations but the constant term is different (i.e. in the usual form of the equations $ax+by+cz=d$ they can be written so that the left hand sides of the two equations are the same but the right hand sides are different). If this is not the case, then for infinitely many solutions the planes must form a sheaf of planes, and for no solutions the planes must form a triangular prism.

Example 4 looks at the three possible outcomes (a unique solution, an infinite number of solutions, no solutions) for three equations in three unknowns.

Example 4

Solve, if possible, each of the following systems of equations:

(i) $\begin{aligned}3x+2y-z&=-1\\2x+z&=4\\x+2y+3z&=5\end{aligned}$ (ii) $\begin{aligned}x+2y+3z&=3\\x+y+z&=2\\y+2z&=1\end{aligned}$ (iii) $\begin{aligned}x-2y+z&=4\\x-3y+3z&=3\\2x-7y+8z&=4\end{aligned}$

Solution

(i) $\begin{aligned}3x+2y-z&=-1\\2x+z&=4\\x+2y+3z&=5\end{aligned} \Rightarrow \begin{pmatrix}3&2&-1\\2&0&1\\1&2&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1\\4\\5\end{pmatrix}$

The inverse matrix is $-\frac{1}{20}\begin{pmatrix}-2&-8&2\\-5&10&-5\\4&-4&-4\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=-\frac{1}{20}\begin{pmatrix}-2&-8&2\\-5&10&-5\\4&-4&-4\end{pmatrix}\begin{pmatrix}-1\\4\\5\end{pmatrix}$$

$$=-\frac{1}{20}\begin{pmatrix}-20\\20\\-40\end{pmatrix}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

There is a unique solution: $x = 1$, $y = -1$, $z = 2$

(ii) $\begin{aligned}x+2y+3z&=3\\x+y+z&=2\\y+2z&=1\end{aligned} \Rightarrow \begin{pmatrix}1&2&3\\1&1&1\\0&1&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix}$

Using a calculator, the determinant of the matrix is zero, so it has no inverse.

$x + 2y + 3z = 3$ ①
$x + y + z = 2$ ②
$y + 2z = 1$ ③

The aim here is to eliminate one of the variables so that you have two equations in two unknowns. You can then see whether these two equations are consistent or not.

Subtracting ② from ① $\Rightarrow$ $y + 2z = 1$
This is the same as equation ③, so the equations are consistent, and there are infinitely many solutions.
The equations form a sheaf of planes.

(iii)
$$\begin{aligned} x - 2y + z &= 4 \\ x - 3y + 3z &= 3 \\ 2x - 7y + 8z &= 4 \end{aligned} \Rightarrow \begin{pmatrix} 1 & -2 & 1 \\ 1 & -3 & 3 \\ 2 & -7 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$$

Using a calculator, the determinant of the matrix is zero, so it has no inverse.

$x - 2y + z = 4$ ①
$x - 3y + 3z = 3$ ②
$2x - 7y + 8z = 4$ ③
Subtracting ③ from 2×① $\Rightarrow 3y - 6z = 4 \Rightarrow y - 2z = \frac{4}{3}$
Subtracting ③ from 2×② $\Rightarrow$ $y - 2z = 2$

$y - 2z$ cannot be both $\frac{4}{3}$ and 2!

The equations are inconsistent.
There are no solutions.
Since none of the planes are parallel, they must form a triangular prism.

Example 4 illustrates the three different types of solution in cases where none of the planes are parallel.

Note that in the case of a sheaf of planes, the planes have a common line, so the solution points form a straight line in three dimensions. In Example 4(ii), the solutions can be expressed in terms of a parameter:

$$x = 1 + \lambda, y = 1 - 2\lambda, z = \lambda$$