# Proof by Induction: Divisibility Proofs

In this type of induction, you are asked to prove that an $n^{th}$ term formula always gives values that are divisible by some given number $p$. The basic structure of the proof is the same as before (base case, induction step, bullet point conclusion). There are two slightly different approaches for the induction step:

- Show directly that if we assume the $k^{th}$ term can be written as a multiple of $p$ then the $k+1^{th}$ term can also be written as a multiple of $p$
- Assume that the $k^{th}$ term can be written as a multiple of $p$ and then show that the difference between the $k+1^{th}$ and $k^{th}$ terms is divisible by $p$

Sometimes the question will guide you to one method or another so I will show both styles of solution for the first example. I prefer the first method and will afterwards stick to that throughout these notes.

The trick to the first method is to write the $k+1^{th}$ term as a multiple of the $k^{th}$ term plus some remainder, then use your assumption and your algebraic skills to take out a factor of $p$.

e.g. Prove that $7^n - 1$ is divisible by 6 for all positive integers $n$

| | | |
|---|---|---|
| **Base case**<br>If $n = 1$ | $7^1 - 1 = 6 = 6 \times 1$<br>i.e. a multiple of 6 | The statement is true for $n = 1$ |
| **Induction step**<br>Assume true for $n = k$<br><br>Then for $n = k + 1$ | i.e. $7^k - 1 = 6p$ for some integer $p$<br><br>$7^{k+1} - 1$<br>$= 7 \times 7^k - 1$<br>$= 7(7^k - 1) + 6$<br>$= 7(6p) + 6$<br>$= 6(7p + 1)$<br>Since $7p + 1$ is an integer, this is a multiple of 6 | If the statement is true for $n = k$…<br>…<br>…<br>…<br>…<br>…<br>…<br>…<br>…then it is also true for $n = k + 1$ |
| **Conclusion** | • True for $n = 1$<br>• If true for $n = k$ then true for $n = k + 1$<br>• Hence true for all positive integers $n$ | |

If you were using the second method, you might write it like this:

Let $f(n) = 7^n - 1$. We need to prove that $f(n)$ is a multiple of 6 for all positive integers $n$.

| | | |
|---|---|---|
| **Base case**<br>If $n = 1$ | $f(1) = 7^1 - 1 = 6 = 6 \times 1$<br>i.e. a multiple of 6 | The statement is true for $n = 1$ |
| **Induction step**<br>Assume true for $n = k$<br><br>Then… | i.e. $f(k)$ is a multiple of 6<br><br>$f(k+1) - f(k)$<br>$= (7^{k+1} - 1) - (7^k - 1)$<br>$= 7^{k+1} - 7^k$<br>$= 7 \times (7^k) - 1 \times (7^k)$<br>$= 6 \times 7^k$<br>Which is a multiple of 6.<br><br>Since $f(k)$ is a multiple of 6 and $f(k+1) - f(k)$ is a multiple of 6 then $f(k+1)$ is a multiple of 6 | If the statement is true for $n = k$…<br>…<br>…<br>…<br>…<br>…<br>…<br>…<br>…then it is also true for $n = k + 1$ |
| **Conclusion** | • True for $n = 1$<br>• If true for $n = k$ then true for $n = k + 1$<br>• Hence true for all positive integers $n$ | |

Here is another example of the first method, just to show you the algebra trick a second time:

e.g. Prove that $n^3 + 2n$ is divisible by 3 for all positive integers $n$

| | | |
|---|---|---|
| **Base case**<br>If $n = 1$ | $1^3 + 2 \times 1 = 3 = 3 \times 1$<br>i.e. a multiple of 3 | The statement is true for $n = 1$ |
| **Induction step**<br>Assume true for $n = k$<br><br>Then for $n = k + 1$ | i.e. $k^3 + 2k = 3p$ for an integer $p$<br><br>$(k+1)^3 + 2(k+1)$<br>$= k^3 + 3k^2 + 3k + 1 + 2k + 2$<br>$= (k^3 + 2k) + 3k^2 + 3k + 3$<br>$= 3p + 3k^2 + 3k + 3$<br>$= 3(p + k^2 + k + 1)$<br>Since $p + k^2 + k + 1$ is an integer, this is a multiple of 3 | If the statement is true for $n = k$…<br>…<br>…<br>…<br>…<br>…<br>…<br>…<br>…then it is also true for $n = k + 1$ |
| **Conclusion** | • True for $n = 1$<br>• If true for $n = k$ then true for $n = k + 1$<br>• Hence true for all positive integers $n$ | |

One last example which is quite a common setup in exams:

e.g. Prove by induction that $2^{3n} - 3^n$ is a multiple of 5 for all positive integers $n$

| | | |
|---|---|---|
| **Base case**<br>If $n = 1$ | $2^3 - 3^1 = 8 - 3 = 5 = 5 \times 1$<br>i.e. a multiple of 5 | The statement is true for $n = 1$ |
| **Induction step**<br>Assume true for $n = k$<br><br>Then for $n = k + 1$ | i.e. $2^{3k} - 3^k = 5p$ for an integer $p$<br><br>$2^{3(k+1)} - 3^{k+1}$<br>$= 2^{3k+3} - 3^{k+1}$<br>$= 2^3 \times 2^{3k} - 3^1 \times 3^k$<br>$= 8 \times 2^{3k} - 3 \times 3^k$<br>$= 3(2^{3k} - 3^k) + 5 \times (2^{3k})$<br>$= 3(5p) + 5 \times 2^{3k}$<br>$= 5(3p + 2^{3k})$<br>Since $3p + 2^{3k}$ is an integer, this is a multiple of 5 | If the statement is true for $n = k$…<br>…<br>…<br>…<br>…<br>…<br>…<br>…<br>…then it is also true for $n = k + 1$ |
| **Conclusion** | • True for $n = 1$<br>• If true for $n = k$ then true for $n = k + 1$<br>• Hence true for all positive integers $n$ | |

The three steps in yellow are the key here.

- The first two lines use the index laws to split the exponential terms so that we can write them as multiples of the two exponential terms which appeared when $n = k$
- In the third line, we reason that since we have 8 of the first exponential term and 3 of the second, we can make 3 pairs (i.e. 3 lots of everything we had when $n = k$) with 5 of the second exponential term left over