Roots of Quartics

Following the same pattern as before:

If $x = \alpha, x = \beta, x = \gamma$ and $x = \delta$ are roots of a quartic equation then the equation may be written: $(x - \alpha)(x - \beta)(x - \gamma)(x - \delta) = 0$.

This expands to $x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + (\alpha\beta\gamma\delta) = 0$

So:

If $x = \alpha, x = \beta, x = \gamma$ and $x = \delta$ are roots of the quartic equation $x^4 + bx^3 + cx^2 + dx + e$ then:

- $\alpha + \beta + \gamma + \delta = -b$
- $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c$
- $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -d$
- $\alpha\beta\gamma\delta = e$

If there the x^4 coefficient $\neq 0$ then all must be $\div a$

We may use shorthand notation as follows:

- $\sum \alpha = -b$
- $\sum \alpha\beta = c$
- $\sum \alpha\beta\gamma = -d$
- $\alpha\beta\gamma\delta = e$

This is an example of sigma notation
e.g. $\sum \alpha\beta$ means "the sum of all possible products of two roots"
This notation expands easily to quintics etc

e.g. Given that $\sum \alpha = 2$, $\sum \alpha\beta = 3$, $\sum \alpha\beta\gamma = -\frac{1}{2}$ and $\alpha\beta\gamma\delta = -\frac{1}{3}$ find the simplest quartic equation with integer coefficients whose roots are α, β, γ and δ

Ignoring the integer coefficient restriction gives us:

$$x^4 - 2x^3 + 3x^2 + \frac{1}{2}x - \frac{1}{3} = 0$$

Multiplying through by 6 gives:

$$6x^4 - 12x^3 + 18x^2 + 3x - 2 = 0$$

e.g. The four positive distinct roots of a quadratic equation $x^4 + bx^3 + cx^2 + dx + 1944 = 0$ are $\alpha, 2\alpha, 3\alpha, 4\alpha$. Find the values of b, c, d.

$$\alpha\beta\gamma\delta = 24\alpha^4 = 1944 \text{ so } \alpha = \sqrt[4]{\frac{1944}{24}} = 3$$

Then

- $\sum \alpha = 3 + 6 + 9 + 12 = 30$
- $\sum \alpha\beta = 18 + 27 + 36 + 54 + 72 + 108 = 315$
- $\sum \alpha\beta\gamma = 162 + 216 + 324 + 648 = 1350$

Putting all of this together gives the equation $x^4 - 30x^3 + 315x^2 - 1350x + 1944 = 0$
i.e. $b = -30$, $c = 315$ and $d = -1350$