$R\sin(\theta + \alpha)$ and $R\cos(\theta + \alpha)$ Form

When two wave forms are added, you might expect the result to be a wobbly mess. Surprisingly, if the sin and cos functions being added have the same period, adding them gives another single trig wave form. The addition formulae for sin and cos can be used to write this as a single trig function.

The Process

- Expand the relevant addition formula
- Equate the matching coefficients of $\sin\theta$ and $\cos\theta$
- Solve simultaneously by:
  - squaring and adding to find R
  - dividing through to find $\tan\alpha$ and hence α

Examples

Write $3\sin\theta + 4\cos\theta$ in the form $R\sin(\theta + \alpha)$ where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$

- $3\sin\theta + 4\cos\theta = R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$
  $3\sin\theta + 4\cos\theta = (R\cos\alpha)\sin\theta + (R\sin\alpha)\cos\theta$
- $R\cos\alpha = 3$ and $R\sin\alpha = 4$
- $R^2\cos^2\alpha + R^2\sin\alpha = 3^2 + 4^2$ so $R^2(\sin^2\theta + \cos^2\theta) = 25$ i.e. $R^2 = 25$ and $R = 5$
  $\frac{R\sin\alpha}{R\cos\alpha} = \frac{4}{3}$ i.e. $\tan\alpha = \frac{4}{3}$ and $\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 0.927$ (3dp)

$3\sin\theta + 4\cos\theta \equiv 5\sin(\theta + 0.927)$

Write $2\cos\theta - 3\sin\theta$ in the form $R\cos(\theta + \alpha)$ where $R > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$ and hence find:
a) the maximum value of $2\sin\theta - 3\cos\theta$ (exact)
b) The lowest positive value of θ for which this maximum occurs (3sf)

- $2\cos\theta - 3\sin\theta = R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$
  $2\cos\theta - 3\sin\theta = (R\cos\alpha)\cos\theta - (R\sin\alpha)\sin\theta$
- $R\cos\alpha = 2$ and $R\sin\alpha = 3$
- $R^2\cos^2\alpha + R^2\sin\alpha = 2^2 + 3^2$ so $R^2(\sin^2\theta + \cos^2\theta) = 13$ i.e. $R^2 = 13$ and $R = \sqrt{13}$
  $\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{2}$ i.e. $\tan\alpha = \frac{3}{2}$ and $\alpha = \tan^{-1}\left(\frac{3}{2}\right) = 0.983$ (3dp)

$2\cos\theta - 3\sin\theta \equiv \sqrt{13}\cos(\theta + 0.983)$

a) Since cos (...) ranges from -1 to 1, the maximum value of $\sqrt{13}\cos(\ldots)$ is $\sqrt{13}$
b) This happens when $\cos(\theta + 0.983) = 1$
   $\theta + 0.983 = \cos^{-1}(1) = 0, 2\pi$ etc
   The lowest positive θ is therefore $\theta = 2\pi - 0.983 = 5.30$ (3sf)