Parametric Equations – Basics

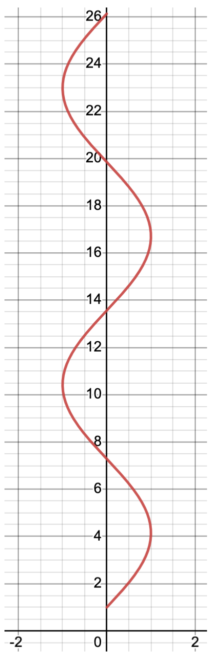
Since Year 7 we have used Cartesian equations for graphs, where we plot all of the points satisfying some equation involving x and y e.g. $x^2 + y^2 = 25$, $y = 15x^3 - \sin(x)$, $y - y_1 = m(x - x_1)$ etc.
These graphs are visual representations of all the pairs of x and y values that make the equation true.

In a system of parametric equations, the x and y coordinates are written as functions of a parameter (usually t but sometimes e.g. θ). Instead of describing the whole curve at once in a relationship between x and y coordinates, we vary the value of the parameter (i.e. substitute in a range of t values) to obtain pairs of x and y coordinates. The effect is rather like tracking the coordinates of a point travelling along a path as time changes. This is particularly useful for modelling movement e.g. in Physics.

e.g. A graph has parametric equations $y = 2t + 1$ and $x = \sin t$, where $0 \leq t \leq 4\pi$ radians

eg when $t = 0$, $x = \sin(0) = 0$ and $y = 2(0) + 1 = 1$ giving the point $(0, 1)$
when $t = \frac{\pi}{6}$, $x = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $y = 2\left(\frac{\pi}{6}\right) + 1 = \frac{\pi}{3} + 1$ giving the point $\left(\frac{1}{2}, \frac{\pi}{3} + 1\right)$
when $t = \frac{\pi}{4}$, $x = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $y = 2\left(\frac{\pi}{4}\right) + 1 = \frac{\pi}{2} + 1$ giving the point $(\frac{\sqrt{2}}{2}, \frac{\pi}{2} + 1)$

If we found the x and y coordinates for every t value between 0 and 2π and then plotted them, we would end up with a graph looking like the one below. In effect, as t varies from 0 towards 4π, the point is travelling from the bottom of the curve towards the top.



It is not always easy to discern the graph shape that will be mapped out by a system of parametric equations, but we will develop a few tools to help.

As a neat checking tool, you can input parametric equations into Desmos by writing the x and y parametric equations inside coordinate brackets, separated by a comma, and then inputting the required limits for t.
The graph above was produced with the following instruction:

$(\sin t, 2t + 1)$
$0 \leq t \leq 4\pi$

Converting to Cartesian by Eliminating the Parameter

The simplest situation is where we rearrange one of the parametric equations to make the parameter the subject of the formula, then substitute that expression into the second parametric equation.

e.g. Convert the parametric equations $x = \frac{t}{2}$ and $y = 3 + t^3$ to Cartesian form

$x = \frac{t}{2} \Rightarrow t = 2x$
Substituting into the y equation gives $y = 3 + (2x)^3$ or $y = 8x^3 + 3$

e.g. Convert the parametric equations $x = 3t - 1$ and $y = 2\sin t$ to Cartesian form

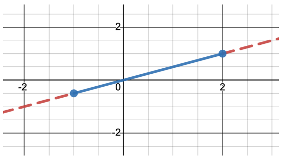
$x = 3t - 1 \Rightarrow t = \frac{x+1}{3}$
Substituting into the y equation gives $y = 2\sin\left(\frac{x+1}{3}\right)$

Converting Domain/Range

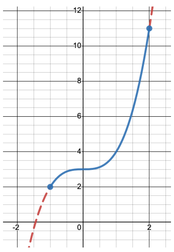
- The range of the x parametric equation gives the domain of the Cartesian equation
- The range of the y parametric equation gives the range of the Cartesian equation

We might like to sketch x and y separately as functions of t, then find the ranges of those graphs.

e.g. State the domain and range of the Cartesian equation equivalent to the parametric equations
$x = \frac{t}{2}$ and $y = 3 + t^3$ $(-1 \le t \le 2)$



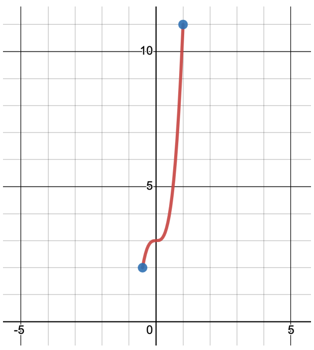
$\mathrm{y} = \frac{\mathrm{t}}{2}$ is a straight line of gradient $\frac{1}{2}$.
When $-1 \le t \le 2$, the range of the x equation is $-\frac{1}{2} \le x \le 1$



$y = 3 + t^3$ is $y = t^3$ translated up by 3, so when $-1 \le t \le 2$ this goes from 2 to 11.
So the range of the y equation is $2 \le y \le 11$

So the domain is $-\frac{1}{2} \le x \le 1$ and the range is $2 \le y \le 11$

Here is the whole parametric graph, for checking purposes:



Notice that x values go between $-\frac{1}{2}$ and 1
while y values go between 2 and 11