Complex Numbers: Modulus-Argument Form

A complex number with modulus r and argument θ may be written in the form $r(\cos\theta + i\sin\theta)$

This **modulus-argument form** of a complex number can be shown by simple trigonometry.

A common sneaky trick!

Just to confuse you, examiners often put a minus sign in the middle of what otherwise looks like a complex number written in modulus-argument form e.g. $2\left(\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right)$.

To resolve this awkward situation, recall that

- sin is an odd function i.e. $\sin(-\theta) = -\sin(\theta)$
- cos is an odd function i.e. $\cos(-\theta) = \cos(\theta)$.

So $r(\cos(\theta) - i\sin(\theta))$ should be rewritten as $r(\cos(-\theta) + i\sin(-\theta))$

e.g. $2\left(\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right)$ can be rewritten as $2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ so $|z| = 2$ and $\arg(z) = -\frac{\pi}{6}$

Products

Modulus-argument form reveals a couple of very useful properties of complex numbers:

- $|z_1 z_2| = |z_1||z_2|$

  i.e. the modulus of the product is the product of the moduli

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

  i.e. the argument of the product is the sum of the arguments

The proof requires knowledge of the trigonometric addition formulae:

- $\sin(A+B) = \sin A\cos B + \cos A\sin B$
- $\cos(A+B) = \cos A\cos B - \sin A\sin B$

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_1 = r_2(\cos\theta_2 + i\sin\theta_2)$ then:

$$
\begin{aligned}
z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\
&= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \text{ as required}
\end{aligned}
$$

e.g. $z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = 5\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$. Find the modulus and argument of $z_1 z_2$

$$
\begin{aligned}
&z_2 = 5\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) \\
&|z_1 z_2| = 2 \times 5 = 10 \\
&\arg(z_1 z_2) = \frac{\pi}{6} + -\frac{\pi}{4} = -\frac{\pi}{12}
\end{aligned}
$$

Quotients

Similarly:

- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

  i.e. the modulus of the quotient is the quotient of the moduli

- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

  i.e. the argument of the quotient is the difference of the arguments

The proof requires use of the complex conjugate to "realize" a denominator, as well as the trigonometric addition formulae.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_1 = r_2(\cos\theta_2 + i\sin\theta_2)$ then:

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\
&= \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \\
&= \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}\frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)} \\
&= \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2} \\
&= \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{1} \\
&= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)) \text{ as required}
\end{aligned}$$

e.g. Express $\frac{5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}$ in the form $a + bi$

$$|z| = \frac{5}{1} = 5$$
$$\arg(z) = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$z = 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{5\sqrt{3}}{2} + \frac{5}{2}i$$