Trig Differentiation

SUPER CRUCIAL NOTE

Whenever calculus is involved, you may assume that you are working in radians unless specifically told otherwise (which NEVER happens!)

First Principles

To differentiate $\sin x$ and $\cos x$ we use both the addition formulae and small angle approximations.

First note that for small h, $\cos(h) \approx 1 - \frac{h^2}{2}$ and $\sin(h) \approx h$ so…

$$\boxed{\lim_{h\to 0}\frac{(\cos(h)-1)}{h}} = \lim_{h\to 0}\frac{\left(1-\frac{h^2}{2}-1\right)}{h} = \lim_{h\to 0}\frac{-h}{2}\boxed{=0}$$

$$\boxed{\lim_{h\to 0}\frac{\sin h}{h}} = \lim_{h\to 0}\frac{h}{h} = \boxed{1}$$

Now if $f(x) = \sin x$ then:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h}+\frac{\cos(x)\sin(h)}{h}\\
&= \sin(x)\boxed{\lim_{h\to 0}\frac{(\cos(h)-1)}{h}}+\cos(x)\boxed{\lim_{h\to 0}\frac{\sin h}{h}}\\
&= \sin(x)\times\boxed{0}+\cos x\times\boxed{1}\\
&= \cos(x)
\end{aligned}$$

Addition formula for sin

Because we treat x as fixed, $\cos(x)$ and $\sin(x)$ are constant and can be brought out to the front of the limit

Using the earlier results

A similar approach will prove that if $f(x) = \cos(x)$ then $f'(x) = -\sin(x)$

The Results

$$\frac{d}{dx}(\sin(kx)) = k\cos(kx)$$

$$\frac{d}{dx}(\cos(kx)) = -k\sin(kx)$$

Example

Find the stationary points on $y = 3x - \cos(3x)$ for $0 \le x \le 2\pi$

$$\frac{dy}{dx} = 3 + 3\sin(3x)$$

$\frac{d}{dx}(\cos(3x)) = -3\sin(3x)$

Then don't forget the double negative!

Stationary points occur when $\frac{dy}{dx} = 0$:

$$\begin{aligned}
3+3\sin(3x) &= 0\\
\sin(3x) &= -1\\
3x &= \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}\\
x &= \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}
\end{aligned}$$

The coordinates are $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{21\pi}{6}\right), \left(\frac{11\pi}{6}, \frac{33\pi}{6}\right)$