Roots of a Polynomial – Related Expressions

Our aim is to rewrite expressions in $\alpha, \beta, \gamma, \delta$ in terms of the coefficients of the polynomial.
You should know (or be able to quickly derive) the following:

Reciprocals

- Quadratic: $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta} = \frac{\sum\alpha}{\alpha\beta}$
- Cubic: $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma} = \frac{\sum\alpha\beta}{\alpha\beta\gamma}$
- Quartic: $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma} = \frac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}$

Products of Powers

- Quadratic: $\alpha^n\beta^n = (\alpha\beta)^n$
- Cubic: $\alpha^n\beta^n\gamma^n = (\alpha\beta\gamma)^n$
- Quartic: $\alpha^n\beta^n\gamma^n\delta^n = (\alpha\beta\gamma\delta)^n$

Sums of Squares

- Quadratic: $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
- Cubic: $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$
- Quartic: $\alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

In all three cases, this may be summarised as: $\sum(\alpha^2) = (\sum\alpha)^2 - 2\sum\alpha\beta$

Sums of Cubes

- Quadratic: $\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$
- Cubic: $\alpha^3+\beta^3+\gamma^3 = (\alpha+\beta+\gamma)^3 - 3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\alpha\gamma) + 3\alpha\beta\gamma$
- Quartic: DON'T NEED TO KNOW FOR A LEVEL!

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e.g. $x^3+3x^2-2x-1=0$ has roots α,β,γ. Find $\alpha^3+\beta^3+\gamma^3$

$\sum\alpha = -3 \qquad \sum\alpha\beta = -2 \qquad \sum\alpha\beta\gamma = 1$

$\alpha^3+\beta^3+\gamma^3 = (-3)^3 - 3(-3)(-2) + 3(1) = -42$

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e.g. $2x^2-3x+5=0$ has roots α,β. Find $(\alpha^2+1)(\beta^2+1)$.

$\alpha+\beta = \frac{3}{2} \qquad \alpha\beta = \frac{5}{2}$

$(\alpha^2+1)(\beta^2+1) = \alpha^2\beta^2 + (\alpha^2+\beta^2) + 1 = \left(\frac{5}{2}\right)^2 + \left(\left(\frac{3}{2}\right)^2 - 2\left(\frac{5}{2}\right)\right) + 1 = \frac{9}{2}$