Linear Transformations of Roots

The question in this section is: how do the coefficients of an equation change if we decide to double the value of all of the roots? Or add 5 to all of the roots? Or double and then add 5 to all of the roots?

There are two methods to approach this type of question:

1. Find $\sum \alpha'$, $\sum \alpha'\beta'$ etc by expanding and grouping together in terms of $\sum \alpha$, $\sum \alpha\beta$ etc
2. Substitute a transformed variable for x into the equation, expand and simplify

e.g. The equation $x^3 + 3x^2 - x + 1 = 0$ has roots α, β, γ
Find the equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$

Method 1:

Originally:

- $\sum \alpha = -3$
- $\sum \alpha\beta = -1$
- $\alpha\beta\gamma = -1$

Now:

- $\begin{aligned}\sum \alpha' &= (2\alpha + 1) + (2\beta + 1) + (2\gamma + 1) \\ &= 2(\alpha + \beta + \gamma) + 3 \\ &= 2\sum \alpha + 3 \\ &= 2(-3) + 3 \\ &= -3\end{aligned}$

Expand and then collect together so that we can use the original coefficient values

- $\begin{aligned}\sum \alpha'\beta' &= (2\alpha + 1)(2\beta + 1) + (2\alpha + 1)(2\gamma + 1) + (2\beta + 1)(2\gamma + 1) \\ &= 4(\alpha\beta + \alpha\gamma + \beta\gamma) + 4(\alpha + \beta + \gamma) + 3 \\ &= 4\sum \alpha\beta + 4\sum \alpha + 3 \\ &= 4(-1) + 4(-3) + 3 \\ &= -13\end{aligned}$

- $\begin{aligned}\alpha'\beta'\gamma' &= (2\alpha + 1)(2\beta + 1)(2\gamma + 1) \\ &= 8(\alpha\beta\gamma) + 4(\alpha\beta + \alpha\gamma + \beta\gamma) + 2(\alpha + \beta + \gamma) + 1 \\ &= 8(-1) + 4(-1) + 2(-3) + 1 \\ &= -17\end{aligned}$

So the equation is $x^3 + 3x^2 - 13x + 17 = 0$

Method 2:

Because all roots are being transformed by $\times 2$ then $+1$

The old equation was $x^3 + 3x^2 - x + 1 = 0$
The transformed variable is $w = 2x + 1$ which rearranges to $x = \frac{w-1}{2}$
Substitute this into the equation i.e. replace x with $\left(\frac{w-1}{2}\right)$:

$\left(\frac{w-1}{2}\right)^3 + 3\left(\frac{w-1}{2}\right)^2 - \left(\frac{w-1}{2}\right) + 1 = 0$

Multiply through by 8 to get rid of the fractions

$\left(\frac{w^3-3w^2+3w-1}{8}\right) + 3\left(\frac{w^2-2w+1}{4}\right) - \left(\frac{w-1}{2}\right) + 1 = 0$

$(w^3 - 3w^2 + 3w - 1) + 6(w^2 - 2w + 1) - 4(w - 1) + 8 = 0$

$w^3 + 3w^2 - 13w + 17 = 0$

It doesn't really matter what letter the equation uses