Sums of Polynomial Series – Basics

Sigma Notation

We can write a series summation in condensed form using sigma notation.

e.g.

$$\sum_{r=3}^{6}(3r^2-1)$$

means:

- Starting with $r=3$
- ...and ending with $r=6$
- ...substitute integer values of r into the expression $3r^2-1$
- ...and add the answers together

i.e. $(3(3)^2-1)+(3(4)^2-1)+(3(5)^2-1)+(3(6)^2-1)$
$=26+47+74+107$
$=254$

The Two Simplest Summation Formulae

$$\sum_{r=1}^{n}1=n$$
$$\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$$

- The first of these is obvious: we are adding 1 + 1 + 1 + ... + 1, n times
- The second can be proven easily using e.g. the arithmetic series proof

e.g.

$$\sum_{r=1}^{5}1=5$$

e.g.

$$\sum_{r=1}^{6}r=\frac{1}{2}(6)(7)=21$$

e.g.

$$\sum_{r=1}^{2k-1}r=\frac{1}{2}(2k-1)(2k)=k(2k-1)$$

Three Useful Rules

$$\sum_{r=1}^{n} kf(r) = k\sum_{r=1}^{n} f(r)$$
$$\sum_{r=1}^{n}\big(f(r)+g(r)\big) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$$
$$\sum_{r=m}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{m-1} f(r)$$

- The first means that we can bring a constant multiple from within a sigma out in front of the sigma
- The second means that if series are added within a sigma, we can sum them separately and add
- The third needs care: if the sum starts from $r = m$ instead of $r = 1$ then we can sum from 1 to n and then cancel the unwanted terms by subtracting the sum from 1 to $m - 1$. WE DO NOT WANT TO CANCEL THE TERM WHERE $r = m$.

e.g.

$$\sum_{r=1}^{20} 7 = 7\sum_{r=1}^{20} 1 = 7(20) = 140$$

e.g.

$$\sum_{r=1}^{8} 2r = 2\sum_{r=1}^{8} r = 2\left(\frac{1}{2}(8)(9)\right) = 72$$

e.g.

$$\sum_{r=1}^{5}(6r-4) = \sum_{r=1}^{5} 6r - \sum_{r=1}^{5} 4 = 6\sum_{r=1}^{5} r - 4\sum_{r=1}^{5} 1 = 6\left(\frac{1}{2}(5)(6)\right) - 4(5) = 70$$

e.g.

$$\sum_{r=4}^{7} r = \sum_{r=1}^{7} r - \sum_{r=1}^{3} r = \frac{1}{2}(7)(8) - \frac{1}{2}(3)(4) = 22$$

e.g.

$$\begin{aligned}
\sum_{r=k}^{2k}(3r+1) &= \sum_{r=1}^{2k}(3r+1) - \sum_{r=1}^{k-1}(3r+1) = \left(3\sum_{r=1}^{2k} r + \sum_{r=1}^{2k} 1\right) - \left(3\sum_{r=1}^{k-1} r + \sum_{r=1}^{k-1} 1\right) \\
&= \left(3\left(\frac{1}{2}(2k)(2k+1)\right) + (2k)\right) - \left(3\left(\frac{1}{2}(k-1)(k)\right) + (k-1)\right) \\
&= 6k^2 + 3k + 2k - \frac{3}{2}k^2 + \frac{3}{2}k - k + 1 \\
&= \frac{9}{2}k^2 + \frac{11}{2}k + 1 \\
&= \frac{1}{2}(9k^2 + 11k + 2) \\
&= \frac{1}{2}(k+1)(9k+2)
\end{aligned}$$