Roots of Polynomials : Cubic Equations

We can repeat our quadratic tricks with cubic equations.

$(x-\alpha)(x-\beta)(x-\gamma)=0$ expands to $x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-(\alpha\beta\gamma)=0$

If α, β and γ are roots of the equation $x^3+bx^2+cx+d=0$ then:

- $\alpha+\beta+\gamma=-b$
- $\alpha\beta+\alpha\gamma+\beta\gamma=c$
- $\alpha\beta\gamma=-d$

Starting to put together a pattern, we may consider that:

- The sum of single roots is $-b$
- The sum of products of two roots is c
- The product of all three roots is $-d$

e.g. If α, β and γ are roots of the equation $x^3+2x^2+3x-1=0$ then

- $\alpha+\beta+\gamma=-2$
- $\alpha\beta+\alpha\gamma+\beta\gamma=3$
- $\alpha\beta\gamma=1$

We may also combine these terms similarly to in the quadratic case eg:

- $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}=\frac{c}{-d}$

We also have a different path to finding a cubic equation when we know its roots.

e.g. Given that a cubic equation $x^3+bx^2+cx+d=0$ has roots $1+2i$ and 3 find b, c, d

Since $1+2i$ is a root, so is $1-2i$

- $\alpha+\beta+\gamma=1+2i+1-2i+3=5$ so $b=-5$
- $\alpha\beta+\alpha\gamma+\beta\gamma=(1+2i)(1-2i)+3(1-2i)+3(1+2i)=11$ so $c=11$
- $\alpha\beta\gamma=(1+2i)(1-2i)(3)=15$ so $d=-15$

The equation is $x^3-5x^2+11x-15=0$

We may be asked to do some simple problem-solving

e.g. The roots of the cubic equation $2x^3+12x^2+cx+d=0$ are $\alpha, 2\alpha$ and 3α. Find c and d.

From the x^2 coefficient, we see that the sum of the roots is $\frac{-12}{2}$

$\alpha+2\alpha+3\alpha=-6$ so $\alpha=-1$ and the three roots are $-1, -2$ and -3

$\frac{c}{2}=(-1)(-2)+(-1)(-3)+(-2)(-3)$ so $c=22$

$\frac{d}{2}=-(-1)(-2)(-3)$ so $d=12$