Differentiating Exponentials and Logs

We met the constant e earlier in the course. It should not be a surprise that $\frac{d}{dx}(e^x) = e^x$. More generally:

$$\boxed{\frac{d}{dx}(e^{kx}) = ke^{kx}}$$

A sneaky bit of log work gives us a further result:

Because $e^{\ln a} = a$

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(\left(e^{\ln a}\right)^x\right) = \frac{d}{dx}\left(e^{(\ln a)x}\right) = \ln a \,.\, e^{(\ln a)x} = \ln a \,.\, a^x$$

Using $\frac{d}{dx}(e^{kx}) = ke^{kx}$

So we have our second result:

Or more generally:

$$\boxed{\begin{gathered}\frac{d}{dx}(a^x) = \ln a \,.\, a^x \\ \frac{d}{dx}(a^{kx}) = k\ln a \,.\, a^{kx}\end{gathered}}$$

Another sneaky bit of rearrangement gives us a further result:

Do "e to the power of" on both sides

Differentiate both sides with respect to y

Because $e^y = x$ from two lines earlier

Handily, the reciprocal of $\frac{dy}{dx}$ is $\frac{dx}{dy}$

$$\begin{gathered}y = \ln x \\ e^y = x \\ \frac{dx}{dy} = e^y \\ \frac{dx}{dy} = x \\ \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{x}\end{gathered}$$

Hence the third result:

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

e.g. Find the gradient of $y = e^{6x} + 2^x + \ln x$ when $x = 2$

$$\frac{dy}{dx} = 6e^{6x} + \ln 2 \,.\, 2^x + \frac{1}{x}$$

$$@x = 2\,, \qquad \frac{dy}{dx} = 6e^{12} + 4\ln 2 + \frac{1}{2}$$

e.g. Find the equation of the tangent to $y = 3^{2x} - \ln x$ at $x = 1$ in the form $y = mx + c$

$$\frac{dy}{dx} = 2\ln 3 \,.\, 3^{2x} - \frac{1}{x}$$

$$@x = 1\,, \qquad \frac{dy}{dx} = 18\ln 3 - 1 \text{ and } y = 9$$

$$y - 9 = (18\ln 3 - 1)(x - 1)$$

$$y = (18\ln 3 - 1)x + (10 - 18\ln 3)$$