Homework 5

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```r
##-----------------------------------------------------
## R code for the Monte Carlo simulations in HW5 math 563
##-----------------------------------------------------
## GOAL: To estimate pi = 3.14159..., the area of the unit circle.
## Strategy 1
pi.est1 <- function(n, reps) {
pi.hat <- numeric(reps)
for(i in 1:reps) pi.hat[i] <- 4 * mean((runif(n)**2 + runif(n)**2) < 1)
return(pi.hat)
}
# TO RUN:
out1 <- pi.est1(n=10000, reps=1000)
print(c(mean(out1), var(out1)))
```

```
## [1] 3.1416720000 0.0002675654
```

```r
# Strategy 2
pi.est2 <- function(n, reps) {
pi.hat <- numeric(reps)
for(i in 1:reps) pi.hat[i] <- 4 * mean(sqrt(1 - runif(n)**2))
return(pi.hat)
}
# TO RUN:
myOutput <- pi.est2(n=10000, reps=1000)
print(c(mean(myOutput), var(myOutput)))
```

```
## [1] 3.141724e+00 7.426008e-05
```

We can see we have smaller variance in 2nd Strategy so π2 (Pi-2) is the best estimators in this case. If you have a choice between two ways to estimate some quantity, choose the method that has the smaller variance. For Monte Carlo estimation, a smaller variance means that you can use fewer Monte Carlo iterations to estimate the quantity.

why the two estimators do estimate π. Both estimators provide a reasonable approximation of pi, but estimate from the 2nd Strategy method is better. More importantly, the standard error for the 2nd Strategy method is lesser then the 1st Strategy.

Home Work # 5

Problem # 1

Ex 7.24.

Solution: we have $X_1, \dots X_n$ be iid λ and λ is gamma(α, β) distribution.

a) The posterior distribution of λ.

we Know

$$P(x; \lambda) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

Now

$$f(x_1, \dots, x_n | \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda}$$

$$= \prod_{i=1}^{n} \left(\frac{1}{x_i!} \right) \lambda^{\sum_{i=1}^{n} x_i} e^{-\lambda}$$

and

$$\pi(\lambda) = \frac{1}{\beta^{\alpha} \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta}$$

Thus the joint distribution of $X_1, \dots X_n$ and λ is

$$f(x_1, \dots, x_n, \lambda) = \prod_{i=1}^{n} \left(\frac{1}{x_i!} \right) \lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda} \pi(\lambda)$$

$$f(x_1,\dots,x_n,\lambda)=\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\lambda^{\sum_{i=1}^{n}x_i}e^{-n\lambda}\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\lambda/\beta}$$

if we let $y=\sum_{i=1}^{n}x_i$ we may write

$$=\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\lambda^{y+\alpha-1}e^{-n\lambda-\lambda/\beta}$$

$$=\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\lambda^{y+\alpha-1}e^{-\frac{\lambda}{\beta/(n\beta+1)}}$$

The marginal distribution of $X_1, X_2, \dots X_n$ becomes

$$m(x_1,\dots x_n)=\int f(x_1,\dots x_n,\lambda)\,d\lambda$$

$$=\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_{0}^{\infty}\lambda^{y+\alpha-1}e^{-\frac{\lambda}{\beta/(n\beta+1)}}\,d\lambda$$

$$=\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}\Gamma(y+\alpha)$$

the posterior distribution of λ is given by

$$f(\lambda\mid x_1\dots x_n)=\frac{f(x_1,\dots x_n,\lambda)}{m(x_1,\dots,x_n)}$$

$$= \frac{\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\lambda^{y+\alpha-1}e^{-\frac{\lambda}{\beta/(n\beta+1)}}}{\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}\Gamma(y+\alpha)}$$

$$= \frac{\lambda^{y+\alpha-1}}{\left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}\Gamma(y+\alpha)} e^{-\frac{\lambda}{\beta/(n\beta+1)}}$$

#

Now

$\lambda | X_1 = x_1, \dots X_n = x_n \sim \text{gamma}(y+\alpha, \beta/(n\beta+1))$

part (B) find the posterior mean and variance :-

mean ⇒

$$E\left(\lambda \mid \sum_{i=1}^{n} X_i = y\right) = (y+\alpha)\frac{\beta}{n\beta+1}$$

$$= \frac{n\beta}{n\beta+1}\frac{y}{n} + \frac{1}{n\beta+1}(\alpha\cdot\beta)$$

#

Variance is $\Rightarrow$

$$var\left(\lambda \mid \sum_{i=1}^{n} X_i = y\right) = (y+\alpha)\left(\frac{\beta}{n\beta+1}\right)^2$$

#

Problem #2

Let $X_1 \ldots X_n$ denote a random sample from a Poisson distribution that has the mean $\theta > 0$.
Show that the MLE of θ is an efficient estimator of θ.

Solution $\Rightarrow$ we know that mean of sample is θ $\quad \theta = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

#

the Joint density is

$$f(x_1 \ldots x_n / \lambda) = \prod_{i=1}^{n}\left(\frac{1}{x_i!}\right) e^{-n\lambda} \prod_{i=1}^{n} \lambda^{x_i}$$

This is the product of $\left[\prod_{i=1}^{n} x_i!\right]^{-1}$.

which does not depend on λ

and

$$e^{-n\lambda} \prod_{i=1}^{n} \lambda^{x_i} = e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}$$

which depends on the random sample only through $\sum_{i=1}^{n} x_i$. So this is a sufficient statistic. or we could multiply it by $1/n$ to see that the sample mean

$$\bar{X}_n = 1/n \sum_{i=1}^{n} x_i$$

is a sufficient statistic.

And the other way using MLE

$$L(\lambda) = \prod_{i=1}^{n} f(x_i | \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$= \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} (x_i!)}$$

$$\text{Log } L(\lambda) = \left(\sum_{i=1}^{n} x_i\right) \log \lambda - n\lambda - \sum_{i=1}^{n} \log(x_i!)$$

$$\frac{\partial \text{ Log } L(\lambda)}{\partial \lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda^2} - n = 0$$

$$\left\{ \hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_i \right\}_{\#} \qquad \therefore \theta = \bar{X}$$

$$\boxed{\lambda = \theta}_{\#}$$

So the efficient estimator is mean $\{\theta = \bar{x}\}$

problem # 3

Ex 7.44.

Solution: Let $X_1, \ldots X_n$ are i.i.d $N(\theta, 1)$ $W = \bar{X} - 1/n$

$$E[W] = \theta^2 + \frac{1}{n} - \frac{1}{n} = \theta^2$$

Since $\bar{X}$ is sufficient and complete

Now let's calculate the un-biased estimator (UMVUE) of θ^2

$$Var(\bar{x}^2 - 1/n) = Var(\bar{x}^2) = E(\bar{X}^4) - [E(\bar{x}^2)]^2$$

using Stein's Sec 3.6

$$EY^4 = E[Y^3(Y-\theta+\theta)] = EY^3(Y-\theta) + EY^3\theta$$

$$= \underbrace{EY^3(Y-\theta)}_{1} + \underbrace{\theta EY^3}_{2}$$

1 $EY^3(Y-\theta) = \sigma^2 E(3Y^2) = \sigma^2 3(\sigma^2+\theta^2) = 3\sigma^4 + 3\theta^2\sigma^2$

$$\theta E Y^3 = \theta(3\theta\sigma^2+\theta^3) = 3\theta^2\sigma^2+\theta^4$$

$$\operatorname{Var} X^2 = 3\sigma^2 + 6\theta^2\sigma^2 + \theta^4 - (\sigma^2+\theta^2)^2$$
$$= 2\sigma^4 + 4\theta^2\sigma^2.$$

Thus,

$$\operatorname{Var}\left(\bar{x}^2 - \frac{1}{n}\right) = \operatorname{Var}\bar{x}^2 = 2\frac{1}{n^2} + 4\theta^2\frac{1}{n}$$

Now we see the Cramer Rao lower bound

$$E_\theta\left(\frac{\partial^2 \log f(x|\theta)}{\partial\theta^2}\right) = E_\theta\left(\frac{\partial^2}{\partial\theta^2}\log\frac{1}{\sqrt{2\pi}}e^{-(x-\theta)^2/2}\right)$$

$$= E_\theta\left(\frac{\partial^2}{\partial\theta^2}\left(\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{(x-\theta)^2}{2}\right)\right)$$

$$= E_\theta\left(\cancel{\frac{\partial}{\partial\theta}0} - \cancel{2}\right.$$

$$= E_\theta\left(0 - \frac{\partial}{\partial\theta}\left(-\frac{2(x-\theta)}{2}\right)\right)$$

$$= E_\theta\left(\frac{\partial}{\partial\theta}(x-\theta)\right) = -1.$$

and $\tau(\theta) = \theta^2$, $[\tau'(\theta)]^2 = (2\theta)^2 = 4\theta^2$

CRLB of estimating θ^2 is

$$= \frac{(\tau'(\theta))^2}{-nE_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right)} = \frac{4\theta^2}{-n(-1)} = \frac{4\theta^2}{n}$$

#

Now we know that

$$Var\left(\bar{x}^2 - \frac{1}{n}\right) = Var\,\bar{x}^2 = \frac{2}{n^2} + \frac{4\theta^2}{n}$$

and CRLB is $\frac{4\theta^2}{n}$.

it is clearly seen that $Var\left(\bar{x}^2 - \frac{1}{n}\right) > \frac{4\theta^2}{n}$

#

problem #4

Ex 7.49

Let $X_1 \ldots X_n$ be iid exponential (λ).

a) find an unbaised estimator of λ based only on $Y = \min(X_1, \ldots X_n)$

we know that:

exponential distribution with density function

$$f(y) = \frac{1}{\theta} e^{-y/\theta}, \quad (y \geq 0)$$

in our case

$Y = \min(X_1, \ldots X_n)$ Let consider $Y = X_1$

then joint pdf.

$$\hat{f}_Y(y) = \frac{n!}{(n-1)!} \frac{1}{\lambda} e^{-y/\lambda} \left(1 - \left(1 - e^{-y/\lambda}\right)\right)^{n-1}$$

$$f_Y(y) = \frac{n}{\lambda} e^{-ny/\lambda}$$

we can say Y is exponetial function of (λ/n)

· so $E[Y] = \lambda/n$

nY = is an unbiased estimator of λ.

b) find a better estimator than the one in part (a)

$$f_X(x) = \prod_{i=1}^{n} \lambda e^{-\lambda x}$$

$$Lf(x) = \lambda^n e^{-\lambda \Sigma x_i}$$

$$\log(Lf(x)) = n \log \lambda - \lambda \Sigma x_i$$

$$\frac{\partial (\log(Lf(x)))}{\partial \lambda} = \frac{n}{\lambda} - \Sigma x_i = 0$$

$$\{ \lambda = n / \Sigma x_i \}$$

Σx_i is a complete sufficient statistic.

for unbiased estimation of λ the best is $E\left(nX_1 \mid \sum_{i}^{n} x_i\right)$

Since we know $E(\sum x_i) = n\lambda$

then we have $E\left(\frac{n X_{(1)}}{\sum_{i=1}^{n} x_i}\right) = \frac{\sum x_i}{n}$

c) 50.1, 70.1, 137.0, 166.9, 170.5, 152.8
80.5, 123.5, 112.6, 148.5, 160.0, 125.4

min

from part (a) $\lambda = n \cdot \min(x)$

$\hat{\lambda} = 12 \times 50.1 = 601.2$ #

from part (b)

$\lambda = \frac{\sum x_i}{n} = \frac{1545.6}{12}$

$\lambda = 128.8$ #

we are not getting good result in exponential model.

Ex 10.1

If W_n is a sequence of estimators of a parameter θ satisfying

i. $\lim_{n\to\infty} \text{Var}_\theta W_n = 0$

ii. $\lim_{n\to\infty} \text{Bias}\, W_n = 0$

for every $\theta \in \Theta$ then W_n is a consistent sequence of estimators of θ.

Solution → Let $X_1, \dots X_n$ be iid with pdf.

$$f(x|\theta) = \frac{1}{2}(1+\theta x) \quad \text{for} \quad -1<x<1, \quad -1<\theta<1$$

$$E[x] = \int_{-1}^{1} x \frac{1}{2}(1+\theta x)\,dx = \frac{1}{2}\int_{-1}^{1}(x+\theta x^2)\,dx = \frac{1}{2}\left[\frac{x^2}{2}+\theta\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{2}+\frac{\theta}{3}-\frac{1}{2}+\frac{\theta}{3}\right] = \frac{1}{2}\left(\frac{2\theta}{3}\right) = \theta/3$$

Similarly

$$E[x^2] = \int_{-1}^{1} x^2 \frac{1}{2}(1+\theta x)\,dx = \frac{1}{2}\int_{-1}^{1}(x^2+\theta x^3)\,dx = \frac{1}{2}\left[\frac{x^3}{3}+\theta\frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{3}+\cancel{\frac{\theta}{4}}+\frac{1}{3}-\cancel{\frac{\theta}{4}}\right] = 1/3$$

$$\text{Var}_\theta X = \frac{1}{3}-\left(\frac{\theta}{3}\right)^2 = \frac{1}{3}\left(1-\frac{\theta^2}{3}\right)$$

The moment estimator is $(\theta = 3\bar{X}_n)$

The estimator is unbiased and

$$\text{Var}_\theta \hat{\theta} = 9\,\text{Var}_\theta \bar{X}_n = 9\frac{1}{3n}\left(1-\frac{\theta^2}{3}\right) = \frac{1}{n}(3-\theta^2) \xrightarrow[n\to\infty]{} 0$$

Hence the estimator is consistent. #

problem #7

Solution⇒ Let $X_1, \dots, X_n$

$$P_\theta(x) = 2e^{-|x-\theta|} \quad x \in R, \ \theta \in R.$$

a) Argue that the MLE of θ is not unique.

likelihood $l(x|\theta) = 2e^{-|x-\theta|}$

$$\text{Log } l(x|\theta) = \log(2e^{-|x-\theta|})$$

$$\text{Log}(l(x|\theta)) = \log 2 - |x-\theta|$$

$$l = \frac{\partial \text{ Log}(l(x|\theta))}{\partial \theta} = 0 + \frac{(x-\theta)}{|x-\theta|}$$

if $x > \theta$ $\quad l = \frac{\partial l(x|\theta)}{\partial \theta} = 1$

if $x < \theta$ $\quad l = \frac{\partial l(x|\theta)}{\partial \theta} = -1$

and θ is also a paramete of x that ~~mean~~ means we have not unique MLE of θ.

Homework#5 7b

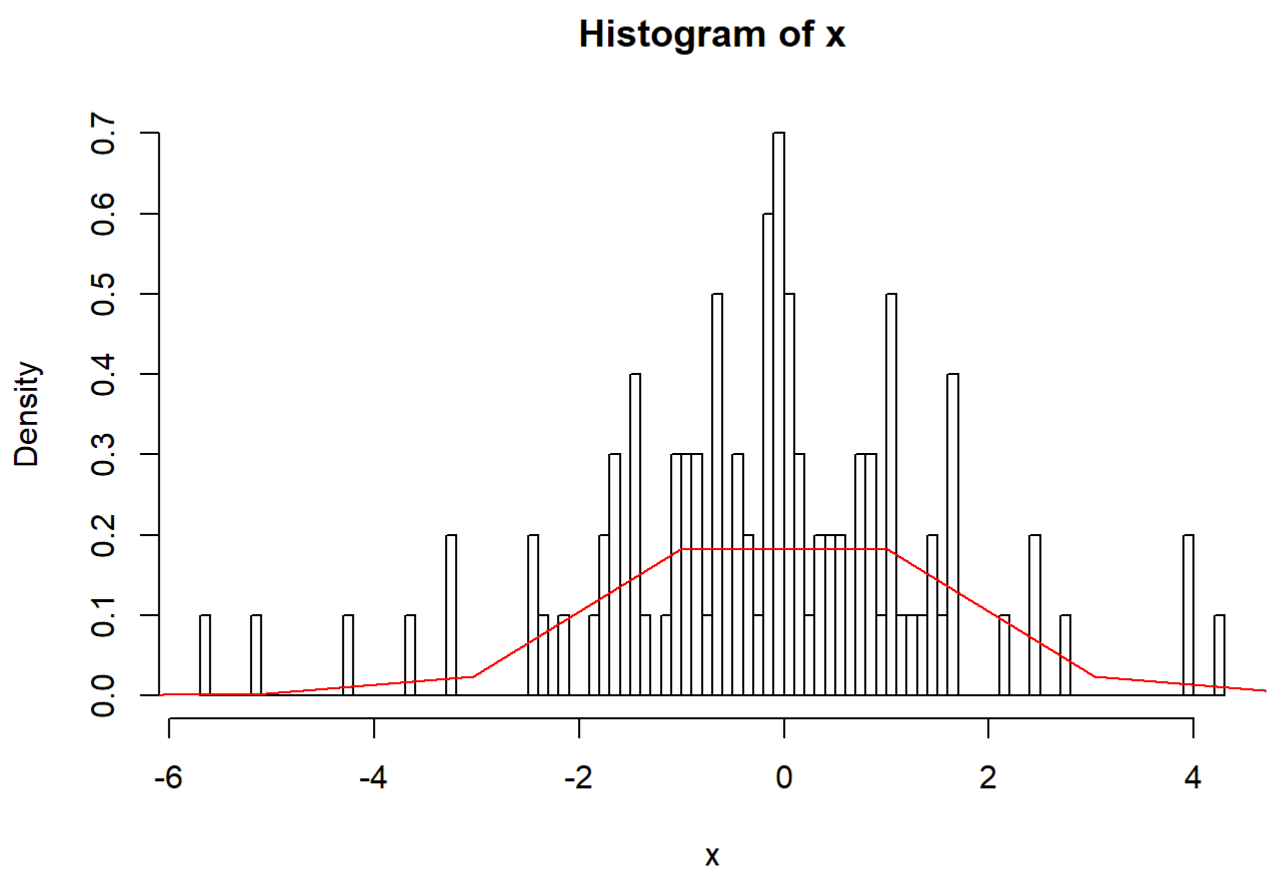
Arinjay Jain

March 9, 2021

```
library(LaplacesDemon)
```

```
## Warning: package 'LaplacesDemon' was built under R version 3.6.3
```

```
x <- rlaplace(100, 0, 1)
hist(x, 100, freq = FALSE)
curve(dlaplace(x, 0, 1), -100, 100, n = 100, col = "red", add = TRUE)
```



we can see the flat peak from -1 to +1

Homework #5

Problem #7-2
part(b)

We have two normal densities

$$P(x,\theta) = \pi\phi(x, \mu, \sigma_1) + (1-\pi)\phi(x, \mu_2, \sigma_2)$$

where $\theta = (\mu_1, \mu_2, \sigma_1, \sigma_2, \pi)$

$\phi \rightarrow$ ~~sound~~ standard density for normal random variable.

Let z_i be the r.v that denotes class membership of observed x_i in given sample.

Z_i is missing data with $z_i \in \{0,1\}$ and

$Z_i \sim$ Bernoulli$\{\pi\}$

$Y = [x, z]$ is complete data density.

$$P\{x_i, z_i|\theta\} = [\pi\phi(x, \mu_1, \sigma)]^{z_i}[(1-\pi)\phi(x, \mu_2, \sigma_2)]^{(1-z_i)}$$

a) $$p(x, z|\theta) = \prod_{i=1}^{n}[\pi\phi(x_i, \mu_1, \sigma_1)]^{z_i}[(1-\pi)\phi(x_2, \mu_2, \sigma_2)]^{(1-z_i)}$$

$$\therefore p(\theta/x, z) = \log p(x, z/\theta)$$

$$= \sum_{i=1}^{n}[z_i\{\log\pi + \log\phi(x_i, \mu, \sigma_1)\}]$$

$$+ [(1-z_i)(\log(1-\pi) + \log\phi(x_i, \mu_2, \sigma_2))]$$

$= \cancel{\log\pi}$

$$= \log \pi \sum_{i=1}^{n} z_i + \sum_{i=1}^{n} z_i \log \phi(x_i, \mu_1, \sigma_1) + \log(1-\pi).$$

$$(n - \sum_{i=1}^{n} z_i) + \sum_{i=1}^{n} (1-z_i) \log \phi(x_i, \mu_2, \sigma_2)$$

Hence this is proved. #

b) we have to find $\phi\{\theta|\theta^{(k)}\}$ in term of

$E\{z_i [x_i, \theta^{(k)}\}$

If the function is in terms of

$\phi(x_i, \mu_1^{(k)}, \sigma_1^{(k)})$.

$$\varphi(\theta|\theta^{(k)}) = \log \pi \sum_{i=1}^{n} E(z_i) + \sum_{i=1}^{n} E(z_i) \log \phi(x_i, \mu_1^{(k)}, \sigma_1^{(k)})$$
$$+ \log(1-\pi)(n - \sum_{i=1}^{n} E(z_i)) + \sum_{i=1}^{n} (1 - E(z_i)) \log \phi(x_i, \mu_2^{(k)}, \sigma_2^{(k)})$$

$$= n\pi \log \pi + \pi \sum_{i=1}^{n} \log \phi(x_i, \mu_1^{(k)}, \sigma_1^{k}) + n(1-\pi) \log$$
$$+ n (1-\pi) \log(1-\pi) + (1-\pi) \sum_{i=1}^{n} \log \phi(x_i, \mu_2^{k}, \sigma_2^{k})$$

$\therefore$ ~~Hence the~~ #