MATH 563 HOMEWORK 2
SOLUTIONS

SONJA PETROVIĆ & TA: HAO LIU

GRADING SCALE: 5 points per problem. About half the points given for correct mathematical result, and the other half for reasoning / explanation of the solution.

Exercise 5.21

5.21 Let m denote the median. Then, for general n we have

$$\begin{aligned} P(\max(X_1, \ldots, X_n) > m) &= 1 - P(X_i \le m \text{ for } i = 1, 2, \ldots, n) \\ &= 1 - [P(X_1 \le m)]^n = 1 - \left(\frac{1}{2}\right)^n. \end{aligned}$$

Exercise 5.24

5.24 Use $f_X(x) = 1/\theta$, $F_X(x) = x/\theta$, $0 < x < \theta$. Let $Y = X_{(n)}$, $Z = X_{(1)}$. Then, from Theorem 5.4.6,

$$f_{Z,Y}(z,y) = \frac{n!}{0!(n-2)!0!}\frac{1}{\theta}\frac{1}{\theta}\left(\frac{z}{\theta}\right)^0\left(\frac{y-z}{\theta}\right)^{n-2}\left(1-\frac{y}{\theta}\right)^0 = \frac{n(n-1)}{\theta^n}(y-z)^{n-2},\ 0 < z < y < \theta.$$

Now let $W = Z/Y$, $Q = Y$. Then $Y = Q$, $Z = WQ$, and $|J| = q$. Therefore

$$f_{W,Q}(w,q) = \frac{n(n-1)}{\theta^n}(q - wq)^{n-2}q = \frac{n(n-1)}{\theta^n}(1-w)^{n-2}q^{n-1},\ 0 < w < 1, 0 < q < \theta.$$

The joint pdf factors into functions of w and q, and, hence, W and Q are independent.

Exercise 5.31

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5.31 We know that $\sigma^2_{\bar{X}} = 9/100$. Use Chebyshev's Inequality to get

$$P\left(-3k/10 < \bar{X} - \mu < 3k/10\right) \geq 1 - 1/k^2.$$

We need $1 - 1/k^2 \geq .9$ which implies $k \geq \sqrt{10} = 3.16$ and $3k/10 = .9487$. Thus

$$P(-.9487 < \bar{X} - \mu < .9487) \geq .9$$

by Chebychev's Inequality. Using the CLT, $\bar{X}$ is approximately $\mathrm{n}(\mu, \sigma^2_{\bar{X}})$ with $\sigma_{\bar{X}} = \sqrt{.09} = .3$ and $(\bar{X} - \mu)/.3 \sim \mathrm{n}(0, 1)$. Thus

$$.9 = P\left(-1.645 < \frac{\bar{X} - \mu}{.3} < 1.645\right) = P(-.4935 < \bar{X} - \mu < .4935).$$

Thus, we again see the conservativeness of Chebychev's Inequality, yielding bounds on $\bar{X} - \mu$ that are almost twice as big as the normal approximation. Moreover, with a sample of size 100, $\bar{X}$ is probably very close to normally distributed, even if the underlying X distribution is not close to normal.

Exercise 5.33

Take any $\epsilon > 0$.

Since X_n converges to X in distribution, for any finite number m, we have

$$\lim_{n\to\infty} P(X_n > -m) = \lim_{n\to\infty} (1 - F_{X_n}(-m)) = 1 - F_X(-m) \tag{1}$$

Since $F_X(\cdot)$ is a distribution function, we have

$$\lim_{x\to-\infty} F_X(x) = 0$$

Therefore, we can choose m such that $F_X(-m) < \frac{\epsilon}{2}$. According to (1), there exists N_1 such that $\forall n > N_1$,

$$P(X_n > -m) > 1 - \frac{\epsilon}{2} \tag{2}$$

Furthermore, it is given that for any finite number c, we have

$$\lim_{n\to\infty} P(Y_n > c + m) = 1$$

. Therefore, there exists N_2 such that $\forall n > N_2$, we have

$$P(Y_n > c + m) > 1 - \frac{\epsilon}{2} \tag{3}$$

Now we take $N = \max(N_1, N_2)$, $\forall n > N$, adding (2)(3) together, we obtain

$$P(X_n > -m) + P(Y_n > c + m) - 1 > 1 - \epsilon$$

By this and by Bonferroni's inequality (*see (1.2.9) in page 11 of our textbook*), we have

$$P(X_n > -m, Y_n > c + m) \geqq P(X_n > -m) + P(Y_n > c + m) - 1 > 1 - \epsilon$$

. Since $\epsilon > 0$ is arbitrary, the proof is completed.