Arinjay Aggarwal
'I', 35

Design and Analysis of Algorithm
Tutorial-1

Ques1 What do you understand by Asymptotic notations. Define different Asymptotic notations with examples.

Ans1→ Asymptotic notations means towards infinity. They are used to tell the complexity of an algorithm having input size very large.
It is priory analysis.
Different types of asymptotic notations are

[i] Big Oh Notation

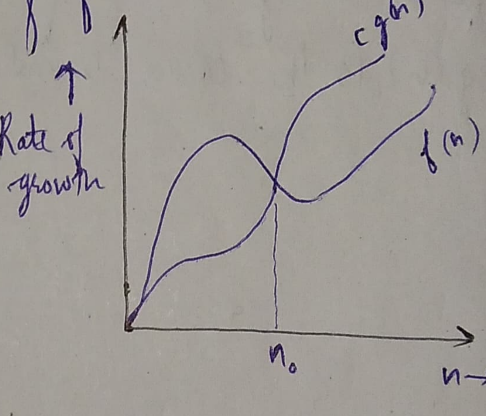
$f(n) = O(g(n))$, if $0 \leq f(n) \leq c(g(n))$ $\forall n \geq n_0$

$g(n)$ is tight upper bound of $f(n)$

Example

```
for(int i=0; i<n; i++)
{
    cout<< i<< endl;
}
```

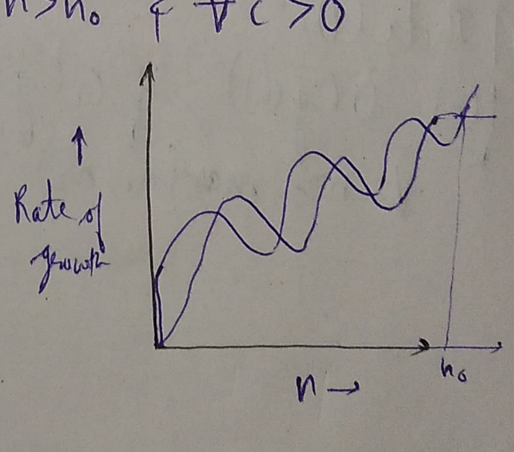
$T(n) = O(n)$



[ii] Small oh Notation

$f(n) = o(g(n))$, if $f(n) < c(g(n))$ $\forall n > n_0$ & $\forall c > 0$

$g(n)$ is upper bound of $f(n)$



[iii] Big Omega(Ω)

$f(n) = \Omega(g(n))$, if $f(n) \geq c(g(n))$ $\forall n \geq n_0$ & some constant $c > 0$

$g(n)$ is tight lower bound of $f(n)$.

Example ⇒ $f(n) = 6n^2 + n + 1$, $g(n) = n^2$

$$0 \le c \cdot g(n) \le f(n)$$

$$0 \le c \cdot n^2 \le 6n^2 + n + 1$$

$$c \le 6 + \frac{1}{n} + \frac{1}{n^2}$$

on putting $n = \infty$, $\frac{1}{n} \Rightarrow \frac{1}{\infty} = 0$

$$c \le 6$$

$6n^2 \le 6n^2 + n + 1 \Rightarrow (n = 1)$

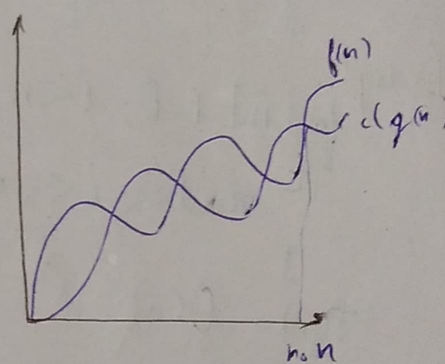
$6 \le 6 + 1 + 1 \Rightarrow 6 \le 8$ True. $\because c > 0$ and $n \ge n_0$ $(n = 1)$ $n_0 = 1$

$$f(n) = \Omega(n^2)$$

[iv] Small Omega (ω)

$f(n) = \omega(g(n))$, if $f(n) > c(g(n))$ $\forall\ n > n_0$ & $\forall\ c > 0$

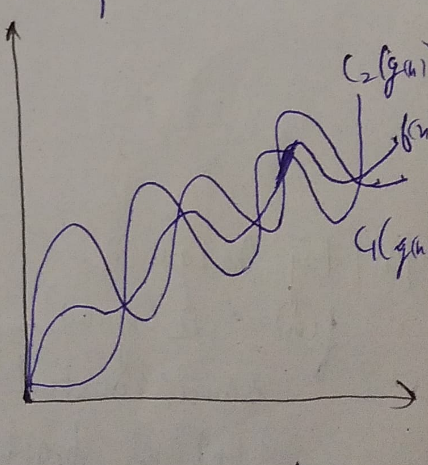
$g(n)$ is the lower bound of $f(n)$.



[v] Theta (θ)

$f(n) = \theta(g(n))$, if $c_1(g(n)) \le f(n) \le c_2(g(n))$

$\forall\ n \ge \max(n_1, n_2)$ and some constant c_1 & $c_2 \ge 0$



Q2. What should be time complexity of
for (i=1 to n){ i = i*2;}

sol^n: i would have 1, 2, 4, 8, 16, ... n
let say these are k terms.
It is a G.P with $a = 1$, $r = 2$
Now, k^{th} term $= t_k = ar^{k-1}$

$$n = 1(2)^{k-1}$$
$$n = 2^{k-1}$$

Taking $\log_2$ on both sides

$$\log_2 n = \log_2(2^{k-1})$$
$$\log_2 n = (k-1)\log_2 2$$
$$\log n = (k-1) \Rightarrow k = 1 + \log_2 n$$
$$T(n) = O(k) = O(1 + \log n) \Rightarrow O(\log n) \text{ Ans}$$

Qu 3. $T(n) = \{3T(n-1)$ if $n > 0$, otherwise $1\}$

$$T(n) = 3T(n-1) \quad -(1)$$

By backward substitution

$$\because T(n) = 3T(n-1)$$
$$T(n-1) = 3T(n-1-1)$$
$$T(n-1) = 3T(n-2) \quad -(2)$$

put (2) in (1)

$$T(n) = 3[3T(n-2)] \Rightarrow T(n) = 9T(n-2) \quad -(3)$$
$$T(n-2) = 3T(n-3)$$
$$T(n) = 27T(n-3)$$

continue for k times

$$T(n) = 3^k T(n-k)$$

assume $n - k = 0 \Rightarrow n = k$

$T(n) = 3^k \, T(0)$ $\qquad$ $(\because T(0) = 1)$

$T(n) = 3^k$

$$\boxed{T(n) = O(3^n)}$$

Ans 4. $T(n) = \{2T(n-1) - 1$, if $n > 0$, otherwise $1\}$

$T(n) = 2T(n-1) - 1 \quad$ — (1)

By using Back ~~ward~~ Substitution Method.

$T(n) = 2T(n-1) - 1$

$T(n-1) = 2T(n-2) - 1$

$T(n-2) = 2T(n-3) - 1$

$T(n) = 2[2T(n-2) - 1] - 1$

$= 2^2 T(n-2) - 2 - 1 \quad$ (2)

$= \cancel{2[2[2T(n-3) - 1] - 1] - 1}$

$= \cancel{2^2[2T(n-3) - 2] - 1}$

$= \cancel{2^3 T(n-3) - 1}$

$= 2^2[2T(n-3) - 1] - 2 - 1$

$= 2^3 T(n-3) - 4 - 2 - 1$

⋮ continue for k times

$T(n) = 2^k T(n-k) - 2^{k-1} - 2^{k-2} - \cdots - 1$

Assume $n - k = 0 \Rightarrow n = k$

$= 2^n T(0) - 2^{n-1} - 2^{n-2} - \cdots - 1$

$T(0) = 1$

$= 2^n - 2^{n-1} - 2^{n-2} - \cdots - 1$

$= 2^n - [\underbrace{2^{n-1} + 2^{n-2} - \cdots + 1}_{\text{G.P. } k \text{ terms}}]$

$a = 2^{n-1}$, $r = 2^{-1} \Rightarrow \frac{1}{2}$

$= \cancel{2^{n-1}(1/2)}$

Sum of G.P

$$a\frac{(1-r^{n-1})}{1-r}$$

$$= \frac{2^{n-1}(1-(1/2)^{n-1})}{1-1/2} = \frac{2^{n-1}\left(\frac{1-(1/2)^n}{1/2}\right)}{1/2}$$

$$= 2^n(1-2(1/2)^n)$$

$$= \cancel{2^n}\frac{(2^n-2)}{\cancel{2^n}} \Rightarrow 2^n-2$$

$$2^n-[2^n-2] \Rightarrow 2 \Rightarrow T(n) = O(2)$$

$T(n) = O(1)$ Ans

Q5, What should be the complexity of

```
int i=1, s=1;
while(s<=n)
{
    i++;
    s=s+i;
    printf("#");
}
```

i	s
1	1
2	3
3	6
4	10
5	15
n	n
	k times

$S = 1, 3, 6, 10, 15, --- n$

let say k terms

k^{th} term $\Rightarrow t_k = t_{k-1} + k$

$k = t_k - t_{k-1}$ — (1)

$k = n - t_{k-1}$

$k = n - c$

$\because t_{k-1}$ would be constant

$$\frac{k(k+1)}{2} = n$$

$k^2 = 2n$

$k = \sqrt{n}$

$\boxed{O(\sqrt{n})}$

Therefore $T(n) = O(n)$ Ans

Q6. Time Complexity of

```
void function (int n)
{   int i, count=0;
    for (i=1; i*i<=n ; i++)
        count++;
}
```

$i*i$

$1\times1 = 1^2$

$2\times2 = 2^2$

$3\times3 = 3^2$

$k\times k = k^2 = n$

$1^2, 2^2, 3^2, \ldots n$

let say k terms.

$t_k = k^2$

$n = k^2 \Rightarrow k = \sqrt{n}$

$$\boxed{T(n) = O(\sqrt{n})}$$

Q7. Time Complexity of

```
void function (int n){
    int i, j, k, count=0;
    for (i=n/2; i<=n; i++)
        for(j=1; j<=n; j=j*2)
            for(k=1; k<=n; k=k*2)
                count++;
}
```

$i = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2 \ldots n$

$= \frac{n}{2}, \frac{n+2}{2}, \frac{n+4}{2} \ldots n$

General form $\Rightarrow$ $\frac{n+0*2}{2} + \frac{n+1*2}{2} + \frac{n+2*2}{2}, \ldots n$

$= \frac{n+k*2}{2}$ $(k = 0, 1, 2 \ldots n)$

Total terms $= k+1$

$t_{k+1} = n$

$$= \frac{n + (k+1)*2}{2} = n \Rightarrow 2n = n + (k+1)*2$$

$$n - 2 = 2k$$

$$k = \frac{n}{2} - 1$$

i	j	k
$\frac{n}{2}$	$\log n$ times	$(\log n)^2$
$\frac{n+2}{2}$	$\log n$ times	$(\log n)^2$
⋮	⋮	
n	$\log n$ times	$(\log n)^2$

$\left(\frac{n}{2} - 1\right)$ times

$$= \left(\frac{n}{2} - 1\right)(\log n)^2$$

$$= \frac{n}{2}\log^2 n - \log^2 n$$

$$\boxed{T(n) = O(n \log^2 n)}$$

Qn 8 Time Complexity of

```
function( int n){
    if (n==1) return;
    for (i=1 to n){
        for (j=1 to n){
            print("*");
        }
    }
    function(n-3);
}
```

10, 7, 4, 1

Function call would be $n, n-3, n-6, n-9 \ldots 1$

let say k terms

AP, $a = n$, $d = -3$,

$a_n = a + (n-1)d$

$1 = n + (k-1)(-3)$

$1 = n - 3k + 3$

$3k = n + 2$

$k = \frac{n+2}{2}$

∴ Function have recursive calls $\frac{n+2}{2}$ times.

Time complexity for two inner loop $= n^2$

$\left(\frac{n+2}{2}\right) n^2 \Rightarrow n^3$

$\boxed{T(n) = O(n^3)}$

Q9 → Time complexity of → void function(int n)

```
{
  for (i = 1 to n) {
     for (j = 1; j <= n; j = j + i)
         print ("*");
  }
}
```

for i (outer loop)

when $i = 1$ → $j = 1, 2, 3, 4 \ldots n \Rightarrow n$

when $i = 2$ → $j = 1, 3, 5, 7 \ldots n \Rightarrow \frac{n}{2}$

when $i = 3$ → $j = 1, 4, 7, \ldots n \Rightarrow \frac{n}{3}$

$\sum_{j=n}^{1} n + \frac{n}{2} + \frac{n}{3} + \cdots + 1$

$\sum_{j=n}^{1} n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$

$O(n \log n)$ Ans

Q10 - For the functions, n^k and c^n, what is the asymptotic relationship b/w these functions?
Assume that $k >= 1$ and $c > 1$ are constants. Find out the value of c and n0 for which relation holds:

As given n^k and c^n
relation b/w n^k and c^n is $n^k = O(c^n)$
as $n^k \leq a c^n$ $\forall n \geq n_0$ for a constant $a > 0$

for $n_0 = 1$
$c = 2$
$1^k \leq a 2^1$
$\therefore n_0 = 1$ and $c = 2$