CPNA Lecture 22 - Solutions to Differential Equations

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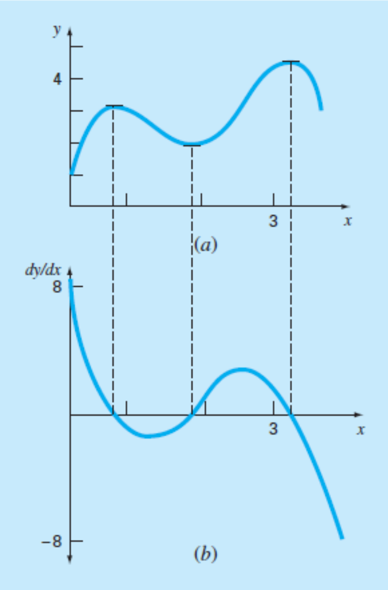
Introduction I

- **Differential Equation**: an equation involving one dependent variable and one or more independent variable along with their differentials
- **Ordinary Differential Equation**: involves only one independent variable (ODE)
- **Partial Differential Equation**: involves two or more independent variables
- **First Order Differential Equation**: a functional relationship between an independent variable x, a dependent variable y and it's first derivative $y' = \frac{dy}{dx}$

$$\phi(x, y, \frac{dy}{dx}) = 0$$

Introduction II

- **Example**:
  (a) The function $y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$
  (b) It's First Order ODE
  $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$



Introduction III

- **Examples from Physical World**:
  - The velocity v of a falling parachutist as a function of time t,

    $$\frac{dv}{dt} = g - \frac{c}{m}v$$

    where, g is the gravitational constant, m is the mass, and c is a drag coefficient
  - The equation describing the position x of a mass-spring system with damping is the second-order equation

    $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

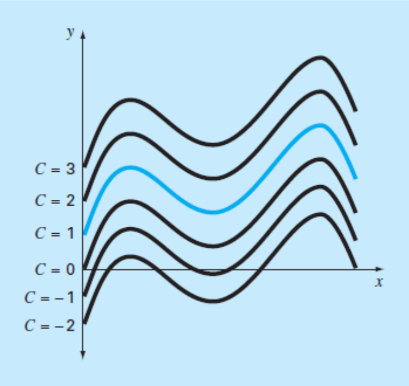
    where c is a damping coefficient and k is a spring constant

Introduction IV

  - The motion of a swinging pendulum under certain simplifying assumptions is described by the second-order differential equation
$$m\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\ \theta = 0$$
where L is the length of the pendulum, g is the gravitational constant, and θ is the angle the pendulum makes with the vertical
- Our objective is to determine the original function given the differential equation
  - We will be given differential equations of the type $\frac{dy}{dx} = f(x, y)$ with an initial condition $y = y_1$ at $x = x_1$
  - $f(x, y)$ may be a general non linear function of (x, y) or may be a table of values
- For our example, we can do that analytically by integrating $y = \int(-2x^3 + 12x^2 - 20x + 8.5) = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + C$

Introduction V

- This is identical to the original function, except the integration constant C
- We can have infinite number of solutions corresponding to infinite number of possible values of C



Introduction VI

- For first order ODE, we need an initial condition to determine a unique solution
  - If the initial condition is $y = 1$ at $x = 0$, then $C = 1$
- In general, for an n^{th}-order differential equation, n initial conditions are required to obtain a unique solution
- Essentially, a solution is a curve $g(x, y)$ in the (x, y) plane whose slope at every point (x, y) in the specified region is given by the equation $\frac{dy}{dx} = f(x, y)$ and also the initial point (x_1, y_1) of the solution curve is given

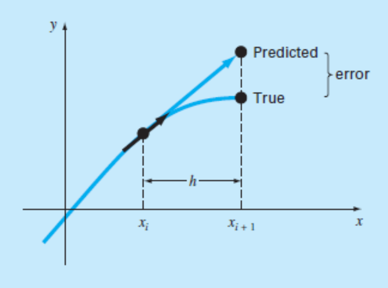
Non-Computer Methods for Solving ODEs

- Using analytical integration techniques
  - Example: $v = \int (g - \frac{c}{m} v) dt$
  - Possible if the indefinite integral can be evaluated in exact equation form
  - $v(t) = \frac{gm}{c}(1 - \exp^{-(\frac{c}{m})t})$
- Exact solutions for many ODEs of practical importance are not available
  - Numerical methods are the only viable alternatives

Euler's Method I

- Given the initial condition $y(x_1) = y_1$, the solution curve may be extrapolated by computing the values $y(x_1 + h)$, $y(x_1 + 2h)$, $\ldots$
- Slope of the solution curve at initial point (x_1, y_1) is $f(x_1, y_1)$
- The point where this slope cuts the vertical line through $x_2 = x_1 + h$ is $y_2 = y_1 + hf(x_1, y_1)$
- In general we have

$$y_{i+1} = y_i + hf(x_i, y_i) \text{ where } x_i = x_1 + ih$$



Euler's Method II

Algorithm 1 Algorithm for Euler's Method

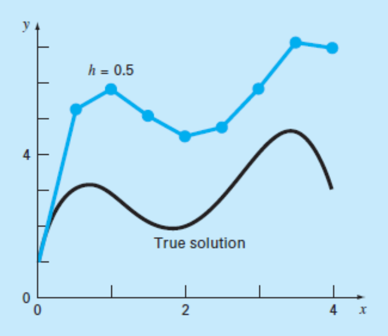
Input: $x_1, y_1, h, x_f, f()$
Output: The values of y_i at x_i, where $x_i = x_1 + ih$ and $x_1 \leq x_i \leq x_f$

while $x_1 \leq x_f$ **do**
    Write x_1, y_1
    $x_2 = x_1 + h$
    $y_2 = y_1 + h * f(x_1, y_1)$
    $x_1 = x_2$
    $y_1 = y_2$
end while

Euler's Method III

Example:

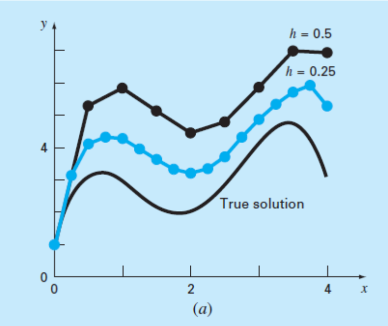
- Comparison of the true solution with a numerical solution using Euler's method for the first order ODE $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$ from $x = 0$ to $x = 4$ with a step size of 0.5
- The initial condition at $x = 0$ is $y = 1$



Euler's Method IV

Example:

- Comparison of two numerical solutions with Euler's method using step sizes of 0.5 and 0.25 for the first order ODE $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$



Euler's Method V

- **Exercise**: Solve the differential equation

$$\frac{dy}{dx} + xy = 0, \;\; y(0) = 1$$

  from $x = 0$ to $x = 0.25$ using Euler's method

- **Solution**: Let us pick $h = 0.05$, $f(x, y) = -xy$
  The Euler formula is $y_{i+1} = y_i + hf(x_i, y_i) = y_i - hx_iy_i$
  At $x_0 = 0, y_0 = 1$
  At $x_1 = 0.05, y_1 = 1 - 0.05 \times 0 \times 1 = 1$
  At $x_2 = 0.1, y_2 = 1 - 0.05 \times 0.05 \times 1 = 0.9975$
  At $x_3 = 0.15, y_3 = 0.9975 - 0.05 \times 0.1 \times 0.9975 = 0.9925$
  At $x_4 = 0.2, y_4 = 0.9925 - 0.05 \times 0.15 \times 0.9925 = 0.985$
  At $x_5 = 0.25, y_5 = 0.985 - 0.05 \times 0.2 \times 0.985 = 0.9752$

Euler's Method VI

The solution is

x_i	0	0.05	0.10	0.15	0.20	0.25
y_i	1	1	0.997	0.992	0.985	0.975

The analytical solution of the differential equation is $y = e^{-x^2/2}$ and is tabulated below

x_i	0	0.05	0.10	0.15	0.20	0.25
y_i	1	0.999	0.995	0.989	0.980	0.969

Modified Euler's Method I

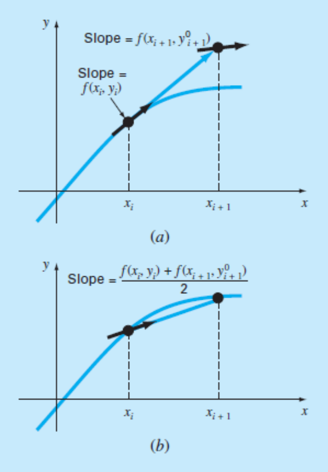
- A fundamental source of error in Euler's method is that the derivative at the beginning of the interval is assumed to apply across the entire interval
- Two simple modifications:
  - Heun's Method
  - Midpoint Method

Modified Euler's Method II

Heun's Method

- Determine two derivatives for the interval — one at the initial point and another at the end point
  Use the slope at the beginning of the interval $y_i' = f(x_i, y_i)$ to extrapolate linearly to $y_{i+1}^0 = y_i + f(x_i, y_i)h$, which is used to estimate slope at the end of the interval as $y_{i+1}' = f(x_{i+1}, y_{i+1}^0)$
- Average them to obtain an improved estimate of the slope for the entire interval and use it to extrapolate linearly from y_i to y_{i+1}

$$y_{i+1} = y_i + h\left[\frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)}{2}\right]$$



Modified Euler's Method III

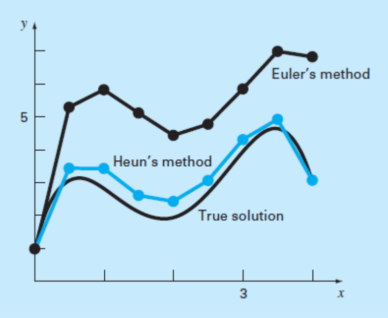
Algorithm 2 Algorithm for Heun's Method

Input: $x_1, y_1, h, x_f, f()$
Output: The values of y_i at x_i, where $x_i = x_1 + ih$ and $x_1 \le x_i \le x_f$

```
while x_1 ≤ x_f do
    Write x_1, y_1
    s_1 = f(x_1, y_1)
    x_2 = x_1 + h
    y_2 = y_1 + h * s_1
    s_2 = f(x_2, y_2)
    y_2 = y_1 + h * (s_1 + s_2)/2
    x_1 = x_2
    y_1 = y_2
end while
```

Modified Euler's Method IV

Comparison of the true solution with a numerical solution using Euler's and Heun's methods for the first order ODE
$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$



Modified Euler's Method V

- **Exercise**: Solve the differential equation

$$\frac{dy}{dx} + xy = 0, \;\; y(0) = 1$$

  from $x = 0$ to $x = 0.25$ using Huen's method

- **Solution**: Here, $f(x, y) = -xy$
  In Huen's method $y_{i+1} = y_i + \frac{h}{2}(s_1 + s_2)$
  where, $s_1 = f(x_i, y_i), s_2 = f(x_i + h, y_i + hs_1), \;\; h = 0.05$

  At $x_1 = 0, y_1 = 1,$

$$\begin{aligned}
&\Rightarrow s_1 = f(x_1, y_1) = 0 \\
&\Rightarrow x_2 = 0.05, y_2^0 = 1 + 0.05 \times 0 = 1, \\
&\Rightarrow s_2 = f(x_2, y_2^0) = -0.05 \times 1 = -0.05 \\
&\Rightarrow y_2 = y_1 + \frac{h}{2}(s_1 + s_2) = 1 + \frac{0.05}{2}(0 - 0.05) = 0.99875
\end{aligned}$$

  At $x_2 = 0.05, y_2 = 0.99875,$

$$\Rightarrow s_2 = f(x_2, y_2) = 0.049938$$

Modified Euler's Method VI

$\Rightarrow x_3 = 0.1, y_3^0 = 0.99875 + 0.05 \times 0.049938 =$
$0.996253,$
$\Rightarrow s_3 = f(x_3, y_3^0) = -0.1 \times 0.996253 = -0.099625$
$\Rightarrow y_3 = y_2 + \frac{h}{2}(s_2 + s_3) =$
$0.99875 + \frac{0.05}{2}(0.049938 - 0.099625) = 0.995011$

Similar way we can find the subsequent y_i's

Modified Euler's Method VII

The solution is

x_i	0	0.05	0.10	0.15	0.20	0.25
y_i	1	0.998750	0.995011	0.988811	0.980196	0.969230

The analytical solution of the differential equation is $y = e^{-x^2/2}$ and is tabulated below

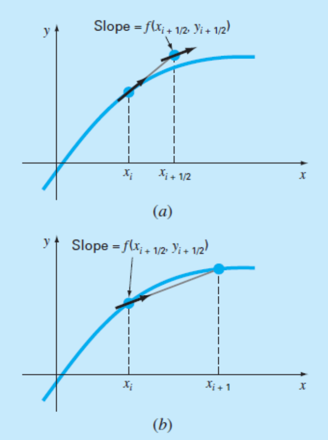
x_i	0	0.05	0.10	0.15	0.20	0.25
y_i	1	0.999	0.995	0.989	0.980	0.969

Modified Euler's Method VIII

The Midpoint (or Improved Polygon) Method

- Uses Euler's method to predict a value of y at the midpoint of the interval as $y_{i+1/2} = y_i + \frac{h}{2} f(x_i, y_i)$
- Then, this predicted value is used to calculate a slope at the midpoint as $y'_{i+1/2} = f(x_{i+1/2}, y_{i+1/2})$, which is assumed to represent a valid approximation of the average slope for the entire interval
- Use this slope to extrapolate linearly from x_i to x_{i+1} as

$$y_{i+1} = y_i + hf(x_{i+1/2}, y_{i+1/2})$$



Modified Euler's Method IX

Algorithm 3 Algorithm for Midpoint Method

Input: $x_1, y_1, h, x_f, f()$
Output: The values of y_i at x_i, where $x_i = x_1 + ih$ and $x_1 \leq x_i \leq x_f$

while $x_1 \leq x_f$ **do**
    Write x_1, y_1
    $x_{h/2} = x_1 + h/2$
    $y_{h/2} = y_1 + h * f(x_1, y_1)/2$
    $x_2 = x_1 + h$
    $y_2 = y_1 + h * f(x_{h/2}, y_{h/2})$
    $x_1 = x_2$
    $y_1 = y_2$
end while

Modified Euler's Method X

- **Exercise**: Solve the differential equation

$$\frac{dy}{dx} + xy = 0, \; y(0) = 1$$

from $x = 0$ to $x = 0.25$ using Midpoint method

- **Solution**: The solution is

x_i	0	0.05	0.10	0.15	0.20	0.25
y_i	1	0.998750	0.995009	0.988806	0.980186	0.969215

The analytical solution of the differential equation is $y = e^{-x^2/2}$ and is tabulated below

x_i	0	0.05	0.10	0.15	0.20	0.25
y_i	1	0.999	0.995	0.989	0.980	0.969

Runge-Kutta Methods I

- Runge-Kutta (RK) methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives
- It's generalized form is

$$y_{i+1} = y_i + h\ \phi(x_i, y_i, h) \tag{1}$$

  where, $\phi(x_i, y_i, h)$ is called an increment function, which can be interpreted as a representative slope over the interval
- The increment function can be written in general form as

$$\phi = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n = \sum_{i=1}^{n} a_i k_i \tag{2}$$

Runge-Kutta Methods II

where the a's are constants and the k's are

$$k_1 = f(x_i, y_i) \tag{3a}$$
$$k_2 = f(x_i + p_2 h, y_i + q_{21} k_1 h) \tag{3b}$$
$$k_3 = f(x_i + p_3 h, y_i + q_{31} k_1 h + q_{32} k_2 h) \tag{3c}$$
$$\dots$$
$$\dots$$
$$k_n = f(x_i + p_n h, y_i + q_{n1} k_1 h + q_{n2} k_2 h + \dots + q_{n,n-1} k_{n-1} h) \tag{3d}$$

In general, $k_i = f(x_i + p_i h, y_i + h \sum_{j=1}^{i-1} q_{ij} k_j)$

the p's and q's are constants

- The k's are in recurrence relationship making RK methods suitable for computer implementation
- The Euler's method is first-order RK method with $n = 1$
- Once n is chosen, values for the a's, p's, and q's are evaluated by setting Eq. 1 equal to terms in a Taylor series expansion

Runge-Kutta Methods III

Tailor's Series for a Function of One Variable

If $f(x)$ is continuous and possesses continuous derivatives of order n in an interval that includes $x = a$, then in that interval

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$
$$+ \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n(x)$$

where $R_n(x)$, the remainder term, can be expressed in the form

$$R_n(x) = \frac{(x-a)^n}{n!}f^{(n)}(\xi)$$

First Order Runge-Kutta Method I

- When $n = 1$, $k_1 = f(x_i, y_i)$
- $y_{i+1} = y_i + ha_1 k_1 = y_i + ha_1 f(x_i, y_i) = y_i + ha_1 y_i'$
- Taylor series expansion

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \dots$$

- Equating the first two terms we get $a_1 = 1$

Second Order Runge-Kutta Method I

- The second order version of Eq. 1 is

$$y_{i+1} = y_i + h(a_1 k_1 + a_2 k_2) \tag{4}$$

where,

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_2 h, y_i + q_{21} k_1 h)$$

- Values of a_1, a_2, p_2, and q_{21} are evaluated by setting Eq. 2 to a Taylor series expansion up to the second order term

$$\begin{aligned} y'' = \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx} f(x, y) \\ &= \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y} \end{aligned}$$

Second Order Runge-Kutta Method II

- Taylor series expansion

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \dots$$

- Taking up to three terms

$$\begin{aligned}
y_{i+1} &= y_i + hy_i' + \frac{h^2}{2!}y_i'' \\
&= y_i + hf(x_i, y_i) + \frac{h^2}{2!}\left[\frac{\partial f(x_i, y_i)}{\partial x} + \frac{\partial f(x_i, y_i)}{\partial y}f(x_i, y_i)\right] \\
&= y_i + hf(x_i, y_i) + \frac{h^2}{2}\frac{\partial f(x_i, y_i)}{\partial x} + \frac{h^2}{2!}f(x_i, y_i)\frac{\partial f(x_i, y_i)}{\partial y}
\end{aligned}$$

$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2}\frac{\partial f(x_i, y_i)}{\partial x} + \frac{h^2}{2!}f(x_i, y_i)\frac{\partial f(x_i, y_i)}{\partial y} \quad (5)$$

Second Order Runge-Kutta Method III

Tailor Series for a function of two independent variables u and v

$$
\begin{aligned}
f(u_{i+1}, v_{i+1}) = f(u_i, v_i) &+ \frac{\partial f}{\partial u}(u_{i+1} - u_i) + \frac{\partial f}{\partial v}(v_{i+1} - v_i) \\
&+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial u^2}(u_{i+1} - u_i)^2 + 2\frac{\partial^2 f}{\partial u \partial v}(u_{i+1} - u_i)(v_{i+1} - v_i) + \frac{\partial^2 f}{\partial v^2}(v_{i+1} - v_i)^2\right] \\
&+ \dots
\end{aligned}
$$

where all partial derivatives are evaluated at the base point i.

- Taking Taylor's series expansion for a two variable function for k_2 (considering only the first order terms)

$$
\begin{aligned}
k_2 &= f(x_i + p_2 h, y_i + q_{21} k_1 h) \\
&= f(x_i, y_i) + p_2 h \frac{\partial f(x_i, y_i)}{\partial x} + q_{21} k_1 h \frac{\partial f(x_i, y_i)}{\partial y}
\end{aligned}
$$

Second Order Runge-Kutta Method IV

- Substituting values of k_1 and k_2, we get

$$\begin{aligned}
y_{i+1} &= y_i + h(a_1 k_1 + a_2 k_2) \\
&= y_i + ha_1 k_1 + ha_2 k_2 \\
&= y_i + ha_1 f(x_i, y_i) + ha_2 \left[f(x_i, y_i) + p_2 h \frac{\partial f(x_i, y_i)}{\partial x} + q_{21} k_1 h \frac{\partial f(x_i, y_i)}{\partial y} \right] \\
&= y_i + hf(x_i, y_i)(a_1 + a_2) + a_2 p_2 h^2 \frac{\partial f(x_i, y_i)}{\partial x} + a_2 q_{21} h^2 f(x_i, y_i) \frac{\partial f(x_i, y_i)}{\partial y}
\end{aligned}$$

$$y_{i+1} = y_i + hf(x_i, y_i)(a_1 + a_2) + a_2 p_2 h^2 \frac{\partial f(x_i, y_i)}{\partial x} + a_2 q_{21} h^2 f(x_i, y_i) \frac{\partial f(x_i, y_i)}{\partial y} \quad (6)$$

- Equating similar terms in equation 5 and 6, we get three equations to evaluate the four unknown constants

$$a_1 + a_2 = 1$$

$$a_2 p_2 = \frac{1}{2}$$

$$a_2 q_{21} = \frac{1}{2}$$

Second Order Runge-Kutta Method V

- We must assume a value of one of the unknowns to determine the other three
- If we specify the value for a_2, then

$$a_1 = 1 - a_2$$

$$p_2 = q_{21} = \frac{1}{2a_2}$$

- We can have infinite number of second order RK methods corresponding to infinite number of values for a_2
- In Heun's method

$$a_2 = \frac{1}{2}$$

$$a_1 = 1 - a_2 = \frac{1}{2}$$

$$p_2 = q_{21} = \frac{1}{2a_2} = 1$$

Second Order Runge-Kutta Method VI

Substituting, these parameters we get

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right) h$$

where,

$$k_1 = f(x_i, y_i), \text{ and } k_2 = f(x_i + h, y_i + k_1 h)$$

- In Midpoint method

$$a_2 = 1$$

$$a_1 = 1 - a_2 = 0$$

$$p_2 = q_{21} = \frac{1}{2a_2} = \frac{1}{2}$$

Substituting, these parameters we get

$$y_{i+1} = y_i + k_2 h$$

Second Order Runge-Kutta Method VII

where,

$$k_1 = f(x_i, y_i), \text{ and } k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

Third Order Runge-Kutta Method I

- For $n = 3$, derivation from equation 1 results in six equations with eight unknowns
- Values of two unknowns must be specified a priori to determine the remaining
- One common version that results is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h \tag{7}$$

where,

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + h, y_i - k_1 h + 2k_2 h)$$

Fourth Order Runge-Kutta Method I

- Most popular of RK methods and it's commonly used form is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \tag{8}$$

where,

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

Fourth Order Runge-Kutta Method II

Algorithm 4 Algorithm for Fourth Order Runge-Kutta Method

Input: x_1, y_1, h, x_f

Output: The values of y_i at x_i, where $x_i = x_1 + ih$ and $x_1 \leq x_i \leq x_f$

while $x_1 \leq x_f$ **do**
    Write x_1, y_1
    $k_1 = f(x_1, y_1)$
    $k_2 = f(x_1 + h/2, y_1 + k_1 * h/2)$
    $k_3 = f(x_1 + h/2, y_1 + k_2 * h/2)$
    $k_4 = f(x_1 + h, y_1 + k_3 * h)$
    $x_2 = x_1 + h$
    $y_2 = y_1 + (k_1 + 2k_2 + 2k_3 + k_4) * h/6$
    $x_1 = x_2$
    $y_1 = y_2$
end while