CPNA Lecture 24 - Solutions to Partial Differential Equations

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Partial Derivative

- Given a function u that depends on both x and y,
  - the partial derivative of u with respect to x at an arbitrary point (x, y) is defined as

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} \tag{1}$$

  - the partial derivative of u with respect to y at an arbitrary point (x, y) is defined as

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} \tag{2}$$

Partial Differential Equation I

- An equation involving partial derivatives of an unknown function of two or more independent variables is called a **partial differential equation**, or **PDE**
- Examples:

$$\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + u = 1 \tag{3}$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} + x\frac{\partial^2 u}{\partial y^2} + 8u = 5y \tag{4}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + 6\frac{\partial^3 u}{\partial x \partial y^2} = x \tag{5}$$

$$\frac{\partial^2 u}{\partial x^2} + xu\frac{\partial u}{\partial y} = x \tag{6}$$

- PDEs have widespread application in engineering

Partial Differential Equation II

- We shall focus on linear second order PDEs with the following general form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0 \tag{7}$$

  where A, B, and C are functions of x and y and D is a function of x, y, u, $\frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$

- Depending on the values of the coefficients of the second-derivative terms — A, B, C equation 7 can be classified into one of three categories

$B^2 - 4AC$	**Category**	**Example**
< 0	Elliptic	Laplace Equation, $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$
$= 0$	Parabolic	Heat Conduction Equation, $\frac{\partial T}{\partial t} = k' \frac{\partial^2 T}{\partial x^2}$
> 0	Hyperbolic	Wave Equation, $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$

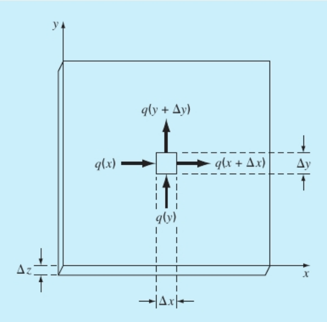
Solution to a Partial Differential Equation

- Solving a PDE means finding the unknown function u
- An analytical (i.e. exact) solution of a PDE is a function that satisfies the PDE and also satisfies any boundary and/or initial conditions given with the PDE
- Most PDEs of interest do not have analytical solutions so a numerical procedure must be used to find an approximate solution

Precomputer Methods for Solving PDEs

- Engineers relied on analytical or exact solutions of partial differential equations
- Many physical systems could not be solved directly but had to be simplified using linearizations, simple geometric representations, and other idealizations

The Laplace Equation I



- Elliptic equations in engineering are typically used to characterize steady-state, boundary-value problems
- We first illustrate how the Laplace equation is derived from a physical problem context
- A thin rectangular plate of thickness *dz* is insulated everywhere but at its edges
- The insulation and the thinness of the plate mean that heat transfer is limited to the x and y dimensions

The Laplace Equation II

- At steady state, the flow of heat into the element over a unit time period dt must equal the flow out

$$q(x)\Delta y\Delta z\Delta t + q(y)\Delta x\Delta z\Delta t = q(x+\Delta x)\Delta y\Delta z\Delta t + q(y+\Delta y)\Delta x\Delta z\Delta t \quad (8)$$

where $q(x)$ and $q(y)$ is the heat fluxes at x and y, respectively $[cal/(cm^2.s)]$
rearranging and taking the limits results

$$\frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} = 0 \quad (9)$$

- Equation 9 is an expression of the conservation of energy for the plate
- However, it cannot be solved unless heat fluxes are specified at the plate's edges
- Equation 9 must be reformulated in terms of temperature as temperature boundary conditions are given

The Laplace Equation III

- Relation between flux and temperature is provided by Fourier's law of heat conduction

$$q_i = -k\rho C \frac{\partial T}{\partial i} \tag{10}$$

where,
q_i = heat flux in the direction of the i dimension $[cal/(cm^2.s)]$,
k = coefficient of thermal diffusivity $[cm^2/s)]$,
ρ = density of the material $[g/cm^3]$,
C = heat capacity of the material $[cal/(g.^\circ C)]$,
T = temperature $(^\circ C)$, which is defined as,

$$T = \frac{H}{\rho C V} \tag{11}$$

where, H = heat (cal) and V = volume (cm^3)

The Laplace Equation IV

Sometimes the term in front of the differential equation in equation 10 is referred to as the coefficient of thermal conductivity $[cal/(s.cm.^{\circ}C)]$

$$k' = k\rho C \tag{12}$$

Substituting equation 10 in equation 9, we get

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{13}$$

which is the **Laplace equation**

Numerical Solution: Finite Difference Method

- The numerical solution of elliptic PDEs such as the Laplace equation proceeds in the reverse manner of the derivation
- Finite differences representations are substituted for the partial derivatives
- PDE is transformed into an algebraic difference equation

Tailor's Series for a Function of One Variable I

If the function f and its first $n+1$ derivatives are continuous on an interval containing x_i and x_{i+1}, then the value of the function at x_{i+1} is given by

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2!}f''(x_i) + \dots$$

$$+ \frac{h^n}{(n)!}f^{(n)}(x_i) + R_n(x_i)$$

where, $h = x_{i+1} - x_i$ and $R_n(x_i)$, the remainder term, can be expressed in the form

$$R_n(x_i) = \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi)$$

where ξ lies between x_i and x_{i+1}

Using Tailor Series to Estimate Truncation Error I

- Finite difference approximation of first derivative
  - First forward finite divided difference
    Truncate the series after the first derivative term

$$f(x_{i+i}) = f(x_i) + hf'(x_i) + R_1$$

or,

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{R_1}{h}$$

We have, $R_n = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \Rightarrow R_1 = \frac{h^2}{(2)!} f^{(2)}(\xi) \Rightarrow \frac{R_1}{h} = O(h)$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

or,

$$f'(x_i) = \frac{\Delta f_i}{h} + O(h)$$

Using Tailor Series to Estimate Truncation Error II

- First backward finite divided difference
  Tailor series expansion for $f(x_{i-1})$ in terms of $f(x_i)$

$$f(x_{i-1}) = f(x_i) - hf'(x_i) + \frac{h^2}{2!}f''(x_i) - \dots$$

Truncating the series after first derivative and rearranging

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(h)$$

or,

$$f'(x_i) = \frac{\nabla f_i}{h} + O(h)$$

Using Tailor Series to Estimate Truncation Error III

- First central finite divided difference
  Tailor series expansion for $f(x_{i+1})$ in terms of $f(x_i)$

  $$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2!}f''(x_i) + \ldots + \frac{h^n}{(n)!}f^{(n)}(x_i) + R_n(x_i)$$

  Tailor series expansion for $f(x_{i-1})$ in terms of $f(x_i)$

  $$f(x_{i-1}) = f(x_i) - hf'(x_i) + \frac{h^2}{2!}f''(x_i) - \ldots + \frac{h^n}{(n)!}f^{(n)}(x_i) + R_n(x_i)$$

  Subtracting (second) from (first)

  $$f(x_{i+1}) - f(x_{i-1}) = 2hf'(x_i) + 2\frac{h^3}{3!}f^3(x_i) + \ldots$$

  or,

  $$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)$$

Using Tailor Series to Estimate Truncation Error IV

- Finite difference approximation of second order derivatives
  - Second forward finite divided difference
    Tailor series expansion for $f(x_{i+2})$ in terms of $f(x_i)$

$$f(x_{i+2}) = f(x_i) + 2hf'(x_i) + \frac{(2h)^2}{2!} f''(x_i) + \ldots + \frac{(2h)^n}{(n)!} f^{(n)}(x_i) + R_n(x_i)$$

Tailor series expansion for $f(x_{i+1})$ in terms of $f(x_i)$

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \ldots + \frac{h^n}{(n)!} f^{(n)}(x_i) + R_n(x_i)$$

Doing (first) - 2 x (second),

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + h^2 f''(x_i) + \ldots$$

or,

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)$$

Using Tailor Series to Estimate Truncation Error V

- Second backward finite divided difference
  Tailor series expansion for $f(x_{i-2})$ in terms of $f(x_i)$

$$f(x_{i-2}) = f(x_i) - 2hf'(x_i) + \frac{(2h)^2}{2!} f''(x_i) + \ldots$$

  Tailor series expansion for $f(x_{i-1})$ in terms of $f(x_i)$

$$f(x_{i-1}) = f(x_i) - hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \ldots$$

  Doing (first) - 2 x (second),

$$f(x_{i-2}) - 2f(x_{i-1}) = -f(x_i) + h^2 f''(x_i) + \ldots$$

  or,

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2} + O(h)$$

Using Tailor Series to Estimate Truncation Error VI

- Second central finite divided difference
  Tailor series expansion for $f(x_{i+1})$ in terms of $f(x_i)$

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2!}f''(x_i) + \ldots$$

  Tailor series expansion for $f(x_{i-1})$ in terms of $f(x_i)$

$$f(x_{i-1}) = f(x_i) - hf'(x_i) + \frac{h^2}{2!}f''(x_i) + \ldots$$

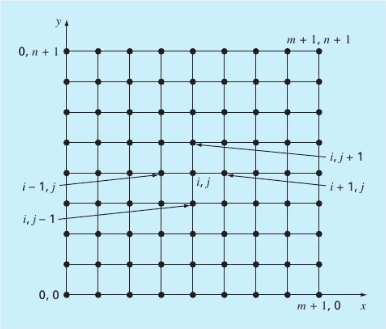
  Doing (first) + (second),

$$f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + h^2 f''(x_i) + O(h^4) \ldots$$

  or,

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h^2)$$

The Laplacian Difference Equation I



- Central differences based on the grid scheme

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} \tag{14}$$

and

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} \tag{15}$$

The Laplacian Difference Equation II

- Substituting in equation 13

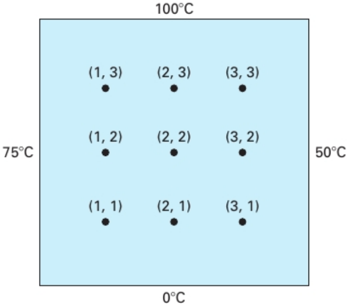
$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} = 0$$

- For the square grid in $\Delta x = \Delta y$, and by collection of terms, the equation becomes

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0 \tag{16}$$

- This relationship holds for all interior points on the plate, is referred to as the **Laplacian difference equation**
- In addition, boundary conditions along the edges of the plate must be specified to obtain a unique solution
- The simplest case is where the temperature at the boundary is set at a fixed value and is called **Dirichlet Boundary Condition**

The Laplacian Difference Equation III



- A balance for node (1, 1) is, according to equation 16

$$T_{2,1} + T_{0,1} + T_{1,2} + T_{1,0} - 4T_{1,1} = 0 \qquad (17)$$

The Laplacian Difference Equation IV

- However, $T_{0,1} = 75$ and $T_{1,0} = 0$, thus equation 17 can be expressed as

$$T_{2,1} + T_{1,2} + -4T_{1,1} = -75 \tag{18}$$

- Similar equations can be developed for the other interior points. The result is the following set of nine simultaneous equations with nine unknowns:

The Laplacian Difference Equation V

$$
\begin{array}{ccccccccccc}
4T_{11} & -T_{21} & & -T_{12} & & & & & & = & 75 \\
-T_{11} & +4T_{21} & -T_{31} & & -T_{22} & & & & & = & 0 \\
 & -T_{21} & 4T_{31} & & & -T_{32} & & & & = & 50 \\
-T_{11} & & & +4T_{12} & -T_{22} & & -T_{13} & & & = & 75 \\
 & -T_{21} & & -T_{12} & +4T_{22} & -T_{32} & & -T_{23} & & = & 0 \\
 & & -T_{31} & & -T_{22} & +4T_{32} & & & -T_{33} & = & 50 \\
 & & & -T_{12} & & & +4T_{13} & -T_{23} & & = & 175 \\
 & & & & -T_{22} & & -T_{13} & +4T_{23} & -T_{33} & = & 100 \\
 & & & & & -T_{32} & & -T_{23} & +4T_{33} & = & 150
\end{array} \quad (19)
$$

This can be solved using methods for solving system of linear equations