CPNM Lecture 16 - Interpolation

Mridul Sankar Barik

Jadavpur University

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Introduction I

- Suppose a function $f(x)$ is not defined explicitly, but its value at some finite number of points x_0, x_1, x_2, ..., x_n is given

x	x_0	x_1	x_2	$\dots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$\dots$	$f(x_n)$

- **Interpolation**: to find $f(x)$ at x in the interval $[x_0, x_n]$, if x is not among the tabulated points
- **Extrapolation**: to find $f(x)$ at x outside the interval $[x_0, x_n]$ with the assumption that behavior of $f(x)$ outside the range $[x_0, x_n]$ is identical to that inside the range

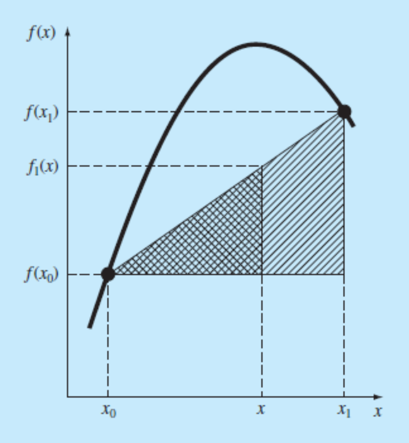
Polynomial Interpolation I

- To find a polynomial $\phi(x)$, such that $f(x)$ agree with $\phi(x)$ at the given set of points
- If we have $n+1$ data points then we can fit a n^{th} degree polynomial

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Linear Interpolation I

- Simplest form used when two data points are available



Linear Interpolation II

- Connect two data points with a straight line

$$\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

or,

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \tag{1}$$

- As the interval decreases, a continuous function will be better approximated by a straight line
- **Exercise**: Estimate the natural logarithm of 2 using linear interpolation. First, perform the computation by interpolating between $\ln 1 = 0$ and $\ln 6 = 1.791759$. Then, repeat the procedure, but use a smaller interval from $\ln 1$ to $\ln 4$ (1.386294). Note that the true value of $\ln 2$ is 0.6931472.

Linear Interpolation III

Solution: Using equation 1 in the interval $x_0 = 1, x_1 = 6$, we get

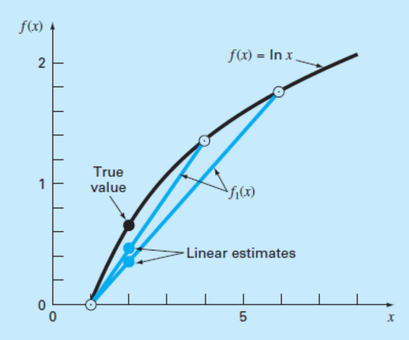
$$f_1(2) = f(1) + \frac{f(6) - f(1)}{6 - 1}(2-1) = 0 + \frac{1.791759 - 0}{5} = 0.3583519$$

Percentage error $\varepsilon_p = \frac{0.6931472 - 0.3583519}{0.6931472} \times 100\% = 48.3\%$
Using the smaller interval $x_0 = 1, x_1 = 4$, we get

$$f_1(2) = f(1) + \frac{f(4) - f(1)}{4 - 1}(2-1) = 0 + \frac{1.386294 - 0}{3} = 0.4620981$$

Percentage error now, $\varepsilon_p = \frac{0.6931472 - 0.4620981}{0.6931472} \times 100\% = 33.3\%$

Linear Interpolation IV



Quadratic Interpolation I

- Use a second order polynomial when three data points are available

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \tag{2}$$

- Where,

$$\begin{aligned} b_0 &= f(x_0) \\ b_1 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ b_2 &= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} \end{aligned} \tag{3}$$

- **Exercise**: Estimate the value of $\ln 2$ Using a second order polynomial and the given three points

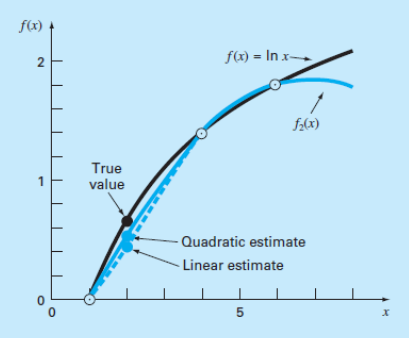
Quadratic Interpolation II

x	$x_0 = 1$	$x_1 = 4$	$x_2 = 6$
$f(x) = \ln x$	$f(x_0) = 0$	$f(x_1) = 1.386294$	$f(x_2) = 1.791759$

- **Solution**: Using equation 3, we get
  - $b_0 = f(x_0) = f(1) = 0$
    $b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(4) - f(1)}{4 - 1} = \frac{1.386294 - 0}{3} = 0.4620981$
    $b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} = \frac{\frac{f(6) - f(4)}{6 - 4} - \frac{f(4) - f(1)}{4 - 1}}{6 - 1} =$
    $\frac{\frac{1.791759 - 1.386294}{2} - \frac{1.386294 - 0}{3}}{5} = -0.0518731$
  - Substituting these values into equation 2 yields the quadratic formula
    $f_2(x) = 0 + 0.4620981(x - 1) - 0.0518731(x - 1)(x - 4)$
  - We evaluate $f_2(x)$ at $x = 2$ for $f_2(2) = 0.5658444$
  - Percentage error $\varepsilon_p = \frac{0.6931472 - 0.5658444}{0.6931472} \times 100\% = 18.4\%$

Quadratic Interpolation III

- The curvature introduced by the quadratic formula improves the interpolation compared with the result obtained using straight lines



General Form of Newton's Interpolating Polynomial I

- To fit an n^{th} order polynomial to $n+1$ data points

$$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \dots$$
$$+ b_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

- The coefficients are evaluated as
  - $b_0 = f(x_0)$
  - $b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \Delta_d f_0$
  - $b_2 = \frac{\Delta_d f_1 - \Delta_d f_0}{x_2 - x_0} = \Delta_d^2 f_0$
  - $\dots$
  - $b_n = \frac{\Delta_d^{n-1} f_1 - \Delta_d^{n-1} f_0}{x_n - x_0} = \Delta_d^n f_0$

General Form of Newton's Interpolating Polynomial II

- In general, if we have $(n+1)$ data points ranging from $(x_0, f(x_0))$ to $(x_n, f(x_n))$
  - 1^{st} finite divided difference $\Delta_d f_i = \frac{(f(x_{i+1})-f(x_i))}{(x_{i+1}-x_i)}$, for $i = 0$ to $(n-1)$
  - 2^{nd} finite divided difference $\Delta_d^2 f_i = \frac{(\Delta_d f_{i+1}-\Delta_d f_i)}{(x_{2+i}-x_i)}$, for $i = 0$ to $(n-2)$
  - 3^{rd} finite divided difference $\Delta_d^3 f_i = \frac{(\Delta_d^2 f_{i+1}-\Delta_d^2 f_i)}{(x_{3+i}-x_i)}$, for $i = 0$ to $(n-3)$
  - 4^{th} finite divided difference $\Delta_d^4 f_i = \frac{(\Delta_d^3 f_{i+1}-\Delta_d^3 f_i)}{(x_{4+i}-x_i)}$, for $i = 0$ to $(n-4)$
  - ...
  - n^{th} finite divided difference $\Delta_d^n f_i = \frac{(\Delta_d^{n-1} f_{i+1}-\Delta_d^{n-1} f_i)}{(x_{n+i}-x_i)}$, for $i = 0$ to $(n-n)$

General Form of Newton's Interpolating Polynomial III

- Divided difference table (for five data points)

x	$f(x)$	$\Delta_d f$	$\Delta_d^2 f$	$\Delta_d^3 f$	$\Delta_d^4 f$
x_0	$f(x_0)$				
		$\Delta_d f_0$			
x_1	$f(x_1)$		$\Delta_d^2 f_0$		
		$\Delta_d f_1$		$\Delta_d^3 f_0$	
x_2	$f(x_2)$		$\Delta_d^2 f_1$		$\Delta_d^4 f_0$
		$\Delta_d f_2$		$\Delta_d^3 f_1$	
x_3	$f(x_3)$		$\Delta_d^2 f_2$		
		$\Delta_d f_3$			
x_4	$f(x_4)$				

General Form of Newton's Interpolating Polynomial IV

- Newton's divided difference interpolating polynomial

$$f(x) = f(x_0) + \Delta_d f_0(x - x_0) + \Delta_d^2 f_0(x - x_0)(x - x_1) + \ldots + \Delta_d^n f_0(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where,

$$\Delta_d^n f_0 = \frac{(\Delta_d^{n-1} f_1 - \Delta_d^{n-1} f_0)}{(x_n - x_0)}$$

- **Exercise**: Estimate the value of $\ln 2$ Using a third order polynomial and the given four points

x	$x_0 = 1$	$x_1 = 4$	$x_2 = 5$	$x_3 = 6$
$f(x) = \ln x$	0	1.386294	1.609438	1.791759

- **Solution**: Divided difference table for the given four data points

General Form of Newton's Interpolating Polynomial V

x	$f(x)$	$\Delta_d f$	$\Delta_d^2 f$	$\Delta_d^3 f$
$x_0 = 1$	0			
		0.462098		
$x_1 = 4$	1.386294		-0.0597385	
		0.223144		0.0078654
$x_2 = 5$	1.609438		-0.0204115	
		0.182321		
$x_3 = 6$	1.791759			

- General form of Newton's divided difference third order formula is

$$f(x) = f(x_0) + \Delta_d f_0 (x - x_0) + \Delta_d^2 f_0 (x - x_0)(x - x_1) + \Delta_d^3 f_0 (x - x_0)(x - x_1)(x - x_2)$$

- Substituting values of respective divided differences, we get

$$f(x) = 0.462098(x-1) - 0.0597385(x-1)(x-4) + 0.0078654(x-1)(x-4)(x-5)$$

General Form of Newton's Interpolating Polynomial VI

- Putting $x = 2$, we get
  $f(2) = 0.462098(2-1) - 0.0597385(2-1)(2-4) + 0.0078654(2-1)(2-4)(2-5) = 0.462098 + 0.119477 + 0.0471924 = 0.6287674$
- Percentage error $\varepsilon_p = \frac{0.6931472 - 0.6287674}{0.6931472} \times 100\% = 9.29\%$

General Form of Newton's Interpolating Polynomial VII

Algorithm 1 Algorithm for Newton's Divided Difference Interpolating Polynomial

Input: Array x and y contains $n+1$ numbers of x and $f(x)$ values respectively, xi is the point where the value of the function f has to be interpolated

Output: The value of $f(xi)$

```
/*fdd is an (n+1) × (n+1) array which stores the finite divided differences*/
for i = 0 to n in steps of 1 do
    fdd[i][0] = y[i]
end for
/*computation of finite divided differences*/
for j = 1 to n in steps of 1 do
    for i = 0 to n − j in steps of 1 do
        fdd[i][j] = (fdd[i+1][j−1] − fdd[i][j−1])/(x[i+j] − x[i])
    end for
end for
/*computation of the value of the function at the point xi */
xterm = 1
yint[0] = fdd[0][0]
for order = 1 to n in steps of 1 do
    xterm = xterm ∗ (xi − x[order − 1])
    yint[order] = yint[order − 1] + fdd[0][order] ∗ xterm
end for
```

Lagrange Interpolation I

- Evaluation of polynomial coefficients become simpler when the following form is considered

$$
\begin{aligned}
f(x) = & \, c_0(x - x_1)(x - x_2)\ldots(x - x_n) \\
& + c_1(x - x_0)(x - x_2)\ldots(x - x_n) \\
& \ldots \\
& + c_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})
\end{aligned}
$$

where,

$$
\begin{aligned}
c_0 &= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} \\
c_1 &= \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} \\
& \vdots \\
c_n &= \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}
\end{aligned}
$$

Lagrange Interpolation II

- This is known as Lagrange Polynomial, general form of which is as follows

$$f(x) = \sum_{i=0}^{n} f(x_i) \prod_{j=0, j \neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

  - It is easier to program in a computer
  - Difficult for hand calculation

Lagrange Interpolation III

Algorithm 2 Algorithm for Lagrange Polynomial Interpolation

Input: Array x and y contains $n+1$ numbers of x and $f(x)$ values respectively, xi is the point where the value of the function f has to be interpolated

Output: The value of $f(xi)$

```
sum = 0
for i = 0 to n in steps of 1 do
    prod = 1
    for j = 0 to n in steps of 1 do
        if j ≠ i then
            prod = prod * (xi − x[j])/(x[i] − x[j])
        end if
    end for
    sum = sum + f[i] * prod
end for
Write xi, sum
```

Lagrange Interpolation IV

- **Exercise**: Estimate the value of $\ln 2$ Using first and second order Lagrange polynomial and the given three points

x	$x_0 = 1$	$x_1 = 4$	$x_2 = 6$
$f(x) = \ln x$	$f(x_0) = 0$	$f(x_1) = 1.386294$	$f(x_2) = 1.791759$

- **Solution**:
  - The first order Lagrange polynomial

$$f_1(x) = f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0}$$
$$= 0 \times \frac{x-4}{1-4} + 1.386294 \times \frac{x-1}{4-1}$$

$$f_1(2) = 1.386294 \times \frac{2-1}{4-1} = 0.462098$$

Lagrange Interpolation V

- The second order Lagrange polynomial

$$
\begin{aligned}
f_2(x) &= f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\
&+ f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= 1.386294 \times \frac{(x-1)(x-6)}{(4-1)(4-6)} + 1.791759 \times \frac{(x-1)(x-4)}{(6-1)(6-4)}
\end{aligned}
$$

$$
\begin{aligned}
f_2(2) &= 1.386294 \times \frac{(2-1)(2-6)}{(4-1)(4-6)} + 1.791759 \times \frac{(2-1)(2-4)}{(6-1)(6-4)} \\
&= 1.386294 \times \frac{2}{3} - 1.791759 \times \frac{1}{5} = 0.924196 - 0.3583518 = 0.5658442
\end{aligned}
$$

Differences of a Polynomial I

- Let $f(x)$ be a polynomial of n^{th} degree

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

- Then we obtain

$$f(x+h)-f(x) = a_1[x+h-x]+a_2[(x+h)^2-x^2]+\ldots+a_n[(x+h)^n-x^n]$$

or

$$f = f(x+h) - f(x) = a_1 h + a_2' x + \ldots + a_{n-1}' x^{n-1}$$

- This is a polynomial of order $(n-1)$
- By repeated application we get
  - $\Delta^2 f$ is a polynomial of degree $(n-2)$
  - $\Delta^3 f$ is a polynomial of degree $(n-3)$

  ......
  - $\Delta^n f$ is a constant
- Thus by inspecting the difference table we can decide the order of the polynomial to be chosen

Newton's Forward Difference Formula I

- Given a table of values (x_i, y_i), $i = 0, 1, 2, \ldots, n$ of any function $y = f(x)$, and the value of x being equally spaced, i.e., $x_i = x_o + ih$, $i = 0, 1, 2, \ldots, n$
- We are required to find values of $f(x)$ [or derivative of $f(x)$] in the range $x_0 \leq x \leq x_n$
- **Forward Differences**
  - Δ: **forward difference operator**
  - **First forward difference**: differences in y values
    - $\Delta f_0 = f(x_1) - f(x_0), \Delta f_1 = f(x_2) - f(x_1), \ldots, \Delta f_{n-1} = f(x_n) - f(x_{n-1})$
  - **Second forward difference**: differences in first forward differences
    - $\Delta^2 f(x_0) = \Delta f(x_1) - \Delta f(x_0), \Delta^2 f(x_1) = \Delta f(x_2) - \Delta f(x_1), \ldots, \Delta^2 f(x_{n-2}) = \Delta f(x_{n-1}) - \Delta f(x_{n-2})$
  - Similarly we can define **third forward difference**, **fourth forward difference** etc.

Newton's Forward Difference Formula II

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	$f(x_0)$					
		Δf_0				
x_1	$f(x_1)$		$\Delta^2 f_0$			
		Δf_1		$\Delta^3 f_0$		
x_2	$f(x_2)$		$\Delta^2 f_1$		$\Delta^4 f_0$	
		Δf_2		$\Delta^3 f_1$		$\Delta^5 f_0$
x_3	$f(x_3)$		$\Delta^2 f_2$		$\Delta^4 f_1$	
		Δf_3		$\Delta^3 f_2$		
x_4	$f(x_4)$		$\Delta^2 f_3$			
		Δf_4				
x_5	$f(x_5)$					

Table 1: Forward Difference Table

- For equal spaced intervals i.e. $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$, the divided differences become
  - $\Delta_d f_0 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \frac{\Delta f_0}{h}$
  - $\Delta_d^2 f_0 = \frac{\Delta_d f_1 - \Delta_d f_0}{(x_2 - x_0)} = \frac{\frac{\Delta f_1}{h} - \frac{\Delta f_0}{h}}{2h} = \frac{\Delta^2 f_0}{2! h^2}$

Newton's Forward Difference Formula III

- $\Delta_d^3 f_0 = \frac{\Delta_d^2 f_1 - \Delta_d^2 f_0}{(x_3 - x_0)} = \frac{\frac{\Delta^2 f_1}{2!h^2} - \frac{\Delta^2 f_0}{2!h^2}}{3h} = \frac{\Delta^3 f_0}{3!h^3}$

  ......
- $\Delta_d^n f_0 = \frac{\Delta^n f_0}{n!h^n}$

- Newton's divided difference interpolating polynomial thus becomes

$$f(x) = f(x_0) + \frac{\Delta f_0}{h}(x - x_0) + \frac{\Delta^2 f_0}{2!h^2}(x - x_0)(x - x_1) \\ + \ldots + \frac{\Delta^n f_0}{n!h^n}(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where,

$$\Delta_d^n f_0 = \frac{(\Delta_d^{n-1} f_1 - \Delta_d^{n-1} f_0)}{nh}$$

- The above form is known as Newton's forward difference interpolating polynomial
- This polynomial can be expressed more concisely by considering $x = x_0 + uh$, so

Newton's Forward Difference Formula IV

- $x - x_0 = uh$
- $x - x_1 = (x_0 + uh) - (x_0 + h) = h(u - 1)$
- $x - x_2 = x - (x_1 + h) = h(u - 1) - h = h(u - 2)$

  ......
- $x - x_{n-1} = h(u - \overline{n - 1})$

- Substituting above into the polynomial we get

$$f(x_0 + uh) = f(x_0) + \Delta f_0 u + \frac{\Delta^2 f_0}{2!} u(u - 1) + \ldots + \frac{\Delta^n f_0}{n!} u(u - 1)(u - 2) \ldots (u - \overline{n - 1})$$

- This is known as Newton Gregory Forward Interpolation Formula
- In general, Newton's forward difference formula is used to compute the approximate value of $f(x)$ when the value of x is near to x_0 of the given table. But, if the value of x is at the end of the table, then this formula gives more error. In this case, Newton's backward formula is used.

Newton's Forward Difference Formula V

Exercise: If $f(x)$ is known at the following data points, then find $f(0.5)$ and $f(1.5)$ using Newton's forward difference formula

x	0	1	2	3	4
$f(x)$	1	7	23	55	109

Solution: The given data satisfies $f(x) = x^3 + 2x^2 + 3x + 1$. It is a degree three polynomial. Hence, third forward differences are constant. Although, from the given five data points, we can fit a degree four polynomial. It is sufficient to approximate the function using a degree three polynomial.

Newton's Forward Difference Formula VI

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	1				
		6			
1	7		10		
		16		6	
2	23		16		0
		32		6	
3	55		22		
		54			
4	109				

Table 2: Forward Difference Table

Newton's Forward Difference Formula VII

Newton's forward difference formula for a degree three polynomial is

$$f(x_0+uh) = f(x_0) + \Delta f_0 u + \frac{\Delta^2 f_0}{2!}u(u-1) + \frac{\Delta^3 f_0}{3!}u(u-1)(u-2)$$

At, $x = 0.5, \; u = \frac{x-x_0}{h} = \frac{0.5-0}{1} = 0.5$
$f(0.5) = 1+0.5\times 6+\frac{0.5\times(0.5-1)\times 10}{2}+\frac{0.5\times(0.5-1)\times(0.5-2)\times 6}{6} = 3.125$

At, $x = 1.5, \; u = \frac{x-x_0}{h} = \frac{1.5-0}{1} = 1.5$
$f(1.5) = 1+1.5\times 6+\frac{1.5\times(1.5-1)\times 10}{2}+\frac{1.5\times(1.5-1)\times(1.5-2)\times 6}{6} = 13.375$

Newton's Backward Difference Formula I

- **Backward Differences**
  - ∇: **backward difference operator**
  - **First backward difference**: differences in $f(x)$ values
    - $\nabla f_n = f(x_n) - f(x_{n-1}), \nabla f_{n-1} = f(x_{n-1}) - f(x_{n-2}), \ldots, \nabla f_2 = f(x_2) - f(x_1), \nabla f_1 = f(x_1) - f(x_0)$
  - **Second backward difference**: differences in first backward differences
    - $\nabla^2 f_n = \nabla f_n - \nabla f_{n-1}, \nabla^2 f_{n-1} = \nabla f_{n-1} - \nabla f_{n-2}, \ldots, \nabla^2 f_3 = \nabla f_3 - \nabla f_2, \nabla^2 f_2 = \nabla f_2 - \nabla f_1$
  - Similarly we can define **third backward difference**, **fourth backward difference** etc.

Newton's Backward Difference Formula II

x	$f(x)$	∇	∇^2	∇^3	∇^4	∇^5
x_0	$f(x_0)$					
		∇f_1				
x_1	$f(x_1)$		$\nabla^2 f_2$			
		∇f_2		∇^{f_3}		
x_2	$f(x_2)$		$\nabla^2 f_3$		$\nabla^4 f_4$	
		∇_{f_3}		$\nabla^3 f_4$		$\nabla^5 f_5$
x_3	$f(x_3)$		$\nabla^2 f_4$		$\nabla^4 f_5$	
		∇f_4		$\nabla^3 f_5$		
x_4	$f(x_4)$		$\nabla^2 f_5$			
		∇f_5				
x_5	$f(x_5)$					

Table 3: Backward Difference Table

Newton's Backward Difference Formula III

- The generic n^{th} order polynomial to fit $n+1$ data points can also be chosen as

$$f_n(x) = b_0 + b_1(x - x_n) + b_2(x - x_n)(x - x_{n-1}) + \dots \\ + b_n(x - x_n)(x - x_{n-1})\dots(x - x_1)$$

- $f(x)$ and $f_n(x)$ must agree at given $n+1$ points
- The coefficients are evaluated as
  - $b_0 = f(x_n)$
  - $b_1 = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = \frac{\nabla f_n}{h}$
  - $b_2 = \frac{\frac{\nabla f_n}{h} - \frac{\nabla f_{n-1}}{h}}{x_n - x_{n-2}} = \frac{\nabla^2 f_n}{2! h^2}$
  - $\dots$
  - $b_n = \frac{\frac{\nabla^{n-1} f_n}{(n-1)! h^{n-1}} - \frac{\nabla^{n-1} f_{n-1}}{(n-1)! h^{n-1}}}{x_n - x_0} = \frac{\nabla^n f_n}{n! h^n}$

Newton's Backward Difference Formula IV

- Thus, the polynomial can be written as

$$f_n(x) = f(x_n) + \frac{\nabla f_n}{h}(x - x_n) + \frac{\nabla^2 f_n}{2!h^2}(x - x_n)(x - x_{n-1}) + \dots$$
$$+ \frac{\nabla^n f_n}{n!h^n}(x - x_n)(x - x_{n-1}) \dots (x - x_1)$$

- This polynomial can be expressed more concisely by considering $x = x_n + uh$, so
  - $x - x_n = uh$
  - $x - x_{n-1} = (x_n + uh) - (x_n - h) = h(u + 1)$
  - $x - x_{n-2} = x - (x_{n-1} - h) = h(u + 1) + h = h(u + 2)$

    $\dots\dots$
  - $x - x_1 = h(u + n - 1)$
- Substituting above into the polynomial we get

$$f_n(x) = f(x_n) + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \dots$$
$$+ u(u+1)\dots(u+n-1)\frac{\nabla^n f_n}{n!}$$

Newton's Backward Difference Formula V

- Backward difference formula is used to interpolate values of the function f nearer to the end of the tabular values
- **Exercise**: Population (in millions) of a city is given in the following table

x	1971	1981	1991	2001	2011
$f(x)$	46	66	81	93	101

  Determine the population in the year 2005.
- **Solution**:

Newton's Backward Difference Formula VI

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
1971	46				
		20			
1981	66		-5		
		15		2	
1991	81		-3		-3
		12		-1	
2001	93		-4		
		8			
2011	101				

Table 4: Backward Difference Table

Newton's Backward Difference Formula VII

Newton's backward difference formula for a degree four polynomial is

$$f(x) = f(x_n) + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 f_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f_n$$

At, $x = 2005$, $u = \frac{x-x_n}{h} = \frac{2005-2011}{10} = -0.6$

$$\begin{aligned} f(2005) &= 101 + -0.6 \times 8 + \frac{-0.6(-0.6+1)}{2} \times -4 \\ &\quad + \frac{-0.6(-0.6+1)(-0.6+2)}{6} \times -1 \\ &\quad + \frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)}{24} \times -3 \\ &= 101 - 4.8 + 0.48 + 0.056 + 0.1008 = 96.8368 \end{aligned}$$