CPNM Lecture 12 - Solutions to Non-Linear Equations

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Nonlinear Systems

- Systems in which the change of output is not proportional to change of input
- Most systems in nature are inherently nonlinear
- Behavior of a nonlinear system is described by a nonlinear system of equations

Outline

- Find roots of equations of the form $f(x) = 0$
  - A common problem in Science and Engineering
  - Algebraic formulae exists when $f(x)$ is quadratic, cubic ...
  - Approximate methods are used when $f(x)$ is a polynomial of higher degree or involve transcendental functions (i.e. e^x, $\sin x$, $\log x$)
  - Various iterative methods exists to obtain an **approximate solution**
    - An approximate solution finds a point $\hat{x}$ for which $f(\hat{x})$ is very near to zero
- Assumptions
  - $f(x)$ is **continuously differentiable** real-valued function of a real variable x, i.e. $f : \mathbb{R} \to \mathbb{R}$
  - The equation $f(x) = 0$ has only **isolated roots**, that is, for each root there is a neighbourhood which does not contain any other roots of the equation

Differentiability and Continuity

- If a function f is differentiable at a point x then f must also be continuous at x
- If a function f is continuous at a point x then f may not be differentiable at x
  - Example: functions having a cusp (a vertical tangent at any point)
- Open Interval: It does not include end points $(a, v) = \{x \mid a < x < b\}$
- Closed Interval: It includes end points $[a, v] = \{x \mid a \leq x \leq b\}$

Bracketing vs. Open Methods

- Bracketing methods
  - Requires two initial guesses of the root, that must bracket or be on either side of the root
- Open methods
  - Start with either a single value or two values that do not necessarily bracket the root
  - When converging, they do so more quickly than bracketing methods

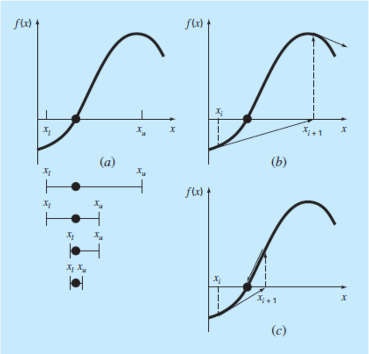
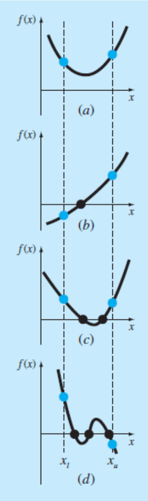
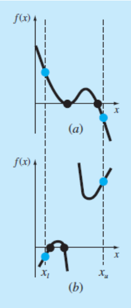


Figure 1: (a) Bracketing, (b) Open Diverging, (c) Open Converging

Number of Roots





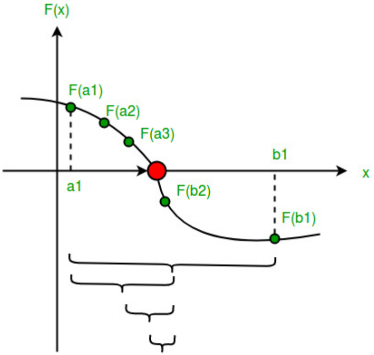
- Same sign $\Rightarrow$ zero or even no. of roots
- Opposite sign $\Rightarrow$ odd no. of roots

Iterative Methods

- The key idea in approximating the isolated real roots consists of two steps:
    1. **Initial guess**: Establishing the smallest possible intervals $[a, b]$ containing one and only one root. Take one point $x_0 \in [a, b]$ as an approximation to the root.
    2. **Improving the value of the root**: If this initial guess x_0 is not in desired accuracy, then devise a method to improve the accuracy.
- This process of improving the value of the root is called the iterative process and such methods are called iterative methods

Bisection Method

- Assumption: $f(x)$ is continuous on a given interval $[x_0, x_1]$ and $f(x_0)f(x_1) < 0$ with $f(x_0) \neq 0$ and $f(x_1) \neq 0$
- So, $f(x)$ has at least one root in $[x_0, x_1]$
- Interval length is halved in each iteration



Bisection Method - Algorithm

Read x_0, x_1, e, n
$y_0 = f(x_0)$
$y_1 = f(x_1)$
for $i = 1$ to n in steps of 1 **do**
    $x_2 = (x_0 + x_1)/2$
    $y_2 = f(x_2)$
    if $| y_2 | \leq e$ **then**
        Write "Convergent solution", x_2, y_2
        Stop
    end if
    if $(y_0 * y_2) > 0$ **then**
        $x_0 = x_2$
        $y_0 = y_2$
    else
        $x_1 = x_2$
        $y_1 = y_2$
    end if
end for
Write "Solution does not converge in n iterations"
Write x_2, y_2

Bisection Method

- Convergence is assured as the root is always kept between the two approximations
- Computational effort: one function evaluation in each iteration
- At the end of n^{th} iteration, length of this interval is $\frac{|x_0 - x_1|}{2^n}$
- Other termination conditions
  - Compute the percentage relative error ε_r, defined as

$$\varepsilon_r = \left| \frac{x_{n+1} - x_n}{x_{n+1}} \right| \times 100\%$$

    and stop when ε_r becomes less than a prescribed tolerance ε_p
  - When number of iterations reaches a specified *maximum*

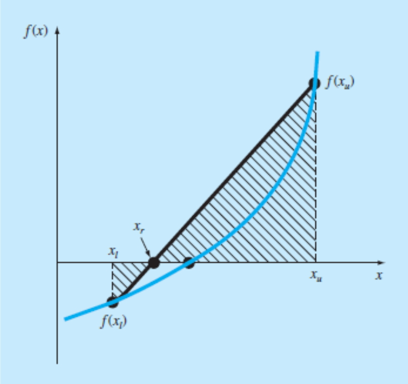
False Position Method I

- Also known as Regula Falsi or Method of Chords
- Choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs
- Equation of chord joining two points $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$ is
$$\frac{y - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$
- Replace the curve between $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$ by the chord joining the two points;
- Take the point of intersection of the chord with x-axis as an approximation of the root

$$x = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

False Position Method II

- Replacement of the curve by a straight line gives a “false position” of the root ⇒ origin of its name (“regula falsi” in Latin)



False Position Method - Algorithm

Read x_0, x_1, e, n
$y_0 = f(x_0)$
$y_1 = f(x_1)$
for $i = 1$ to n in steps of 1 **do**
    $x_2 = (x_0 * y_1 - x_1 * y_0)/(y_1 - y_0)$
    $y_2 = f(x_2)$
    if $\mid y_2 \mid \leq e$ **then**
        Write "Convergent solution", x_2, y_2
        Stop
    end if
    if $y_0 * y_2 < 0$ **then**
        $x_1 = x_2$
        $y_1 = y_2$
    else
        $x_0 = x_2$
        $y_0 = y_2$
    end if
end for
Write "Solution does not converge in n iterations"
Write x_2, y_2

Newton-Raphson Method I

- We want to approximate the solution to $f(x) = 0$ and an initial approximation to this solution is x_0
- Get the tangent line to $f(x)$ at x_0

$$y = f(x_0) + f'(x_0)(x - x_0)$$

- Let's call the point where the tangent at x_0 crosses the x-axis $(x_1, 0)$
- Plug this point into the tangent line and solve for x_1 as follows,

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

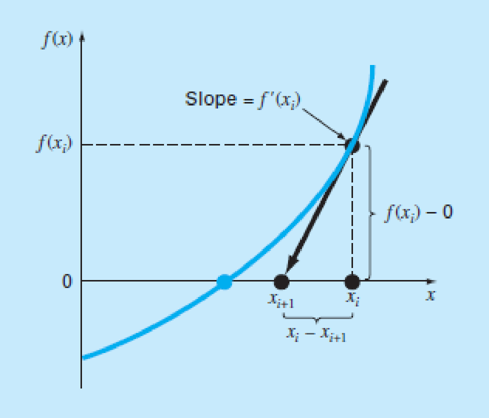
Newton-Raphson Method II

- We can find the new approximation provided the derivative isn't zero at the original approximation
- If x_n is an approximation a solution of $f(x) = 0$ and if $f'(x) \neq 0$ the next approximation is given by,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- We repeat the whole process to find an even better approximation

Newton-Raphson Method III



Newton Raphson Method - Algorithm

Read x_0, *epsilon*, *delta*, n
for $i = 1$ to n in steps of 1 **do**
    $y_0 = f(x_0)$
    $y_0' = f'(x_0)$
    if $\mid y_0' \mid \leq delta$ **then**
        Write "Slope too small", x_0, y_0'
        Stop
    end if
    $x_1 = x_0 - (y_0/y_0')$
    if $\mid (x_1 - x_0)/x_1 \mid \leq epsilon$ **then**
        Write "Convergent Solution", $x_1, f(x_1)$
        Stop
    end if
    $x_0 = x_1$
end for
Write "Solution does not converge in n iterations"
Write y_0, y_0', x_0, x_1

Examples of Poor Convergence of Newton Raphson Method