On the Embeddedness of Minimal Surfaces

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No one will read this except you and me.

— F. Schulze

1 Introduction

In the summer of 2024, I completed a research project at the University of Warwick under the supervision of Professor Felix Schulze. This document contains a write-up of my work during this time.

In particular, we present an exposition on the research paper [EWW02] by Tobias Ekholm, Brian White, and Daniel Wienholtz. This paper proves a sufficient condition for the embeddedness of minimal surfaces, and moreover imposes certain restrictions on the topology of such surfaces. In this section we examine these results in greater detail, and explore some of their widespread applications.

The present document is a shortened version of my full write-up, and was created for the purposes of public engagement. Interested readers can access a far more detailed version

2 On the Embeddedness of Minimal Surfaces

2.1 Overview

We now provide an exposition on the research paper [EWW02] by Tobias Ekholm, Brian White, and Daniel Wienholtz. To better motivate the results presented below, we make relevant historical remarks throughout. A more detailed overview of the history is also presented in the paper itself, and interested readers are encouraged to consult it for further details.

Before we begin, let us first lay down some basic definitions and notational conventions that will be used throughout this section. A *simple closed curve* is the image of a circle under a continuous and injective map. Similarly, a *disc* is the image of the set $\overline{D} \coloneqq \{x \in \mathbb{R}^2 : |x| \leqslant 1\}$ under a continuous map F. Here we do not insist that F be injective, meaning that discs in general can have overlaps and self-intersections. A *minimal* disc is one that minimises its surface area among all discs with the same boundary. Note that we need an additional condition on F to ensure that "area" is well-defined. One possibility is to assume that F is locally Lipschitz.[1]

The existence of minimal discs is guaranteed by the Douglas–Radó theorem:

[1] Recall the *area formula*, and convince yourself that it remains true if we merely assume that f is locally Lipschitz.

Theorem 2.1 (Douglas–Radó, [Whi16, Theorem 35])**.** *Let Γ be a simple closed curve in $\mathbb{R}^n$. Moreover, let $\mathcal{C}$ be the class of continuous maps $F : \overline{D} \to \mathbb{R}^n$ such that the restriction to the open disc D is locally Lipschitz, and such that $F\big|_{\partial D}$ is a monotonic parametrisation of Γ. Then $\mathcal{C}$ contains a map F that minimises the mapping area, and whose image is hence a minimal disc.*

Note that such a minimal disc will, in general, contain self-intersections and branch points. Even so, a minimal disc is the simplest example of a *minimal surface*. The latter is the image of a compact 2-manifold under a continuous and conformal harmonic map into $\mathbb{R}^n$, such that the restriction to the boundary is one-to-one. This is the definition presented in [EWW02], although many equivalent ones exist. Much like minimal discs, minimal surfaces can contain self-intersections and branch points.

A natural next step is to find sufficient conditions which guarantee that a minimal surface is smoothly embedded. Meeks and Yau proved in [MY82] that if Γ lies on the boundary of a convex set, then the minimal disc obtained from the Douglas–Radó theorem must be smoothly embedded.

In this section, we show that Γ having total curvature at most 4π is another such sufficient condition. For this we will need a key result, which is an extension of the familiar monotonicity theorem for minimal submanifolds of $\mathbb{R}^{n+k}$.

Theorem 2.2 (Extended Monotonicity, [Whi16, Theorem 5])**.** *Suppose that M is a compact n-dimensional minimal submanifold of $\mathbb{R}^{n+k}$ with rectifiable boundary Γ, and that $p \in \mathbb{R}^{n+k}$. Let $E = E(\Gamma, p)$ denote the* exterior cone *with vertex p over Γ:*

$$E = \bigcup_{q\in\Gamma} \{tq + (1-t)p : t \geqslant 1\}, \tag{2.1.1}$$

and let $M' := M \cup E$. Then the density ratio

$$\frac{\mathcal{H}^n(M' \cap B(p,r))}{\omega_n r^n} \tag{2.1.2}$$

is an increasing function of r for all $r > 0$. That is,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathcal{H}^n(M' \cap B(p,r))}{\omega_n r^n}\right) \geqslant 0, \tag{2.1.3}$$

with equality if and only if M' is a cone.

2.2 Interior Regularity

We can now prove that the interior of a minimal surface M is embedded and free of branch points. This is a proof by contradiction, and relies on the following three facts:

1. The density of M at an interior point p is bounded above by the density at p of the cone subtended by ∂M.

2. The density at p of this cone is at most $1/2\pi$ times the total curvature of ∂M.

3. The density of M at any interior branch point or self-intersection point is at least 2.

The first statement is a consequence of the extended monotonicity theorem. The second statement follows from the Gauß–Bonnet theorem, and the third is a well-known fact. Later we will encounter analogous facts for points on the boundary.

Theorem 2.3 ([EWW02, Theorem 1.3])**.** *Let M be a minimal surface in $\mathbb{R}^n$ with rectifiable boundary $\Gamma = \partial M$, and let p be a point in $\mathbb{R}^n$. Then*

$$\Theta^2(M, p) \leqslant \Theta^2(\mathrm{Cone}(\Gamma, p), p), \tag{2.2.1}$$

with equality if and only if $M = \mathrm{Cone}(\Gamma, p)$.

Theorem 2.4 ([EWW02, Theorem 1.1])**.** *Let Γ be a closed curve in $\mathbb{R}^n$, and let p be a point not in Γ. Then*

$$\mathrm{Length}(\Pi_p \Gamma) \leqslant \mathrm{TotalCurvature}(\Gamma), \tag{2.2.2}$$

where Π_p is the radial projection to the unit sphere centred at p. That is,

$$\begin{aligned} \Pi_p &: \mathbb{R}^n \setminus \{p\} \longrightarrow \partial B(p, 1); \\ \Pi_p(x) &:= p + \frac{x - p}{|x - p|}. \end{aligned}$$

Equivalently,

$$\Theta^2(\mathrm{Cone}(\Gamma, p), p) \leqslant \frac{1}{2\pi} \mathrm{TotalCurvature}(\Gamma). \tag{2.2.3}$$

Two proofs can be found in [EWW02], one using the Gauß–Bonnet theorem, and a second using the integral geometric formulas from [Mil50].

We can now combine these intermediate results to improve on the interior regularity of our surface:

Theorem 2.5 ([EWW02], Theorem 2.1)**.** *Let Γ be a simple closed curve in $\mathbb{R}^n$ with total curvature at most 4π, and let M be a minimal surface with boundary Γ. Then the interior of M is embedded, and contains no branch points.*

From Theorem 2.5, the Fáry–Milnor theorem follows as a simple corollary. This is a fundamental result linking the geometry and topology of a simple closed curve in $\mathbb{R}^3$. It was proven independently by István Fáry in 1949 and by John Milnor in 1950.

Corollary 2.6 (Fáry–Milnor, [Far49], [Mil50])**.** *Let Γ be a simple closed curve in $\mathbb{R}^3$ with total curvature at most 4π. Then Γ is unknotted.*

Proof. Let $F : B(0, 1) \to \mathbb{R}^3$ be the least-area disc bounded by Γ. (That is, the Douglas–Radó solution to the Plateau problem.) By Theorem 2.5, this disc is smoothly embedded on the interior of $B(0, 1)$. In particular, the function $r \mapsto F(\partial B(0, r))$ describes an isotopy of curves for $r \neq 0$. When $r = 1$, the curve is Γ. When r is small, the curve is nearly circular and therefore unknotted. $\square$

2.3 Boundary Regularity

We wish to extend the conclusions of Theorem 2.5 to the boundary of M. Of course we continue to assume an upper bound of 4π on the total curvature.

Our proof will have a similar structure to that of Theorem 2.5. In particular, we recall that the density of M at a boundary branch point or self-intersection point must be at least $3/2$.

We begin with the following variant of Theorem 2.4:

Theorem 2.7 ([EWW02, Theorem 3.1]). *Let Γ be a simple closed curve in $\mathbb{R}^n$ with finite total curvature, and let p be a point in Γ. Then*

$$\mathrm{Length}(\Pi_p\Gamma) \leqslant \mathrm{TotalCurvature}(\Gamma) - \pi - \theta, \tag{2.3.1}$$

where θ is the exterior angle to Γ at p.

We are now ready to prove regularity at the boundary. We first do this for *smooth* boundary curves, before dealing with more general curves.

Theorem 2.8 ([EWW02, Theorem 3.2]). *Let Γ be a smooth simple curve in $\mathbb{R}^n$ with total curvature at most 4π, and let M be a minimal surface with boundary Γ. Then M is a smoothly embedded manifold with boundary.*

2.4 Boundaries with Corners

We now consider boundaries with less regularity, such as ones that include corners. In the proof of the theorem below, we make use of the following fact: any tangent cone to a 2-dimensional minimal variety (such as a stationary integral varifold) intersects the unit sphere in a finite collection of geodesic arcs.

Theorem 2.9 ([EWW02], Theorem 4.1). *Let Γ be a simple closed curve in $\mathbb{R}^n$ with total curvature at most 4π, and let M be a minimal surface with boundary Γ.*

(i) If p is a point in Γ with exterior angle θ, then the density $\Theta(M, p)$ is either $\frac{1}{2} + \frac{\theta}{2\pi}$ or $\frac{1}{2} - \frac{\theta}{2\pi}$.

(ii) If p is a cusp point (that is, $\theta = \pi$), then the density $\Theta(M, p)$ is 0 unless Γ lies in a plane.

(iii) M is embedded up to and including the boundary.

2.5 Nondisc-type Surfaces

We know from the Douglas–Radó theorem that a simple closed curve Γ in $\mathbb{R}^n$ bounds a minimal disc. One might ponder whether such a Γ can bound other minimal surfaces and, if so, whether those are also smoothly embedded. In this section, [EWW02] gives an example of such a Γ in $\mathbb{R}^3$ that bounds at least two other minimal surfaces, namely Möbius strips. The construction given is as follows:

Consider two convex polygons which lie entirely in the halfplane $\{(x, y, 0) \in \mathbb{R}^3 : y \geqslant 0\}$ such that each polygon has exactly one vertex, namely the origin, lying on the x-axis. For simplicity, we may assume that we have two copies of the same regular n-gon. Give both polygons the positive (that is, anticlockwise) orientation. We now rotate one polygon about the x-axis through a small positive angle, and the other through a small negative angle. Note that the two polygons no longer lie on a common plane, but still share the origin as a common vertex.

Consider the closed connected curve Γ which starts at the origin, traces out one polygon, followed by the other. We claim that Γ has total curvature less than 4π. A sketch of the proof is presented below, which relies on the integral-geometric formula of Milnor [Mil50, Theorem 3.1].

For a generic unit vector $\mathbf{v}$, the function given by $f_{\mathbf{v}}(x) = \mathbf{v} \cdot x$ will not be constant on any segment of Γ. Such a function $f_{\mathbf{v}}$ can have at most four local extrema (two on each polygon). However, the set of such vectors $\mathbf{v}$ for which $f_{\mathbf{v}}$ has only two local extrema contains vectors arbitrarily close to $(0,0,1)$, and hence is open and non-empty. Therefore, the average number of local extrema is less than 4, which implies that the total curvature of Γ is less than 4π.

The curve Γ is not embedded, since it contains a self-intersection at the origin. Note however that Γ can be made embedded, and even analytic, following a suitable perturbation. Furthermore, we may assume that Γ is arbitrarily close to a curve traversing the unit circle in the xy-plane twice. We may also dilate Γ so that it lies outside the unit cylinder $B(0,1) \times \mathbb{R}$.

A disc bounded by such a curve Γ must then have area at least 2π, since its projection to the xy-plane must cover the unit disc twice. But clearly Γ bounds a Möbius strip of much smaller area, and hence it bounds a minimal Möbius strip.

Following this construction, [EWW02] makes the following conjecture:

Conjecture. Let Γ be a smooth simple closed curve in $\mathbb{R}^n$ with total curvature at most 4π. Then, in addition to a unique minimal disc, Γ bounds either:

(i) no other minimal surfaces, or

(ii) exactly one minimal Möbius strip and no other minimal surfaces, or

(iii) exactly two minimal Möbius strips and no other minimal surfaces.

In case (ii), the strip has index 0 and nullity 1. In case (iii), both strips have nullity 0, one has index 1 and the other has index 0.

Note that the minimal surfaces in the above conjecture are assumed to be *classical* minimal surfaces. In fact Γ can also bound other minimal varieties, with one such example being provided in [EWW02, Section 7].

To the author's knowledge, this conjecture remains open at the time of writing.

2.6 Disconnected Boundaries

We now consider the case of a minimal surface M with more than one boundary component. We assume as usual that the total boundary curvature (that is, the sum of the boundary curvatures of each component) is at most 4π. Appealing to Borsuk's extension of Fenchel's theorem, each component has boundary curvature at least 2π, with equality if and only if ∂M is a plane convex curve. Thus ∂M must consist of exactly two components Γ_1 and Γ_2, each of which is a plane convex curve.

If M is a cone then it must be locally planar, since its scalar and mean curvatures both vanish. That is, M is the union of two planar regions R_1 and R_2 bounded by two plane curves Γ_1 and Γ_2. Note that the vertex of the cone must belong to both regions, implying that R_1 and R_2 intersect. If M is not a cone, then all the conclusions of Theorems 2.5, 2.7, and 2.9 hold, with exactly the same proofs.

2.7 Two basic properties of curves with finite total curvature

First, a sufficient condition for the rectifiability of a curve in $\mathbb{R}^n$:

Theorem 2.10 ([EWW02], Theorem 10.1)**.** *If Γ is a compact connected curve in $\mathbb{R}^n$ with finite total curvature, then it has finite length.*

Proof. Let γ be a parametrisation of Γ. We may assume that this is closed (if not, we close it up with a straight line segment). If $\mathbf{u}$ is a unit vector, then the total variation of $t \mapsto \gamma(t) \cdot \mathbf{u}$ is at most the diameter of Γ times the number of local extrema of $\gamma(t) \cdot \mathbf{u}$. Averaging both sides of this inequality over all unit vectors $\mathbf{u}$ gives

$$c_n \operatorname{Length}(\Gamma) \leqslant \operatorname{diam}(\Gamma) \cdot \operatorname{TotalCurvature}(\Gamma), \tag{2.7.1}$$

where the constant $c_n > 0$ depends only on the dimension n. $\square$

Lemma 2.11 ([EWW02], Lemma 10.2)**.** *Suppose $\gamma : [a, b] \to \mathbb{R}^n$ is an injective curve of finite total curvature. For $\xi < \eta$ in $[a, b]$, let*

$$T_{\xi\eta} := \frac{\gamma(\eta) - \gamma(\xi)}{|\gamma(\eta) - \gamma(\xi)|}$$

be the unit vector pointing from $\gamma(\xi)$ to $\gamma(\eta)$.

If $a < x \leqslant y < b$, then the angle $\angle(T_{ax}, T_{yb})$ between T_{ax} and T_{yb} is at most the total curvature of γ restricted to the open interval (a, b).

Proof. By the triangle inequality for geodesic distances in the unit sphere, we have that

$$\begin{aligned} \operatorname{TotalCurvature}(\gamma|_{(a,b)}) &\geqslant \angle(T_{ax}, T_{xy}) + \angle(T_{xy}, T_{yb}) \\ &\geqslant \angle(T_{ax}, T_{yb}). \end{aligned}$$

$\square$

Theorem 2.12 ([EWW02], Theorem 10.3)**.** *Suppose $\gamma : [A, B] \to \mathbb{R}^n$ is an injective curve of finite total curvature. Then the strong one-sided tangents*

$$T^+(a) = \lim_{\substack{a \leqslant x < y \\ y \to a}} T_{xy} \quad \text{and} \quad T_-(b) = \lim_{\substack{x < y \leqslant b \\ x \to b}} T_{xy} \tag{2.7.2}$$

exist for every $a \in [A, B)$ and every $b \in (A, B]$. Furthermore, $T^+(x)$ and $T_-(x)$ both approach $T^+(a)$ as x approaches a from the right, and they both approach $T_-(b)$ as x approaches b from the left.

Proof. Let T be the limit of T_{ax} as $x \to a$ with $x > a$. Applying Lemma 2.11 to $\gamma|_{[a,y]}$, we get that

$$\angle(T, T_{xy}) \leqslant \operatorname{TotalCurvature}(\gamma|_{(a,y)}) \tag{2.7.3}$$

for $a < x < y < b$. Notice that as $y \to a$ with $y > a$, the right-hand side goes to 0. This proves that $T^+(a) = T$ exists. Likewise, T_- exists at every point.

Letting $x \to y$ and then $y \to a$ in the definition of $T^+(a)$, we immediately read off that $T_-(y) \to T^+(a)$ as $y \to a$. If however, we let $y \to x$ and then $x \to a$, we get that $T^+(x) \to T^+(a)$. The convergence to $T_-(b)$ from the left is used analogously. $\square$

3 References

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