Gauss' Theorema Egregium

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Abstract

In 1827, the German mathematician Carl Friedrich Gauss proved a major result in the field of Differential Geometry. This theorem, which would become known as *Gauss' Theorema Egregium*[1], is the focus of this essay. We develop the theory leading up to the theorem, and conclude with some of its interesting applications.

Contents

[1] Latin for *Gauss' Remarkable Theorem.*

1 The First Fundamental Form

1.1 Surfaces and Tangent Planes

Surfaces in $\mathbb{R}^3$ were first introduced in MA259: Multivariable Calculus. Here, we recall some basic definitions before moving on to more advanced concepts.

Definition 1.1. *A set $\mathcal{S} \subset \mathbb{R}^3$ is called a* surface *if, for all points $p \in \mathcal{S}$, there exist open sets $V \subset \mathbb{R}^2$ and $W \subset \mathbb{R}^3$ such that $p \in W$ and the sets V and $\mathcal{S} \cap W$ are homeomorphic.*

These homeomorphisms $\sigma : V \to \mathcal{S} \cap W$ are known as *surface patches*. They can be represented visually as follows:

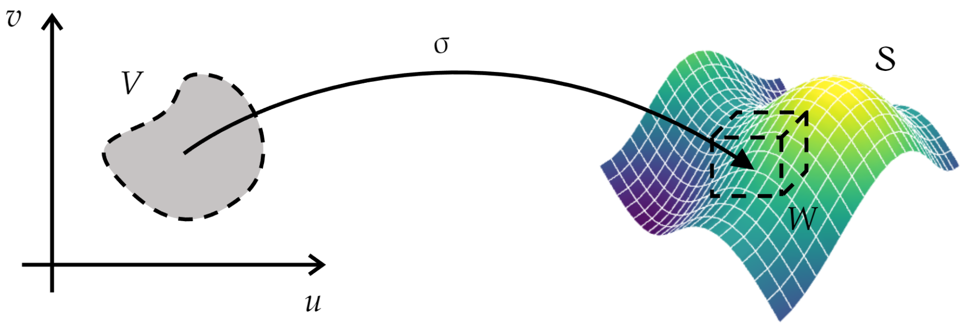

Figure 1: Visual representation of a surface patch.

Where there is no possibility for ambiguity, we will use the terms surface patch and *parametrisation* interchangeably.

For the purposes of this essay, we are especially interested in surface patches that are *regular*.

Definition 1.2. *Suppose $\sigma : V \to \mathcal{S} \cap W$ is a surface patch given by*

$$\sigma(u, v) := (x(u, v), y(u, v), z(u, v))$$

Then σ is called smooth *if it has continuous partial derivatives of all orders. It is called* regular *if it is smooth, and its first partial derivatives, denoted by σ_u and σ_v, are linearly independent for all $p = (u, v) \in V$.*[2]

With this definition in mind, we will say that a surface $\mathcal{S}$ is regular if for every point $p \in \mathcal{S}$, there exists a neighbourhood of p that is parametrised by a regular surface patch.

Much like we defined tangent lines for regular curves, we can also define *tangent planes* for regular surfaces:

[2]Throughout this essay, whenever we write σ_u or σ_v we actually mean $\sigma_u(p)$ and $\sigma_v(p)$ respectively.

Definition 1.3. *Given a regular surface $\mathcal{S} \subset \mathbb{R}^3$ and a point $p \in \mathcal{S}$, we define the* tangent plane *at p as the set of all vectors tangent to $\mathcal{S}$ at p. We denote this plane by $T_p\mathcal{S}$.*

The above definition is not very useful in practice. As such, we seek to characterise tangent planes in terms of surface patches, which we can do by the following lemma:

Lemma 1.1. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a regular surface. For an arbitrary point $p \in \mathcal{S}$, let σ be a surface patch whose image contains p. It follows that $T_p\mathcal{S} = \text{span}\{\sigma_u, \sigma_v\}$.*

Proof. Consider a curve $\mathcal{C}$ on the surface $\mathcal{S}$ which passes through p, and is parametrised regularly by $\gamma : I \to \mathbb{R}^3$. If we think of u and v as functions, we may write $\gamma(t) = \sigma(u(t), v(t))$, which we then differentiate using the chain rule:

$$\gamma'(t) = u'(t)\sigma_u + v'(t)\sigma_v$$

Since $p \in \mathcal{C}$, there exists $t_0 \in I$ such that $\gamma(t_0) = p$. Hence, $\gamma'(t_0)$ is a tangent vector to $\mathcal{C}$ at p, and so $\gamma'(t_0) \in T_p\mathcal{S}$. Moreover, $\gamma'(t_0) = u'(t_0)\sigma_u + v'(t_0)\sigma_v$, so this tangent vector is a linear combination of σ_u and σ_v. The set of all such linear combinations is precisely the span of σ_u and σ_v, so $\gamma'(t_0) \in \text{span}\{\sigma_u, \sigma_v\}$. Since $\mathcal{S}$ is regular, this span will be two-dimensional, that is, a plane. As we vary over all such curves $\mathcal{C}$, we obtain all the tangent vectors at p (up to scaling). Therefore, $T_p\mathcal{S} = \text{span}\{\sigma_u, \sigma_v\}$, as required.

□

1.2 The First Fundamental Form

We reach the crux of this section: the first fundamental form. This arises naturally as the canonical inner product on the tangent plane of a regular surface. More formally:

Definition 1.4. *Given a regular surface $\mathcal{S}$ and a point $p \in \mathcal{S}$, the* first fundamental form *of $\mathcal{S}$ at p is the inner product denoted by* I *and defined by:*[3]

$$\begin{aligned} \mathrm{I} : T_p\mathcal{S} \times T_p\mathcal{S} &\to \mathbb{R} \\ (\mathbf{x}, \mathbf{y}) &\mapsto \langle \mathbf{x}, \mathbf{y} \rangle \end{aligned}$$

Since our ground field is $\mathbb{R}$, this inner product is a symmetric bilinear form. Hence, for a given basis of $T_p\mathcal{S}$, it can be represented by a symmetric 2×2 matrix. In general, we have:

$$\mathrm{I}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \mathbf{y}, \tag{1}$$

[3]Many sources define the first fundamental form as the quadratic form $Q : T_p\mathcal{S} \to \mathbb{R}$ given by $Q(\mathbf{x}) := \mathrm{I}(\mathbf{x}, \mathbf{x})$. The two definitions are essentially equivalent.

where E, F and G are called the *coefficients* of the first fundamental form. More specifically, suppose that a point $p \in \mathcal{S}$ is parametrised by the surface patch σ. By lemma (1.1), the set $\{\sigma_u, \sigma_v\}$ is a basis of $T_p\mathcal{S}$, and therefore gives rise to a corresponding coefficient matrix.

The following lemma allows us to derive expressions for these coefficients in terms of σ:

Lemma 1.2. *Let* I *be the first fundamental form of a regular surface* $\mathcal{S} \subset \mathbb{R}^3$, *and let* $p \in \mathcal{S}$ *be a point parametrised by* σ. *The coefficients of* I *at* p *are given by:*

$$E = \langle \sigma_u, \sigma_u \rangle, \qquad F = \langle \sigma_u, \sigma_v \rangle, \qquad G = \langle \sigma_v, \sigma_v \rangle$$

Proof. Let $\mathbf{w}_1, \mathbf{w}_2 \in T_p\mathcal{S}$ be arbitrary. By lemma (1.1), there exist scalars $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\mathbf{w}_1 = \alpha\sigma_u + \beta\sigma_v$ and $\mathbf{w}_2 = \gamma\sigma_u + \delta\sigma_v$. Therefore,

$$\begin{aligned}
\mathrm{I}(\mathbf{w}_1, \mathbf{w}_2) &= \langle \mathbf{w}_1, \mathbf{w}_2 \rangle \\
&= \langle \alpha\sigma_u + \beta\sigma_v, \gamma\sigma_u + \delta\sigma_v \rangle \\
&= \alpha\gamma\langle \sigma_u, \sigma_u \rangle + (\alpha\delta + \beta\gamma)\langle \sigma_u, \sigma_v \rangle + \beta\delta\langle \sigma_v, \sigma_v \rangle \\
&= \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} \langle \sigma_u, \sigma_u \rangle & \langle \sigma_u, \sigma_v \rangle \\ \langle \sigma_u, \sigma_v \rangle & \langle \sigma_v, \sigma_v \rangle \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\
&= \mathbf{w}_1^T \begin{pmatrix} \langle \sigma_u, \sigma_u \rangle & \langle \sigma_u, \sigma_v \rangle \\ \langle \sigma_u, \sigma_v \rangle & \langle \sigma_v, \sigma_v \rangle \end{pmatrix} \mathbf{w}_2
\end{aligned}$$

Comparing this with the expression in (1), we obtain the desired result. □

It may not be obvious from its definition why the first fundamental form is so important. As we will soon prove, it is preserved by certain transformations known as *local isometries*, which we now define:

Definition 1.5. *Let* $\mathcal{S}$ *and* $\tilde{\mathcal{S}}$ *be two surfaces, and let* γ *be a curve in* $\mathcal{S}$. *A* local isometry *is a function* $d : \mathcal{S} \to \tilde{\mathcal{S}}$ *which maps* γ *to a curve* $\tilde{\gamma} = d \circ \gamma$ *of the same length in* $\tilde{\mathcal{S}}$. *If such a function* d *exists, the surfaces* $\mathcal{S}$ *and* $\tilde{\mathcal{S}}$ *are called* locally isometric.

Theorem 1.1. *Local isometries preserve the first fundamental form.*

Proof. Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be two isometric surfaces which are parametrised by σ and $\tilde{\sigma}$ respectively. Moreover, let $d : \mathcal{S} \to \tilde{\mathcal{S}}$ be an isometry between the two surfaces. By definition of d, a curve in $\mathcal{S}$ is mapped to a curve of the same length in $\tilde{\mathcal{S}}$. Therefore, following the discussion in section B.1 we have that

$$\int_I \left[E(u'(t))^2 + 2Fu'(t)v'(t) + G(v'(t))^2\right]^{1/2} dt$$

and

$$\int_I \left[\tilde{E}(u'(t))^2 + 2\tilde{F}u'(t)v'(t) + \tilde{G}(v'(t))^2\right]^{1/2} dt$$

are equal for all curves and for all intervals. Therefore, we differentiate to obtain that

$$\left[E(u'(t))^2 + 2Fu'(t)v'(t) + G(v'(t))^2\right]^{1/2}$$

and

$$\left[\tilde{E}(u'(t))^2 + 2\tilde{F}u'(t)v'(t) + \tilde{G}(v'(t))^2\right]^{1/2}$$

are equal. By simplifying and equating coefficients:

$$E = \tilde{E}, \qquad F = \tilde{F}, \qquad G = \tilde{G}$$

Therefore, the first fundamental forms of $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are the same. $\square$

1.3 Intrinsic Properties of a Surface

A number of properties—the so-called *intrinsic properties*—are completely determined by the first fundamental form of a surface. One of these in particular, the Gaussian curvature, will be the main focus of this essay in sections 2.4 and 3. However, a number of other intrinsic properties are also interesting in their own right, which is why we've dedicated Appendix B to exploring them further.

2 Curvature of a Surface

2.1 Normal Vectors, Orientability, and the Gauss Map

In MA134: Geometry & Motion we saw that a curve $\mathcal{C}$, parametrised by $\gamma : [a, b] \to \mathbb{R}$, can be thought of as having a certain *orientation*: that is, starting from $\gamma(a)$ and ending at $\gamma(b)$. In particular, that orientation was closely related to its principal normal vectors $\mathbf{N}$. We wish to generalise this notion to surfaces.

Definition 2.1. *Let $\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit 2-sphere, and let $\mathcal{S} \subset \mathbb{R}^3$ be a regular surface. Then $\mathcal{S}$ is called* orientable *if there exists a continuous vector field $N : \mathcal{S} \to \mathbb{S}^2$, such that $N(p)$ is normal to $T_p\mathcal{S}$ for all $p \in \mathcal{S}$. This vector field N is called the* Gauss map *acting on $\mathcal{S}$.*

The requirement that N be continuous allows us to consistently assign a unit normal vector to each point in $\mathcal{S}$. In particular, if $p \in \mathcal{S}$ is parametrised by σ, then there are two such vectors at p, given by

$$\pm \frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}.$$

We are free to choose either one as the value of $N(p)$, provided that the resulting vector field is continuous.

An example of a surface where such an assignment is *not* possible is the Möbius strip [Figure 2].[4]

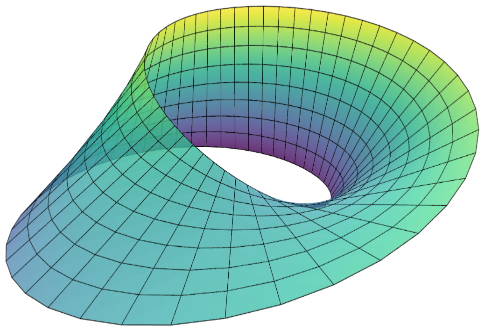

Figure 2: A Möbius strip: the simplest non-orientable surface.

Note that, given an orientable surface $\mathcal{S}$, the choice of orientation is not unique: indeed, if $p \mapsto N(p)$ is a Gauss map, then $p \mapsto -N(p)$ is also a Gauss map. To avoid any ambiguity, we introduce the distinction between *orientable* and *oriented* surfaces:

Definition 2.2. *A surface $\mathcal{S} \subset \mathbb{R}^3$ is called* oriented *if it comes equipped with a Gauss map N.*

We also make the following observation: for a regular and oriented surface $\mathcal{S} \subset \mathbb{R}^3$, the vectors σ_u, σ_v and N are all linearly independent, and hence form a basis of $\mathbb{R}^3$.

The Gauss map is also special in a different way: since it maps points to normal vectors, its Jacobian matrix defines a linear endomorphism on the tangent plane of a surface. This detail will be crucial in defining the Weingarten map in the following section.

2.2 The Weingarten Map[5]

Having established rigorous definitions of normal vectors and orientation, we now move on to defining curvature. Returning to our previous analogy, we observe that the curvature of a space curve is closely related to the rate of change of its normal vectors. This is evident, for example, from the Frenet–Serret formulæ corresponding to that curve.

Definition 2.3. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a regular and oriented surface with Gauss map N, and let $p \in \mathcal{S}$ be arbitrary. The* Weingarten map *of $\mathcal{S}$ at p is the function $\mathcal{W}_{p,\mathcal{S}} : T_p\mathcal{S} \to T_p\mathcal{S}$ defined by:*

[4] Readers are encouraged to craft their own Möbius strip from a rectangular piece of paper and convince themselves of this fact.

[5] Named after the German mathematician Julius Weingarten (1836–1910). He made important contributions to the differential geometry of surfaces.

$$\mathcal{W}_{p,\mathcal{S}}(\mathbf{v}) := -\partial N(p)\mathbf{v},$$

where $\partial N(p)$ *is the Jacobian matrix of* N *at* p.[6]

The Weingarten map is clearly a linear map. Therefore, a natural next step is to examine its eigenvalues, its determinant, and its trace. As we will see, each of these corresponds to a different type of curvature. Before we do this however, it is necessary to establish that its eigenvalues are actually real numbers.

2.3 The Second Fundamental Form

The aim of this section is to prove that the eigenvalues of the Weingarten map are always real. To do this, we first define another bilinear form on the tangent plane of a surface: the *second fundamental form.*

Definition 2.4. *Given a regular, oriented surface* $\mathcal{S}$ *with Weingarten map* $\mathcal{W}_{p,\mathcal{S}}$*, the* second fundamental form *of* $\mathcal{S}$ *at* p *is the bilinear form denoted by* $\mathrm{I\!I}$ *and defined by:*[7]

$$\begin{aligned} \mathrm{I\!I} : T_p\mathcal{S} \times T_p\mathcal{S} &\to \mathbb{R} \\ (\mathbf{x}, \mathbf{y}) &\mapsto \mathrm{I}(\mathcal{W}_{p,\mathcal{S}}(\mathbf{x}), \mathbf{y}), \end{aligned}$$

where I *is the first fundamental form of* $\mathcal{S}$ *at* p.

We asserted in its definition that $\mathrm{I\!I}$ is a bilinear form, so let's actually prove it:

Lemma 2.1. *The second fundamental form is a bilinear form.*

Proof. Let $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in T_p\mathcal{S}$ and $\alpha, \beta \in \mathbb{R}$ be arbitrary. We have that:

$$\begin{aligned} \mathrm{I\!I}(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2, \mathbf{v}) &= \mathrm{I}(\mathcal{W}_{p,\mathcal{S}}(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2), \mathbf{v}) \\ &= \mathrm{I}(\alpha\mathcal{W}_{p,\mathcal{S}}(\mathbf{v}_1) + \beta\mathcal{W}_{p,\mathcal{S}}(\mathbf{v}_2), \mathbf{v}) \\ &= \alpha\,\mathrm{I}(\mathcal{W}_{p,\mathcal{S}}(\mathbf{v}_1), \mathbf{v}) + \beta\,\mathrm{I}(\mathcal{W}_{p,\mathcal{S}}(\mathbf{v}_2), \mathbf{v}) \\ &= \alpha\,\mathrm{I\!I}(\mathbf{v}_1, \mathbf{v}) + \beta\,\mathrm{I\!I}(\mathbf{v}_2, \mathbf{v}) \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{I\!I}(\mathbf{v}, \alpha\mathbf{v}_1 + \beta\mathbf{v}_2) &= \mathrm{I}(\mathcal{W}_{p,\mathcal{S}}(\mathbf{v}), \alpha\mathbf{v}_1 + \beta\mathbf{v}_2) \\ &= \alpha\,\mathrm{I}(\mathcal{W}_{p,\mathcal{S}}(\mathbf{v}), \mathbf{v}_1) + \beta\,\mathrm{I}(\mathcal{W}_{p,\mathcal{S}}(\mathbf{v}), \mathbf{v}_2) \\ &= \alpha\,\mathrm{I\!I}(\mathbf{v}, \mathbf{v}_1) + \beta\,\mathrm{I\!I}(\mathbf{v}, \mathbf{v}_2) \end{aligned}$$

□

[6] The minus sign in the definition is simply a matter of convention: it has become standard in the literature because it makes some calculations slightly easier.

[7] The curious reader might wonder if there is a *third fundamental form.* Indeed there is, but it is not as important as the first two, and we will not be dealing with it here.

Lemma 2.2. *Let $p \in \mathcal{S}$ be a point parametrised by σ. The following expressions hold:*

$$\begin{aligned}
\mathrm{I\!I}(\sigma_u, \sigma_u) &= \langle N(p), \sigma_{uu} \rangle \\
\mathrm{I\!I}(\sigma_u, \sigma_v) &= \langle N(p), \sigma_{uv} \rangle \\
\mathrm{I\!I}(\sigma_v, \sigma_u) &= \langle N(p), \sigma_{vu} \rangle \\
\mathrm{I\!I}(\sigma_v, \sigma_v) &= \langle N(p), \sigma_{vv} \rangle
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^3$.

Proof. From the definition of the Gauss map, it follows that

$$\langle N(p), \sigma_u \rangle = 0 = \langle N(p), \sigma_v \rangle$$

Using the product rule, we differentiate the first equality with respect to u to obtain

$$\langle N_u(p), \sigma_u \rangle + \langle N(p), \sigma_{uu} \rangle = 0$$

Therefore:

$$\begin{aligned}
\langle N(p), \sigma_{uu} \rangle &= -\langle N_u(p), \sigma_u \rangle \\
&= \langle -\partial N(p)\sigma_u, \sigma_u \rangle \\
&= \langle \mathcal{W}_{p,\mathcal{S}}(\sigma_u), \sigma_u \rangle \\
&= \mathrm{I}(\mathcal{W}_{p,\mathcal{S}}(\sigma_u), \sigma_u) \\
&= \mathrm{I\!I}(\sigma_u, \sigma_u)
\end{aligned}$$

This proves the first expression. The other three follow similarly by differentiating each of the two equalities with respect to u and v. □

Corollary 2.1. *The second fundamental form is symmetric.*

Proof. We have assumed that $\mathcal{S}$ is regular, so its second-order partial derivatives exist and are continuous. Clairaut's theorem, together with lemma 2.2, implies that $\mathrm{I\!I}(\sigma_u, \sigma_v) = \mathrm{I\!I}(\sigma_v, \sigma_u)$. Now, suppose that $\mathbf{v}_1, \mathbf{v}_2 \in T_p\mathcal{S}$ are arbitrary tangent vectors. By lemma 1.1, $\{\sigma_u, \sigma_v\}$ is a basis of $T_p\mathcal{S}$, so there exist scalars $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\mathbf{v}_1 = \alpha\sigma_u + \beta\sigma_v$ and $\mathbf{v}_2 = \gamma\sigma_u + \delta\sigma_v$. Therefore:

$$\begin{aligned}
\mathrm{I\!I}(\mathbf{v}_1, \mathbf{v}_2) &= \mathrm{I\!I}(\alpha\sigma_u + \beta\sigma_v, \gamma\sigma_u + \delta\sigma_v) \\
&= \alpha\gamma\, \mathrm{I\!I}(\sigma_u, \sigma_u) + \alpha\delta\, \mathrm{I\!I}(\sigma_u, \sigma_v) + \beta\gamma\, \mathrm{I\!I}(\sigma_v, \sigma_u) + \beta\delta\, \mathrm{I\!I}(\sigma_v, \sigma_v) \\
&= \alpha\gamma\, \mathrm{I\!I}(\sigma_u, \sigma_u) + \alpha\delta\, \mathrm{I\!I}(\sigma_v, \sigma_u) + \beta\gamma\, \mathrm{I\!I}(\sigma_u, \sigma_v) + \beta\delta\, \mathrm{I\!I}(\sigma_v, \sigma_v) \\
&= \mathrm{I\!I}(\gamma\sigma_u + \delta\sigma_v, \alpha\sigma_u + \beta\sigma_v) \\
&= \mathrm{I\!I}(\mathbf{v}_2, \mathbf{v}_1)
\end{aligned}$$

Hence $\mathrm{I\!I}$ is symmetric, as required. □

Corollary 2.2. *The Weingarten map $\mathcal{W}_{p,\mathcal{S}}$ is a self-adjoint linear operator with respect to the first fundamental form.*

Proof. Let $\mathbf{v}, \mathbf{w} \in T_p\mathcal{S}$ be arbitrary. We have that:

$$\mathrm{I}(\mathcal{W}_{p,\mathcal{S}}(\mathbf{v}), \mathbf{w}) = \mathrm{I\!I}(\mathbf{v}, \mathbf{w}) = \mathrm{I\!I}(\mathbf{w}, \mathbf{v}) = \mathrm{I}(\mathcal{W}_{p,\mathcal{S}}(\mathbf{w}), \mathbf{v}) = \mathrm{I}(\mathbf{v}, \mathcal{W}_{p,\mathcal{S}}(\mathbf{w}))$$

□

It follows that the eigenvalues of $\mathcal{W}_{p,\mathcal{S}}$ are real numbers, since all self-adjoint linear operators have real eigenvalues.

Much like the first fundamental form, the second fundamental form can also be represented by a symmetric 2×2 matrix:

$$\mathrm{I\!I}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} \mathbf{y}, \tag{2}$$

where L, M and N are called the *coefficients* of the second fundamental form.

Given a specific parametrisation σ, we can then find explicit expressions for L, M and N:

Lemma 2.3. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a regular and oriented surface with Gauss map N. Suppose further that $p \in \mathcal{S}$ is parametrised by σ. The coefficients of $\mathrm{I\!I}$ at p are given by:*

$$L = -\langle N_u, \sigma_u \rangle, \qquad M = -\langle N_u, \sigma_v \rangle = -\langle N_v, \sigma_u \rangle, \qquad N = -\langle N_v, \sigma_v \rangle$$

Proof. Our proof will mimic that of lemma 1.2. Let $\mathbf{w}_1, \mathbf{w}_2 \in T_p\mathcal{S}$ be arbitrary. By lemma 1.1, there exist scalars $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\mathbf{w}_1 = \alpha\sigma_u + \beta\sigma_v$ and $\mathbf{w}_2 = \gamma\sigma_u + \delta\sigma_v$. Therefore,

$$\begin{aligned} \mathrm{I\!I}(\mathbf{w}_1, \mathbf{w}_2) &= \mathrm{I\!I}(\alpha\sigma_u + \beta\sigma_v, \gamma\sigma_u + \delta\sigma_v) \\ &= \alpha\gamma\, \mathrm{I\!I}(\sigma_u, \sigma_u) + (\alpha\delta + \beta\gamma)\, \mathrm{I\!I}(\sigma_u, \sigma_v) + \beta\delta\, \mathrm{I\!I}(\sigma_v, \sigma_v) \end{aligned}$$

By lemma 2.2, this is equal to

$$\begin{aligned} &= +\alpha\gamma\langle N(p), \sigma_{uu}\rangle + (\alpha\delta + \beta\gamma)\langle N(p), \sigma_{uv}\rangle + \beta\delta\langle N(p), \sigma_{vv}\rangle \\ &= -\alpha\gamma\langle N_u(p), \sigma_u\rangle - (\alpha\delta + \beta\gamma)\langle N_u(p), \sigma_v\rangle - \beta\delta\langle N_v(p), \sigma_v\rangle \\ &= \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} -\langle N_u(p), \sigma_u\rangle & -\langle N_v(p), \sigma_u\rangle \\ -\langle N_u(p), \sigma_v\rangle & -\langle N_v(p), \sigma_v\rangle \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\ &= \mathbf{w}_1^T \begin{pmatrix} -\langle N_u(p), \sigma_u\rangle & -\langle N_v(p), \sigma_u\rangle \\ -\langle N_u(p), \sigma_v\rangle & -\langle N_v(p), \sigma_v\rangle \end{pmatrix} \mathbf{w}_2 \end{aligned}$$

Comparing this with the expression in (2), we obtain the desired result.

□

2.4 Principal, Mean, and Gaussian Curvatures

In what follows, $\mathcal{S}$ is a regular and oriented surface with Gauss map N and Weingarten map $\mathcal{W}_{p,\mathcal{S}}$ at the point $p \in \mathcal{S}$.[8]

Definition 2.5 (Principal Curvatures)**.** *The* principal curvatures *of $\mathcal{S}$ at p are denoted by κ_1 and κ_2, and are precisely the eigenvalues of $\mathcal{W}_{p,\mathcal{S}}$. By* corollary (2.2)*, they are both real numbers.*

Definition 2.6 (Mean Curvature)**.** *The* mean curvature *of $\mathcal{S}$ at p is denoted by κ_H and is defined by:*[9]

$$\kappa_H := \frac{1}{2}\,\mathrm{trace}(\mathcal{W}_{p,\mathcal{S}}) = \frac{\kappa_1 + \kappa_2}{2}$$

Definition 2.7 (Gaussian Curvature)**.** *The* Gaussian curvature *of $\mathcal{S}$ at p is denoted by κ_G and is defined by:*

$$\kappa_G := \det(\mathcal{W}_{p,\mathcal{S}}) = \kappa_1\kappa_2$$

Note that we have not specified a choice of basis for the Weingarten map. Since this map is a linear endomorphism, it turns out that both its trace and determinant are independent of the choice of basis. This ensures that the curvatures defined are indeed unique.

When first defining the Weingarten map, we noted that the minus sign in its definition was a matter of convention. Here we see that, as far as the Gaussian curvature is concerned, it makes no difference. In fact, this important point allows us to define Gaussian curvature even for certain surfaces that are not orientable!

3 Gauss' Remarkable Theorem

In this section we work up to, and eventually prove, Gauss' celebrated theorem. First, we state two preliminary results. Their proofs involve several tedious—but not complicated—calculations, which we omit. The interested reader can find detailed proofs in [1], under propositions 7.4.4 and 10.1.2 respectively.

For simplicity, we assume throughout that $\mathcal{S} \subset \mathbb{R}^3$ is a regular and oriented surface with Gauss map $\mathbf{N}$. Moreover, we assume that the point $p \in \mathcal{S}$ is parametrised by σ, and that the first and second fundamental forms of $\mathcal{S}$ at p are given by (1) and (2) respectively.

[8] We have suppressed the point p from the notations used for the curvatures. This will be clear from the context when necessary.

[9] Although not the focus of this essay, the mean curvature is important in the study of so-called *minimal surfaces*, such as soap films.

Lemma 3.1. *Let $\mathcal{S}$ be the surface described above. The second partial derivatives of σ are given by the equations:*

$$\begin{aligned}\sigma_{uu} &= \Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + L\mathbf{N}\\ \sigma_{uv} &= \Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_v + M\mathbf{N}\\ \sigma_{vv} &= \Gamma^1_{22}\sigma_u + \Gamma^2_{22}\sigma_v + N\mathbf{N},\end{aligned}$$

where the Γ^k_{ij} are the six Christoffel[10] symbols*, which are defined by:*

$$\begin{aligned}\Gamma^1_{11} &:= \frac{GE_u - 2FF_u + FE_u}{2(EG-F^2)} & \Gamma^2_{11} &:= \frac{2EF_u - EE_u - FE_u}{2(EG-F^2)}\\ \Gamma^1_{12} &:= \frac{GE_v - FG_u}{2(EG-F^2)} & \Gamma^2_{12} &:= \frac{EG_u - FE_v}{2(EG-F^2)}\\ \Gamma^1_{22} &:= \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)} & \Gamma^2_{22} &:= \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}\end{aligned}$$

Before we move on, let's make a few remarks about this lemma. First, observe that $\{\sigma_u, \sigma_v, \mathbf{N}\}$ is a basis of $\mathbb{R}^3$. This lemma thus allows us to find explicit expressions for the coefficients of $\sigma_{uu}, \sigma_{uv} = \sigma_{vu}$ and σ_{vv} in this basis. Second, the Christoffel symbols are only defined when $EG - F^2 \neq 0$, that is, when the first fundamental form has non-zero determinant. If, however, $EG - F^2$ *is* equal to zero, one can show that the Gaussian curvature at the point p is equal to zero. Finally, the Christoffel symbols only depend on the coefficients of the first fundamental form (and their partial derivatives). This fact will become important later.
We also establish the following set of equations:

Lemma 3.2. *Let $\mathcal{S}$ be the surface described above, and let κ_G be its Gaussian curvature at the point $p \in \mathcal{S}$. We have that:*

$$\begin{aligned}\kappa_G E &= (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2\\ \kappa_G F &= (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^1_{22}\\ \kappa_G F &= (\Gamma^2_{12})_v - (\Gamma^2_{22})_u + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^1_{22}\\ \kappa_G G &= (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{11}\Gamma^1_{22} + \Gamma^1_{12}\Gamma^2_{22} - \Gamma^2_{12}\Gamma^1_{22} - (\Gamma^1_{12})^2.\end{aligned}$$

With the above lemmas, we are now ready to state and prove Gauss' Remarkable Theorem!

Theorem 3.1 (Theorema Egregium)**.** *The Gaussian curvature of a regular and orientable surface is an intrinsic property of that surface.*

[10]Named after the German mathematician Elwin Bruno Christoffel (1829–1900). Among his many contributions was the laying out of the mathematical foundation for general relativity.

Proof. If $E = F = G = 0$, then the determinant of I at p equals zero, so $\kappa_G = 0$. Suppose that at least one of E, F, G is non-zero, and consider the corresponding equation established in lemma (3.2). Next, substitute the expressions for the Christoffel symbols defined in lemma (3.1) and solve for κ_G. Since the Christoffel symbols only depend on the coefficients of I, the same is true for κ_G. Therefore, κ_G is an intrinsic property of the surface, according to the definition in section 1.3. $\square$

One might wonder if we can get multiple expressions for the Gaussian curvature using different equations from lemma (3.2). It turns out that all of these are equal to each other. This unique expression is known as the *Brioschi*[11] *formula*, and is given by:

$$\kappa_G = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}$$

4 Applications & Conclusion

In the final section of this essay, we present two interesting applications of Gauss' theorem. These illustrate the wide-ranging applicability of this theorem, both in pure and applied mathematics.

4.1 Maps of the Earth

We use the unit sphere $\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ to model the Earth, and the $xy-$plane $\Pi_z := \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ to model a flat sheet of paper.
Let $\sigma_1(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$ and $\sigma_2(u, v) = (u, v, 0)$. We assign Gauss maps to both surfaces as follows:

$$N_1(p) := p \qquad \text{and} \qquad N_2(p) := (0, 0, 1)$$

Next, we find the matrices for the Weingarten maps. Consider $\mathbb{S}^2$ first: let γ_1 be an arbitrary smooth curve on $\mathbb{S}^2$ such that $\gamma_1(0) = p$ and $\gamma_1'(0) = \mathbf{v}$. It follows that:

$$\mathcal{W}_{p,\mathbb{S}^2}(\mathbf{v}) = -\frac{d}{dt}N_1(\gamma_1(t))\Big|_{t=0} = -\gamma_1'(0) = -\mathbf{v}.$$

Therefore,

$$\mathcal{W}_{p,\mathbb{S}^2}(\mathbf{v}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{v}.$$

[11]Named after the Italian mathematician Francesco Brioschi (1824–1897).

Now for Π_z, note that N_2 is constant, so the matrix for $\mathcal{W}_{p,\Pi_z}$ will be the zero matrix:

$$\mathcal{W}_{p,\Pi_z}(\mathbf{v}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0}.$$

Therefore, the Gaussian curvatures will be equal to 1 and 0, for the sphere and the plane respectively.

Suppose there were a local isometry between $\mathbb{S}^2$ and Π_z. Both surfaces would then have the same first fundamental form. Therefore, they would also have the same Gaussian curvature, by Gauss' Theorema Egregium. This is a contradiction, so the two surfaces are not locally isometric.

This explains why it's not possible to make a map of the Earth on a flat sheet of paper which preserves distances between all points.

4.2 The Three Classical Geometries

In the previous application, we saw that the unit sphere has Gaussian curvature equal to 1, and that a plane has Gaussian curvature equal to 0. Now, consider the 2-dimensional hyperbolic plane defined by:[12]

$$\mathcal{H}^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1 \wedge z > 0\}.$$

Following a similar calculation as before, we can deduce that this surface has constant Gaussian curvature equal to -1.

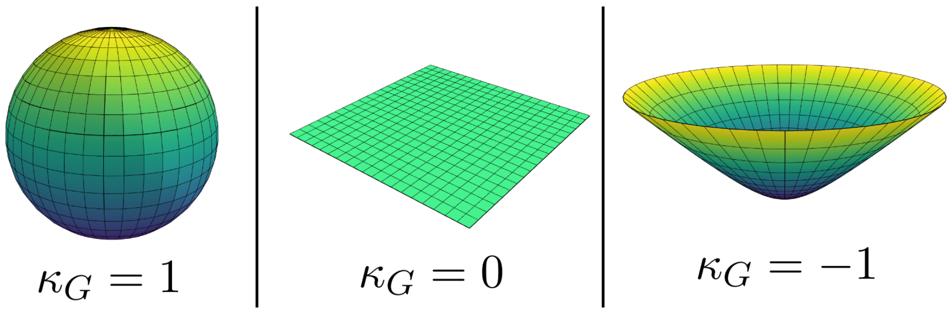

Figure 3: The three types of curvature

The three surfaces are pictured above. The three values for curvature create a trichotomy of sorts: surfaces with positive Gaussian curvature are called *spherical*, those with zero Gaussian curvature are called *developable* (or *parabolic*), and those with negative Gaussian curvature are called *hyperbolic*.

The concept of Gaussian curvature may be generalised to higher-dimensional manifolds. In this broader framework, this trichotomy forms the basis for the

[12] In the study of quadratic forms, this set is the upper-half of a *hyperboloid of two sheets*.

three classical geometries: spherical, Euclidean, and hyperbolic geometry respectively.

Appendix A Necessary Background

In this first appendix, we provide a brief review of several basic definitions and results that should be familiar from first- and second-year modules. Additionally, we establish a consistent notation that will be used throughout this essay.

A.1 Linear and Multilinear Algebra

Let V be a vector space over some ground field K.

Definition A.1. *A* bilinear form *on V is a map $\tau : V \times V \to K$ satisfying the following two properties:*

(i) $\tau(\alpha v_1 + \beta v_2, v) = \alpha\tau(v_1, v) + \beta\tau(v_2, v)$,

(ii) $\tau(v, \alpha v_1 + \beta v_2) = \alpha\tau(v, v_1) + \beta\tau(v, v_2)$

for all $v, v_1, v_2 \in V$ and for all $\alpha, \beta \in K$.

Especially useful in our study of surfaces are the so-called symmetric bilinear forms:

Definition A.2. *A bilinear form is called* symmetric *if, in addition to properties (i) and (ii), it also satisfies:*

(iii) $\tau(v_1, v_2) = \tau(v_2, v_1)$

for all $v_1, v_2 \in V$.

Given a bilinear form, one can then define the corresponding quadratic form as follows:

Definition A.3. *Let $\tau : V \times V \to K$ be a bilinear form. The corresponding* quadratic form *is the map $q : V \to K$ defined by:*

$$q(x) := \tau(x, x)$$

A.2 Topology

Definition A.4. *A set $U \subseteq \mathbb{R}^n$ is defined to be* open *if $\forall\, x \in U,\ \exists\, \varepsilon > 0$ such that $y \in \mathbb{R}^n \wedge |x - y| < \varepsilon \implies y \in U$.*

Definition A.5. *A set $X \subseteq \mathbb{R}^n$ is defined to be* closed *if the set $\mathbb{R}^n \setminus X$ is open.*

Definition A.6. *Two topological spaces* $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$ *are said to be* homeomorphic *if there exists a function* $f : X \to Y$*, called a* homeomorphism*, such that* f *is continuous, invertible, and* $f^{-1} : Y \to X$ *is also continuous.*

A.3 Multivariable Calculus and Differential Geometry

Definition A.7. *Let* U *be an open subset of* $\mathbb{R}^n$*, and let* $f : U \to \mathbb{R}^k$ *be a vector-valued function defined by* $f(x) := (f_1(x), \ldots, f_n(x))^T$*, whose partial derivatives exist and are continuous. The* Jacobian matrix *of* f *at* x *is then denoted by* $\partial f(x)$ *and defined by:*

$$\partial f(x) := \begin{pmatrix} \partial_1 f_1(x) & \cdots & \partial_n f_1(x) \\ \vdots & & \vdots \\ \partial_1 f_k(x) & \cdots & \partial_n f_k(x) \end{pmatrix}$$

Moreover, the *Jacobian determinant* is defined to be the determinant of the Jacobian matrix.

Theorem A.1 (Clairaut's Theorem)**.** *Let* U *be an open subset of* $\mathbb{R}^n$*, and let* $f : U \to \mathbb{R}$ *be a function whose partial derivatives up to order 2 all exist and are continuous. Then* $\partial_i \partial_j f(x) = \partial_j \partial_i f(x)$ *for all* $i, j \in \{1, \ldots, n\}$*.*

Definition A.8. *Let* $I \subseteq \mathbb{R}$ *be an interval, and let* $\gamma : I \to \mathbb{R}^k$ *be a continuous vector-valued function defined by* $\gamma(t) := (\gamma_1(t), \ldots, \gamma_k(t))$*, where each component* γ_i *is a continuous real-valued function. Then the image of* γ*, denoted by* $\operatorname{im}(\gamma)$*, is called a* curve *in* $\mathbb{R}^k$*, and the function* γ *is called its* parametrisation.[13]

Theorem A.2 (Frenet–Serret Formulæ)**.** *For a unit-speed curve in* $\mathbb{R}^3$ *with non-zero curvature, we have the following system of equations:*

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix},$$

where $\mathbf{T}, \mathbf{N}$*, and* $\mathbf{B}$ *are the* unit tangent, principal normal, and binormal *vectors respectively, and* κ *and* τ *are the* curvature *and* torsion*, respectively. Together,* $\mathbf{T}, \mathbf{N}$*, and* $\mathbf{B}$ *form an orthonormal basis of* $\mathbb{R}^3$*.*

[13] For practical purposes, we may refer to γ as both the parametrisation and the curve itself. In such cases, we really mean that $\operatorname{im}(\gamma)$ is the curve. That is, we think of curves as subsets of $\mathbb{R}^k$.

Appendix B Intrinsic Properties

In section 1.3, we defined the intrinsic properties of a surface to be those that are completely determined by the first fundamental form. In this second appendix, we present some examples of such properties and explain why they are so important.

B.1 Lengths of Curves

Recall that the length of a smooth curve parametrised regularly by $\gamma : [a, b] \to \mathbb{R}^3$ is given by the integral

$$\int_a^b |\gamma'(t)| dt.$$

Suppose now that this curve lies on a regular surface $\mathcal{S}$, parametrised by σ. We can then write

$$\gamma(t) = \sigma(u(t), v(t)),$$

from which it follows that

$$\gamma'(t) = \sigma_u(t)u'(t) + \sigma_v(t)v'(t).$$

Hence,

$$\begin{aligned}|\gamma'(t)|^2 &= \langle\gamma'(t), \gamma'(t)\rangle \\ &= \langle\sigma_u, \sigma_u\rangle(u'(t))^2 + 2\langle\sigma_u, \sigma_v\rangle u'(t)v'(t) + \langle\sigma_v, \sigma_v\rangle(v'(t))^2.\end{aligned}$$

Therefore, the length of the curve is equivalently given by

$$\int_a^b \left[E(u'(t))^2 + 2Fu'(t)v'(t) + G(v'(t))^2\right]^{1/2} dt.$$

Since this expression only depends on the coefficients of I, we conclude that the length of a curve on a surface is an intrinsic property.

B.2 Areas of Regions

Let $\mathcal{S}$ be a regular surface. Recall again that the area of a region $\mathcal{S} \cap W$ parametrised by $\sigma : U \to \mathcal{S} \cap W$ is given by the surface integral

$$\iint_U |\sigma_u \times \sigma_v| dudv.$$

We use the following identity from vector calculus:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

Suppose now that $\mathbf{a} = \mathbf{c} = \sigma_u$ and $\mathbf{b} = \mathbf{d} = \sigma_v$. It follows that

$$|\sigma_u \times \sigma_v|^2 = EG - F^2$$

Therefore, the length of the region $\mathcal{S} \cap W$ is equivalently given by

$$\iint_U \sqrt{EG - F^2} dudv.$$

Once again, this expression only depends on the coefficients of I, so the area of a region on a surface is also an intrinsic property.
The above identity also shows that $EG - F^2 \geq 0$, so no issues arise when considering its square root.

B.3 Angles Between Curves

Consider a regular surface $\mathcal{S}$ and two intersecting curves, γ_1 and γ_2, which lie on $\mathcal{S}$. Suppose that their point of intersection, say p, is parametrised by σ. We wish to find the angle at which these two curves intersect. This is equal to the angle between the two tangent vectors at p. Moreover, these tangent vectors can be expressed as a linear combination of σ_u and σ_v, so it suffices to find the angle between these vectors. Let this angle be θ. We have:

$$\begin{aligned} \langle \sigma_u, \sigma_v \rangle &= |\sigma_u||\sigma_v| \cos\theta \\ \implies F &= \sqrt{EG} \cos\theta \end{aligned}$$

Since $|\sigma_u|, |\sigma_v| \neq 0$,

$$\cos\theta = \frac{F}{\sqrt{EG}},$$

and we are done. This last assumption is justified because the concept of an angle is only well-defined for non-zero vectors.
Note that theorem (1.1) implies that these three properties are preserved between locally isometric surfaces. This often gives us elegant ways of calculating them; for example, by comparing one surface with a second one which is locally isometric, but simpler to work with.

This essay draws inspiration from the following sources, but does not take the approach of any one in particular.

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All images and diagrams were produced using the Python programming language.