

Homework 5

Question 1. [30 marks] Prove that (Q3) is an obsolete axiom. Namely, solve Exercise 3.1.3, from **Week 5** of the notes.

[*Hint.* You are allowed to use instances of propositional tautologies as “propositional axioms”. The following tautologies may be useful $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$ and $(A \rightarrow B) \rightarrow ((B \leftrightarrow C) \rightarrow (A \rightarrow C))$.]

Question 2. [40 marks] Write the proof of the Soundness Theorem. This involves the following steps:

- (1) Exercise 2.1.1,
- (2) Exercise 2.2.1
- (3) Proving that the deduction rules are sound, namely, for any sentence ϕ :
 - If $T \models \phi$ and $T \models \phi \rightarrow \psi$ then $T \models \psi$.
 - If $T \models \phi$ then $T \models (\forall x)\phi$.

[*Hint.* The second item here is not that hard.]

Question 3. [30 marks] Let \mathcal{L} be a language and $F(\mathcal{L})$ the set of all \mathcal{L} -formulas.

- (1) Show that there exists at least one map $\text{val} : F(\mathcal{L}) \rightarrow \{T, F\}$ such that:
 - (a) If $\phi \in F(\mathcal{L})$ is of the form $(\forall x)\psi$ then $\text{val}(\phi) = F$.
 - (b) If $\phi \in F(\mathcal{L})$ is of the form $(\exists x)\psi$ then $\text{val}(\phi) = T$.
 - (c) If $\phi \in F(\mathcal{L})$ is of the form $\neg\psi$ then $\text{val}(\phi) = f_{\neg}(\text{val}(\psi))$.
 - (d) If $\phi \in F(\mathcal{L})$ is of the form $\psi \Box \chi$ then $\text{val}(\phi) = f_{\Box}(\text{val}(\psi), \text{val}(\chi))$, where $\Box \in \{\wedge, \vee, \rightarrow\}$.

Now, let val be as above.

- (2) Show that if ϕ is an instance of an axiom (A1)-(A3), (Q1), (Q2), (Q4) (we have already seen (Q3) is obsolete), then $\text{val}(\phi) = T$.
- (3) Show that if $\text{val}(\phi \rightarrow \chi) = T$ and $\text{val}(\phi) = T$, then $\text{val}(\chi) = T$.
- (4) Show that if there exists a derivation of ϕ which does not use (Gen), then $\text{val}(\phi) = T$. Conclude that there are formulas that are derivable but are not derivable without (Gen).

- (5) *Extra:* Can you adapt the argument above to show that the axiom (Q2) is indispensable?¹

¹Explaining perhaps typos in a previous version of this HW.