Practice Final Exam 1

This exam consists of 6 questions. The first four questions count for 80% of the exam and cover material you have seen in the lectures. Questions 5 and 6 count only for 20% of the exam and *build* on the material that you have seen in class. I recommend you start with the first four questions.

Question 1 – Basic Definitions

(a) [5 marks] Let T be a first-order $\mathcal{L}$-theory, and ϕ an $\mathcal{L}$-sentence. Define what $T \vdash \phi$ means.

(b) [10 marks] Give examples of first-order theories T_1, T_2 and first-order sentences ϕ_1, ϕ_2 such that: $T_1 \vDash \phi_1$ and $T_2 \nvDash \phi_2$. [If your examples are from the course, you do not need to justify them. Otherwise, you do.]

(c) [5 marks] Let T be a first-order $\mathcal{L}$-theory. Define the following terms:

- T is *complete*.
- T is *decidable*.

(d) [20 marks] Give examples of (i) a complete theory, (ii) an incomplete theory, (iii) a decidable theory, (iv) an undecidable theory. Briefly justify your answers.

Question 2 – Completeness and Compactness

For part (c) of this question, you may assume that $\mathcal{L}$ be a countable language.

(a) [5 marks] State the *Gödel's Completeness theorem* of first-order logic.

(b) [15 marks] Briefly discuss the main ideas involved in the proof of the Gödel's Completeness theorem.

(c) [5 marks] State the *Compactness theorem* of first-order logic.

(e) [15 marks] You may assume the compactness theorem for arbitrary (i.e. not necessarily countable) languages. Prove that there is no first-order theory T such that every model of T is uncountable. [You may **not** use the Löwenheim-Skolem Theorems in this question.]

Question 3 – Recursion Theory

(a) [5 marks] Define the sets of primitive recursive and recursive functions.

(b) [10 marks] Prove (directly from the definition) that the function $f : \mathbb{N}^2 \to \mathbb{N}$ sending (x, y) to $x + y$ is primitive recursive.

(c) [5 marks] Define what it means for a set $A \subseteq \mathbb{N}$ to be (i) recursive and (ii) recursively enumerable.

(d) [20 marks] Prove that every recursive set is recursively enumerable, but not vice versa. [You may use the results we proved during the course, other than the Halting problem.]

Question 4 – Incompleteness

(a) [5 marks] Let $\mathcal{N} \models T_{PA_0}$. Define what it means for an element of the base set N to be standard.

(b) [15 marks] Prove that there are models of T_{PA_0} with non-standard elements. Argue that if $\mathcal{N} \models T_{PA}$ contains at least one non-standard element, then it contains infinitely many.

(c) [5 marks] Define what it means for a function $f : \mathbb{N} \to \mathbb{N}$ to be representable. State the relationship between representable functions and recursive functions?

(d) [10 marks] Prove that if $f, g : \mathbb{N} \to \mathbb{N}$ are representable functions then so is $f \circ g : \mathbb{N} \to \mathbb{N}$ (this is the map sending x to $f(g(x))$).

(e) [5 marks] State Gödel's first incompleteness theorem.

Question 5

(a) [10 marks] Recall that the finite spectrum of an $\mathcal{L}$-sentence ϕ is the set

$$\mathsf{Sp}(\phi) = \{n \in \mathbb{N} : \text{ there is an } \mathcal{L}\text{-structure } \mathcal{M} \text{ of size } n \text{ s.t. } \mathcal{M} \models \phi\}$$

i. Is there a first-order sentence whose finite spectrum is the set of all non-zero composite numbers? Justify your answer.

ii. Let ϕ be a first-order sentence. Prove that if $\mathsf{Sp}(\phi)$ is infinite, then ϕ has at least one infinite model.

(b) [10 marks] Let $\mathcal{L}$ be a first-order language, and write Fml_1 for the set of $\mathcal{L}$-formulas in at least one free variable. Given an $\mathcal{L}$-structure $\mathcal{M}$ and some $a \in M$, the *type* of a in $\mathcal{M}$ is the following set:

$$\mathsf{tp}^{\mathcal{M}}(a) := \{\theta(x) : \theta(x) \in \mathsf{Fml}_1, \text{ and } \mathcal{M} \models \theta(a)\}$$

i. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. Suppose that for all $a, a' \in A$ we have that $\mathsf{tp}^{\mathcal{M}}(a) = \mathsf{tp}^{\mathcal{M}}(a')$. Show that for all formulas $\phi(x) \in \mathsf{Fml}_1$, the set:

$$\phi(M) := \{b \in M : \mathcal{M} \models \phi(b)\}$$

either contains A or is disjoint from A.

ii. Let $\mathcal{L}$ be a countable language. Let T be an $\mathcal{L}$-theory with at least one infinite model. Prove that there is a model $\mathcal{M} \models T$ such that M contains at least two elements that have the same type.

[*Hint.* If $\mathcal{L}$ is a countable language, show that (for any $\mathcal{M} \models T$) the number of possible types of elements of $\mathcal{M}$ is at most $2^{\aleph_0}$. Then use Löwenheim-Skolem and the Pigeonhole principle.]

Question 6

(a) [10 marks] Let $\mathcal{M} \vDash T_{PA}$ be a non-standard model and $\phi(x)$ an $\mathcal{L}_{Peano}$-formula. Show that if $\mathcal{M} \vDash \phi(\underline{n})$ for all $n \in \mathbb{N}$, then there is a non-standard $c \in M$ such that $\mathcal{M} \vDash \phi(c)$.

[*Hint.* It is important here that $\mathcal{M} \vDash T_{PA}$ and not just T_{PA_0}.]

(b) [10 marks] Let $\mathcal{L}$ be a language with a single unary function symbol f and T an $\mathcal{L}$-theory asserting that f is a bijection with no cycles. Find two countable models of T which are not isomorphic.