

## Homework 4

**Question 1.** [30 marks] Solve the following exercises from the **Week 4** notes to complete the proof of the Substitution Lemma:

- (1) Exercise 3.2.6.
- (2) Exercise 3.2.8.

**Question 2.** [20 marks]. Prove Lemma 3.2.9, namely, show that if  $\phi$  is an  $\mathcal{L}$ -formula,  $y \in \text{Var} \setminus \text{Var}(\phi)$  and  $x \in \text{Var}$ , then  $(\phi[y/x])[x/y]$  is equal to  $\phi$ .

**Question 3.** [40 marks] Let  $\mathcal{L}$  be a first-order language,  $\mathcal{M} = (M; \dots)$  an  $\mathcal{L}$ -structure and  $N \subseteq M$  such that:

- For all  $\underline{c} \in \text{Const}(\mathcal{L})$  we have that  $c^{\mathcal{M}} \in N$ .
- For all  $n$ -ary  $\underline{f} \in \text{Fun}(\mathcal{L})$  and all  $a_1, \dots, a_n \in N$  we have  $f^{\mathcal{M}}(a_1, \dots, a_n) \in N$ .

Define an  $\mathcal{L}$ -structure on  $N$  by setting:

- $c^{\mathcal{N}} = c^{\mathcal{M}}$ , for all  $\underline{c} \in \text{Const}(\mathcal{L})$ ;
- $f^{\mathcal{N}} = f^{\mathcal{M}} \upharpoonright_{N^n}$ , for all  $n$ -ary  $\underline{f} \in \text{Fun}(\mathcal{L})$ ;
- $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^n$  for all  $n$ -ary  $\underline{R} \in \text{Rel}(\mathcal{L})$ .

You are asked to do the following:

- (1) Briefly explain why  $\mathcal{N} = (N; \dots)$  is an  $\mathcal{L}$ -structure.
- (2) Show that if  $A \subseteq M$  is non-empty then there is a unique substructure  $\langle A \rangle$  of  $\mathcal{M}$  whose universe contains  $A$  and such that every substructure of  $\mathcal{M}$  whose universe contains  $A$  also contains the universe of  $\langle A \rangle$ . We call  $\langle A \rangle$  the **substructure generated by  $A$** .
- (3) Show that if  $A = \emptyset$  there may not exist such a substructure of  $\mathcal{M}$ . Give an example, however, in which it exists.
- (4) Suppose that  $\mathcal{L}$  is **relational** (i.e. it contains no function symbols). What is the universe of  $\langle A \rangle$ ?
- (5) We say that a substructure  $\mathcal{N}$  of  $\mathcal{M}$  is **finitely generated** if it is of the form  $\langle A \rangle$  for a finite set  $A \subseteq M$ . Let  $\phi$  be a formula of the form  $(\forall x)\psi$ . Show that  $\mathcal{M} \models \phi$  if and only if for every finitely generated substructure  $\mathcal{N}$  of  $\mathcal{M}$  we have that  $\mathcal{N} \models \phi$ .
- (6) Show that the result from 5 need not hold when  $\phi$  is of the form  $(\exists x)\psi$ .

**Question 4.** [10 marks] Let  $\mathcal{L}$  be a language and  $\phi$  an  $\mathcal{L}$ -sentence. The **finite spectrum** of  $\phi$  is the set  $\{n \in \mathbb{N} : \text{there is } \mathcal{M} = (M; \dots) \models \phi, |M| = n\}$ , i.e. the set of cardinalities of finite models of  $\phi$ . For each of the following subsets of  $\mathbb{N}$  find a formula whose spectrum is exactly that set, or explain why it's not possible.

- (1)  $\emptyset$ .
- (2)  $\mathbb{N}$ .
- (3)  $\mathbb{N}_{>1}$
- (4)  $\{n\}$ , for  $n \in \mathbb{N}_{>1}$