CHAPTER 4

Sounds like things are complete (Cont'd)

5. Building models

5.1. Henkin Models (Cont'd). Now for the second step:

Proposition 5.1.5. *Let T be a consistent $\mathcal{L}$-theory, then, there is a language $\mathcal{L}' \supseteq \mathcal{L}$ and an $\mathcal{L}'$-theory $T' \supseteq T$ such that:*

(1) T' is has Henkin witnesses;

(2) T' is complete.

Proof. Recall that we assume that our language $\mathcal{L}$ is countable. In particular, the set of all $\mathcal{L}$-formulas is also countable. Let $C = \{\underline{c}_1, \underline{c}_2, \dots\}$ be countable set of "fresh" constant symbols (i.e. $C \cap \mathsf{Const}(\mathcal{L}) = \emptyset$). Clearly, the language $\mathcal{L} \cup C$ is still countable, hence we can enumerate all the $\mathcal{L} \cup C$-formulas. Fix an enumeration $\phi_1, \phi_2, \phi_3, \dots$. We build a chain $T_0 \subseteq T_1 \subseteq \cdots$ of theories in respective languages $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \cdots$ such that:

- T_n is consistent, as an $\mathcal{L}_n$-theory.
- $\phi_n \in T_{n+1}$ or $\neg\phi_n \in T_{n+1}$.
- $\mathcal{L}_{n+1} \setminus \mathcal{L}$ is finite.
- If ϕ_n is of the form $\exists x \psi$ and $\phi_n \in T_{n+1}$, then there is some $\underline{c} \in \mathcal{L}_{n+1}$ such that $\psi[\underline{c}/x] \in T_{n+1}$.

To get us started, we set $T_0 = T$ and $\mathcal{L}_0 = \mathcal{L}$.

Once T_n and $\mathcal{L}_n$ have been defined, we construct T_{n+1} and $\mathcal{L}_{n+1}$ as follows:

- Step 1. Let $C_n \subseteq \mathsf{Const}(\mathcal{L}) \cup C$ be the constants that appear in ϕ_{n+1}. Then $\mathcal{L}'_{n+1} = \mathcal{L}_n \cup C_n$.
- Step 2. T_n is consistent as an $\mathcal{L}'_{n+1}$-theory. We've already shown this in Corollary 4.2.3, since by induction hypothesis, T_n is consistent as an $\mathcal{L}_n$-theory.

- Step 3. If $T_n \cup \{\phi_{n+1}\}$ is consistent as an $\mathcal{L}'_{n+1}$-theory, then we set $T'_{n+1} = T_n \cup \{\phi_{n+1}\}$, and jump to Step 5.
- Step 4. If $T_n \cup \{\phi_{n+1}\}$ is inconsistent as an $\mathcal{L}'_{n+1}$-theory, then, by Corollary 4.1.8, $T \vdash \neg\psi$. In this case we set $T_{n+1} = T_n \cup \{\neg\phi_n\}$, $\mathcal{L}_{n+1} = \mathcal{L}'_{n+1}$ and stop here.
- Step 5. If $\phi_n \in T'_{n+1}$ and ϕ_{n+1} is not of the form $(\exists x)\psi$ we set $T_{n+1} = T_n \cup \{\neg\phi_n\}$, $\mathcal{L}_{n+1} = \mathcal{L}'_{n+1}$ and stop here, otherwise we go on to Step 6.
- Step 6. If $\phi_n \in T'_{n+1}$ and ϕ_{n+1} is of the form $(\exists x)\psi$ we choose a constant symbol $\underline{c}$ in $(\mathcal{L} \cup C) \setminus \mathcal{L}'_{n+1}$ (which is possible because C is infinite and we have made sure that $\mathcal{L}_n$ and hence $\mathcal{L}'_{n+1}$ are finite expansions of $\mathcal{L}$. Then we set: $\mathcal{L}_{n+1} = \mathcal{L}'_{n+1} \cup \{\underline{c}\}$ and $T_{n+1} := T'_{n+1} \cup \{\psi[\underline{c}/x]\}$.

At this point we're almost done with our construction, but we still have to prove that T_{n+1} is consistent as an $\mathcal{L}_{n+1}$-theory. This is obvious, unless we've ended up here after going through Step 6. Suppose then that we have gone through Step 6 and ended up with an inconsistent $\mathcal{L}_{n+1}$-theory. Then, by Corollary 4.1.8 we have that:

$$T_n \cup \{\exists x\psi\} \vdash_{\mathcal{L}_{n+1}} \neg\psi[\underline{c}/x].$$

But then, by Lemma 4.2.1 we have that:

$$T_n \cup \{\exists x\psi\} \vdash_{\mathcal{L}'_{n+1}} (\forall x)\neg\psi.$$

But, by (Q4) and (MP) we have that:

$$T_n \cup \{\exists x\psi\} \vdash_{\mathcal{L}'_{n+1}} \neg(\forall x)\neg\psi,$$

which means that $T_n \cup \{\exists x\psi\} = T_n \cup \{\phi_{n+1}\} = T'$ is inconsistent, as an $\mathcal{L}'_n$-theory, contradicting Step 3.

So the induction goes through! Now, let $\mathcal{L}' = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n$ and $T' = \bigcup_{n\in\mathbb{N}} T_n$. We claim that:

- T' is consistent as an $\mathcal{L}'$-theory: This was Exercise 4.1.7.
- T' is complete: Let ψ be an $\mathcal{L}'$-sentence. Then, it is an $\mathcal{L} \cup C$-sentence, so $\psi = \phi_n$ for some $n \in \mathbb{N}$. Thus, either $\psi \in T_n$ or $\neg\psi \in T_n$.
- T' has Henkin witnesses: Let ψ be an $\mathcal{L}'$-formula in one free variable x. Then, $(\exists x)\phi$ is an $\mathcal{L} \cup C$-sentence, and is thus ϕ_n for some $n \in \mathbb{N}$. Then, we either have that $(\exists x)\psi, \psi[c/x] \in T_{n+1}$ or $\neg(\exists)\psi \in T_{n+1}$. In either case, we have that $T_{n+1} \vdash_{\mathcal{L}_{n+1}} (\exists x)\psi \implies \psi[c/x]$, which (by completeness and consistency of T') means that $T' \vdash_{\mathcal{L}'} (\exists x)\psi \implies \psi[c/x]$. $\square$

5.2. Putting our blocks and our models together. Now we have pretty much everything we could have asked for:

PROOF OF GÖDEL'S COMPLETENESS THEOREM. Given a consistent $\mathcal{L}$-theory T we use Proposition 5.1.5 to build a complete consistent $\mathcal{L}'$-theory T' which has Henkin witnesses. Then, using Proposition 5.1.3, we can build a model of T'. The **reduct** (i.e. the structure obtained by forgetting the symbols in $\mathcal{L}' \setminus \mathcal{L}$) is a model of T. (This is the proof of Adequacy, from here, we conclude as we did in the propositional case.) $\square$

Exercise 5.2.1. There is a step missing! Prove that if $\mathcal{L}' \supseteq \mathcal{L}$ are languages, T is an $\mathcal{L}$-theory and $\mathcal{M}'$ is an $\mathcal{L}'$-structure such that $\mathcal{M}' \vDash T$, then $\mathcal{M} \vDash T$, where $\mathcal{M}$ is the reduct of $\mathcal{M}'$ to $\mathcal{L}$.

We will prove a compactness theorem, once again:

THEOREM 5.2.2 (Compactness). *Let T be an $\mathcal{L}$-theory. Then, the following are equivalent:*

(1) T has a model.

(2) Every finite subset of T has a model.

PROOF. One direction is obvious. For the other direction, observe that if every finite subset of T has a model, then every finite subset of T is consistent (essentially soundness) and therefore T is consistent, so T has a model (essentially completeness). $\square$

The first time I heard this theorem, two thoughts crossed my mind:

- Why is it called compactness?
- Why do we care?

Many people have given non-answers to the questions above, but I think Poizat's non-answer is one of the best:

"The compactness theorem [...] is due to Gödel, in fact [...], the theorem was for Gödel a simple corollary (we could even say an unexpected corollary, a rather strange remark!) of his "completeness theorem" of logic, in which he showed that a finite system of rules of inference is sufficient to express the notion of consequence. [...]

This unfortunate compactness theorem was brought in by the back door, and we might say that its original modesty still does it wrong in logic textbooks. In my opinion it is *a much more essential and primordial result (and thus also less sophisticated) than Gödel's completeness theorem [...]; it is an error of method to deduce it from the latter.*[1]

If we do it this way, it is by a very blind fidelity to the historic conditions that witnessed its birth. [...] This approach-deducing Compactness from the possibility of axiomatising the notion of deduction-once applied to the propositional calculus[2] *gives the strangest proof on record of the compactness of* 2^ω*!*

It is undoubtedly more "logical," but it is inconvenient, to require the student to absorb a system of formal deduction, ultimately quite arbitrary, which can be justified only much later when we can show that it indeed represents the notion of semantic consequence. We should not lose sight of the fact that the formalisms have no raison d'être except insofar as they are adequate for representing notions of substance."

That sure was a non-answer. Anyway...

Completeness completed – some summary

Let's summarise what's happened so far. We proved the following theorem:

THEOREM. Let $\mathcal{L}$ be a first-order language and T an $\mathcal{L}$-theory. For any $\mathcal{L}$-formula ϕ, the following are equivalent:

(1) $T \vDash \phi$.

(2) $T \vdash_{\mathcal{L}} \phi$.

The implication (2) $\implies$ (1) is the *Soundness Theorem* for $\vdash_{\mathcal{L}}$, and it was the "easy" implication. To prove it all, we had to do was check that every axiom of our proof system was universally valid (here we had to use the Substitution Lemma) and that the deduction rules of our proof system preserve universal validity.

The implication (1) $\implies$ (2) is the *Completeness Theorem* for $\vdash_{\mathcal{L}}$ and it was the really hard one. The crux of the proof was the following statement:

If T is consistent, then T is satisfiable.

In full, this says that:

If there is no $\mathcal{L}$-sentence ϕ such that $T \vdash_{\mathcal{L}} \phi$ and $T \vdash_{\mathcal{L}} \neg\phi$ then there is an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \vDash T$.

[1] Oops.

[2] Like we did in the previous part of the course!

To prove this we had to build a model for a consistent theory T. The steps we took were as follows:

(1) Step 1. The Deduction Theorem: $T \vdash_{\mathcal{L}} \phi \to \psi$ if and only if $T \cup \{\phi\} \vdash_{\mathcal{L}} \psi$.

(2) Step 2. Deduce from the Deduction Theorem that $T \vdash_{\mathcal{L}} \phi$ if and only if $T \cup \{\neg\phi\}$ is inconsistent.

(3) Step 3. Prove that reducts preserve satisfaction.

(4) Step 4. Prove that expansions by constants don't affect provability. More precisely $T \vdash_{\mathcal{L}} (\forall x)\phi$ if and only if $T \vdash_{\mathcal{L} \cup \{\underline{c}\}} \vdash \phi[\underline{c}/x]$, for any constant $\underline{c} \notin \mathsf{Const}(\mathcal{L})$.

(5) Step 5. Prove that consistency is preserved under expansions by constants, more precisely: If T is a consistent $\mathcal{L}$-theory, then T is a consistent $\mathcal{L} \cup C$ theory, where C is any set of constant symbols such that $C \cap \mathsf{Const}(\mathcal{L}) = \emptyset$.

(6) Step 6. Prove that complete theories with Henkin witnesses have models.

(7) Step 7. Prove that every consistent theory can be completed (in an expansion) to a complete theory with Henkin witnesses.

The Completeness Theorem is our main positive result. Things from now on will start getting more and more negative. This is a good point for you to make sure you understand the structure of the proof described above. Namely, for each step in the proof, write out the full statement that we proved, go back to the statement, read its proof and if there are steps missing, do them (see also HW6).

5.3. A note on uncountable languages. Recall that from the beginning of our journey through first-order logic we always assumed that the language $\mathcal{L}$ that we were working with was countable. The ONLY place where we used this was in the proof of Proposition 5.1.5, where we assumed that there is an enumeration of the formulas of $\mathcal{L}$ and did induction. Proposition 5.1.5 is still true when $\mathcal{L}$ is uncountable, but the proof needs to be amended.

The general fact that we can prove rather easily using Zorn's lemma is the following:

Proposition 5.3.1. *Let T be a consistent $\mathcal{L}$-theory. Then, there exists a consistent $\mathcal{L}$-theory Σ which is complete and contains T.*

PROOF. Let $\mathcal{T}$ be the following set:

$$\mathcal{T} := \{T' : T' \text{ is a consistent } \mathcal{L}\text{-theory containing } T\},$$

By assumption, $\mathcal{T}$ is non-empty (indeed, it contains T). Given any chain $T_1' \subseteq T_2' \subseteq \cdots$ in $\mathcal{T}$, its union is an upper bound for it (and is clearly still consistent, by Exercise 4.1.7). Thus, by Zorn's lemma, $\mathcal{T}$ has a maximal element Σ. Now, this maximal Σ must be complete. Indeed, let ϕ be any formula. It $\phi \notin \Sigma$ then $T' \cup \{\phi\}$ must be inconsistent, by maximality of Σ. But then, by the Deduction Theorem, $T' \vdash \neg\phi$. $\square$

Given this, one just has to run the inductive construction we carried out in the countable case transfinitely (with a fixed enumeration of formulas and constants). This will build an $\mathcal{L}'$-theory T' with Henkin witnesses that may or may not be complete. Invoking the lemma above gives a completion of T', and since in that lemma the language stayed the same, the completion we get still has Henkin witnesses.

This was a bit sketchy, but I invite you to think about it for at least 20 minutes.

End of digression

Homework 6