Maths 410 – Homework 1

Due Feb. 6, 2026 – Beginning of class.

Question 1. Prove that if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $r \in \mathbb{Q} \setminus \{0\}$ then $r + x, rx \in \mathbb{R} \setminus \mathbb{Q}$. Deduce that $\sqrt{12} \in \mathbb{R} \setminus \mathbb{Q}$. [*Hint.* $12 = 4 \times 3$.]

Question 2. Let $A \subseteq \mathbb{R}$ be non-empty and bounded below. Define $-A := \{-x \in \mathbb{R} : x \in A\}$. Show that $\inf(A) = -\sup(-A)$.

Question 3. Let $b \in \mathbb{R}_{>1}$ and $y \in \mathbb{R}_{>0}$. The goal of this question is to prove that there is a unique $x \in \mathbb{R}$ such that $b^x = y$ (this x is called the *logarithm of* y *base* b).

(1) Show that if $n \in \mathbb{N}_{>0}$, then $b^n - 1 \geq n(b-1)$. [*Hint.* You can prove this by induction on n, but there is also a direct proof.]

(2) Deduce that $b - 1 \geq n\left(b^{\frac{1}{n}} - 1\right)$.

(3) Deduce, now that if $t \in \mathbb{R}_{>1}$ and $n > \frac{b-1}{t-1}$, then $b^{\frac{1}{n}} < t$.

(4) Suppose that $w \in \mathbb{R}$ is such that $b^w < y$. Show that there is some $N \in \mathbb{N}$ such that if $n \geq N$ then $b^{w+\frac{1}{n}} < y$. [*Hint.* Use the previous part with $t = yb^{-w}$, recall that $b^{x+y} = b^x b^y$ (which is proved in the extra credit problem, below).]

(5) Suppose that $w \in \mathbb{R}$ is such that $b^w > y$. Show that there is some $N \in \mathbb{N}$ such that if $n \geq N$ then $b^{w-\frac{1}{n}} < y$. [*Hint.* Try $t = \frac{1}{y}b^{-w}$.]

(6) Let $A = \{w \in \mathbb{R} : b^w < y\}$. Show that $\sup(A)$ exists.

(7) Let A be as above. Show that $x = \sup(A)$ satisfies $b^x = y$. Deduce that this x is unique.

Extra Credit. Let $b \in \mathbb{R}_{>1}$.

(1) If $m, n, p, q \in \mathbb{Z}$ with $n, q \neq 0$, and $r = \frac{m}{n} = \frac{p}{q}$, prove that:

$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}.$$

Hence, it makes sense to define $b^r = (b^m)^{\frac{1}{n}}$.

(2) If $x \in \mathbb{R}$ let $B(x) := \{b^t : t \in \mathbb{Q} \text{ and } t \leq x\}$. Prove that, if $r \in \mathbb{Q}$ then $b^r = \sup B(r)$. Hence, it makes sense to define $b^x := \sup B(x)$.

(3) Prove that if $r, s \in \mathbb{Q}$ then $b^{r+s} = b^r b^s$. Argue that for all $x, y \in \mathbb{R}$ and all $t \in \mathbb{Q}$, if $t < x + y$ then there are $r, s \in \mathbb{Q}$ such that $r < x, s < y$ such that $t = r + s$. Finally, prove that $b^{x+y} = b^x b^y$, for any $x, y \in \mathbb{R}$.