Homework 2

Question 1. [30 marks] Solve the following exercises, from **Weeks 2 and 3(a)** of the notes:

- Exercise 2.3.4
- Exercise 2.4.19
- Exercise 1.5.5

In the following two exercises, you are asked to show that a proof system is *sound* but *incomplete*. This time we will be working with formulas that involve the logical connectives $\rightarrow, \vee$ and $\neg$.

The axioms of this new system K are the following:

(A1) $(\phi \rightarrow (\psi \rightarrow \phi))$

(A2) $((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)))$

(K3) $\phi \rightarrow (\phi \vee \psi)$

(K4) $\phi \rightarrow (\psi \vee \phi)$

(K5) $(\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \vee \psi) \rightarrow \chi))$

(K6) $(\phi \rightarrow \neg\psi) \rightarrow ((\phi \rightarrow \psi) \rightarrow \neg\phi))$

(K7) $\neg\phi \rightarrow (\phi \rightarrow \psi)$

The only deduction rule of this system is still MP.

Let Γ be a set of formulas. A **formal proof in** K of formula ϕ, from Γ is a finite sequence:

$$(\phi_1, \ldots, \phi_n)$$

of formulas such that $\phi_n = \phi$ and for each $i \leq n$, one of the following holds:

- $\phi_i \in \Gamma$;
- Or ϕ_i is an instance of (A1),(A2),(K3)-(K7);
- Or ϕ_i can be deduced from an instance of (MP) for some $j, k < i$.

If there is a formal proof in K of ϕ from Γ, we write $\Gamma \vdash_K \phi$. If $\Gamma = \emptyset$ we write $\vdash_K \phi$ and call ϕ a **theorem of** K.

Question 2. [30 marks] Prove the soundness theorem for this proof system (i.e. prove that if $\Gamma \vdash \phi$ then $\Gamma \vDash_K \phi$). [You don't need to show that (A1) and (A2) are instances of tautologies again.]

Question 3. [40 marks] We will use a method based on invariants to show that K is not complete. For a minute, we will change our semantics, and allow the logical connectives as functions from $\{F, M, T\}$ (to some power) to $\{F, M, T\}$ with the following "truth tables":

x	$f_\neg(x)$
F	T
M	M
T	F

x	y	$f_\wedge(x,y)$
F	F	F
F	M	F
F	T	F
M	F	F
M	M	M
M	T	M
T	F	F
T	M	M
T	T	T

x	y	$f_\vee(x,y)$
F	F	F
F	M	M
F	T	T
M	F	M
M	M	M
M	T	T
T	F	T
T	M	T
T	T	T

x	y	$f_\to(x,y)$
F	F	T
F	M	T
F	T	T
M	F	M
M	M	T
M	T	T
T	F	F
T	M	M
T	T	T

You can intuitively think of M as "*Maybe*". For this new semantics, we will consider assignments $\mathsf{Var} \to \{F, M, T\}$. Given an assignment $\mathcal{A} : \mathsf{Var} \to \{F, M, T\}$ and a formula ϕ we define $\mathsf{val}_\mathcal{A}(\phi)$ as in Definition 2.2.2. We write $\mathcal{A} \vDash \phi$ if $\mathsf{val}_\mathcal{A}(\phi) = T$

Definition. We say that ϕ is a *strong tautology* if for all assignments $\mathcal{A} : \mathsf{Var} \to \{F, M, T\}$ we have that $\mathcal{A} \vDash \phi$.

(1) Show that axioms (A1),(A2), (K3)-(K7) are strong tautologies.

(2) Prove that if ϕ and $\phi \to \psi$ are strong tautologies, then so is ψ.

(3) Prove that $A \vee \neg A$ is not a strong tautology.

(4) Deduce that the proof system K is not complete.

[*Hint.* You must find a formula ϕ such that $\vDash \phi$ but $\nvdash_K \phi$ (i.e. it is not the case that $\vdash_K \phi$).]