CHAPTER 4

Sounds like things are complete

1. Tautologies are still true!

1.1. Universally valid formulas...

Definition 1.1.1. An $\mathcal{L}$-formula ϕ is called *universally valid* if for every $\mathcal{L}$-structure $\mathcal{M}$ and every assignment $\alpha : \mathsf{Var} \to \mathcal{M}$, we have that $\mathcal{M} \vDash \phi[\alpha]$. A *tautology* is a universally valid $\mathcal{L}$-sentence. We write $\vDash \phi$ to indicate that ϕ is universally valid.

Example 1.1.2. Let $\mathcal{L}$ be a language with a single binary relation symbol $\underline{R}$. Then:

(1) $(x = y) \to (\underline{R}(x, y) \leftrightarrow \underline{R}(y, x)$ is universally valid.

(2) $((\exists x)(\forall y)\underline{R}(x, y)) \to (\forall y)(\exists x)\underline{R}(x, y)$ is a tautology.

(3) The sentence $(\exists x)(x = x)$ is a tautology.

(4) The formula $(\exists x)x \neq y$ is not universally valid.

Exercise 1.1.3. Prove (1)-(4) in the previous example carefully.

The following exercise connects tautologies with universally valid formulas. Since $\forall$ is sometimes referred to as **universal quantification**, it also justifies our terminology.

Exercise 1.1.4. Show that the formula $\phi(x_1, \ldots, x_n)$ is universally valid if and only if the sentence $(\forall x_1) \cdots (\forall x_n)\phi$ is a tautology. The sentence $(\forall x_1) \cdots (\forall x_n)\phi$ is called the **universal closure** of $\phi(x_1, \ldots, x_n)$.

An important fact about universally valid formulas is that their universal validity is preserved under **expansions** (i.e. under the operation of adding more symbols to our language $\mathcal{L}$ to obtain a new language $\mathcal{L}' \supseteq \mathcal{L}$ and interpreting, in an $\mathcal{L}$-structure all symbols of $\mathcal{L}$ as we did previously).

Lemma 1.1.5. *Let ϕ be an $\mathcal{L}$ formula and consider a language $\mathcal{L}' \supseteq \mathcal{L}$. Then, ϕ is universally valid as an $\mathcal{L}$-formula if and only if it is universally valid as an $\mathcal{L}'$-formula.*

PROOF. It's enough to note that any $\mathcal{L}$-structure has an expansion to an $\mathcal{L}'$-structure. □

Exercise 1.1.6. Why, in the proof of the previous lemma, is it enough to note that any $\mathcal{L}$-structure has an expansion to an $\mathcal{L}'$-structure?

1.2. ...and the Propositional Calculus (again). In all honesty if when we dropped the propositional logic stuff and started all of this first-order logic business you felt like all of that hard work was for nothing, well, you were somewhat right, but as a matter of fact, not *totally* right. Propositional logic gives us a way of generating universally valid formulas!

First, a little lemma connecting propositional logic with first-order logic in a totally expected way.

Lemma 1.2.1. *Let ϕ be a propositional formula, with $\mathsf{Var}(\phi) = \{A_1, \ldots, A_n\}$. Let $\psi_1, \ldots, \psi_n$ be $\mathcal{L}$-formulas. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\alpha : \mathsf{Var} \to \mathcal{M}$ an assignment. Let $\mathcal{A}_\alpha : \mathsf{Var} \to \{T, F\}$ be the propositional assignment defined by:*

$$\mathcal{A}_\alpha : \mathsf{Var} \to \{T, F\}$$
$$A_i \mapsto \begin{cases} T & \text{if } \mathcal{M} \vDash \psi_i[\alpha] \\ F & \text{otherwise} \end{cases}$$

Then, the following are equivalent:

(1) $\mathcal{M} \vDash (\phi[\psi_1/A_1, \ldots, \psi_n/A_n])[\alpha]$.[1]

(2) $\phi[\mathcal{A}_\alpha] = T$.

PROOF. We argue by induction on the structure of ϕ. Indeed, if ϕ is a propositional variable, then this is immediate. Now, if ϕ is of the form $\phi_1 \wedge \phi_2$, then we have:

$$\begin{aligned}(\phi_1 \wedge \phi_2)[\mathcal{A}_\alpha] = T &\text{ iff } \phi_1[\mathcal{A}_\alpha] \text{ and } \phi_2[\mathcal{A}_\alpha] = T \\ &\text{ iff } \mathcal{M} \vDash (\phi_1[\psi_1/A_1, \ldots, \psi_n/A_n])[\alpha] \text{ and } \mathcal{M} \vDash (\phi_2[\psi_1/A_1, \ldots, \psi_n/A_n])[\alpha] \\ &\text{ iff } \mathcal{M} \vDash (\phi_1 \wedge \phi_2[\psi_1/A_1, \ldots, \psi_n/A_n])[\alpha].\end{aligned}$$

The other cases follow similarly (see next exercise). CRUCIALLY we are doing induction on the structure of a PROPOSITIONAL formula, so we never need to consider quantifiers! □

[1]Here, of course, I'm a notation abuser who is writing $\phi[\psi_1/A_1, \ldots, \psi_n/A_n]$ to mean the first-order formula obtained by replacing every instance of A_i in ϕ by ψ_i.

Exercise 1.2.2. Finish the proof of the previous lemma.

In a sense, what this tells us is that if a first-order formula is built up in a propositional way (e.g. if it is quantifier-free), then to evaluate satisfaction, we just need to do a truth table!

Corollary 1.2.3. *In the notation of the previous lemma, if ϕ is a propositional tautology, then $\phi[\psi_1/A_1, \dots, \psi_n/A_n]$ is a universally valid formula.*

PROOF. For any $\mathcal{L}$-structure and any assignment α, we have that $\phi[\mathcal{A}_\alpha] = T$ (as in the previous lemma) is true, and thus we're done. □

Exercise 1.2.4. Show that the converse of this corollary is not true. More precisely, write down a first-order formula χ which is of the form $\phi[\psi_1/A_1, \dots, \psi_n/A_n]$, for some propositional formula ϕ and first-order formulas $\psi_1, \dots, \psi_n$, and such that χ is universally valid, but ϕ is not.

The upshot of this is that we have everything we discussed in the previous chapter concerning tautologies. So even in the first-order world, we may abbreviate $\wedge$ and $\vee$ and $\neg$ and all other logical connectives using only $\rightarrow$. We have already shown that $(\exists x)\phi$ semantically is just an abbreviation for $\neg(\forall x)\neg\phi$ and thus, it follows that, up to logical equivalence, all first-order formulas can be written using the logical symbols $\rightarrow$, $\neg$ and $\forall$. Syntactically, this is not quite that obvious (for starters, our syntax does not quite know that structures are supposed to be non-empty). Thus, for the syntactic business below, we will also keep track of $\exists$ (this can be handled in various ways, but that's what we're going with here).

2. New day, new axiome

First-order logic is a much more complicated beast than propositional logic. The old axioms will not suffice. Throughout this chapter, we will fix a first-order language $\mathcal{L}$, and everything we will be doing will be happening in $\mathcal{L}$ (or some expansion of it).

2.1. Axioms for Equality. We need to make sure that our proof system understands that the symbol $\doteq$ behaves like equality. The best (and only) way to do this is to hardcode it:

(E1) Reflexivity: $(\forall x)(x \doteq x)$.

(E2) Symmetry: $(\forall x)(\forall y)(x \doteq y \rightarrow y \doteq x)$.

(E3) Transitivity: $(\forall x)(\forall y)(\forall z)(x \doteq y \wedge y \doteq z \rightarrow x \doteq z)$.

(E4) For each n-ary relation symbol $\underline{R} \in \mathsf{Rel}(\mathcal{L})$ a "congruence" axiom:

$$(\forall x_1)\ldots(\forall x_n)(\forall y_1)\ldots(\forall y_n)\left(\bigwedge x_i = y_i \to (\underline{R}(x_1,\ldots,x_n) \to \underline{R}(y_1,\ldots,y_n))\right).$$

(E5) For each n-ary function symbol $\underline{f} \in \mathsf{Fun}(\mathcal{L})$ a "congruence" axiom:

$$(\forall x_1)\ldots(\forall x_n)(\forall y_1)\ldots(\forall y_n)\left(\bigwedge x_i = y_i \to (\underline{f}(x_1,\ldots,x_n) \doteq \underline{f}(y_1,\ldots,y_n))\right).$$

Exercise 2.1.1. Show that (E1)-(E5) are tautologies.

2.2. Quantifier Axioms. We also need some axioms that tell our proof system what the deal with quantifier is. We'll be somewhat economical here:

(Q1) For every $\mathcal{L}$-formula ϕ such that $x \notin \mathsf{Free}(\phi)$ and every $\mathcal{L}$-formula ψ, the axiom:

$$((\forall x)(\phi \to \psi) \to (\phi \to (\forall x)\psi).$$

(Q2) For every $\mathcal{L}$-formula ϕ and every $\mathcal{L}$-term t the axiom:

$$(\forall x)\phi \to \phi[t/x].$$ [2]

(Q3) For every $\mathcal{L}$-formula ϕ and every $\mathcal{L}$-term t the axiom:

$$\phi[t/x] \to (\exists x)\phi$$

(Q4) For every $\mathcal{L}$-formula ϕ, the axiom:

$$\neg(\forall x)\neg\phi \leftrightarrow \exists\phi.$$

Exercise 2.2.1. Show that (Q1)-(Q4) are tautologies.

[*Hint.* If you've proved that (Q2) and (Q3) are tautologies without having used the Substitution Lemma, then something's gone wrong.]

[2] This is where the Substitution Lemma comes in really handy. If we had defined substitutions in a more naive way, then we would need to restrict instances of this axiom to terms t such that there is no free occurrence of x in ϕ which lies within the scope of a quantification that binds a variable in t. We don't need to worry about that though!

2.3. Propositional Axioms. Well we can't get away from propositional tautologies, can we. Let's recall our three axioms from back in the day, but now where we allow ϕ, ψ and χ to be arbitrary $\mathcal{L}$-formulas:

(A1) $(\phi \to (\psi \to \phi))$

(A2) $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$

(A3) $(\neg\phi \to \neg\psi) \to ((\neg\phi \to \psi) \to \phi))$

Since these are substitutions into propositional tautologies, we have the following, right away:

Remark 2.3.1. All instances of (A1)-(A3) are universally valid.

3. So what's a proof, again?

A proof system is the same thing it was last time we discussed proof systems. It'd be funny if the concept had changed. That being said, this time around for first-order logic, our proof system will have all the axioms we listed above. We'll need to throw in a new scary looking rule, too. Let's summarise:

- Axioms: (E1)-(E5), (Q1)-(Q4), and (A1)-(A3)
- Deduction Rules:

(MP) Given: $\phi \to \psi$ and ϕ
Deduce: ψ

(Gen) Given: ϕ
Deduce: $(\forall x)\phi$

A formal proof of formula ϕ is still a finite sequence:

$$(\phi_1, \dots, \phi_n)$$

of formulas such that $\phi_n = \phi$ and for each $i \leq n$, one of the following holds:

- Either ϕ_i is an instance of an axiom;
- Or ϕ_i can be deduced from an instance of (MP) or (Gen) for some $j, k < i$.

If there is a formal proof of ϕ, we again write $\vdash_{\mathcal{L}} \phi$ (Indeed, if we were working in some $\mathcal{L}' \supseteq \mathcal{L}$ then the notion of $\vdash_{\mathcal{L}'}$ could be different, it'll turn out that it's not but okay we are trying to be formal right now). In this case, as before we shall call ϕ a **theorem** (of $\mathcal{L}$), More generally (and just like before) let T be an $\mathcal{L}$-theory. We say that ϕ is **deducible** (in $\mathcal{L}$) from T if there is a finite sequence:

$$(\phi_1, \dots, \phi_n)$$

of formulas such that $\phi_n = \phi$ and for each $i \leq n$, one of the following holds:

- Either ϕ_i is an instance of an axiom;
- Or $\phi_i \in T$
- Or ϕ_i can be deduced from an instance of (MP) or (Gen) for some $j, k < i$.

In this case, we write $T \vdash_{\mathcal{L}} \phi$.

3.1. Simplifying derivations. We showed in the previous chapter that from (A1)-(A3) and (MP), all propositional tautologies can be proved. Thus, we may assume that all first-order instances of propositional tautologies are in our axiom list.

Given the following remark, and me feeling a little splurgy, how about, this time around, since we already have all propositional tautologies that we forgo some formalities and keep all binary connectives in our language (algorithmically, we can substitute them out before a formal proof and then substitute them back in in every step of the proof). This will make our lives rather easy:

Example 3.1.1. Here are some easy yet important deductions:

(1) $\{\phi, \psi\} \vdash_{\mathcal{L}} \phi \wedge \psi$

$$\begin{aligned} \delta_1 :& \phi \to (\psi \to (\phi \wedge \psi)) && \text{(Taut)} \\ \delta_2 :& \psi \to (\phi \wedge \psi) && \text{(MP)} \\ \delta_3 :& \phi \wedge \psi && \text{(MP)} \end{aligned}$$

This would have been possible, but a real pain if we didn't shortcut the derivation of instances of propositional tautologies from (A1)-(A3). Thankfully, all the work we did when showing completeness of propositional logic allows us to do that.

(2) Suppose that $y \notin \mathsf{Var}(\phi)$. Then, $\vdash_{\mathcal{L}} (\forall y)\phi[y/x] \to \forall x\phi$. This is immediate, by how we've set up our axioms, since:

$$
\begin{aligned}
&\delta_1 : (\forall y)\phi[y/x] \to (\phi[y/x])[x/y] && \text{(Q2)}\\
&\delta_2 : (\forall x)\,((\forall y)\phi[y/x] \to (\phi[y/x])[x/y]) && \text{(Gen)}\\
&\delta_3 : ((\forall x)\,((\forall y)\phi[y/x] \to (\phi[y/x])[x/y]) \to ((\forall y)\phi[y/x] \to (\forall x)((\phi[y/x])[x/y])) && \text{(Q1)}\\
&\delta_4 : ((\forall y)\phi[y/x] \to (\forall x)(\phi[y/x][x/y]) && \text{(MP)}\\
&\delta_5 : (\forall x)(\phi[y/x][x/y]) && \text{(MP)}
\end{aligned}
$$

But, by Lemma 3.2.9, we have that $\phi[y/x][x/y] = \phi$. Hopefully, this starts to justify why we needed all those annoying syntactic lemmas in the previous chapter!

(3) $\vdash_{\mathcal{L}} (\forall x)\phi \to \phi$.

$$\delta_1 : (\forall x)\phi \to \phi[x/x] \quad \text{(Q2)}$$

But $\phi[x/x] = \phi$.

The following lemma gives us an important derivation:

Lemma 3.1.2. *If $\phi \to \psi$ is an instance of a propositional tautology, then*

$$\vdash (\forall x)\phi \to (\forall x)\psi.$$

Proof. This is a simple derivation:

$$
\begin{aligned}
&\delta_1 : && (\forall x)\phi \to \phi && \text{Example 3.1.1(3)}\\
&\delta_2 : && \phi \to \psi && \text{(Ass/Taut)}\\
&\delta_3 : && ((\forall x)\phi \to \phi) \to ((\phi \to \psi) \to ((\forall x)\phi \to \psi)) && \text{(Taut)}\\
&\delta_4 : && ((\phi \to \psi) \to ((\forall x)\phi \to \psi)) && \text{(MP)}\\
&\delta_5 : && (\forall x)\phi \to \psi) && \text{(MP)}\\
&\delta_6 : && (\forall x)[(\forall x)\phi \to \psi] && \text{(Gen)}\\
&\delta_7 : && (\forall x)[(\forall x)\phi \to \psi] \to [(\forall x)\phi \to (\forall x)\psi] && \text{(Q1)}\\
&\delta_8 : && (\forall x)\phi \to (\forall x)\psi && \text{(MP).}
\end{aligned}
$$

The tautology we used for δ_3 is of course:

$$(A \to B) \to ((B \to C) \to (A \to C)).$$

$\square$

Exercise 3.1.3. Show that (Q3) is derivable from the rest of the axioms and deduction rules. More precisely, let ϕ be any formula and t any term. Show that there exists a sequence $(\phi_1, \ldots, \phi_n)$, with ϕ_n being $(\forall x)\phi \to \phi[t/x]$, where for each $i \leq n$, one of the following holds:

- Either ϕ_i is an instance of an axiom (E1)-(E5), (Q1),(Q2),(Q4), (A1)-(A3);[3]
- Or ϕ_i can be deduced from an instance of (MP) or (Gen) for some $j, k < i$.

Observe that if $\mathcal{L}$ is a theory with no constant symbols, then

3.2. Sounds sound. Once again, the minimal requirement from $\vdash_{\mathcal{L}}$ is that it only produces deductions of universally valid formulas. More generally:

THEOREM 3.2.1 (Soundness). *Let T be an $\mathcal{L}$ theory and ϕ an $\mathcal{L}$-formula. If $T \vdash_{\mathcal{L}} \phi$ then $T \vDash \phi$.*

PROOF. Combine Exercises 2.1.1 and 2.2.1 and Remark 2.3.1. Then prove that:

- If $T \vDash \phi$ and $T \vDash \phi \to \psi$ then $T \vDash \psi$.
- If $T \vDash \phi$ then $T \vDash (\forall x)\phi$.

Conclude by induction on the length of derivations. □

That sure was fast. Well don't worry about it, that's what HW is for!

[3] By the previous discussion, you can use any propositional tautology here.