

Maths 410 – Homework 2

Due Feb. 16, 2026 – Beginning of class.

Question 1. [10 points] Let $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ denote the set of all irrational numbers. Explain why \mathbb{I} is uncountable.

Question 2. [15 points] Carefully prove that the function:

$$f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$$
$$(x, y) \mapsto \frac{(x+y)(x+y+1)}{2} + y$$

is a bijection. [*Remark.* This completes the proof that \mathbb{Q} is countable.]

Question 3. [15 points] Construct a set of real numbers with exactly 2 limit points.

Question 4. [20 points] Let X be a set. Recall that the *discrete metric* is given by:

$$d : X \times X \rightarrow \mathbb{R}$$

$$(x, y) \mapsto \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

Prove that the discrete metric is a metric. Prove that for all $x \in X$, the set $\{x\}$ is open. Deduce that all subsets of X are both open and closed. Conclude that the only compact subsets of X are the finite ones.

Question 5. [40 points] Let X be a metric space and $E \subseteq X$. Write E° for the set of all interior points of E . We call E° the *interior* of E .

- (1) Prove that E° is always open.
- (2) Prove that E is open if, and only if $E = E^\circ$.
- (3) Prove that if $G \subseteq E$ and G is open, then $G \subseteq E$.
- (4) Prove that the complement of E° is the closure of the complement of E .
- (5) Do E and \overline{E} always have the same interior?
- (6) Do E and E° always have the same closures?

In the last two parts, justify your answers.

Extra Credit. A number $x \in \mathbb{R}$ is called *real algebraic* if it is a root of a polynomial with integer coefficients. Prove that the set of all real algebraic numbers is countable.