

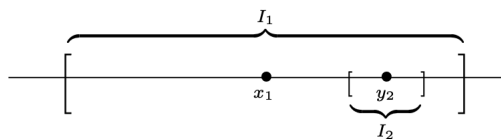
## Maths 410 – Homework 4

Due March 2, 2026 – Beginning of class.

**Definition.** Let  $(X, d)$  be a metric space and  $E \subseteq X$ . We say that  $x \in E$  is an *isolated point* of  $E$  if it is not a limit point of  $E$ . We say that  $E$  is *perfect* if  $E$  is closed and contains no isolated points.

**Question 1.** [50 points] The goal of this question is to prove that every non-empty perfect subset of  $\mathbb{R}$  is uncountable. Let  $P \subseteq \mathbb{R}$  be a non-empty perfect set.

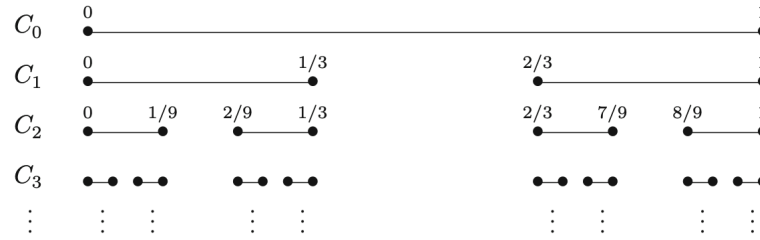
- (1) Carefully prove that  $P$  is infinite.
- (2) Suppose towards a contradiction that  $P = \{x_n : n \in \mathbb{N}\}$  is countable. Let  $I_1 \subseteq \mathbb{R}$  be a closed and bounded interval such that  $x_1 \in I_1^\circ$ . Prove that there is some  $y_2 \in P \setminus \{x_1\}$  such that  $y_2$  is an interior point of  $I_1$ . [Here you need to use that  $x_1$  is not isolated.]
- (3) Suppose that  $I_1 = [a, b]$ , for  $a < b \in \mathbb{R}$ , Construct a closed interval  $I_2 \subseteq I_1$  such that  $y_2 \in I_2$  but  $x_1 \notin I_2$ . *Hint.* Consider the following picture:



- (4) Explain why we may continue this process so that for all  $n \in \mathbb{N}$  we can find a closed interval  $I_n$  such that  $I_{n+1} \subseteq I_n$ ,  $x_n \notin I_{n+1}$ , and  $I_n \cap P \neq \emptyset$ . [*Hint.* The idea is that we may always insist that  $y_{n+1} \neq x_n$ .]
- (5) Let  $K_n = I_n \cap P$ . Show that  $\bigcap_{n \in \mathbb{N}} K_n = \emptyset$  and derive a contradiction.

**Question 2.** [50 points] The goal of this question is to build an uncountable subset of  $\mathbb{R}$  which contains no open intervals. The set  $C$  we will construct is known as the *Cantor Set*, and the construction goes as follows: Let  $C_0 = [0, 1]$ . Let  $C_1$  be the subset of  $C_0$  obtained by removing the open interval in the “middle third”, i.e.  $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ . Let  $C_2$  be obtained by removing the middle third open interval from each of the two intervals of  $C_1$  (see picture). Continuing like this gives us a sequence  $C_0 \supseteq C_1 \cdots \supseteq C_n \supseteq \cdots$ , where each  $C_n$  consists of  $2^n$  disjoint closed intervals, each having length  $\frac{1}{3^n}$ .

Here's the picture:



- (1) Prove that  $C_n$  is compact, for each  $n \in \mathbb{N}$ . Deduce that:

$$C := \bigcap_{n \in \mathbb{N}} C_n$$

is non-empty. [*Remark.* In fact, it is clear that  $C$  is non-empty, as it contains all the endpoints of the intervals that appear in the construction.]

- (2) Explain why, for all  $k, m \in \mathbb{N}$  we have that:

$$C \cap \left( \frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right) = \emptyset.$$

Deduce that for all  $a < b \in \mathbb{R}$  we have that  $(a, b)$  is not a subset of  $C$ .

- (3) Let  $x \in C$ . Prove that  $x$  is not isolated. Deduce that  $C$  is perfect. Using Question 1, explain why  $C$  contains more points than the endpoints of the intervals of  $C_n$ .

**Extra Credit.** Construct a perfect set  $P \subseteq \mathbb{R}$  such that  $P \cap \mathbb{Q} = \emptyset$ .

[*Hint.* Redo the construction of the Cantor set, but with irrational endpoints. Fix an enumeration of the rationals within the endpoints you picked, and at stage  $n$  remove make sure to remove the  $n$ -th rational number in your enumeration.]