Maths 410 – Homework 2

Due Feb. 16, 2026 – Beginning of class.

Question 1. [15 points] Let $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ denote the set of all irrational numbers. Explain why $\mathbb{I}$ is uncountable.

Question 2. [20 points] Carefully prove that the function:

$$\begin{aligned} f : \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (x, y) &\mapsto \frac{(x+y)(x+y+1)}{2} + y \end{aligned}$$

is a bijection. [*Remark.* This completes the proof that $\mathbb{Q}$ is countable.]

Question 3. [35 points] Let $n \in \mathbb{N}_{>0}$. Construct a set of real numbers with exactly n limit points.

Question 4. [30 points] Let X be a set. Recall that the *discrete metric* is given by:

$$\begin{aligned} d : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Prove that the discrete metric is a metric. Prove that for all $x \in X$, the set $\{x\}$ is open. Deduce that all subsets of X are both open and closed.

Extra Credit. A number $x \in \mathbb{R}$ is called *real algebraic* it is a root of a polynomial with integer coefficients. Prove that the set of all real algebraic numbers is countable.