

## Homework 2

**Question 1.** [30 marks] Solve the following exercises, from **Weeks 2 and 3(a)** of the notes:

- Exercise 2.3.4
- Exercise 2.4.19
- Exercise 1.5.5

In the following two exercises, you are asked to show that a proof system is *sound* but *incomplete*. This time we will be working with formulas that involve the logical connectives  $\rightarrow$ ,  $\vee$  and  $\neg$ .

The axioms of this new system  $K$  are the following:

- (A1)  $(\phi \rightarrow (\psi \rightarrow \phi))$
- (K2)  $\phi \rightarrow (\phi \vee \psi)$
- (K3)  $\phi \rightarrow (\psi \vee \phi)$
- (K4)  $(\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \vee \psi) \rightarrow \chi)$
- (K5)  $\neg\phi \rightarrow (\phi \rightarrow \psi)$

The only deduction rule of this system is still MP.

Let  $\Gamma$  be a set of formulas. A **formal proof in  $K$**  of formula  $\phi$ , from  $\Gamma$  is a finite sequence:

$$(\phi_1, \dots, \phi_n)$$

of formulas such that  $\phi_n = \phi$  and for each  $i \leq n$ , one of the following holds:

- $\phi_i \in \Gamma$ ;
- Or  $\phi_i$  is an instance of (A1),(K2)-(K5);
- Or  $\phi_i$  can be deduced from an instance of (MP) for some  $j, k < i$ .

If there is a formal proof in  $K$  of  $\phi$  from  $\Gamma$ , we write  $\Gamma \vdash_K \phi$ . If  $\Gamma = \emptyset$  we write  $\vdash_K \phi$  and call  $\phi$  a **theorem of  $K$** .

**Question 2.** [30 marks] Prove the soundness theorem for this proof system (i.e. prove that if  $\Gamma \vdash \phi$  then  $\Gamma \models_K \phi$ ). [You don't need to show that (A1) and (A2) are instances of tautologies again.]

**Question 3.** [40 marks] We will use a method based on invariants to show that  $K$  is not complete. For a minute, we will change our semantics, and allow the

logical connectives as functions from  $\{F, M, T\}$  (to some power) to  $\{F, M, T\}$  with the following “truth tables”:

|     |               |     |     |                    |     |     |                  |     |     |                         |
|-----|---------------|-----|-----|--------------------|-----|-----|------------------|-----|-----|-------------------------|
| $x$ | $f_{\neg}(x)$ | $x$ | $y$ | $f_{\wedge}(x, y)$ | $x$ | $y$ | $f_{\vee}(x, y)$ | $x$ | $y$ | $f_{\rightarrow}(x, y)$ |
| $F$ | $T$           | $F$ | $F$ | $F$                | $F$ | $F$ | $F$              | $F$ | $F$ | $T$                     |
| $M$ | $M$           | $F$ | $M$ | $F$                | $F$ | $M$ | $M$              | $F$ | $M$ | $T$                     |
| $T$ | $F$           | $F$ | $T$ | $F$                | $F$ | $T$ | $T$              | $F$ | $T$ | $T$                     |
|     |               | $M$ | $F$ | $F$                | $M$ | $F$ | $M$              | $M$ | $F$ | $M$                     |
|     |               | $M$ | $M$ | $M$                | $M$ | $M$ | $M$              | $M$ | $M$ | $T$                     |
|     |               | $M$ | $T$ | $M$                | $M$ | $T$ | $T$              | $M$ | $T$ | $T$                     |
|     |               | $T$ | $F$ | $F$                | $T$ | $F$ | $T$              | $T$ | $F$ | $F$                     |
|     |               | $T$ | $M$ | $M$                | $T$ | $M$ | $T$              | $T$ | $M$ | $M$                     |
|     |               | $T$ | $T$ | $T$                | $T$ | $T$ | $T$              | $T$ | $T$ | $T$                     |

You can intuitively think of  $M$  as “*Maybe*”. For this new semantics, we will consider assignments  $\mathbf{Var} \rightarrow \{F, M, T\}$ . Given an assignment  $\mathcal{A} : \mathbf{Var} \rightarrow \{F, M, T\}$  and a formula  $\phi$  we define  $\text{val}_{\mathcal{A}}(\phi)$  as in Definition 2.2.2. We write  $\mathcal{A} \models \phi$  if  $\text{val}_{\mathcal{A}}(\phi) = T$ .

**Definition.** We say that  $\phi$  is a *strong tautology* if for all assignments  $\mathcal{A} : \mathbf{Var} \rightarrow \{F, M, T\}$  we have that  $\mathcal{A} \models \phi$ .

- (1) Show that axioms (A1),(A2), (K3)-(K6) are strong tautologies.
- (2) Prove that if  $\phi$  and  $\phi \rightarrow \psi$  are strong tautologies, then so is  $\psi$ .
- (3) Prove that  $A \vee \neg A$  is not a strong tautology.
- (4) Deduce that the proof system  $K$  is not complete.

[*Hint.* You must find a formula  $\phi$  such that  $\models \phi$  but  $\not\models_K \phi$  (i.e. it is not the case that  $\vdash_K \phi$ ).]