MODEL THEORY AND COMBINATORICS
OR
(THE UNEXPECTED VIRTUE OF TAMENESS)

ARIS PAPADOPOULOS

Abstract. Hrushovski once called model theory the “geography of tame mathematics”; a quote which may make no sense if you’re not a model theorist (or if you are, in fact, a geographer). I will try to explain this point of view, by discussing how model theorists draw “dividing lines” separating the tame theories from the wild ones and illustrating this through examples from combinatorics.

The aim of this talk is to provide a gentle introduction to some recent developments in the area. Perhaps ambitiously, I will try to (briefly and informally) discuss three very different topics: (i) SzemerÃĺdi Regularity in stable (due to M. Malliaris and S. Shelah), NIP (due to J. Fox, J. Pach, and A. Suk), and distal structures (due to A. Chernikov and S. Starchenko); (ii) Algorithmic tameness in hereditary classes of relational structures (joint work with S. Braunfeld, A. Dawar, and I. Eleftheriadis); and (iii) Zarankiewicz’s problem in semilinear (due to A. Basit, A. Chernikov, S. Starchenko, T. Tao and C.-M. Tran) and semibounded (joint work with P. Eleftheriou) o-minimal structures.

1. A Biased Introduction to Model Theory

I started my abstract by quoting Hrushovski who said that model theory is the *“geography of tame mathematics”*. Given this, I should start my talk by drawing the picture of a map:

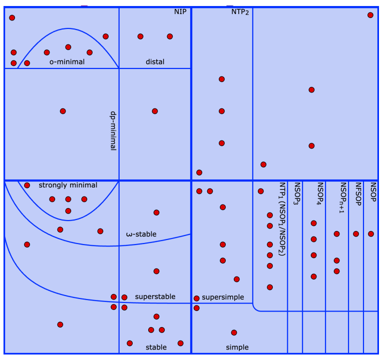

Figure 1. The Map of the Universe, from [Con].

Perhaps a stupid picture to draw out of context, so let’s take a step back. It’s hard to define *model theory*, but here’s Hodges’ attempt [Hod93]:

> “Model theory is the study of the *construction* and *classification* of structures within *specified classes* of structures”

These classes of structures are specified using *sets of axioms* written in first-order logic.

These are notes for a talk in a PhD seminar in Leuven, given on Wednesday, the 26th of April 2023.

Aside: First-order Logic. The logician's point of view is that a lot of mathematics boils down to statements that can be expressed using the same sort of general structure think maybe "for all $\epsilon > 0$ there is a $\delta > 0$" etc.

To make this (slightly) more precise we fix a countable vocabulary $\mathcal{L}$ (a countable set of relation, function, and constant symbols) which always includes $=$. We use this to construct formulas using *conjunction* ($\wedge$), *disjunction* ($\vee$), *implication* ($\rightarrow$), *negation* ($\neg$), *universal quantifiers* ($\forall$), and *existential quantifiers* ($\exists$). Given a fixed vocabulary $\mathcal{L}$ an $\mathcal{L}$-structure is one where all the symbols in $\mathcal{L}$ have been given meaning. Satisfaction (truth) in an $\mathcal{L}$-structure is evaluated in the obvious (to be illustrated in the examples that follow) way.

Given an $\mathcal{L}$-sentence ϕ and an $\mathcal{L}$-structure M we will write $M \vDash \phi$ to indicate that M is a *model* of ϕ. By a *theory* we mean a set of formulas that has a model. We say that a theory T is *complete* if there is some $\mathcal{L}$-structure M such that $T = \{\phi : M \vDash \phi\}$.

Example 1. Let $\mathcal{L}$ be a language with a single binary relation E symbol $\mathcal{L}$ and a constant symbol c. An example of such $\mathcal{L}$-structure is: Let M be the structure above. Then, for instance

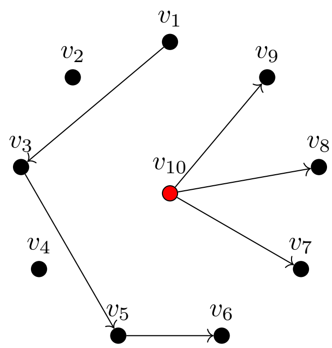

FIGURE 2. The $\{R, c\}$-structure with: $R = \{(v_1, v_3), (v_3, v_5), (v_5, v_6), (v_{10}, v_7), (v_{10}, v_8), (v_{10}, v_9)\}$ and $c = v_{10}$.

$$M \vDash \text{"there are 3 distinct elements that } c \text{ points to"}$$

or even:

$$M \vDash \text{"there is a directed path of length 3"}.$$

Here are some more things we can always say (in any language):

- For any $n \in \mathbb{N}$ there are at least n distinct elements:
$$(\exists x_1) \dots (\exists x_n) \bigwedge_{i<j\leq n} x_i \neq x_j.$$
  Let's call the sentence above ϕ_n
- For any $n \in \mathbb{N}$ there are at most n distinct elements: $\neg\phi_{n+1}$.
- For any $n \in \mathbb{N}$ there are exactly n distinct elements: $\phi_n \wedge \neg\phi_{n+1}$

Let's illustrate this with a couple of examples:

Example 2 (Graphs)**.** Perhaps the easiest example to consider is that of *graphs*. A graph is just a set of vertices with some edges between them. We want the edge relation to be *symmetric* (i.e. graphs are *undirected*: if there is an edge from v to w then there is an edge from w to v) and *irreflexive* (i.e. graphs are *simple*: there are no edges with a single endpoint). For example:

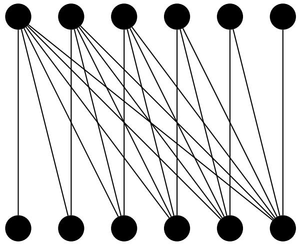

FIGURE 3. The *half-graph* of size 12, H_6.

The graph in Figure 3 above is the so-called *half-graph* with 6 vertices on each side, denoted H_6. TO construct H_n we consider two disjoint sets of n vertices each, say $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ and draw an edge between v_i and w_j if, and only if $i < j$. Keep this graph in mind, we'll come back to it later...

We make a choice for the language (Sieve of Eratosthenes: Choose the simplest possible language), so we choose a language with a single *binary* relation symbol E. What can we say in this language?

- *Symmetric*: $(\forall x)(\forall y)(E(x,y) \to E(y,x))$.
- *Irreflexive*: $(\forall x)(\neg E(x,x))$.

In particular, the class of graphs is precisely the class of all structures in the language $\{E\}$ which satisfy the two formulas above. We say that these formulas *axiomatise* the class of all graphs.

Example 3 (Groups)**.** Here we have more choice in the language, but I think we should go with a language that has a single constant symbol id, and two function symbols, one binary, say $\times$, and one unary, say $\cdot^{-1}$. We can axiomatise, amongst the structures in this language the ones that are groups by saying:

- *Associativity*:$(\forall x)(\forall y)(\forall z)[(x \times y) \times z = x \times (y \times z)]$.
- *Identity*: $(\forall x)[(x \times \mathsf{id} = x) \wedge (\mathsf{id} \times x = x)]$.
- *Inverses*:$(\forall x)[(x \times x^{-1} = \mathsf{id}) \wedge (x^{-1} \times x = \mathsf{id})]$.

End of Aside. $\dashv$

Theorem 1.1 (Löwenheim-Skolem, rather informally)**.** *If a theory T has an infinite model, then it has at least one model of every infinite cardinality.*

All of the above leads to a very natural question:

Question. *Let T be a complete theory with infinite models in a countable language $\mathcal{L}$ and κ an infinite cardinal. How many non-isomorphic models of cardinality κ does T have?*

We define the *spectrum function of T* to be the following function:

$$I(T, \kappa) := \#\text{non-isomorphic models of } T \text{ of cardinality } \kappa,$$

for each infinite cardinal κ.

Fact. *Let T as above. For all infinite cardinals κ we have that $1 \leq I(T, \kappa) \leq 2^\kappa$.*

Example 4. Here are some spectrum functions:

- *Infinite Sets*: Let $\mathcal{L} = \emptyset$ (i.e. the language of *pure equality*) and T_∞ the theory of an *infinite set*. Then, clearly $I(T_\infty, \kappa) = 1$ for any infinite cardinal κ.

- *Algebraically Closed Fields*: Two algebraically closed fields of the same characteristic are isomorphic if, and only if, they have the same *transcendence degree*. It follows that, up to isomorphism, there are $\aleph_0$-many countable models of ACF_p, i.e. that $I(\mathsf{ACF}_p, \aleph_0) = \aleph_0$, but for each uncountable cardinal κ, only one model of ACF_p of cardinality κ, $I(\mathsf{ACF}_p, \kappa) = 1$, for each $\kappa \geq \aleph_1$.
- *Vector Spaces over* $\mathbb{Q}$: Again, two vector spaces over the same field are isomorphic if, and only if, they have the same *dimension*. It follows that, up to isomorphism, there are $\aleph_0$-many countable vector spaces over $\mathbb{Q}$ i.e. that $I(\mathsf{VS}_{\mathbb{Q}}, \aleph_0) = \aleph_0$, but for each uncountable cardinal κ, only one model of $\mathsf{VS}_{\mathbb{Q}}$ of cardinality κ, $I(\mathsf{VS}_{\mathbb{Q}}, \kappa) = 1$, for each $\kappa \geq \aleph_1$.
- *Dense Linear Orders*: By a theorem of Cantor the theory DLO has, up to isomorphism, only one model of cardinality $\aleph_0$), i.e. $I(\mathsf{DLO}, \aleph_0) = 1$. On the other hand, it is again easy to make *many* uncountable models of DLO, in fact, $I(\mathrm{DLO}, \kappa) = 2^\kappa$ for each $\kappa \geq \aleph_1$.

Conjecture (Morley)**.** *For any complete, countable theory T and any uncountable cardinals $\kappa > \lambda \geq \aleph_1$ we have that:*

$$I(T, \kappa) \geq I(T, \lambda),$$

i.e. the uncountable spectrum of a(ny) theory T is non-decreasing.

This is now a theorem due to *Shelah*, who took a rather bold approach. His method of proving the conjecture can be summarised as follows:

Describe all the possible spectrum functions $I(T, \kappa)$, as T varies, and show that each of these is non-decreasing...

What would distinguish the possible spectrum fucntions of T, according to Shelah would be certain **dividing lines**.

Definition 1.1 (Dividing line – **very** informal)**.** A *dividing line* is some abstract property P such that, any theory satisfying P can be shown to be *tame* and any theory satisfying $\neg P$ can be shown to be *wild*, where *tameness* and *wildness* correspond to some *structure* and *non-structure* results about T, respectively.]

The map in Figure 1 contains many such dividing lines, we will discuss some of them in the sequel, but first, a quote from [She13]:

> "(...) I have been attracted to trying to find some order in the darkness, more specifically, finding meaningful dividing lines among general families of structures. This means that there are meaningful things to be said on both sides of the divide: characteristically, understanding the tame ones and giving evidence of being complicated for the chaotic ones. It is expected that this will eventually help in understanding even specific classes and even specific structures. Some others see this as the aim of model theory, not so for me. Still, I expect and welcome such applications and interactions."

The general "*dividing line methodology*" can thus be summarised as follows:

Step 1 Find a test problem (e.g. having "many"/"few" models of every cardinality).

Step 2 Identify a property P and prove that:

- Step 2(a) The existence of P yields a "positive" solution to the test problem.
- Step 2(b) The absence of P yields a "negative" solution to the test problem.

Step 3 Repeat.

The amazing thing that I hope to convince you about in the next sections is that the "good" dividing lines may have astounding applications in many regions of maths, far removed from the original test problem these dividing lines were designed to solve.

2. Dividing Lines

First, I will try to briefly explain what some "good" dividing lines look like.

Aside: Definable Sets. A very important notion in model theory that I've not mentioned yet is that of a *definable set*. A definable sets is just the "solution sets" to a formula (with free variables) in the same way that an algebraic variety is the solution set to a system of polynomial equations. In general, given a structure M, a subset $D \subseteq M^n$ is definable if there is a formula $\phi(x_1, \dots, x_n)$ in n free variables (i.e. placeholders for elements of the structure) such that:

$$D = \{(a_1, \dots, a_n) \in M^n : M \vDash \phi(a_1, \dots, a_n)\}.$$

Given a formula $\phi(x_1, \dots, x_n)$ we can partition the free variables into disjoint groups, say $\bar{y} = (x_1, \dots, x_m)$ and $\bar{z} = (x_{m+1}, \dots, x_n)$, for some $m < n$. Then, we can view $\phi(\bar{y}, \bar{z})$ as the edge relation of a bipartite graph on $M^m \sqcup M^{n-m}$.

End of Aside. $\dashv$

Perhaps the two most important dividing lines are stability and NIP. For more information and proper definitions, see [She78].

Definition 2.1. A theory T is *stable* if for every formula $\phi(\bar{x}; \bar{y})$ there is some n, such that, in every model of T the graph defined by $\phi(\bar{x}; \bar{y})$ does not contain H_n as a subgraph.

This seems like a very strange definition. Indeed, this is also a strange way of phrasing it. Intuitively, if the graph whose edge relation is given by a formula $\phi(\bar{x}; \bar{y})$ contains as subgraph H_n for all $n \in \mathbb{N}$ then, by **??** (in some model of T) it contains as a subgraph H_∞, and therefore (after a small syntactic manipulation) we can find a formula $\phi'(\bar{x}; \bar{y})$ which defines a linear order on an infinite set.

Motto: Stable theories don't have an order on infinite sets!

Here are some examples of stable theories that you may be aware of:

(1) The theory of an infinite set.
(2) The theory of algebraically closed fields (in a fixed characteristic).
(3) The theory of vector spaces over $\mathbb{Q}$.

Generally (but take this with a pinch of salt) stable theories are fairly easy to understand. Of course, I haven't told you why we care about stability. The answer goes back to $I(T, \kappa)$. In particular, if T is unstable then for all $\kappa > \aleph_0$ we have that $I(T, \kappa) = 2^\kappa$ (i.e. T has the maximum number of models of each given cardinality that it could have). There is more to be said here, especially about the tame side, but maybe that's a story for another time.

Another, fairly similar dividing line is NIP. The definition is the following:

Definition 2.2. A theory T is *NIP* (*No Independence Property*) if for every formula $\phi(\bar{x}; \bar{y})$ there is some finite bipartite graph G, such that in every model of T the graph defined by $\phi(\bar{x}; \bar{y})$ does not contain G as a subgraph.

So clearly, by definition, every stable theory is NIP. Intuitively, a theory T is NIP if the definable subsets in models of T are "combinatorially simple".

Motto: Graphs definable in NIP theories are "combinatorially simple". (Combinatorially simple, for a graph, here means that its edge relation has finite VC-dimension).

The class of NIP theories is a much larger class of theories and it contains examples such as:

(1) Dense Linear Orders.
(2) The p-adics.
(3) Real closed fields $(\mathbb{R}, +, \times, <, 0, 1)$

I'll mention also, in passing, a few other relevant lines:

- *o-Minimality*: Introduced in [], o-minimallity has a very different flavour than the previous ones. A theory T is o-minimal if it has an order $<$ and every definable subset of 1-space is a finite union of points and intervals.
- *Distality*: Introduced in [], distality is a generalisation of o-minimality, and says (very informally) that all definable sets admit a "combinatorially nice" cell decomposition.
- *monadic NIP*: Monadic NIP is a strengthening of NIP, which says that no matter how we colour subsets of the domain of our structure it remains NIP.

Given all of the above, the aim of the remainder of the talk is to convince you that somehow these "abstract nonsense" definitions can have applications in normal mathematics. There are many other areas of mathematics that I could have focused on, but I chose to focus on applications of dividing lines in combinatorics.

3. Szémeredi's Regularity Lemma

Széredi's regularity lemma is a powerful tool from finite combinatorics, proved by Szémeredi in 1978, as part of his proof of the celebrated Szémeredi theorem (which says that any subset of $\mathbb{N}$ with positive upper density contains arbitrarily large arithmetic progressions). He commented then that

Rather informally this is the statement of the regularity lemma:

Theorem 3.1 (Szémeredi Regularity, informally)**.** *For every $\epsilon > 0$ there is some $N = N(\epsilon) \in \mathbb{N}$ such that: For all graphs $G = (V, E)$ there is a partition of V into at most N parts $V_1, \ldots, V_n$ such that "most (i.e. all but an ϵ-fraction) pairs of behave like random graphs with some fixed edge density, up to ϵ".*

This statement raises an obvious question:

Q. What does it mean to behave like a random graph with some edge density?
A. Intuitively (and that's as far as I want to mention here) it means that the proportion of edges between any two large enough subsets over the total possible number of edges is always the same (the edge density).

We can illustrate this as follows:

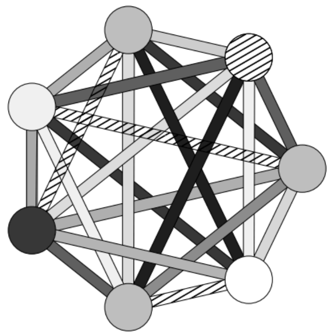

FIGURE 4. Szémeredi's Regularity Lemma for arbitrary graphs.

Some things to keep in mind:

- The *irregular pairs* are necessary. That is, we can not change the word *most* to *all.*
- The number of parts in the partition can be *very* large (a tower of powers of 2 of height a power of $\frac{1}{\epsilon}$).
- The regularity lemma does not tell us how nicely the edges are distributed within the parts.
- The regularity lemma does not tell us anything about the edge densities between the regular pairs.

Aside: Szémeredi's Regularity Lemma, formally. To be able to state a formal version of Theorem 3.1, we need to introduce several definitions. Throughout this section, let $G = (V, E)$ be a graph. Given disjoint subsets $X, Y \subseteq V$ we define $e_G(X, Y) = |(X \times Y) \cap E|$. From this, we can define the *edge density* between X and Y to be

$$d_G(X, Y) := \frac{e_G(X, Y)}{|X \times Y|}.$$

Fix $\epsilon > 0$. We say that X, Y is an *ϵ-regular pair* if for every $A \subseteq X$ with $|A| \geq \epsilon|X|$ and every $B \subseteq Y$ with $|B| \geq \epsilon|Y|$ we have that $|d_G(A, B) - d_G(X, Y)| < \epsilon$. We say that a partition of G into $V_1, \ldots, V_n$ is *ϵ-regular* if:

$$\sum_{\substack{(i,j)\in[n]^2, \\ (V_i,V_j) \text{ is not } \epsilon\text{-regular}}} |V_i||V_j| \leq \epsilon|V|^2,$$

i.e at most an ϵ-fraction of the vertices of G are in ϵ-irregular pairs.

Theorem 3.2 (Szémeredi's Regularity Lemma)**.** *Fix $\epsilon > 0$. Then, there is some $N = N(\epsilon) \in \mathbb{N}$ such that: For all graphs $G = (V, E)$ there is an ϵ-regular partition of V into at most N parts.*

End of Aside. $\dashv$

We will now restrict our attention to graphs definable (i.e. their edge relation is a formula) in tame structures. In particular:

(1) *NIP regularity lemma* [JFS18] (and [CS21], for hypergraphs): Here the number of parts is polynomial in $\frac{1}{\epsilon}$, there could still be irregular pairs, but for all the regular pairs the edge density is either at least $1-\epsilon$ or at most ϵ (in other words, all pairs are *ϵ-homogeneous*).

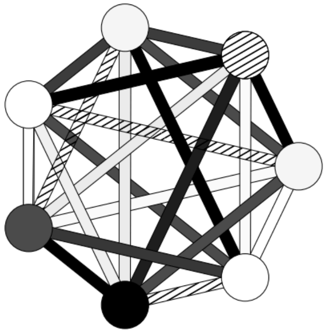

FIGURE 5. Szémeredi's Regularity Lemma for graphs definable in *NIP* structures.

(2) *Stable Regularity Lemma* [MS14]: Here, again the number of parts is polynomial in $\frac{1}{\epsilon}$, and this time there are *no irregular pairs*. Moreover, for every part A, all the vertices of G have degree either at most $\epsilon|A|$ in A or at least $(1-\epsilon)|A|$ in A (in other words, all parts are ϵ-*excellent*), and every pair (A, B) is 2ϵ-homogeneous (something more is true, in fact, every pair is ϵ-*uniform*, but that's not terribly important)

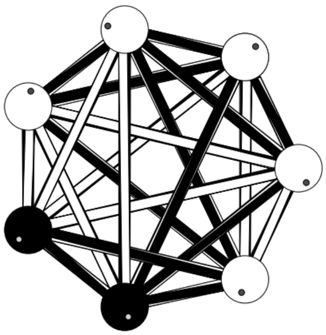

FIGURE 6. Szémeredi's Regularity Lemma for graphs definable in *stable* structures.

(3) *Distal regularity lemma* [CS18]: Again the number of parts is polynomial in $\frac{1}{\epsilon}$. In this case thoug, for all regular pairs either all edges are present or no edge is present (in other words the regular pairs are *homogeneous*).

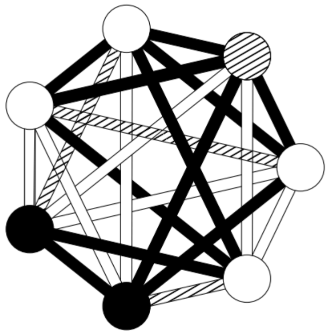

FIGURE 7. Szémeredi's Regularity Lemma for graphs definable in *distal* structures.

All illustrations in this section are taken from [JNdMS20, Table 1].

4. Algorithmic Tameness for Classes of Relational Structures

It's important to note that Szémeredi's Regularity Lemma does not tell us much for *sparse* graphs (intuitively, we could partition a very sparse graph so that all of its edges are between irregular pairs of the partition). So now we take a very sharp left turn and discuss very briefly an application of model theory to *classes* of relational structures which are, in a very precise sense, sparse.

I will only talk about classes of structures which are closed under taking induced substructures (that is, *monotone* classes of structures).

Let's start what it means for a graph class to be *nowhere-dense*:

Definition 4.1. A class of graphs $\mathcal{C}$ is *nowhere-dense* if there is some $r \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ if $G \in \mathcal{C}$ then G does not contain K_n^r (the r-subdivided n-clique) as a subgraph.

Intuitively, a complete graph (or even a subdivided complete graph, if the number of subdivisions is much smaller than the number of edges) is a dense graph, so a nowhere-dense class of graphs is not allowed to contain any such object.

Examples of nowhere-dense classes of graphs you may have heard include the class of *planar graphs* (graphs that can be drawn on the plane without their edges crossing each other), and more generally every class of graphs that can be drawn with a bounded number of crossings. Connecting to the previous work, as observed by [AA14], it follows from a result of [PZ78] that all stable classes of graphs are nowhere-dense.

Nowhere-dense graphs were introduced in [NdM11] and enjoy a very rich and interesting theory. In particular, for monotone classes of graphs nowhere-density coincides with *tractability* (i.e. the existence of a relatively fast algorithm that decides the first-order properties of the graphs).

It had been an open question to determine what the natural analogue of nowhere-density was for monotone classes of higher-arity structures, which we answered in a recent preprint, using model theoretic dividing lines, in particular, monadic NIP:

Theorem 4.1 (Braunfeld, Dawar, Eleftheriadis, P. [BDEP23]). *Let $\mathcal{C}$ be a monotone class of relational structures. Then $\mathcal{C}$ is* tractable *if, and only if* $\mathsf{Th}(\mathcal{C})$ *is monadically NIP if, and only if, the class of Gaifman graphs of $\mathcal{C}$ is nowhere dense.*

5. Zarankiewicz's Problem

I'll take another sharp turn now and briefly discuss an interesting application of model theory to *extremal graph theory.* A central question in classical extremal graph theory is the following.

Question. *Let H be a fixed graph. What is the maximum number of edges of a graph G on n vertices which does not contain H as a subgraph?*

To fix notation, given a graph H and a natural number $n \in \mathbb{N}$ define $\mathsf{ex}(n, H)$ to be the maximum number of edges in an n-vertex graph G which does not contain H as a subgraph. Let $K_{s,s}$ denote the complete bipartite graph with s vertices in each side. For example:

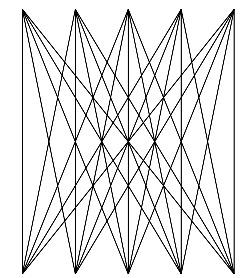

Figure 8. $K_{5,5}$, the complete bipartite graph with 5 vertices on each side.

Zarankiewicz's Problem is asks for $\mathsf{ex}(n, K_{s,s})$. This is of significant importance, because for all bipartite graphs H there is some $s \in \mathbb{N}$ such that H is a subgraph of $K_{s,s}$. In particular, every graph that omits H must also omit $K_{s,s}$, and therefore:

$$\mathsf{ex}(H, n) \leq \mathsf{ex}(K_{s,s}, n),$$

so, essentially, Zarankiewicz's problem asks for an upper bound on the extremal number of general bipartite graphs.

Theorem 5.1 (Kővari-Sós-Turán)**.** *For every $s \in \mathbb{N}$ there is some $C = C(t) \in \mathbb{N}$ such that:*

$$\mathsf{ex}(n, K_{s,s}) \leq Cn^{2-\frac{1}{s}},$$

for all (large enough) $n \in \mathbb{N}$.

To know that an upper bound for $\mathsf{ex}(H, n)$ is *optimal* we need to show that it coincides with a lower bound for $\mathsf{ex}(H, n)$.

Motto: We want *small upper bounds* and *large lower bounds* for extremal numbers

However, stronger bounds are known for restricted families of graphs arising in geometric settings:

Example 5 (Point-Line Incidences)**.** Let H be the *incidence graph* of a set of n points and n lines in $\mathbb{R}^2$ (this is a bipartite graph where on the one part we have the points $\{p_1, \dots, p_n\}$ and on the other part we have the lines $\{l_1, \dots, l_n\}$, and we draw an edge between p_i and l_j if, and only if, $p_i \in l_j$). This graph is always $K_{2,2}$-free (since we cannot have two points which belong to two distinct lines), and so Theorem 5.1 implies that there is some constant C such that $|E(H)| \leq Cn^{\frac{3}{2}}$ for all $n \in \mathbb{N}$.

However, the *Szémeredi-Trotter theorem* "improves" this showing that there is some constant C such that $|E(H)| \leq Cn^{\frac{4}{3}}$ for all $n \in \mathbb{N}$, which is actually optimal. (What does this mean? It means that there is a family of incidence graphs of points and lines which achieves this bound, i.e. the upper bound could not be lowered).

There are two very important approaches in computing lower bounds, the so-called *probabilistic method* and *algebraic constructions*. I want to very briefly mention the importance of model theory

Theorem 5.2 ([BCS+21, Corollary 5.11], informal)**.** *Let $\mathcal{M}$ be an o-minimal structure. Then, the following are equivalent:*

(1) There is no infinite field definable in $\mathcal{M}$.
(2) For every definable graph $G = (V, E)$ which is $K_{s,s}$-free there is some $\beta < \frac{4}{3}$ we have that $|E| = \mathcal{O}(|G|^{\beta})$.

The real point of this theorem is that *algebra is necessary* to construct lower bounds for Zarankiewicz's problem. Observe that Theorem 5.2 does not tell us anything about the "size" of the field that could exist in $\mathcal{M}$ – it could be a field inside some interval or on the whole of $\mathcal{M}$, but the theorem does not give this information. In [EP23] (still in-progress) we extend this result, to be able to extract this additional information.

References

[AA14] H. Adler and I. Adler. Interpreting nowhere dense graph classes as a classical notion of model theory. *European Journal of Combinatorics*, 36:322–330, 2014.

[BCS+21] A. Basit, A. Chernikov, S. Starchenko, T. Tao, and C.-M. Tran. ZarankiewiczâĂŹs problem for semilinear hypergraphs. *Forum of Mathematics, Sigma*, 9:e59, 2021.

[BDEP23] S. Braunfeld, A. Dawar, I. Eleftheriadis, and A. Papadopoulos. Monadic nip in monotone classes of relational structures, 2023.

[Con] G. Conant. Forking and dividing.

[CS18] A. Chernikov and S. Starchenko. Regularity lemma for distal structures. *Journal of the European Mathematical Society*, 20(10):2437–2466, July 2018.

[CS21] A. Chernikov and S. Starchenko. Definable Regularity Lemmas for NIP Hypergraphs. *The Quarterly Journal of Mathematics*, 72(4):1401–1433, 03 2021.

[EP23] P. Eleftheriou and A. Papadopoulos. ZarankiewiczâĂŹs problem for eventually linear hypergraphs. *Work in Progress*, 2023+.

[Hod93] Wilfrid Hodges. *Model Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1993.

[JFS18] J. Pach J. Fox and A. Suk. Erdős-hajnal conjecture for graphs with bounded VC-dimension. *Discrete & Computational Geometry*, 61(4):809–829, November 2018.

[JNdMS20] Y. Jiang, J. Nešetřil, P. Ossona de Mendez, and S. Siebertz. Regular partitions of gentle graphs. *Acta Mathematica Hungarica*, 161(2):719–755, July 2020.

[MS14] M. Malliaris and S. Shelah. Regularity lemmas for stable graphs. *Transactions of the American Mathematical Society*, 366(3):1551–1585, 2014.

[NdM11] J. Nešetřil and P. Ossona de Mendez. On nowhere dense graphs. *European Journal of Combinatorics*, 32(4):600–617, 2011.

[PZ78] K.-P. Podewski and M. Ziegler. Stable graphs. *Fund. Math*, 100(2):101–107, 1978.

[She78] S. Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1978.

[She13] S. Shelah. Response to the award of the 2013 steele prize for seminal research, 2013.