Numerically solving integral equations of wave ensembles

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Abstract

Allows for a broad frequency range, and to easily test different statistical assumptions. Assumptions such as the pair-correlation and QCA.

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1 Effective waves for uniformly distributed species

We consider a halfspace $x > 0$ filled with S types of inclusions (species) that are uniformly distributed. The fields are governed by the scalar wave equation:

$$\nabla^2 u + k^2 u = 0, \quad \text{(in the background material)} \tag{1}$$

$$\nabla^2 u + k_j^2 u = 0, \quad \text{(inside the } j\text{-th scatterer)}, \tag{2}$$

The background and species material properties are summarised in Table 1. The goal is to calculate how a medium with these scatterers, randomly uniformly distributed, reflects and transmits waves in an ensemble average sense.

For simplicity we will consider that all particles are cylindrical, though it is easy to extend the results to any smooth particle by using Waterman's T-matrixWaterman (1971); Varadan et al. (1978); Mishchenko et al. (1996).

Background properties:	wavenumber k	density ρ	sound speed c	
Specie properties:	number density $\mathfrak{n}_j$	density ρ_j	sound speed c_j	radius a_j
total number density $\mathfrak{n}$	effective wavenumber k_*	species min. distance $a_{j\ell} > a_j + a_\ell$		

Table 1: Summary of material properties and notation. The index j refers to properties of the j-th species. Note a typical choice for $a_{j\ell}$ is $a_{j\ell} = c(a_j + a_\ell)$, where $c = 1.01$.

2 Cylindrical species

We consider an incident wave

$$u_{\text{in}} = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \quad \text{with} \quad \mathbf{k}\cdot\mathbf{x} = kx\cos\theta_{\text{in}} + ky\sin\theta_{\text{in}}, \tag{3}$$

and angle of incidence θ_{in} from the x-axis, exciting a material occupying the halfspace $x > 0$.

Combining equations (3.6) and then quasicrystalline approximation (3.10) from (Gower et al., 2018), we arrive at

$$\mathfrak{n} \sum_{n=-\infty}^{\infty} \int_{\mathcal{S}} \int_{\substack{x_2>0 \\ \|\mathbf{x}_1-\mathbf{x}_2\|>a_{21}}} \mathcal{A}^n(\mathbf{x}_2, \mathbf{s}_2) F_{n-m}(k\mathbf{x}_1 - k\mathbf{x}_2, k) d\mathbf{x}_2 d\mathbf{s}_2^n$$
$$+ \mathcal{A}^m(\mathbf{x}_1, \mathbf{s}_1) + \mathrm{e}^{\mathrm{i}\mathbf{x}_1\cdot\mathbf{k}} \mathrm{e}^{\mathrm{i}m(\pi/2-\theta_{\text{in}})} = 0, \quad \text{for} \quad x_1 > 0, \quad (4)$$

where

$$F_{n-m}(k\mathbf{x}_1 - k\mathbf{x}_2, k) = \mathrm{e}^{\mathrm{i}(n-m)\Theta_{21}} H_{n-m}(kR_{21}) g(\mathbf{x}_1 - \mathbf{x}_2 | \mathbf{s}_1, \mathbf{s}_2), \quad (5)$$

$p(\mathbf{s}_1)$ is the probability density function of picking a species in $\mathcal{S}$ and we assumed statistical independence $p(\mathbf{s}_1, \mathbf{s}_2) = p(\mathbf{s}_1)p(\mathbf{s}_2)$. The function $g(\mathbf{x}_1 - \mathbf{x}_2|\mathbf{s}_1, \mathbf{s}_2)$ is the pair-correlation, assuming the particle centred at $\mathbf{x}_1$ ($\mathbf{x}_2$) is of type $\mathbf{s}_1$ ($\mathbf{s}_2$). If we were to use whole correction, then $g(\mathbf{x}_1 - \mathbf{x}_2|\mathbf{s}_1, \mathbf{s}_2) = 1$.

In terms of the notation from (Gower et al., 2018):

$$|\mathcal{R}_N| p(\mathbf{\Lambda}_2|\mathbf{\Lambda}_1) = |\mathcal{R}_N|^2 \frac{p(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2)}{p(\mathbf{s}_1)} = |\mathcal{R}_N|^2 p(\mathbf{s}_2) p(\mathbf{x}_1, \mathbf{x}_2|\mathbf{s}_1, \mathbf{s}_2) = p(\mathbf{s}_2) g(\mathbf{x}_1 - \mathbf{x}_2|\mathbf{s}_1, \mathbf{s}_2). \quad (6)$$

We also borrow equation (4.1) from (Gower et al., 2018) to substitute

$$\mathcal{A}^m(x_1, y_1, \mathbf{s}_1) = \mathcal{A}^m(x_1, \mathbf{s}_1) \mathrm{e}^{\mathrm{i}y_1 k \sin\theta_{\text{in}}},$$

which is a result of the symmetry present in (4). substituting the above into (4) results in

$$\int_{\substack{x_2>0\\ \|\mathbf{x}_2-\mathbf{x}_1\|>a_{21}}} \mathcal{A}^n(x_2,\mathbf{s}_2)\mathrm{e}^{\mathrm{i}y_2 k\sin\theta_{\mathrm{in}}} F_{n-m}(k\mathbf{x}_1-k\mathbf{x}_2,k)d\mathbf{x}_2 =$$

$$\int_{x_2>0} \mathcal{A}^n(x_2,\mathbf{s}_2)\int_{y\notin B} \mathrm{e}^{\mathrm{i}y_2 k\sin\theta_{\mathrm{in}}} F_{n-m}(k\mathbf{x}_1-k\mathbf{x}_2,k)dy_2 dx_2, \quad (7)$$

where B is the interval

$$B = \begin{cases} [y_1-\sqrt{a_{21}^2-(x_2-x_1)^2}, y_1+\sqrt{a_{21}^2-(x_2-x_1)^2}], & |x_2-x_1|\leq a_{21} \\ [0,0], & |x_2-x_1|>a_{21}. \end{cases}$$

$$\mathfrak{n}\sum_{n=-\infty}^{\infty}\int_{\mathcal{S}}\int_{\substack{x_2>0\\ \|\mathbf{x}_2-\mathbf{x}_1\|>a_{21}}} \mathcal{A}^n(x_2,\mathbf{s}_2)\mathrm{e}^{\mathrm{i}y_2 k\sin\theta_{\mathrm{in}}} F_{n-m}(k\mathbf{x}_1-k\mathbf{x}_2,k)d\mathbf{x}_2 d\mathbf{s}_2^n$$

$$+\mathcal{A}^m(x_1,\mathbf{s}_1)\mathrm{e}^{\mathrm{i}y_1 k\sin\theta_{\mathrm{in}}}+\mathrm{e}^{\mathrm{i}\mathbf{x}_1\cdot\mathbf{k}}\mathrm{e}^{\mathrm{i}m(\pi/2-\theta_{\mathrm{in}})}=0, \quad \text{for} \quad x_1>0, \quad (8)$$

References

Gower, A. L., M. J. A. Smith, W. J. Parnell, and I. D. Abrahams (2018). "Reflection from a multi-species material and its transmitted effective wavenumber". In: *Proc. R. Soc. A* 474(2212), p. 20170864. DOI: `10.1098/rspa.2017.0864`. URL: `http://rspa.royalsocietypublishing.org/content/474/2212/20170864` (visited on 04/22/2018).

Mishchenko, M. I., L. D. Travis, and D. W. Mackowski (1996). "T-matrix computations of light scattering by nonspherical particles: A review". In: *Journal of Quantitative Spectroscopy and Radiative Transfer*. Light Scattering by Non-Spherical Particles

55(5), pp. 535–575. DOI: 10.1016/0022-4073(96)00002-7. URL: http://www.sciencedirect.com/science/article/pii/0022407396000027.

Varadan, V. K., V. V. Varadan, and Y.-H. Pao (1978). "Multiple scattering of elastic waves by cylinders of arbitrary cross section. I. SH waves". In: *The Journal of the Acoustical Society of America* 63(5), pp. 1310–1319. URL: http://scitation.aip.org/content/asa/journal/jasa/63/5/10.1121/1.381883 (visited on 09/14/2016).

Waterman, P. C. (1971). "Symmetry, Unitarity, and Geometry in Electromagnetic Scattering". In: *Physical Review D* 3(4), pp. 825–839. DOI: 10.1103/PhysRevD.3.825. URL: https://link.aps.org/doi/10.1103/PhysRevD.3.825.