

# Numerically solving integral equations of wave ensembles

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## **Abstract**

Allows for a broad frequency range, and to easily test different statistical assumptions. Assumptions such as the pair-correlation and QCA.

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# 1 Effective waves for uniformly distributed species

We consider a halfspace  $x > 0$  filled with  $S$  types of inclusions (species) that are uniformly distributed. The fields are governed by the scalar wave equation:

$$\nabla^2 u + k^2 u = 0, \quad (\text{in the background material}) \quad (1)$$

$$\nabla^2 u + k_j^2 u = 0, \quad (\text{inside the } j\text{-th scatterer}), \quad (2)$$

The background and species material properties are summarised in Table 1. The goal is to calculate how a medium with these scatterers, randomly uniformly distributed, reflects and transmits waves in an ensemble average sense.

For simplicity we will consider that all particles are cylindrical, though it is easy to extend the results to any smooth particle by using Waterman's T-matrix Waterman (1971); Varadan et al. (1978); Mishchenko et al. (1996).

Background properties:	wavenumber $k$	density $\rho$	sound speed $c$
Specie properties:	number density $\mathbf{n}_j$	density $\rho_j$	sound speed $c_j$ radius $a_j$
total number density $\mathbf{n}$	effective wavenumber $k_*$	species min. distance $a_{j\ell} > a_j + a_\ell$	

Table 1: Summary of material properties and notation. The index  $j$  refers to properties of the  $j$ -th species. Note a typical choice for  $a_{j\ell}$  is  $a_{j\ell} = c(a_j + a_\ell)$ , where  $c = 1.01$ .

## 2 Cylindrical species

We consider an incident wave

$$u_{\text{in}} = e^{i\mathbf{k} \cdot \mathbf{x}} \quad \text{with} \quad \mathbf{k} \cdot \mathbf{x} = kx \cos \theta_{\text{in}} + ky \sin \theta_{\text{in}}, \quad (3)$$

and angle of incidence  $\theta_{\text{in}}$  from the  $x$ -axis, exciting a material occupying the halfspace  $x > 0$ .

Combining equations (3.6) and the quasicrystalline approximation (3.10) from (Gower et al., 2018), we arrive at

$$\sum_{n=-\infty}^{\infty} \int_{\mathcal{S}} \int_{\substack{x_2 > 0 \\ \|\mathbf{x}_1 - \mathbf{x}_2\| > a_{12}}} \mathcal{A}_n(k\mathbf{x}_2, \mathbf{s}_2) F_{n-m}(k\mathbf{x}_2 - k\mathbf{x}_1, k) d\mathbf{x}_2 d\mathbf{s}_2^n \\ + \mathcal{A}_m(k\mathbf{x}_1, \mathbf{s}_1) + e^{i\mathbf{x}_1 \cdot \mathbf{k}} e^{im(\pi/2 - \theta_{\text{in}})} = 0, \quad \text{for } x_1 > 0, \quad (4)$$

where

$$d\mathbf{s}_2^n = \mathbf{n} Z_n(\mathbf{s}_2) p(\mathbf{s}_2) d\mathbf{s}_2, \quad (5)$$

$$F_n(\mathbf{X}, k) = (-1)^n e^{in\Theta} H_n(R) (1 + g(R/k; \mathbf{s}_1, \mathbf{s}_2)), \quad (6)$$

with  $(R, \Theta)$  being the polar coordinates of  $\mathbf{X} = (X, Y)$ ,  $p(\mathbf{s}_1)$  is the probability density function of picking a species in  $\mathcal{S}$  and we assumed statistical independence  $p(\mathbf{s}_1, \mathbf{s}_2) = p(\mathbf{s}_1)p(\mathbf{s}_2)$ . Note we included  $k$  in the argument of  $\mathcal{A}_m$  for convenience, as later we will non-dimensionalise. The function  $g(R; \mathbf{s}_1, \mathbf{s}_2)$  is the pair-correlation, assuming  $R$  is the distance between two particles, one centred of type  $\mathbf{s}_1$  and another of type  $\mathbf{s}_2$ . If we were to use whole correction, then  $g(R; \mathbf{s}_1, \mathbf{s}_2) = 0$ . For most random systems we expect that rapidly  $g(R; \mathbf{s}_1, \mathbf{s}_2) \rightarrow 0$  as  $R \rightarrow \infty$ , so we will assume

$$g(R; \mathbf{s}_1, \mathbf{s}_2) = 0, \quad \text{for } R > \bar{a}_{12}. \quad (7)$$

In terms of the notation from (Gower et al., 2018):

$$\begin{aligned}
|\mathcal{R}_N|p(\mathbf{\Lambda}_2|\mathbf{\Lambda}_1) &= |\mathcal{R}_N|^2 \frac{p(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2)}{p(\mathbf{s}_1)} = |\mathcal{R}_N|^2 p(\mathbf{s}_2) p(\mathbf{x}_1, \mathbf{x}_2 | \mathbf{s}_1, \mathbf{s}_2) \\
&= p(\mathbf{s}_2) (1 + g(\|\mathbf{x}_1 - \mathbf{x}_2\|; \mathbf{s}_1, \mathbf{s}_2)). \quad (8)
\end{aligned}$$

For computational efficiency, we will change variables to

$$X = k\mathbf{x}_2 - k\mathbf{x}_1, \quad (x, y) = (kx_1, ky_1), \quad (9)$$

We also borrow equation (4.1) from (Gower et al., 2018) to substitute

$$\mathcal{A}_m(kx_1, ky_1, \mathbf{s}) = \mathcal{A}_m(kx_1, \mathbf{s}) e^{iky_1 \sin \theta_{\text{in}}}, \quad (10)$$

which is due to the symmetry of (4). Substituting the above into (4), we can rewrite the integrated term:

$$\begin{aligned}
&\int_{\substack{x_2 > 0 \\ \|\mathbf{x}_2 - \mathbf{x}_1\| > a_{12}}} \mathcal{A}_n(kx_2, \mathbf{s}_2) e^{iy_2 k \sin \theta_{\text{in}}} F_{n-m}(k\mathbf{x}_2 - k\mathbf{x}_1, k) d\mathbf{x}_2 = \\
&\quad \frac{e^{iy \sin \theta_{\text{in}}}}{k^2} \int_{X > -x} \mathcal{A}_n(X + x, \mathbf{s}_2) \int_{Y^2 > k^2 a_{12}^2 - X^2} e^{iY \sin \theta_{\text{in}}} F_{n-m}(\mathbf{X}, k) dY dX,
\end{aligned}$$

then we split the integral on the right in the form

$$\int_{Y^2 > k^2 a_{12}^2 - X^2} e^{iY \sin \theta_{\text{in}}} F_{n-m}(\mathbf{X}, k) dY = \chi_{\{|X| < ka_{12}\}} B_{n-m}(X, k) + \chi_{\{|X| > ka_{12}\}} S_{n-m}(X, k),$$

where by using (7), equation (B.3) from Gower et al. (2018),

$$S_n(X, k) = \int_{-\infty}^{\infty} e^{iY \sin \theta_{\text{in}}} F_n(\mathbf{X}, k) dY = G_n(X, k) + \frac{2}{\cos \theta_{\text{in}}} \begin{cases} i^n e^{-in\theta_{\text{in}}} e^{iX \cos \theta_{\text{in}}} & X \geq 0, \\ (-i)^n e^{in\theta_{\text{in}}} e^{-iX \cos \theta_{\text{in}}} & X < 0, \end{cases} \quad (11)$$

with

$$\begin{aligned} G_n(X, k) &= (-1)^n \int_{-\sqrt{k^2 \bar{a}_{12}^2 - X^2}}^{\sqrt{k^2 \bar{a}_{12}^2 - X^2}} e^{iY \sin \theta_{\text{in}}} e^{in\Theta} H_n(R) g(R/k; \mathbf{s}_1, \mathbf{s}_2) dY \\ &= 2(-1)^n \int_0^{\sqrt{k^2 \bar{a}_{12}^2 - X^2}} \cos(Y \sin \theta_{\text{in}} + n\Theta) H_n(R) g(R/k; \mathbf{s}_1, \mathbf{s}_2) dY \end{aligned} \quad (12)$$

The term

$$\begin{aligned} B_n(X, k) &= \int_{-\infty}^{\infty} \chi_{\{Y^2 > k^2 a_{12}^2 - X^2\}} e^{iY \sin \theta_{\text{in}}} F_n(\mathbf{X}, k) dY \\ &= 2(-1)^n \int_{\sqrt{k^2 a_{12}^2 - X^2}}^{\infty} \cos(Y \sin \theta_{\text{in}} + n\Theta) H_n(R) (1 + g(R/k; \mathbf{s}_1, \mathbf{s}_2)) dY. \end{aligned} \quad (13)$$

It is difficult to numerically integrate the above because the integrand tends to zero very slowly as  $Y$  increases. To numerically integrate the above we use

$$\cos(Y \sin \theta_{\text{in}} + n\Theta) = \cos((n\pi)/2 + Y \sin(\theta_{\text{in}})) + \mathcal{O}(X/Y), \quad (14)$$

$$H_n(R) = -(-1)^{3/4} e^{-in\pi/2} \sqrt{\frac{2}{\pi Y}} + \mathcal{O}(X^{3/2}/Y^{3/2}), \quad (15)$$

which we use to rewrite

$$\begin{aligned}
B_n(X, k) &= 2(-1)^n \int_{\sqrt{k^2 a_{12}^2 - X^2}}^{Y_1} \cos(Y \sin \theta_{\text{in}} + n\Theta) H_n(R) (1 + g(R/k; \mathbf{s}_1, \mathbf{s}_2)) dY \\
&+ \frac{1+i}{\sqrt{\pi Y_1} \cos(\theta_{\text{in}})^2} e^{iY_1(1-\sin(\theta_{\text{in}}))} [1 + (-1)^n e^{2iY_1 \sin(\theta_{\text{in}})} (1 - \sin(\theta_{\text{in}})) + \sin(\theta_{\text{in}})] + \mathcal{O}(X/Y),
\end{aligned} \tag{16}$$

where we assumed that  $g(R/k; \mathbf{s}_1, \mathbf{s}_2) = 0$  for  $Y > Y_1$ .

Written in full the governing equation (4) now becomes

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \int_{\mathcal{S}} \int_{X>0} \mathcal{A}_n(X, \mathbf{s}_2) K(X-x, k) dX d\mathbf{s}_2^n \\
&+ k^2 \mathcal{A}_m(x, \mathbf{s}_1) + k^2 e^{ix \cos \theta_{\text{in}}} e^{im(\pi/2 - \theta_{\text{in}})} = 0, \quad \text{for } x_1 > 0, \tag{17}
\end{aligned}$$

where I swapped the integration variable  $X \rightarrow X - x$ . where

$$K(X-x, k) = \chi_{\{|X-x|>ka_{12}\}} S_{n-m}(X-x, k) + \chi_{\{|X-x|<ka_{12}\}} B_{n-m}(X-x, k). \tag{18}$$

## 2.1 Numerical solution

To start we consider only one single species and consider only whole correction pair-correlation. After trying methods based on Chebyshev and function approximation, I've decided they are too computationally intense. For this reason I'm going with the simple discretisation:  $\mathcal{A}_n^j = \mathcal{A}_n(x^j)$  for  $x^j = jh$  and  $j = 0, \dots, J$ . A regular spaced mesh is best for convolutions. With analogous notation for the other fields, let the vectors:

$$\mathcal{A}_n = (\mathcal{A}_n^j)_j, \quad \mathcal{S}_n = (S_n^j)_j, \quad \mathcal{B}_n = (B_n^j)_j, \quad \mathbf{b}_n = -k^2 (e^{ix^j \cos \theta_{\text{in}}} e^{in(\pi/2 - \theta_{\text{in}})})_j. \tag{19}$$

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} \int_{\mathcal{S}} \int_{X>0} \mathcal{A}_n(X) S_{n-m}(X-x) dX d\mathbf{s}_2^n \\
& + \sum_{n=-\infty}^{\infty} \int_{\mathcal{S}} \int_{\substack{|X-x|<ka_{12} \\ X>0}} \mathcal{A}_n(X) (B_{n-m}(X-x, k) - S_{n-m}(X-x)) dX d\mathbf{s}_2^n \\
& + k^2 \mathcal{A}_m(x) + k^2 e^{ix \cos \theta_{\text{in}}} e^{im(\pi/2 - \theta_{\text{in}})} = 0, \quad \text{for } x_1 > 0 \quad (20)
\end{aligned}$$

**Art:** as the integrals are convolutions, it may be possible to use a Fourier of Laplace transform to solve this.

Now considering only one species (to reduce the number of sums)

$$\mathbf{n} \sum_{n,j \geq 0} Z_n \sigma_j \left( S_{n-m}^{j-\ell} + (B_{n-m}^{j-\ell} - S_{n-m}^{j-\ell}) \chi_{\{|j-\ell| \leq p\}} \right) \mathcal{A}_n^j + k^2 \mathcal{A}_m^\ell = b_m^\ell, \quad (21)$$

**Art:** the above discrete integrals have been checked against Mathematica brute force numerical evaluation.

for  $\ell > 0$ , where  $p = \lfloor ka_{12}/h \rfloor$ , and  $\sigma_j$  represents the discrete integral. In matrix form,

$$\sum_n \mathcal{E}_{nm} + \mathbf{n} \sum_n Z_n (\mathbf{P}_{n-m} + \mathbf{Q}_{n-m}) \mathcal{A}_n + k^2 \mathbf{I} \mathcal{A}_m = \mathbf{b}_m, \quad (22)$$

where the components of matrices are

$$P_n^{\ell j} = \sigma_j S_n^{j-\ell}, \quad Q_n^{\ell j} = \sigma_j (B_n^{j-\ell} - S_n^{j-\ell}) \chi_{\{|j-\ell| \leq p\}}, \quad (23)$$

with  $j, \ell = 0, 1, \dots, J$ , and

$$\mathcal{E}_{nm}^\ell = \mathbf{n} Z_n \int_{X>x^J} \mathcal{A}_n(X) S_{n-m}(X-x^\ell) dX. \quad (24)$$

If we do not include  $\mathcal{E}_{nm}^\ell$ , then the solution of (22) would be valid for a plate occupying the

region  $0 \leq x \leq x^J$ . If we wish to calculate the backscattering from the whole halfspace, then we can calculate  $\mathcal{E}_{nm}^\ell$  by approximating  $\mathcal{A}_n(x)$  as a sum of plane waves, shown below.

## 2.2 Effective waves

The average scattering coefficients can be approximated by a sum of plane waves:

$$\mathcal{A}_n(X) = i^n \sum_p e^{-in\theta_p} A_n^p e^{i(X-\bar{x})\frac{k_p}{k} \cos \theta_p} \quad \text{for } X > \bar{x}, \quad (25)$$

where  $\bar{x}$  is for now undetermined, except  $\bar{x} < x^J$ , so that the above has a region of overlap with the discrete form (19). For each  $p$  the terms  $A_n^p$  and  $k_p$  satisfy the dispersion equation

$$\sum_n D_{nm}(k_p) A_n^p = 0, \quad (26)$$

for all  $m$ . We can then determine  $k_p$  by solving  $\det(D_{nm}(k_p)) = 0$ , where we require  $\text{Im } k_p > 0$  to guarantee that the integral (24) converges, and to lead to a physically reasonable solution. The components  $D_{nm}$  are given by... Once  $k_p$  is determined, then (26) can be used to write the vector  $\mathbf{A}^p = [\dots, A_{-n}^p, A_{1-n}^p, \dots, A_{n-1}^p, A_n^p, \dots]$  in the form

$$\mathbf{A}^p = \alpha^p \mathbf{a}^p, \quad (27)$$

where  $\mathbf{a}^p$  is determined from (26) and  $\alpha^p$  is unknown. With  $k_p$ , we can also determine the angle  $\theta_p$  by using Snell's law:  $k_p \sin \theta_p = k \sin \theta_{\text{in}}$ . From here onwards, we assume that  $k_p$ ,  $\theta_p$  and the  $\mathbf{a}^p$  have been determined.



The average wave (25) also needs to satisfy (extinction of the incident wave)

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{2k\mathbf{n}Z_n}{\cos\theta} \sum_p A_n^p \frac{e^{in(\theta_{\text{in}}-\theta_p)} e^{-i\bar{x}\cos\theta}}{k_p \cos\theta_p - k \cos\theta} = \\
ik^2 + \frac{2i\mathbf{n}}{\cos\theta} \sum_{n=-\infty}^{\infty} \int_0^{\bar{x}} \mathcal{A}_n(X) Z_n (-i)^n e^{in\theta_{\text{in}}} e^{-iX \cos\theta_{\text{in}}} dX \\
= ik^2 + i\mathbf{n} \sum_{n=-\infty}^{\infty} Z_n \int_0^{\bar{x}} \mathcal{A}_n(X) S_n(-X) dX. \quad (28)
\end{aligned}$$

For direct backscattering,  $\theta_{\text{in}} = \theta_p = 0$ , we reach a simpler form:

$$\sum_{n=-\infty}^{\infty} 2k\mathbf{n}Z_n \sum_p A_n^p \frac{e^{-i\bar{x}}}{k_p - k} = ik^2 + i\mathbf{n} \sum_{n=-\infty}^{\infty} Z_n \int_0^{\bar{x}} \mathcal{A}_n(X) S_n(-X) dX. \quad (29)$$

Using (27), the discrete version of (28) becomes

$$\sum_p W^p \alpha^p - i\mathbf{n} \sum_n Z_n \sum_{j=0}^L \sigma_j S_n^{-j} \mathcal{A}_n^j = ik^2 \implies \quad (30)$$

$$\mathbf{W} \cdot \boldsymbol{\alpha} - \sum_n \mathbf{V}_n \cdot \boldsymbol{\mathcal{A}}_n = ik^2, \quad (31)$$

where  $x^L = hL = \bar{x} < x^J$  and

$$W^p = \sum_{n=-\infty}^{\infty} \frac{2k\mathbf{n}Z_n}{\cos\theta} \frac{e^{in(\theta_{\text{in}}-\theta_p)} e^{-i\bar{x}\cos\theta}}{k_p \cos\theta_p - k \cos\theta} \alpha_n^p, \quad V_n^j = i\mathbf{n}Z_n \sum_{j=0}^L \sigma_j S_n^{-j}. \quad (32)$$

To determine the  $\alpha^p$  we have so far only the scalar equation (31). To completely determine the  $\alpha^p$  (assuming there is more than one), we need to match the ansatz (25) with the numerical solution introduced in Section 2.1.

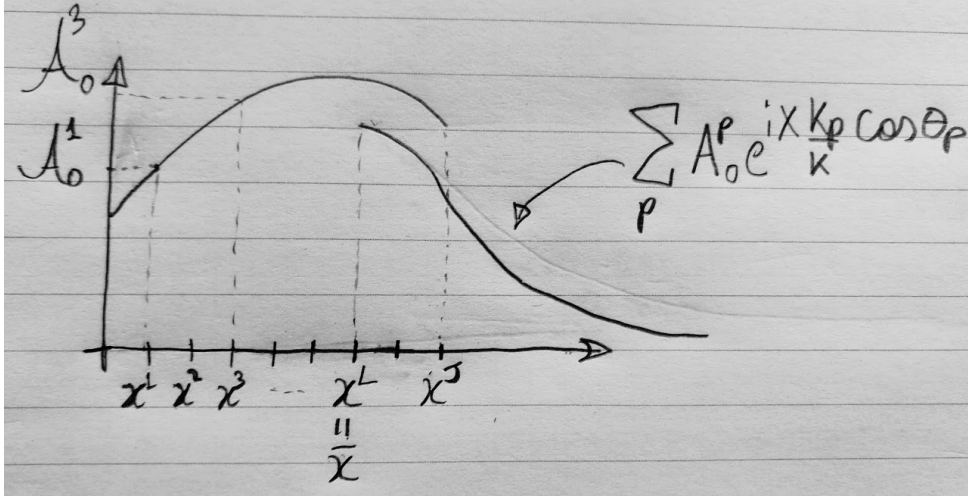


Figure 1: Shows the overlap region of the numerical solution and the ansatz approach.

### 2.3 Connecting the integral and ansatz approach

Need to substitute:

$$A_n^p \rightarrow A_n^p e^{-i\bar{x} \frac{k_p}{k} \cos \theta_p}$$

Substituting (25) into (24) and using (11) we arrive at

$$\begin{aligned} \mathcal{E}_{nm}^\ell &= \frac{2nZ_n}{\cos \theta_{\text{in}}} (-1)^n i^{-m} e^{i(m-n)\theta_{\text{in}}} \sum_p e^{-in\theta_p} A_n^p \int_{X > x^J} e^{i\frac{X}{k}(k_p \cos \theta_p + k \cos \theta_{\text{in}})} e^{-i x^\ell \cos \theta_{\text{in}}} dX \\ &= \frac{2knZ_n}{\cos \theta_{\text{in}}} (-1)^n i^{-m} \sum_p A_n^p e^{i(m-n)\theta_{\text{in}} - in\theta_p} \frac{ie^{i\frac{x^J}{k}(k_p \cos \theta_p + k \cos \theta_{\text{in}}) - i x^\ell \cos \theta_{\text{in}}}}{k_p \cos \theta_p + k \cos \theta_{\text{in}}}, \quad (33) \end{aligned}$$

where we used  $X - x^\ell > x^J - x^\ell \geq 0$ , for  $\ell < J$ , when substituting  $S_{n-m}(X - x^\ell)$  with (11). Above we can see that the effective waves with the smallest  $\text{Im } k_p$ , and largest  $A_n^p$ , contribute the most to  $E_{nm}^\ell$ . The case of direct backscattering,  $\theta_{\text{in}} = \theta_p = 0$ , leads to the simpler form

$$\mathcal{E}_{nm}^\ell = 2knZ_n (-1)^n i^{-m} \sum_p A_n^p \frac{ie^{i\frac{x^J}{k}(k_p + k) - i x^\ell}}{k_p + k}. \quad (34)$$

Using (27), we write (33) in the matrix form

$$\sum_n \mathcal{E}_{nm}^\ell = \sum_n \sum_p E_{nm}^{\ell p} a_n^p \alpha^p = (\mathbf{E}_m \boldsymbol{\alpha})^\ell \quad \text{with} \quad \boldsymbol{\alpha} = [\alpha^1, \alpha^2, \dots]. \quad (35)$$

Equation (32), together with (22) and (35) combined:

$$\mathbf{E}_m \boldsymbol{\alpha} + \mathbf{n} \sum_n Z_n (\mathbf{P}_{n-m} + \mathbf{Q}_{n-m}) \mathcal{A}_n + k^2 \mathbf{I} \mathcal{A}_m = \mathbf{b}_m, \quad (36)$$

valid for all  $m$ , are the available equations to determine the unknowns  $\mathcal{A}_n^j$  and  $\alpha^p$ . If the  $\alpha^p$  were all known, then we could use the above to determine the  $\mathcal{A}_n$ . However, there is only (32), one scalar equation, to determine the  $\alpha^p$

Finally in block-matrix form:  $\mathbf{M} \mathcal{A} = \mathbf{b}$ , where

$$\mathbf{M}_{mn} = \mathbf{n} Z_n (\mathbf{P}_{n-m} + \mathbf{Q}_{n-m}) + k^2 \delta_{mn} \mathbf{I}, \quad (37)$$

$$\mathcal{A} = [\mathcal{A}_{-M}, \mathcal{A}_{1-M}, \dots, \mathcal{A}_{M-1}, \mathcal{A}_M]^T, \quad (38)$$

$$\mathbf{b} = [\mathbf{b}_{-M}, \mathbf{b}_{1-M}, \dots, \mathbf{b}_{M-1}, \mathbf{b}_M]^T. \quad (39)$$

### 3 Reflection

Using equations (3.7-3.8) from Gower et al. (2018), we have that the averaged reflected wave is

$$u_R(kx_R, ky_R) = \mathbf{n} \sum_{m=-\infty}^{\infty} Z_m \int \mathcal{A}_m(kx_1, ky_1) F_m(k\mathbf{x}_1 - k\mathbf{x}_R, k) dy_1 dx_1, \quad (40)$$

which can be rewritten by using (10) and  $\mathbf{X} = k\mathbf{x}_1 - k\mathbf{x}_R$  (assuming  $k > 0$ ) to arrive at

$$\begin{aligned}
u_R(kx_R, ky_R) &= \mathbf{n} \sum_{m=-\infty}^{\infty} Z_m \frac{e^{iky_R \sin \theta_{\text{in}}}}{k} \int_0^{\infty} \mathcal{A}_m(kx_1) \int_{-\infty}^{\infty} e^{iY \sin \theta_{\text{in}}} F_m(\mathbf{X}, k) dY dx_1, \\
&= \frac{\mathbf{n}}{k} e^{iky_R \sin \theta_{\text{in}}} \sum_{m=-\infty}^{\infty} Z_m \int_0^{\infty} \mathcal{A}_m(kx_1) \left[ G_n(kx_1 - kx_R, k) + \frac{2i^m e^{-im\theta_{\text{in}}} e^{i(kx_1 - kx_R) \cos \theta_{\text{in}}}}{\cos \theta_{\text{in}}} \right] dx_1,
\end{aligned} \tag{41}$$

where we used (11) and that  $kx_1 - kx_R > 0$ , because  $x_1$  is in the halfspace  $x_1 > 0$  and the receiver position  $x_R$  satisfies  $x_R < 0$ . If we use whole correction, then  $G_n = 0$ , which leads to

$$u_R(kx_R, ky_R) = e^{ik(-x_R \cos \theta_{\text{in}} + y_R \sin \theta_{\text{in}})} \frac{\mathbf{n}}{k} \sum_{m=-\infty}^{\infty} \frac{2i^m e^{-im\theta_{\text{in}}}}{\cos \theta_{\text{in}}} Z_m \int_0^{\infty} \mathcal{A}_m(kx_1) e^{ikx_1 \cos \theta_{\text{in}}} dx_1. \tag{42}$$

The above matches exactly equation (7.12) from Gower et al. (2018), when using their ansatz (4.3).

To numerically evaluate the reflection coefficient above, we use  $x = kx_1$  and discretise in the same way as before to get

$$R = \sum_{m=-\infty}^{\infty} \frac{2\mathbf{n}i^m e^{-im\theta_{\text{in}}}}{k^2 \cos \theta_{\text{in}}} Z_m \sum_j \sigma_j \mathcal{A}_m^j e^{ix^j \cos \theta_{\text{in}}}. \tag{43}$$

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