Notes on effective waves in a multi-species material

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Abstract

Brief self-contained notes on how to calculate effective wavenumbers for uniformly distributed multi-species inclusions. The formulas for multi-species cylinders and spheres are given, as well as reflection from a halfspace filled with cylinders. Code to implement the formulas is given in `https://github.com/arturgower/EffectiveWaves.jl`. For detailed derivations see our paper "Reflection from a multi-species material and its transmitted effective wavenumber", which shows how to introduce a pair-correlation between the species.

Keywords: polydisperse, multiple scattering, multi-species, effective waves, quasicrystalline approximation, statistical methods

1 Effective waves for uniformly distributed species

We consider a halfspace $x > 0$ filled with S types of inclusions (species) that are uniformly distributed. The fields are governed by the scalar wave equation:

$$\nabla^2 u + k^2 u = 0, \quad \text{(in the host and outside all scatterers)} \tag{1}$$

$$\nabla^2 u + k_j^2 u = 0, \quad \text{(inside the } j\text{-th scatterer)}, \tag{2}$$

The host and species material properties are summarised in Table 1. The goal is to find an effective homogeneous medium with wavenumber k_*, where waves propagate, in an ensemble average sense, with the same speed and attenuation as they would in a material filled with scatterers. See Gower (2017) for the code that implements the formulas below.

Below we present the effective wavenumber, for any incident wavenumber and moderate number fraction, when the species are either all cylinders or spheres*. For cylindrical inclusions we also present the reflection of a plane wave from this multi-species material.

Host material properties:	wavenumber k	density ρ	sound speed c	
specie material properties:	number density $\mathfrak{n}_j$	density ρ_j	sound speed c_j	radius a_j
total number density $\mathfrak{n}$	effective wavenumber k_*		specie min. distance $a_{j\ell} > a_j + a_\ell$	

Table 1: Summary of material properties and notation. The index j refers to properties of the j-th specie. Note a typical choice for $a_{j\ell}$ is $a_{j\ell} = c(a_j + a_\ell)$, where $c = 1.01$.

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*In principal these formulas can be extended to include different shaped scatterers by using Waterman's T-matrix Waterman (1971); Varadan et al. (1978); Mishchenko et al. (1996)

1.1 Cylindrical species

We consider an incident wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} \quad \text{with} \quad \mathbf{k}\cdot\mathbf{x} = kx\cos\theta_{\text{in}} + ky\sin\theta_{\text{in}}, \tag{3}$$

and angle of incidence θ_{in} from the x-axis, exciting a material occupying the halfspace $x > 0$. Then, assuming low number density $\mathfrak{n}$ (or low volume fraction $\sum_\ell \pi a_\ell^2 \mathfrak{n}_\ell$), the effective transmitted wavenumber k_* becomes

$$k_*^2 = k^2 - 4i\mathfrak{n}\langle f_\circ\rangle(0) - 4i\mathfrak{n}^2\langle f_{\circ\circ}\rangle(0) + \mathcal{O}(\mathfrak{n}^3), \tag{4}$$

with $\langle f_\circ\rangle$ and $\langle f_{\circ\circ}\rangle$ given by (8). The above reduces to Linton and Martin (2005) equation (81) for a single species, which has been confirmed by several independent methods Martin et al. (2010); Martin and Maurel (2008); Chekroun et al. (2012); Kim (2010) in the low frequency limit.

The ensemble-average reflected wave measured at $x < 0$ is given by

$$\frac{\mathfrak{n}}{\alpha^2}\left[R_1 + \mathfrak{n}R_2\right]e^{-i\alpha x + i\beta y} + \mathcal{O}(\mathfrak{n}^3), \tag{5}$$

where

$$R_1 = i\langle f_\circ\rangle(\theta_{\text{ref}}), \quad \theta_{\text{ref}} = \pi - 2\theta_{\text{in}}, \tag{6}$$

$$R_2 = \frac{2\langle f_\circ\rangle(0)}{k^2\cos^2\theta_{\text{in}}}\left[\langle f_\circ\rangle'(\theta_{\text{ref}})\sin\theta_{\text{in}}\cos\theta_{\text{in}} - \langle f_\circ\rangle(\theta_{\text{ref}})\right] + i\langle f_{\circ\circ}\rangle(\theta_{\text{ref}}), \tag{7}$$

which reduces to Martin (2011) equations (40-41) for a single species, which they show agrees with other known results small k.

The ensemble-average far-field pattern and multiple-scattering pattern are[†]

$$\langle f_\circ \rangle(\theta) = -\sum_{\ell=1}^{S} \sum_{n=-\infty}^{\infty} \frac{\mathfrak{n}_\ell}{\mathfrak{n}} Z_\ell^n \mathrm{e}^{\mathrm{i}n\theta},$$

$$\langle f_{\circ\circ} \rangle(\theta) = -\pi \sum_{\ell,j=1}^{S} \sum_{m,n=-\infty}^{\infty} a_{\ell j}^2 d_{n-m}(ka_{\ell j}) \frac{\mathfrak{n}_\ell \mathfrak{n}_j}{\mathfrak{n}^2} Z_\ell^n Z_j^m \mathrm{e}^{\mathrm{i}n\theta}, \tag{8}$$

where $d_m(x) = J'_m(x)H'_m(x) + (1 - (m/x)^2)J_m(x)H_m(x)$, the J_m are Bessel functions, the H_m are Hankel functions of the first kind and $a_{\ell j} > a_\ell + a_j$ is some fixed distance. The Z_j^m describe the type of scatterer:

$$Z_j^m = \frac{q_j J'_m(ka_j)J_m(\gamma ka_j) - J_m(ka_j)J'_m(\gamma ka_j)}{q_j H'_m(ka_j)J_m(\gamma ka_j) - H_m(ka_j)J'_m(\gamma ka_j)} = Z_j^{-m}, \tag{9}$$

with $q = (\rho_j c_j)/(\rho c)$ and $\gamma = c/c_j$. For instance, taking the limits $q \to 0$ or $q \to \infty$, recovers Dirichlet or Neumann boundary conditions, respectively.

1.2 Effective wavenumber for multi-species spheres

The results here are derived by applying the theory in our paper... to the results in Linton and Martin (2006). We omit the details as the result follows by direct analogy.

For spherical inclusions the transmitted wavenumber becomes,

$$k_*^2 = k^2 - \mathfrak{n}\frac{4\pi\mathrm{i}}{k}\langle F_\circ \rangle(0) + \mathfrak{n}^2 \frac{(4\pi)^2}{k^4}\langle F_{\circ\circ} \rangle + \mathcal{O}(\mathfrak{n}^3), \tag{10}$$

where for spheres we define the ensemble-average far-field pattern and multiple-scattering

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[†]Note we introduced the terminology "multiple-scattering pattern".

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pattern,

$$\langle F_\circ \rangle(\theta) = -\sum_{n=0}^{\infty} P_n(\cos\theta) \sum_{j=1}^{S} (2n+1) \zeta_j^n \frac{\mathfrak{n}_j}{\mathfrak{n}}, \tag{11}$$

$$\langle F_{\circ\circ} \rangle = \frac{i(4\pi)^2}{2} \sum_{n,p=0}^{\infty} \sum_{j,\ell=1}^{S} \sum_q \frac{\sqrt{(2n+1)(2p+1)}}{(4\pi)^{3/2}} \sqrt{2q+1}\, \mathcal{G}(n,p,q) k a_{j\ell} D_q(k a_{j\ell}) \zeta_j^n \zeta_\ell^p \frac{\mathfrak{n}_j \mathfrak{n}_\ell}{\mathfrak{n}^2},$$

where

$$D_m(x) = x j_m'(x)(x h_m'(x) + h_m(x)) + (x^2 - m(m+1)) j_m(x) j_m(x),$$

P_n are the Legendre polynomials, j_m are the spherical Bessel functions, h_m are spherical Hankel functions of the first kind and

$$\zeta_j^m = \frac{q_j j_m'(k a_j) j_m(\gamma k a_j) - j_m(k a_j) j_m'(\gamma k a_j)}{q_j h_m'(k a_j) j_m(\gamma k a_j) - h_m(k a_j) j_m'(\gamma k a_j)} = \zeta_j^{-m}, \tag{12}$$

with $q = (\rho_j c_j)/(\rho c)$ and $\gamma = c/c_j$, where the $\mathcal{G}$ is a Gaunt coefficient and is equal to

$$\mathcal{G}(n,p,q) = \frac{\sqrt{(2n+1)(2p+1)(2q+1)}}{2\sqrt{4\pi}} \int_0^\pi P_n(\cos\theta) P_p(\cos\theta) P_q(\cos\theta) \sin\theta d\theta.$$

See Caleap et al. (2012) for details on reflection from a single species, although, to our knowledge, a rigours formula for reflection from a single species valid for moderate number fraction and any wavenumber has not yet been deduced.

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