THE YOKONUMA–TEMPERLEY–LIEB ALGEBRA

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Abstract. In this paper we introduce the Yokonuma–Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra over a two-sided ideal generated by an expression analogous to the one of the classical Temperley-Lieb algebra. The main theorem provides necessary and sufficient conditions for the Markov trace defined on the Yokonuma–Hecke algebra to pass through to the quotient algebra, leading to a sequence of knot invariants which coincide with the Jones polynomial.

Introduction

The Temperley–Lieb algebra appeared originally in Statistical Mechanics and is important in several areas of Mathematics. In his seminal work V.F.R. Jones [15] constructed a Markov trace on the Temperley–Lieb algebra, leading to unexpected applications in knot theory as well as to a fertile interaction between Knot theory and Representation theory. In algebraic terms, the Temperley–Lieb algebra can be defined as a quotient of the Iwahori–Hecke algebra.

In [8] the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ (defined originally in [20]) has been defined as a quotient of the modular framed braid group $\mathcal{F}_{d,n}$, which comprises framed braids with framings modulo d, over a quadratic relation (Eq. 13) involving the framing generators t_i by means of certain weighted idempotents e_i (Eq. 9). Setting $d = 1$, the algebra $\mathrm{Y}_{1,n}(u)$ coincides with the Iwahori–Hecke algebra $\mathrm{H}_n(u)$. The Yokonuma–Hecke algebras have been studied in [20, 8, 10, 19, 3]. Further, in [8] the second author found an inductive linear basis for the algebras $\mathrm{Y}_{d,n}(u)$ and constructed a unique Markov trace tr on these algebras depending on parameters $z, x_1, \ldots, x_{d-1}$. Aiming to extracting framed link invariants from tr, as it turned out in [11], tr does not re-scale directly according to the framed braid equivalence, leading to conditions that have to be imposed on the trace parameters $x_1, \ldots, x_{d-1}$; namely, they had to satisfy a non-linear system of equations, the E–*system* (Eq. 17). The x_i's being d^{th} roots of unity is one obvious solution. Gérardin found in [11, Appendix] the full set of solutions of the E–system. Given now any solution of the E–system, 2–variable isotopy invariants for framed, classical and singular links were constructed in [11, 12, 13] respectively, which are studied further in [1, 4].

In this paper we define an analogue for the Temperley–Lieb algebra in the context of framed braids, *the Yokonuma–Temperley–Lieb algebra*, denoted by $\mathrm{YTL}_{d,n}(u)$. It is defined as a quotient of the Yokonuma–Hecke algebra over a two-sided ideal I (Eq. 22 and Definition 2), analogous to the classical case. For $d = 1$ the algebra $\mathrm{YTL}_{1,n}(u)$ coincides with the Temperley–Lieb algebra. We first show that I is a principal ideal (Lemma 4) and we give a presentation for $\mathrm{YTL}_{d,n}(u)$ with non-invertible generators, analogous to the classical case (Proposition 2). We then give

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a spanning set $\Sigma_{d,n}$ for $\mathrm{YTL}_{d,n}(u)$, where each word in $\Sigma_{d,n}$ contains the highest and lowest index braiding generator exactly once (Proposition 4). Moreover, any word in $\Sigma_{d,n}$ inherits the splitting property from $\mathrm{Y}_{d,n}(u)$, that is, it splits into the framing part and the braiding part. We also present the results of Chlouveraki and Pouchin [2] on the dimension (Proposition 5) and a linear basis for $\mathrm{YTL}_{d,n}(u)$ (Theorem 3). From the spanning set $\Sigma_{d,n}$, they extracted an explicit basis for $\mathrm{YTL}_{d,n}(u)$ by describing a set of linear dependence relations among the framing parts for each fixed element in the braiding part. Finally, using the dimension results of [2] we find a basis for $\mathrm{YTL}_{2,3}(u)$ different than the basis in [2].

Next, we seek conditions such that the trace tr, defined on the algebras $\mathrm{Y}_{d,n}(u)$, passes to the quotient algebras $\mathrm{YTL}_{d,n}(u)$. More precisely, we compute first the values of the trace parameter z that annihilate the generator of the defining ideal I, which are the roots of a quadratic equation (Eq. 47). Then we annihilate the traces of all elements of $\mathrm{Y}_{d,n}(u)$ that lie in I and so we end up with a system (Σ) of quadratic equations in z (Eqs. 55a–55c). If we demand that (Σ) has both roots of Eq. 47 as common solutions, which is essential for discussing link invariants, we end up with sufficient conditions for the trace tr to pass to the quotient algebras $\mathrm{YTL}_{d,n}(u)$ (Theorem 5). More precisely, Theorem 5 states that the trace tr passes to the quotient algebra $\mathrm{YTL}_{d,n}(u)$ if the trace parameters are d^{th} roots of unity $x_1, \ldots, x_{d-1}$ and $z = -\frac{1}{u+1}$ and $z = -1$. Note that these two values for z are precisely the ones that Jones computed such that the Ocneanu trace on $\mathrm{H}_n(u)$ passes to the quotient, the Temperley–Lieb algebra $\mathrm{TL}_n(u)$. If we also let (Σ) to have one common solution for z we obtain the necessary and sufficient conditions for the trace tr to pass through to the quotient algebras $\mathrm{YTL}_{d,n}(u)$ (Theorem 6). More precisely, Theorem 6 states that the trace tr passes to the quotient algebras $\mathrm{YTL}_{d,n}(u)$ if and only if either the conditions of Theorem 5 are satisfied or the trace parameters $x_1, \ldots, x_{d-1}$ comprise a solution of the E–system (other than d^{th} roots of unity) or $z = -\frac{1}{2}$. This is our main result.

In [1] it is shown that if the trace parameters $x_1, \ldots, x_{d-1}$ are d^{th} roots of unity, then the classical link invariants derived from the algebra $\mathrm{Y}_{d,n}(u)$ coincide with the 2–variable Jones or Homflypt polynomial. Using Theorem 6 and the results in [1], we obtain from the invariants for framed and classical links in [11, 12] related to $\mathrm{Y}_{d,n}(u)$, 1–variable framed and classical link invariants through the algebras $\mathrm{YTL}_{d,n}(u)$. As we show, these invariants coincide with the Jones polynomial for the case of classical links and they are framed analogues of the Jones polynomial for the case of framed links.

The paper is organized as follows: In Section 1 we recall the definition and basic properties of the classical Temperley–Lieb algebra and the Yokonuma–Hecke algebra. In Section 2 we define the Yokonuma–Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra over a two-sided ideal (Eq. 22 and Definition 2), which we show that is a principal ideal (Lemma 4). Finally, we give a presentation for $\mathrm{YTL}_{d,n}(u)$ with non-invertible generators (Proposition 2). In Section 3 we present a spanning set for $\mathrm{YTL}_{d,n}(u)$ and the results of Chlouveraki and Pouchin [2] on the dimension and a linear basis for $\mathrm{YTL}_{d,n}(u)$. Then we give a basis for $\mathrm{YTL}_{2,3}(u)$. Section 4 focuses on the necessary and sufficient conditions under which the trace tr on $\mathrm{Y}_{d,n}(u)$ passes to the quotient algebra $\mathrm{YTL}_{d,n}(u)$ (Theorems 5 and 6). Finally, in Section 5 we discuss the invariants for classical and framed links that can be constructed through the trace tr and we recover the Jones polynomial.

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1. Preliminaries

1.1. *Notations.* Throughout the paper we shall fix the following notation. By the term algebra we mean an associative unital (with unity 1) algebra over the field $\mathbb{C}(u)$, where u is an indeterminate. The following two positive integers are also fixed: d and n.

As usual we denote by B_n the braid group on n strands, that is the group generated by the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i is the positive crossing between the i^{th} and the $(i+1)^{st}$ strand, satisfying the well-known braid relations: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$.

We denote by S_n the symmetric group on n symbols. Let s_i be the elementary transposition $(i, i+1)$. We denote by l the length function on S_n with respect to the s_i's.

Denote by $C_d = \langle t \,|\, t^d = 1\rangle$ the cyclic group of order d. Let $t_i = (1, \ldots, t, 1, \ldots, 1) \in C_d^n$, where t is in the i^{th} position. Denote also by $C_{d,n} := C_d^n \rtimes S_n$, where the action is defined by permutation on the indices of the t_i's, namely: $s_i t_j = t_{s_i(j)} s_i$.

Finally, we denote by $\mathrm{H}_n(u)$ the Iwahori–Hecke algebra, that is, the $\mathbb{C}(u)$–algebra defined by generators $1, v_1, \ldots, v_{n-1}$ who satisfy the following relations:

$$\begin{gathered} v_iv_jv_i = v_jv_iv_j, \quad |i-j| = 1 \\ v_iv_j = v_jv_i, \quad |i-j| > 1 \\ v_i^2 = (u-1)v_i + u. \end{gathered}$$

1.2. *The Temperley–Lieb algebra.* Originally, the Temperley–Lieb algebra, over $\mathbb{C}$, was defined by generators $1, f_1, \ldots, f_{n-1}$ subject to the following relations:

$$\begin{array}{rcl} f_i^2 & = & f_i \\ f_if_jf_i & = & \delta f_i, \quad |i-j| = 1 \\ f_if_j & = & f_jf_i, \quad |i-j| > 1 \end{array}$$

where δ is a non–zero complex number (see [6],[14],[15]). The generators f_i are non-invertible; one can define the Temperley–Lieb algebra with the following invertible generators (see [14]):

$$h_i := (u+1)f_i - 1 \tag{1}$$

where u is defined via the relation $\delta^{-1} = 2 + u + u^{-1}$. The Temperley algebra $\mathrm{TL}_n(u)$, over $\mathbb{C}(u)$, is defined by generators $h_1, \ldots, h_{n-1}$ under the relations:

$$h_ih_jh_i = h_jh_ih_j, \quad |i-j| = 1 \tag{2}$$
$$h_ih_j = h_jh_i, \quad |i-j| > 1 \tag{3}$$
$$h_i^2 = (u-1)h_i + u \tag{4}$$
$$h_ih_jh_i + h_jh_i + h_ih_j + h_i + h_j + 1 = 0, \quad |i-j| = 1. \tag{5}$$

Note that if $n > 3$ relations (5) are symmetric with respect to the indices i, j, so relations (2) follow from relations (5). Relations (2)–(4) are the well–known defining relations of the Iwahori–Hecke algebra $\mathrm{H}_n(u)$. Therefore, $\mathrm{TL}_n(u)$ can be considered as a quotient of $\mathrm{H}_n(u)$ via the morphism $v_i \mapsto h_i$. It turns out that the set:

$$\left\{ (h_{j_1}h_{j_1-1}\ldots h_{j_1-k_1})\,(h_{j_2}h_{j_2-1}\ldots h_{j_2-k_2})\ldots\left(h_{j_p}h_{j_p-1}\ldots h_{j_p-k_p}\right)\right\}$$

where $1 \le j_1 < j_2 < \ldots < j_p \le n-1$ and $1 \le j_1 - k_1 < j_2 - k_2 < \ldots < j_p - k_p$, furnishes a linear basis for $\mathrm{TL}_n(u)$ and the dimension of $\mathrm{TL}_n(u)$ is equal to the n^{th} Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$ [14, 15]. Recall finally, that in [5], Ocneanu constructed a unique Markov trace on the algebras $\mathrm{H}_n(u)$:

Theorem 1 (Ocneanu)**.** *For any $\zeta \in \mathbb{C}^{\times}$ there exists a linear trace τ on $\cup_{n=1}^{\infty} \mathrm{H}_n(u)$ uniquely defined by the inductive rules:*

(1) $\tau(ab) = \tau(ba), \quad a, b \in \mathrm{H}_n(u)$
(2) $\tau(1) = 1$
(3) $\tau(av_n) = \zeta\, \tau(a), \quad a \in \mathrm{H}_n(u)$.

Jones' methods for redefining his Markov trace on the Temperley–Lieb algebra as factoring of the Ocneanu trace on the Iwahori–Hecke algebra [14] tells us that the least requirement is that the Ocneanu trace respects the defining relations (5). This requirement implies:

$$\zeta = -\frac{1}{u+1} \quad \text{or} \quad \zeta = -1. \tag{6}$$

The Ocneanu trace is used in [14] for constructing the Homflypt polynomial invariant for classical knots and links. Then, by specializing ζ to $-\frac{1}{u+1}$ the Jones polynomial was recovered.

1.3. *The Yokonuma–Hecke algebra.* The group $\mathbb{Z}^n$ is generated by the "framing generators" $t_1, \ldots, t_n$, the standard multiplicative generators of $\mathbb{Z}^n$. In this notation an element $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ in the additive notation can be expressed as $t_1^{a_1} \ldots t_n^{a_n}$. The *framed braid group* on n strands is then defined as:

$$\mathcal{F}_n = \mathbb{Z}^n \rtimes B_n,$$

where the action of B_n on $\mathbb{Z}^n$ is given by the permutation induced by a braid on the indices:

$$\sigma_i t_j = t_{s_i(j)} \sigma_i. \tag{7}$$

In particular, $\sigma_i t_i = t_{i+1}\sigma_i$ and $t_{i+1}\sigma_i = \sigma_i t_i$. A word w in $\mathcal{F}_n$ has thus the "splitting property", that is, it splits into the "framing" part and the "braiding" part:

$$w = t_1^{a_1} \ldots t_n^{a_n}\, \sigma,$$

where $\sigma \in B_n$ and $a_i \in \mathbb{Z}$. So w is a classical braid with an integer attached to each strand. Topologically, an element of $\mathbb{Z}^n$ is identified with a framed identity braid on n strands, while a classical braid in B_n is viewed as a framed braid with all framings 0. The multiplication in $\mathcal{F}_n$ is defined by placing one braid on top of the other and collecting the total framing of each strand to the top.

For a fixed positive integer d, the *d-modular framed braid group* on n strands, $\mathcal{F}_{d,n}$, is defined as the quotient of $\mathcal{F}_n$ over the *modular relations*:

$$t_i^d = 1 \quad (i = 1, \ldots, n). \tag{8}$$

Thus, $\mathcal{F}_{d,n} = C_d^n \rtimes B_n$, where C_d^n is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^n$ but with multiplicative notation. Framed braids in $\mathcal{F}_{d,n}$ have framings modulo d.

Passing now to the group algebra $\mathbb{C}\mathcal{F}_{d,n}$, we have the following elements $e_i \in \mathbb{C}C_d^n$ (see [10] for diagrammatic interpretations), which are idempotents (cf. [10, Lemma 4]):

$$e_i := \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}, \qquad i = 1, \ldots, n-1. \tag{9}$$

The definition of the idempotent e_i can be generalized in the following way. For any indices i, j and any $m \in \mathbb{Z}/d\mathbb{Z}$, we define the following elements in $\mathbb{C}C_d^n$:

$$e_{i,j} := \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_j^{-s}, \tag{10}$$

and:

$$e_i^{(m)} := \frac{1}{d}\sum_{s=0}^{d-1} t_i^{m+s} t_{i+1}^{-s}. \tag{11}$$

(notice that $e_i = e_{i,i+1} = e_i^{(0)}$). The following lemma collects some of the relations among the e_i's, the t_i's and the σ_i's. These relations will be used in the paper.

Remark 1. Later on we are going to use the elements defined above inside the group algebras $\mathbb{C}(u)G = \mathbb{C}(u) \otimes \mathbb{C}G$, where G could be one of the following groups: $C_{d,n}$, $\mathcal{F}_{d,n}$, S_n. We will use the same symbols along this article by some abuse of notation.

Lemma 1. *For the idempotents e_i and for $1 \leq i,\ j \leq n-1$ the following relations hold:*

$$\begin{array}{rcl} t_j e_i & = & e_i t_j \\ e_{i+1}\sigma_i & = & \sigma_i e_{i,i+2} \\ e_i \sigma_j & = & \sigma_j e_i, \quad \text{for } j \neq i-1, i+1 \\ e_j \sigma_i \sigma_j & = & \sigma_i \sigma_j e_i \quad \text{for } |i-j| = 1 \\ e_i e_{i+1} & = & e_i e_{i,i+2} \\ e_i e_{i+1} & = & e_{i,i+2} e_{i+1}. \end{array}$$

Proof. All relations are immediate consequences of the definitions. The proofs for the first four relations can be found, for example, in [13, Lemma 2.1]. For the sixth relation we have:

$$\begin{aligned} e_i e_{i+1} &= \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s} \, \frac{1}{d}\sum_{m=0}^{d-1} t_{i+1}^m t_{i+2}^{-m} \\ &= \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{m=0}^{d-1} t_i^s t_{i+1}^{m-s} t_{i+2}^{-m}. \end{aligned} \tag{12}$$

Setting now $k = m - s$ we obtain:

$$\begin{aligned} (12) &= \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} t_i^s t_{i+1}^{k} t_{i+2}^{-k-s} \\ &= \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_{i+2}^{-s} \, \frac{1}{d}\sum_{k=0}^{d-1} t_{i+1}^k t_{i+2}^{-k} \\ &= e_{i,i+2} e_{i+1}. \end{aligned}$$

The fifth relation is proved in an analogous way. □

The *Yokonuma–Hecke algebra* $\mathrm{Y}_{d,n}(u)$ is defined [8, 10] as the quotient of the group algebra $\mathbb{C}(u)\mathcal{F}_{d,n}$ over the two-sided ideal generated by the elements:

$$\sigma_i^2 - 1 - (u-1)\, e_i - (u-1)\, e_i\, \sigma_i, \quad \text{for all } i. \tag{13}$$

Let g_i be the image of σ_i in the quotient of $\mathbb{C}(u)\mathcal{F}_{d,n}$ by the two–sided ideal defined above. The ideal relations imply the following quadratic relations in $\mathrm{Y}_{d,n}(u)$:

$$g_i^2 = 1 + (u-1)\, e_i + (u-1)\, e_i\, g_i \tag{14}$$

(see [10] for diagrammatic interpretations). Since the quadratic relations do not change the framing we have $\mathbb{C}C_d^n \subset \mathrm{Y}_{d,n}(u)$ and we keep the same notation for the elements of $\mathbb{C}C_d^n$ and for the elements e_i in $\mathrm{Y}_{d,n}(u)$. The elements g_i are invertible:

$$g_i^{-1} = g_i + (u^{-1} - 1)\, e_i + (u^{-1} - 1)\, e_i\, g_i.$$

For $d = 1$ we have $t_j = 1$ and $e_i = 1$, and in this case the quadratic relations (14) become $g_i^2 = (u-1)g_i + u$, which are the quadratic relations of the Iwahori–Hecke algebra $\mathrm{H}_n(u)$. So, $\mathrm{Y}_{1,n}(u)$ coincides with the algebra $\mathrm{H}_n(u)$. Further, there is an obvious epimorphism of the Yokonuma-Hecke algebra $\mathrm{Y}_{d,n}(u)$ onto the algebra $\mathrm{H}_n(u)$ via the map:

$$\begin{array}{rcl} g_i & \mapsto & v_i \\ t_j & \mapsto & 1. \end{array} \tag{15}$$

We can alternatively define the algebra $\mathrm{Y}_{d,n}(u)$ as a u–deformation of the algebra $\mathbb{C}C_{d,n}$. More precisely, let $w \in S_n$ and let $w = s_{i_1} \dots s_{i_k}$ be a reduced expression for w. Since the generators g_i of $\mathrm{Y}_{d,n}(u)$ satisfy the same braiding relations as the generators of S_n, then together with the well-known theorem of Matsumoto [16], it follows that $g_w := g_{i_1} \dots g_{i_k}$ is well defined. Notice that the defining generators g_i correspond to g_{s_i}. We have the following multiplication rule in $\mathrm{Y}_{d,n}(u)$ (see Proposition 2.4[7]):

$$g_{s_i} g_w = \begin{cases} g_{s_i w} & \text{for } l(s_i w) > l(w) \\ g_{s_i w} + (u-1)e_i g_{s_i w} + (u-1)e_i g_w & \text{for } l(s_i w) < l(w) \end{cases} \tag{16}$$

We also write g_{t_i} for t_i and we define: $g_{t_i w} = g_{t_i} g_w = t_i g_w$. Using the above multiplication formulas the second author proved in [8] that $\mathrm{Y}_{d,n}(u)$ has the following standard basis:

$$\{t_1^{a_1} \cdots t_n^{a_n} g_w \mid a_i \in \mathbb{Z}/d\mathbb{Z},\, w \in S_n\}.$$

Further, we have an inductive basis of the Yokonuma–Hecke algebra, which is used in the proof of the main theorem.

Proposition 1 ([8] Proposition 8)**.** *Every element in* $\mathrm{Y}_{d,n+1}(u)$ *is a unique linear combination of words, each of one of the following types:*

$$\mathfrak{m}_n g_n g_{n-1} \dots g_i t_i^k \quad \textit{or} \quad \mathfrak{m}_n t_{n+1}^k,$$

where $k \in \mathbb{Z}/d\mathbb{Z}$ *and* $\mathfrak{m}_n$ *is a word in the inductive basis of* $\mathrm{Y}_{d,n}(u)$.

1.4. *A Markov trace on* $\mathrm{Y}_{d,n}(u)$**.** Let d be a positive integer. We will write the elements of the additive group $\mathbb{Z}/d\mathbb{Z}$ by $\{0, 1, \dots, d-1\}$.

Using the above basis, the second author constructed in [8] a linear Markov trace on the algebra $\mathrm{Y}_{d,n}(u)$. Namely:

Theorem 2 ([8] Theorem 12)**.** *Let d a positive integer.*

For indeterminates z, x_i, where $i \in \mathbb{Z}/d\mathbb{Z}, i \neq 0$, there exists a unique linear Markov trace tr*:*

$$\mathrm{tr} : \cup_{n=1}^{\infty} \mathrm{Y}_{d,n}(u) \longrightarrow \mathbb{C}(u)[z, x_1, \dots, x_{d-1}]$$

defined inductively on n by the following rules:

$$\begin{array}{rcll} \mathrm{tr}(ab) & = & \mathrm{tr}(ba) & \\ \mathrm{tr}(1) & = & 1 & \\ \mathrm{tr}(a g_n) & = & z\,\mathrm{tr}(a) & (\textit{Markov property}) \\ \mathrm{tr}(a t_{n+1}^s) & = & x_s \mathrm{tr}(a) & (s = 1, \dots, d-1) \end{array}$$

where $a, b \in \mathrm{Y}_{d,n}(u)$.

Note that the first rule of tr is the standard rule for a trace function, the second rule is the basis of the inductive computation of tr, the third rule is the so–called *Markov property* that takes care of the highest index braiding generator in the word, whilst the fourth rule takes care of the highest index framing generator in the word.

Remark 2. We will define $x_0 := 1$, so x_i is defined for all $i \in \mathbb{Z}/d\mathbb{Z}$. In order to make the presentation simpler, we will use the symbol x_i for every $i \in \mathbb{Z}$, by assuming that $x_i = x_r$ where $0 \leq r < d$, is the residue of the division of i by d.

By direct computation, $\mathrm{tr}(e_i)$ takes the same value for all i. We denote this value by E, that is:

$$E := \mathrm{tr}(e_i) = \frac{1}{d}\sum_{s=0}^{d-1} x_s x_{d-s}.$$

For all $m \in \mathbb{Z}/d\mathbb{Z}, m \neq 0$ or equivalently for all $0 \leq m \leq d-1$, we also define:

$$E^{(m)} := \mathrm{tr}(e_i^{(m)}) = \frac{1}{d}\sum_{s=0}^{d-1} x_{m+s} x_{d-s},$$

where $e_i^{(m)}$ is defined in (11). Notice that $E = E^{(0)}$.

1.5. *The* E*–system.* In order for an invariant for framed knots and links to be constructed through the trace on $\mathrm{Y}_{d,n}(u)$, tr should be normalized and rescaled properly. In [11] it is proved that such a rescaling is possible if the trace parameters x_i are solutions of a non-linear system of equations, the so–called E–system.

Definition 1. We say that the set of complex numbers $\{x_0, x_1, \ldots, x_{d-1}\}$ (where x_0 is always equal to 1) satisfies the E*–condition* if $x_1, \ldots, x_{d-1}$ satisfy the following E*–system* of non–linear equations in $\mathbb{C}$:

$$E^{(m)} = x_m E \qquad (1 \leq m \leq d-1)$$

or equivalently:

$$\sum_{s=0}^{d-1} x_{m+s} x_{d-s} = x_m \sum_{s=0}^{d-1} x_s x_{d-s} \qquad (1 \leq m \leq d-1). \tag{17}$$

In [11, Appendix] it is proved that the solutions of the E–system are the functions $x_{_S}$, from $\mathbb{Z}/d\mathbb{Z}$ to $\mathbb{C}$, parametrized by the non–empty subsets S of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ as follows:

$$x_{_S} = \frac{1}{|S|}\sum_{s\in S} \exp_s, \tag{18}$$

where $\exp_s(k) = \exp(2i\pi sk/d)$ and exp denotes the usual complex exponential function.

Remark 3. It is worth noting that the solution of the E–system can be interpreted as a generalization of the Ramanujan's sum. Indeed, by taking the subset P of $\mathbb{Z}/d\mathbb{Z}$ consisting of the numbers coprimes to d, then the solution parametrized by P is, up to the factor $|P|$, the Ramanujan's sum $c_d(k)$ (see [17]).

Equivalently, $x_{_S}$ can be seen as an element in $\mathbb{C}C_d$, namely:

$$x_{_S} = \sum_{k=0}^{d-1} x_k t^k, \tag{19}$$

where $x_k = \frac{1}{|S|}\sum_{s\in S} \chi_s(t^k)$, $k = 0, \ldots, d-1$, and χ_s is the character of C_d defined as $\chi_s : t^m \mapsto \exp_s(m)$. So, the coefficient x_k of t^k in (19) corresponds to $x_{_S}(k)$ in (18).

Recall now that on the group algebra $\mathbb{C}G$ of the finite group G, we have two products, one of them is the multiplication coordinate-wise, also called the multiplications of the values, which is defined as:

$$\left(\sum_{g\in G} a_g g\right)\cdot\left(\sum_{g\in G} b_g g\right)=\sum_{g\in G} a_g b_g g.$$

and the other product is the convolution product:

$$\left(\sum_{g\in G} a_g g\right)*\left(\sum_{h\in G} b_h h\right)=\sum_{g\in G}\sum_{h\in G} a_g b_h gh=\sum_{g\in G}\left(\sum_{h\in G} a_h b_{gh^{-1}}\right) g. \tag{20}$$

Lemma 2. *In $\mathbb{C}C_d$ consider the element $x=\sum_{0\leq k\leq d-1} x_k t^k$. We have:*

$$x*x=d\sum_{0\leq \ell\leq d-1} E^{(\ell)}t^{\ell}$$

and

$$x*x*x=d^2\sum_{0\leq \ell\leq d-1}\mathrm{tr}(e_1^{\ell}e_2)t^{\ell}.$$

Proof. The expression for $x*x$ follows immediately by direct computation. For the second expression we have that:

$$\begin{aligned}
x*x*x &= d\sum_{0\leq \ell\leq d-1} E^{(\ell)}t^{\ell}*x\\
&= d\sum_{0\leq \ell\leq d-1} E^{(\ell)}t^{\ell}*\sum_{0\leq k\leq d-1} x_k t^k\\
&= d\sum_{0\leq \ell,k\leq d-1} E^{(\ell)}x_k t^{\ell+k}\\
&= d\sum_{0\leq \ell,k,s\leq d-1} x_s x_{\ell-s}x_k t^{\ell+k}\\
&= d\sum_{0\leq \ell,k,s\leq d-1} x_s x_{\ell-s-k}x_k t^{\ell}\\
&= d^2\mathrm{tr}(e_1^{(\ell)}e_2).
\end{aligned}$$

□

For each $a\in\mathbb{Z}/d\mathbb{Z}$ the character χ_a defines, with respect to the convolution product, an element $\mathbf{i}_a$ of $\mathbb{C}C_d$,

$$\mathbf{i}_a:=\sum_{0\leq s\leq d-1}\chi_a(s)t^s.$$

One can verify that

$$\mathbf{i}_a*\mathbf{i}_b=\begin{cases} d\,\mathbf{i}_a & \text{if } a=b\\ 0 & \text{if } a\neq b\end{cases}$$

that is, $\mathbf{i}_a/d$ is an idempotent element. On the other hand, regarding $\delta_a:=t^a$ as element in $\mathbb{C}C_d$, it is clear that,

$$\delta_a \cdot \delta_b = \begin{cases} \delta_a & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}.$$

The connection between the two products on $\mathbb{C}C_d$ is given by the *Fourier transform*. More precisely, the Fourier transform is the linear automorphism on $\mathbb{C}C_d$, defined as:

$$x := \sum_{0 \leq s \leq d-1} a_s t^s \mapsto \widehat{x} := (x * \mathbf{i}_s)(0) = \sum_{0 \leq \ell \leq d-1} a_\ell \chi_s(d - \ell). \tag{21}$$

With the above notation we have:

Lemma 3. *The following hold in* $\mathbb{C}C_d$*:*

$$\widehat{x * y} = \widehat{x} \cdot \widehat{y}, \qquad \widehat{x \cdot y} = d^{-1} \widehat{x} * \widehat{y},$$

$$\widehat{\delta_a} = \mathbf{i}_{-a}, \qquad \widehat{\mathbf{i}}_a = d\delta_a, \qquad \widehat{\widehat{x}}(u) = dx(-u).$$

Proof. The proof is just a straightforward computation (see [18]). □

2. The Yokonuma–Temperley–Lieb Algebra

In this section we define a Temperley–Lieb analogue, in the case of framing, as quotient of $\mathrm{Y}_{d,n}(u)$ over an appropriate two–sided ideal.

2.1. *The Yokonuma–Temperley–Lieb algebra.* The Hecke algebra, $\mathrm{H}_n(u)$, can be considered as a u–deformation of the $\mathbb{C}S_n$, while $\mathrm{TL}_n(u)$ is the quotient of $\mathrm{H}_n(u)$ over the two–sided ideal:

$$J = \langle v_{i,j} \,;\, \text{for all}\, i, j \,\text{such that}\; |i - j| = 1 \rangle,$$

where $v_{i,j}$'s are the Steinberg elements

$$v_{i,j} := 1 + v_i + v_j + v_i v_j + v_j v_i + v_i v_j v_i.$$

It is well-known that that J is a principal ideal. Indeed,

$$J = \langle v_{1,2} \rangle.$$

Notice now that $v_{i,j}$ can be rewritten as

$$v_{i,j} = \sum_{\alpha \in W_{i,j}} v_\alpha,$$

where $W_{i,j}$ is the subgroup of S_n generated by s_i and s_j (clearly, $W_{i,j}$ is isomorphic to S_3). On the other hand $\mathrm{Y}_{d,n}(u)$ can be regarded as a u–deformation of $\mathbb{C}[C_d^n \rtimes S_n]$. The symmetric group S_n can be considered as a subgroup of $C_d^n \rtimes S_n$, therefore the subgroups $W_{i,j}$ of S_n can be also regarded as subgroups of $C_d^n \rtimes S_n$. Thus, in analogy to the ideal J of $\mathrm{H}_n(u)$, it is natural to consider the following ideal I of $\mathrm{Y}_{d,n}(u)$:

$$I := \langle g_{i,j} \,;\, \text{for all}\, i, j \,\text{such that}\; |i - j| = 1 \rangle, \tag{22}$$

where

$$g_{i,j} := \sum_{\alpha \in W_{i,j}} g_\alpha = 1 + g_i + g_j + g_i g_j + g_j g_i + g_i g_j g_i. \tag{23}$$

We then define:

Definition 2. For $n \geq 3$, the *Yokonuma–Temperley–Lieb* algebra, $\mathrm{YTL}_{d,n}(u)$, is defined as the quotient:

$$\mathrm{YTL}_{d,n}(u) = \frac{\mathrm{Y}_{d,n}(u)}{I}.$$

In other words, the algebra $\mathrm{YTL}_{d,n}(u)$ can be presented by the generators $1, g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$ (by abuse of notation), subject to the following relations:

$$g_i g_j = g_j g_i, \quad |i-j| > 1 \tag{24}$$

$$g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i \tag{25}$$

$$g_i^2 = 1 + (u-1)e_i + (u-1)e_i g_i \tag{26}$$

$$t_i t_j = t_j t_i, \quad \text{for all } i, j \tag{27}$$

$$t_i^d = 1, \quad \text{for all } i \tag{28}$$

$$g_i t_i = t_{i+1} g_i \tag{29}$$

$$g_i t_{i+1} = t_i g_i \tag{30}$$

$$g_i t_j = t_j g_i, \quad \text{for } j \neq i, \text{ and } j \neq i+1 \tag{31}$$

$$g_i g_{i+1} g_i + g_i g_{i+1} + g_{i+1} g_i + g_i + g_{i+1} + 1 = 0. \tag{32}$$

We shall refer to relations (32) as *the Steinberg relations.*

Notice that relations (24)–(31) are the defining relations of the algebra $\mathrm{Y}_{d,n}(u)$. Note also that relations (32) are symmetric with respect to the indices i, $i+1$, i.e.:

$$g_i g_{i+1} g_i = -g_i g_{i+1} - g_{i+1} g_i - g_{i+1} - g_i - 1 = g_{i+1} g_i g_{i+1}.$$

so in the case $n > 3$ relations (25) follow from relations (32).

Remark 4. In analogy to the Yokonuma–Hecke algebra, $\mathrm{YTL}_{1,n}(u)$ coincides with the algebra $\mathrm{TL}_n(u)$. Further, the epimorphism (15) induces an epimorphism of the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$ onto the algebra $\mathrm{TL}_n(u)$. Clearly, by relations (29) and (30), any monomial in $\mathrm{YTL}_{d,n}(u)$ inherits the *splitting property* of $\mathrm{Y}_{d,n}(u)$, that is, it can be written in the form:

$$w = t_1^{a_1} \ldots t_n^{a_n} g_{i_1} \ldots g_{i_k}, \tag{33}$$

where: $a_1, \ldots, a_n \in \mathbb{Z}/d\mathbb{Z}$.

We now have the following:

Lemma 4. *The ideal I is principal.*

Proof. Observe first that (σ_1, σ_2) is conjugate to (σ_i, σ_{i+1}) in the braid group, hence also in $\mathcal{F}_{d,n}$. This proves that the pairs (g_1, g_2) and (g_i, g_{i+1}) are conjugate in $\mathrm{Y}_{d,n}$. This conjugation maps the elements $g_{1,2}$ to $g_{i,i+1}$, and the ideal I is principal.

□

Corollary 1. $\mathrm{YTL}_{d,n}(u)$ *is the $\mathbb{C}(u)$–algebra generated by the set $\{1, t_1, \ldots, t_n, g_1, \ldots, g_{n-1}\}$ whose elements are subject to the defining relations of $\mathrm{Y}_{d,n}(u)$ and the relation:*

$$g_{1,2} = 0.$$

Proof. The result follows using the multiplication rule defined on $\mathrm{Y}_{d,n}(u)$ and Lemma 4. □

2.2. *A presentation with non-invertible generators.* In analogy with Eq. 1 one can obtain a presentation for the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$ with the non-invertible generators:

$$l_i := \frac{1}{u+1}(g_i + 1). \tag{34}$$

In particular we have:

Proposition 2. $\mathrm{YTL}_{d,n}(u)$ *can be viewed as the algebra generated by the elements:*

$$1, l_1, \ldots l_{n-1}, t_1, \ldots, t_n,$$

which satisfy the following defining relations:

$$\begin{aligned}
t_i^d &= 1, \quad \text{for all } i && (35)\\
t_i t_j &= t_j t_i, \quad \text{for all } i, j && (36)\\
l_i t_j &= t_j l_i, \quad \text{for } j \neq i \text{ and } j \neq i+1 && (37)\\
l_i t_i &= t_{i+1} l_i + \frac{1}{u+1}(t_i - t_{i+1}) && (38)\\
l_i t_{i+1} &= t_i l_i + \frac{1}{u+1}(t_{i+1} - t_i) && (39)\\
l_i^2 &= \frac{(u-1)e_i + 2}{u+1}\, l_i && (40)\\
l_i l_j &= l_j l_i, \quad |i-j| > 1 && (41)\\
l_i l_{i+1} l_i &= \frac{(u-1)e_i + 1}{(u+1)^2}\, l_i. && (42)
\end{aligned}$$

Proof. Obviously, $\mathrm{YTL}_{d,n}(u)$ is generated by the l_i's and the t_i's. It is a straightforward computation to see that relations (24)–(32) are transformed into the relations (35) – (42). However, we shall show here how it works for the quadratic relations (35) and the Steinberg relations (42). From Eq. 34 we obtain:

$$g_i = (u+1)l_i - 1. \tag{43}$$

We then have that:

$$g_i^2 = ((u+1)l_i - 1)^2,$$

which is equivalent to:

$$1 + (u-1)e_i + (u-1)e_i g_i = (u+1)^2 l_i^2 - 2(u+1)l_i + 1$$

or equivalently:

$$(u-1)(u+1)e_i l_i = (u+1)^2 l_i^2 - 2(u+1)l_i,$$

which leads to:

$$l_i^2 = \frac{(u-1)e_i + 2}{u+1} l_i.$$

which is Eq. 40.

For the Steinberg elements $g_{i,i+1}$ using Eq. 43 we have that:

$$g_{i,i+1} = g_i g_{i+1} g_i + g_{i+1} g_i + g_i g_{i+1} + g_{i+1} + g_i + 1 = (u+1)^3 l_i l_{i+1} l_i - (u+1)^2 l_i^2 + (u+1) l_i.$$

From the Steinberg relation (32) and Eq. 40 we have that:

$$(u+1)^2 l_i l_{i+1} l_i = ((u-1)e_i + 1) l_i$$

or equivalently:

$$l_i l_{i+1} l_i = \frac{(u-1)e_i + 1}{(u+1)^2}\, l_i,$$

which is Eq. 42. □

Remark 5. Setting $d = 1$ in the presentation of $\mathrm{YTL}_{d,n}(u)$ in Proposition 2, one obtains the classical presentation of $\mathrm{TL}_n(u)$, as discussed in Subsection 1.2. Note also that, substituting in the braid relation (25) the g_i's using Eq. 43, we obtain the equation:

$$l_i l_{i+1} l_i - \frac{(u-1)e_i + 1}{(u+1)^2}\, l_i = l_{i+1} l_i l_{i+1} - \frac{(u-1)e_{i+1} + 1}{(u+1)^2}\, l_{i+1}$$

which becomes superfluous, since it can be deduced from Eq. 42. This was to be expected, since the braid relations (25) were also superfluous.

3. A spanning set for The Yokonuma–Temperley–Lieb algebra

In this section we discuss various properties of a word in $\mathrm{YTL}_{d,n}(u)$ and we present a spanning set for $\mathrm{YTL}_{d,n}(u)$ (Proposition 4). Furthermore, using the work of Chlouveraki and Pouchin in [2] we give their formula for the dimension of $\mathrm{YTL}_{d,n}(u)$ (Proposition 5) and we also discuss their results on the linear basis of $\mathrm{YTL}_{d,n}(u)$ (Theorem 3). We finally compute a basis for $\mathrm{YTL}_{2,3}(u)$ different than the one of Theorem 3.

3.1. We have the following definition:

Definition 3. In $\mathrm{YTL}_{d,n}(u)$ we define a length function l as follows:

$$l(t^a g_{i_1} \ldots g_{i_k}) := l'(s_{i_1} \ldots s_{i_k}),$$

where l' is the usual *length function* of S_n and $t^a := t_1^{a_1} \ldots t_n^{a_n} \in C_d^n$. A word in $\mathrm{YTL}_{d,n}(u)$ of the form (33) shall be called *reduced* if it is of minimal length with respect to relations (24)–(26), (32).

Proposition 3. *Each word in* $\mathrm{YTL}_{d,n}(u)$ *can be written as a sum of monomials, where the highest and lowest index of the generators* g_i *appear at most once.*

Proof. An analogous statement holds for the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ where only the highest index generators appear at most once [8, Proposition 8]. Since $\mathrm{YTL}_{d,n}(u)$ is a quotient of the algebra $\mathrm{Y}_{d,n}(u)$ the highest index property passes through to the algebra $\mathrm{YTL}_{d,n}(u)$. The idea is analogous to [15, Lemma 4.1.2] and it is based on induction on the length of reduced words, use of the braid relations and reduction of length using the quadratic relations (26). For the case of the lowest index generator g_i we use induction on the length of reduced words and the Steinberg relations (32). Indeed, clearly, the statement is true for all words of length ≤ 2, namely for words of the form t^a, $t^a g_i$ $t^a g_i g_j$

For words of length 3: Let $w = t^a g_i g_j g_i$. Applying relation (25) will violate the highest index property of the word, so we must use the Steinberg relation (32) and we have:

$$t^a g_i g_j g_i \quad = \quad -t^a g_j g_i - t^a g_i g_j - t^a g_j - t^a g_i - t^a.$$

We assume that the lowest index generator appears at most once in all reduced words of length $\leq r$, and we will show the lowest index property for words of length $r+1$. Let $w = t^a g_{i_1} g_{i_2} \cdots g_{i_{r+1}}$ be a reduced word in $\mathrm{YTL}_{d,n}(u)$ of length $r+1$, and $l = \min\{i_1, \ldots, i_{r+1}\}$.

Let first $w = t^a w_1 g_l w_2 g_l w_3$, and suppose that w_2 does not contain g_l. We then have two possibilities:

If w_2 does not contain g_{l+1}, then g_l commutes with all the g_i's in w_2 and since there cannot be a g_l^2 term in a reduced word, we have, using the induction hypothesis, that:

$$\begin{aligned} w &= t^a w_1 g_l w_2 g_l w_3 \\ &= t^a w_1 w_2 g_l^2 w_3 \\ &= t^a w_1 w_2 (1 + (u-1)e_l + (u-1)e_l g_l)) w_3 \\ &= t^a w_1 w_2 w_3 + (u-1) t^a w_1 w_2 e_l w_3 + (u-1) t^a w_1 w_2 e_l g_l w_3. \end{aligned}$$

If w_2 does contain g_{l+1}, then, by the induction hypothesis w_2 has the form $w_2 = v_1 g_{l+1} v_2$, where in v_1, v_2 the lowest index generator is at least g_{l+2}, hence:

$$\begin{aligned} w &= t^a w_1 g_l v_1 g_{l+1} v_2 g_l w_3 \\ &= t^a w_1 v_1 g_l g_{l+1} g_l v_2 w_3. \end{aligned}$$

Applying now the Steinberg relation (32) we obtain a linear combination of words each of which has at least one less occurrence of g_l than w. Note also that in the case where $l+1 = m$, where $m = \max\{i_1, \dots i_{r+1}\}$, no contradiction is caused with respect to the highest index generator. Continuing in the same manner for all possible pairs of g_l in the word we reduce to having g_l at most once. □

The following proposition gives us a precise spanning set for $\mathrm{YTL}_{d,n}(u)$.

Proposition 4. *The following set of reduced words*

$$\Sigma_{d,n} = \left\{ t^a (g_{i_1} g_{i_1-1} \dots g_{i_1-k_1})(g_{i_2} g_{i_2-1} \dots g_{i_2-k_2}) \dots (g_{i_p} g_{i_p-1} \dots g_{i_p-k_p}) \right\}, \tag{44}$$

where

$$t^a = t_1^{a_1} \dots t_n^{a_n} \in C_d^n, \quad 1 \le i_1 < i_2 < \dots < i_p \le n-1,$$

and

$$1 \le i_1 - k_1 < i_2 - k_2 < \dots < i_p - k_p,$$

spans the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$*. The highest index generator is* g_{i_p} *of the rightmost cycle and the lowest index generator is* $g_{i_1-k_1}$ *of the leftmost cycle of a word in* $\Sigma_{d,n}$*.*

Proof. Through relations (24) – (32) any word is a linear combination of words of the form $t^a g_{i_1} \dots g_{i_k}$, where $g_{i_1} \dots g_{i_k}$ is a the image of a fully commutative word of the braid monoid and it is well known that a fully commutative word can be written under the form given in the statement of Proposition 4. □

M. Chlouveraki and G. Pouchin in [2] have computed the dimension for $\mathrm{YTL}_{d,n}(u)$ by using the representation theory of the Yokonuma–Hecke algebra [3]. More precisely, they proved the following result.

Proposition 5 (cf. Proposition 4 [2])**.** *The dimension of the Yokonuma–Temperley–Lieb algebra is:*

$$\dim(\mathrm{YTL}_{d,n}(u)) = dc_n + \frac{d(d-1)}{2} \sum_{k=1}^{n-1} \binom{n}{k}^2,$$

where c_n *is the* n^{th} *Catalan number.*

3.2. To find an explicit basis for $\mathrm{YTL}_{d,n}(u)$ Chlouveraki and Pouchin in [2] worked as follows: As mentioned in Remark 4 each word in $\mathrm{YTL}_{d,n}(u)$ inherits the splitting property. For each fixed element in the braiding part, they described a set of linear dependence relations among the framing parts (see [2, Proposition 5]). Using these relations they extracted from $\Sigma_{d,n}$ (recall Eq. 44) a smaller spanning set for $\mathrm{YTL}_{d,n}(u)$ and showed that the cardinality of this smaller spanning set is equal to the dimension of the algebra. Thus, it is a basis for $\mathrm{YTL}_{d,n}(u)$. Before describing this basis, we will need the following notations:

Let $\underline{i}$ and $\underline{k}$ be the following p–tuples:

$$\underline{i} = (i_1, \ldots, i_p) \quad \text{and} \quad \underline{k} = (k_1, \ldots, k_p)$$

and let $\mathcal{I}$ be the set of pairs $(\underline{i}, \underline{k})$ such that:

$$1 \leq i_1 < \ldots < i_p \leq n-1 \quad \text{and} \quad 1 \leq i_1 - k_1 < \ldots < i_p - k_p \leq n-1.$$

We also denote by $g_{\underline{i},\underline{k}}$ the element:

$$g_{\underline{i},\underline{k}} := (g_{i_1} g_{i_1-1} \ldots g_{i_1-k_1})(g_{i_2} g_{i_2-1} \ldots g_{i_2-k_2}) \ldots (g_{i_p} g_{i_p-1} \ldots g_{i_p-k_p}).$$

Under these notations the set $\Sigma_{d,n}$ can be written as:

$$\Sigma_{d,n} = \{t_1^{r_1} \ldots t_n^{r_n}\, g_{\underline{i},\underline{k}} \,|\, r_1, \ldots, r_n \in \mathbb{Z}/d\mathbb{Z}, (\underline{i}, \underline{k}) \in \mathcal{I}\}.$$

The *degree of a word* $w = t_1^{r_1} \ldots t_n^{r_n} g_{i_1} \ldots g_{i_m}$ in $\mathrm{Y}_{d,n}(u)$, denoted $\deg(w)$, is defined to be the integer m. Set:

$$\Sigma_{d,n}^{<w} := \{s \in \Sigma_{d,n} \,|\, \deg(s) < \deg(w)\}.$$

The group algebra $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ is isomorphic to the subalgebra of $\mathrm{Y}_{d,n}(u)$ that is generated by the t_i's but not to the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's. Further, the group algebra $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ has a natural basis, $B_{d,n}$, given by monomials in $t_1, \ldots, t_n$, the following:

$$B_{d,n} = \{t_1^{r_1} \ldots t_n^{r_n} \,|\, r_1, \ldots, r_n \in \mathbb{Z}/d\mathbb{Z}\}.$$

Thus, any element of $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ can be written as a linear combination of words in $B_{d,n}$. There is a surjective algebra morphism from $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ to the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's. We will denote the image of an element $b \in B_{d,n}$ into the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's with $\bar{b}$. We then have the following theorem:

Theorem 3 (Chlouveraki and Pouchin, cf. [2], Theorem 2)**.** *The following set is a linear basis for* $\mathrm{YTL}_{d,n}(u)$*:*

$$S_{d,n} = \{\bar{b}_{\underline{i},\underline{k}}\, g_{\underline{i},\underline{k}} \,|\, (\underline{i}, \underline{k}) \in \mathcal{I},\, b_{\underline{i},\underline{k}} \in \mathcal{B}_{d,n}(g_{\underline{i},\underline{k}})\},$$

where $\mathcal{B}_{d,n}(g_{\underline{i},\underline{k}})$ *is a proper subset of* $B_{d,n}$ *such that:*

$$\{b_{\underline{i},\underline{k}} + R(g_{\underline{i},\underline{k}}) \,|\, b_{\underline{i},\underline{k}} \in \mathcal{B}_{d,n}(g_{\underline{i},\underline{k}})\}$$

is a basis of the quotient space $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n/R(g_{\underline{i},\underline{k}})$*, and where* $R(w)$ *is the following ideal of* $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$*:*

$$R(w) = \{\mathfrak{m} \in \mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n \,|\, \bar{\mathfrak{m}}\, w \in \mathrm{Span}_{\mathbb{C}(u)}(\Sigma_{d,n}^{<w})\}.$$

3.3. For $d = 2$, $n = 3$ it is relatively easy to find a basis for $\mathrm{YTL}_{2,3}(u)$. We will give here a basis different than the one in Theorem 3. Before continuing, we need the following technical lemma that will be also used in the proof of Theorem 5.

Lemma 5 (cf. Lemma 7.5 [9])**.** *For the element $g_{1,2}$ we have in $\mathrm{Y}_{d,n}(u)$ (recall (10) for $e_{1,3}$):*

$$\begin{array}{lrcl}
(1) & g_1g_{1,2} & = & [1+(u-1)e_1]g_{1,2} \\
(2) & g_2g_{1,2} & = & [1+(u-1)e_2]g_{1,2} \\
(3) & g_1g_2g_{1,2} & = & [1+(u-1)e_1+(u-1)e_{1,3}+(u-1)^2e_1e_2]g_{1,2} \\
(4) & g_2g_1g_{1,2} & = & [1+(u-1)\,e_2+(u-1)e_{1,3}+(u-1)^2e_1e_2]g_{1,2} \\
(5) & g_1g_2g_1g_{1,2} & = & [1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1e_2]g_{1,2}.
\end{array}$$

Analogous relations hold for multiplications with $g_{1,2}$ from the right.

Proof. The idea is to expand the left–hand side of each equation and then use Eq. 26 and Lemma 1. We will demonstrate the proof for the indicative cases (1) and (4). The other cases are proved similarly.

For case (1) we have:

$$\begin{aligned}
g_1g_{1,2} &= g_1+g_1^2+g_1g_2+g_1^2g_2+g_1g_2g_1+g_1^2g_2g_1 \\
&= g_1+[1+(u-1)e_1+(u-1)e_1g_1] \\
&\quad +g_1g_2+[g_2+(u-1)e_1g_2+(u-1)e_1g_1g_2] \\
&\quad +g_1g_2g_1+[g_2g_1+(u-1)e_1g_2g_1+(u-1)e_1g_1g_2g_1] \\
&= g_{1,2}+(u-1)e_1g_{1,2}.
\end{aligned}$$

Case (2) is completely analogous. In order to prove case (4) we will use cases (1) and (2):

$$\begin{aligned}
g_2g_1g_{1,2} &= g_2\,(g_{1,2}+(u-1)e_1g_{1,2}) \\
&= g_2g_{1,2}+(u-1)e_{1,3}g_2g_{1,2} \quad \text{(Lemma 1)} \\
&= [1+(u-1)e_2]\,g_{1,2}+(u-1)e_{1,3}(1+(u-1)e_2)g_{1,2} \\
&= [1+(u-1)e_2]\,g_{1,2}+(u-1)e_{1,3}g_{1,2}+(u-1)^2e_{1,3}e_2g_{1,2} \quad \text{(Lemma 1)} \\
&= \left[1+(u-1)\,e_2+(u-1)e_{1,3}+(u-1)^2e_1e_2\right]g_{1,2}.
\end{aligned}$$

□

To find a basis for $\mathrm{YTL}_{2,3}(u)$: From Proposition 5 we have that $\dim(\mathrm{YTL}_{2,3}(u)) = 28$. On the other hand the spanning set $\Sigma_{2,3}$ of $\mathrm{YTL}_{2,3}(u)$ of Proposition 4, contains 40 elements. Thus, any relation $w_1g_{1,2}w_2 = 0$ with w_1, $w_2 \in \mathrm{Y}_{2,3}(u)$ reduces to having w_1, $w_2 \in \Sigma_{2,3}$. Further, if any of w_1, w_2 contain braiding generators, then by Lemma 5 (after pushing framing generators in w_2 to the right) these get absorbed by $g_{1,2}$. Thus, and since $e_{i,j} = \frac{1}{2}(1+t_it_j)$ for $d = 2$, it suffices to consider the following system of equations:

$$w_1\,g_{1,2}w_2 = 0 \qquad w_1, w_2 \in \mathcal{T}, \tag{45}$$

where $\mathcal{T} := \{1, t_1, t_2, t_3, t_1t_2, t_1t_3,\ t_2t_3, t_1t_2t_3\}$. For finding all possible linear dependencies in $\Sigma_{2,3}$, after substituting $g_1g_2g_1$ with $-1-g_1-g_2-g_1g_2-g_2g_1$ in Eq. 45, note that some of these 64 equations reduce trivially to $g_{1,2} = 0$; for example if $w_2 = 1$ or $w_2 = t_1t_2t_3$ (since it commutes with $g_{1,2}$). From the rest one can extract 12 linearly independent equations which, applied on the spanning set $\Sigma_{2,3}$ lead to the following basis for $\mathrm{YTL}_{2,3}(u)$:

$$\begin{aligned}
\mathcal{S}_{2,3} = \{&1, t_1, t_2, t_1t_2, g_1, t_2g_1, t_3g_1, t_2t_3g_1, g_2, t_1g_2, t_3g_2, t_1t_3g_2, \\
&g_1g_2, t_1g_1g_2, t_2g_1g_2, t_3g_1g_2, t_1t_2g_1g_2, t_1t_3g_1g_2, t_2t_3g_1g_2, t_1t_2t_3g_1g_2, \\
&g_2g_1, t_1g_2g_1, t_2g_2g_1, t_3g_2g_1, t_1t_2g_2g_1, t_1t_3g_2g_1, t_2t_3g_2g_1, t_1t_2t_3g_2g_1\}\,.
\end{aligned}$$

4. A Markov trace on $\mathrm{YTL}_{d,n}(u)$

The following section is dedicated to finding the necessary and sufficient conditions for the trace tr on $\mathrm{Y}_{d,n}(u)$ to pass to the quotient algebra $\mathrm{YTL}_{d,n}(u)$, in analogy to the classical case, where the Ocneanu trace on $\mathrm{H}_n(u)$ passes to the quotient algebra $\mathrm{TL}_n(u)$ if the trace parameter ζ satisfies some appropriate condition.

4.1. It is clear by now that if the trace pass to $\mathrm{YTL}_{d,n}(u)$ then it has to kill the generator $g_{1,2}$ of the principal ideal through which the quotient is defined, that is, if $\mathrm{tr}(g_{1,2}) = 0$. We have the following lemma:

Lemma 6. *For the element $g_{1,2}$ we have:*

$$\mathrm{tr}(g_{1,2}) = (u+1)z^2 + ((u-1)E+3)\, z + 1. \tag{46}$$

Proof. The proof is a straightforward computation:

$$\begin{aligned}
\mathrm{tr}(g_{1,2}) &= \mathrm{tr}(1) + \mathrm{tr}(g_1) + \mathrm{tr}(g_2) + \mathrm{tr}(g_1g_2) + \mathrm{tr}(g_2g_1) + \mathrm{tr}(g_1g_2g_1) \\
&= 1 + 2z + 2z^2 + z + (u-1)Ez + (u-1)z^2 \\
&= (u+1)z^2 + ((u-1)E+3)\, z + 1.
\end{aligned}$$

□

Lemma 6, together with the equation:

$$\mathrm{tr}(g_{1,2}) = (u+1)z^2 + ((u-1)E+3)\, z + 1 = 0 \tag{47}$$

gives us the following values for z:

$$z_{\pm} = \frac{-\left((u-1)E+3\right) \pm \sqrt{\left((u-1)E+3\right)^2 - 4(u+1)}}{2(u+1)}. \tag{48}$$

We shall do now the analysis for all conditions that must be imposed on the trace parameters in order that tr passes to $\mathrm{YTL}_{d,n}(u)$. Having in mind Corollary 1 and the linearity of tr, it follows that tr passes to $\mathrm{YTL}_{d,n}(u)$ if and only if the following equations are satisfied for all monomials $\mathfrak{m}$ in the inductive basis of $\mathrm{Y}_{d,n}(u)$. Namely:

$$\mathrm{tr}(\mathfrak{m}\, g_{1,2}) = 0. \tag{49}$$

Let us first consider the case $n = 3$. By Proposition 1 the elements in the inductive basis of $\mathrm{Y}_{d,3}(u)$ are of the following forms:

$$t_1^a t_2^b t_3^c, \quad t_1^a g_1 t_1^b t_3^c, \quad t_1^a t_2^b g_2 g_1 t_1^c, \quad t_1^a t_2^b g_2 t_2^c, \quad t_1^a g_1 t_1^b g_2 t_2^c, \quad t_1^a g_1 t_1^b g_2 g_1 t_1^c. \tag{50}$$

Using Lemma 5 and the following notations:

$$\begin{aligned}
Z_{a,b,c} &:= (u+1)z^2 x_{a+b+c} + \left((u-1)E^{(a+b+c)} + x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b}\right) z + x_a x_b x_c \\
V_{a,b+c} &:= (u+1)z^2 x_{a+b+c} + (u+1)zE^{(a+b+c)} + z\, x_a x_{b+c} + x_a E^{(b+c)} \\
W_{a,b,c} &:= (u+1)z^2 x_{a+b+c} + (u+2)zE^{(a+b+c)} + \mathrm{tr}\left(e_1^{(a+b+c)} e_2\right)
\end{aligned}$$

we obtain by (49) and (50) the following equations, for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$:

$$Z_{a,b,c} = 0 \tag{51}$$

$$Z_{a,b,c} + (u-1)V_{a,b+c} = 0 \tag{52}$$

$$Z_{a,b,c} + (u-1)\left[V_{a,b+c} + V_{b,a+c} + W_{a,b,c}\right] = 0 \tag{53}$$

$$Z_{a,b,c} + (u-1)\left[V_{a,b+c} + V_{b,a+c} + V_{c,a+b} + W_{a,b,c}\right] = 0. \tag{54}$$

Equations 51–54 reduce to the following system of equations of $z, x_1, \ldots, x_{d-1}$ for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$:

$$(\Sigma)\begin{cases} Z_{a,b,c} = 0 & \text{(55a)} \\ V_{a,b+c} = 0 & \text{(55b)} \\ W_{a,b,c} = 0 & \text{(55c)} \end{cases}$$

Notice that for $a = b = c = 0$ Eq. 51 becomes Eq. 47. If, now, we require both solutions in (48) to participate in the solutions of (Σ), then we are led to sufficient conditions for tr to pass to $\mathrm{YTL}_{2,3}(u)$ (Section 4.2). If not then we are led to necessary and sufficient conditions for tr to pass to $\mathrm{YTL}_{2,3}(u)$ (Section 4.3).

4.2. Suppose that both solutions for z from Eq. 48 participate in the solution set of (Σ). We have the following proposition:

Proposition 6. *If the trace parameters x_i are d^{th} roots of unity $x_i = x_1^i$, $1 \leq i \leq d-1$ and $z = -\frac{1}{u+1}$ or $z = -1$, then the trace* tr *defined on $\mathrm{Y}_{d,3}(u)$ passes to the quotient $\mathrm{YTL}_{d,3}(u)$.*

Proof. Suppose that (Σ) has both solutions for z from Eq. 48. This implies that there exist λ in $\mathbb{C}(u)(x_1 \ldots, x_{d-1})$ such that:

$$Z_{a,b,c} = \lambda Z_{0,0,0}.$$

From this we deduce that:

$$\lambda = x_{a+b+c}$$

$$x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b} = 3x_{a+b+c}$$

$$E^{(a+b+c)} = x_{a+b+c}E \tag{56}$$

$$x_{a+b+c} = x_a x_b x_c. \tag{57}$$

Since this holds for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$, by taking $b = c = 0$ in Eq. 56 we have that:

$$E^{(a)} = x_a E, \tag{58}$$

which is exactly the E–system. Moreover, by taking $c = 0$ in Eq. 57 we obtain:

$$x_a x_b = x_{a+b}. \tag{59}$$

This implies that the x_i's are d^{th} roots of unity which is equivalent to $E = 1$ [11, Appendix]. In order to conclude the proof it is enough to verify that these conditions for the x_i's satisfy also (55b)–(55c) of (Σ). Since the x_i's are solutions of the E–system, Eq. 55b is immediately satisfied. We will finally check Eq. 55c. One has that $\mathrm{tr}(e_1^{(m)}e_2) = x_m E^2$ as soon as the x_s satisfy the E–system. Once this has been noticed, Eq. 55c becomes the same as Eq. 51 using Eq. 57 and $E = 1$. $\square$

Using induction on n one can prove the general case of the sufficient conditions for tr to pass to $\mathrm{YTL}_{d,n}(u)$. Indeed we have:

Theorem 4. *If the trace passes to the quotient for $n = 3$ then it passes for all $n > 3$.*

Proof. By induction on n. In Proposition 6 we proved the case where $n = 3$. Assume that the statement holds for all $\mathrm{YTL}_{d,k}(u)$, where $k \leq n$, that is:

$$\mathrm{tr}(a_k\, g_{1,2}) = 0$$

for all $a_k \in \mathrm{Y}_{d,k}(u)$, $k \leq n$. We will show the statement for $k = n+1$. It suffices to prove that the trace vanishes on any element in the form $a_{n+1}g_{1,2}$, where a_{n+1} belongs to the inductive basis of $\mathrm{Y}_{d,n+1}(u)$ (recall Proposition 1), given the conditions of the Theorem. Namely:

$$\mathrm{tr}(a_{n+1}\, g_{1,2}) = 0.$$

Since a_{n+1} is in the inductive basis of $\mathrm{Y}_{d,n+1}(u)$, it is of one of the following forms:

$$a_{n+1} = a_n g_n \dots g_i t_i^k \quad \text{or} \quad a_{n+1} = a_n t_{n+1}^k,$$

where a_n is in the inductive basis of $\mathrm{Y}_{d,n}(u)$. For the first case we have:

$$\mathrm{tr}(a_{n+1}\, g_{1,2}) = \mathrm{tr}(a_n g_n \dots g_i t_i^k\, g_{1,2}) = z\, \mathrm{tr}(a_n g_{n-1} \dots g_i t_i^k\, g_{1,2})$$

and the result follows by induction. Therefore the statement is proved. The second case is proved similarly. Hence, the proof is concluded. □

The above theorem allows us to see that:

Theorem 5. *For $n \geq 3$, the trace* tr *defined on $\mathrm{Y}_{d,n}(u)$ passes to the quotient $\mathrm{YTL}_{d,n}(u)$ if the trace parameters x_i are d^{th} roots of unity $x_i = x_1^i$, $1 \leq i \leq d-1$ and $z = -\frac{1}{u+1}$ or $z = -1$.*

4.3. Moving on, we investigate the possibility of the x_i's being solutions of the E–system, other than d^{th} roots of unity. We have the following:

Theorem 6. *The trace* tr *passes to the quotient $\mathrm{YTL}_{d,n}(u)$ if and only if the x_i's are solutions of the* E*–system and one of the two cases holds:*

(i) *For some $0 \leq m_1 \leq d-1$ the x_ℓ's are expressed as:*

$$x_\ell = \exp_{m_1}(\ell) \quad (0 \leq \ell \leq d-1).$$

In this case the x_ℓ's are d^{th} roots of unity and $z = -\frac{1}{u+1}$ or $z = -1$.

(ii) *For some $0 \leq m_1, m_2 \leq d-1$, where $m_1 \neq m_2$ the x_ℓ's are expressed as:*

$$x_\ell = \frac{1}{2}\left(\exp_{m_1}(\ell) + \exp_{m_2}(\ell)\right) \quad (0 \leq \ell \leq d-1).$$

In this case we have $z = -\frac{1}{2}$.

Note that case (i) captures Theorem 5.

Proof. Observe that the x_ℓ's expressed by (i) are indeed solutions of the system (Σ). We will now assume that our solutions are not of this form. This implies that $x_a \neq E^{(a)}$ for some $0 \leq a \leq d-1$, and this will allow us to have this quantity in denominators later.

We will use induction on n. We will first prove the case $n = 3$. Suppose that trace tr passes to the quotient algebra $\mathrm{YTL}_{d,3}(u)$. This means that (Σ) has solutions for z any one of those in Eq. 48, for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$. Subtracting Eq. 55a from Eq. 55b we obtain:

$$\left(x_a x_{b+c} + x_b x_{a+c} - 2E^{(a+b+c)}\right) z = -\left(x_a x_b x_c - x_c E^{(a+b)}\right). \tag{60}$$

For $b = c = 0$ in Eq. 60 and since we assumed that there is an a such that $x_a \neq E^{(a)}$ we obtain: $z = -\frac{1}{2}$. On the other hand, subtracting Eqs. 55a and 55b from Eq. 55c we have:

$$\left(3E^{(a+b+c)} - x_a x_{b+c} - 2x_c x_{a+b}\right) z = x_a x_b x_c + x_c E^{(a+b)} - x_b E^{(a+c)} - \mathrm{tr}(e_1^{(a+b+c)} e_2). \tag{61}$$

For the value a such that $x_a - E^{(a)} \neq 0$ and for $b = c = 0$ in Eq. 61 we obtain:

$$z = -\frac{x_a - \mathrm{tr}(e_1^{(a)} e_2)}{3(x_a - E^{(a)})}. \tag{62}$$

By combining Eqs. 60 and 62 we have that:

$$\frac{1}{2} = \frac{x_a - \mathrm{tr}(e_1^{(a)} e_2)}{3(x_a - E^{(a)})}$$

or equivalently:

$$3(x_a - E^{(a)}) = 2(x_a - \mathrm{tr}(e_1^{(a)} e_2)).$$

Using Lemma 2, this is equivalent to:

$$3x - \frac{3}{d} x * x = 2x - \frac{2}{d^2} x * x * x.$$

By taking the Fourier transform (see Lemma 3) we arrive at:

$$\frac{2}{d^2}\widehat{x}^3 - \frac{3}{d}\widehat{x}^2 + \widehat{x} = 0.$$

Assuming that $\widehat{x} = \sum_{0 \le \ell \le d-1} y_\ell t^\ell$ we have the following expression for the coefficients y_ℓ in the expansion of $\widehat{x}$:

$$y_\ell \left(\frac{2}{d^2} y_\ell^2 - \frac{3}{d} y_\ell + 1 \right) = 0.$$

So either $y_\ell = 0$ or $y_\ell = d$ or $y_\ell = \frac{1}{2}d$. So if we take a partition of the set $\{\ell : 0 \le \ell \le d-1\}$ into sets S_0, S_1, $S_{\frac{1}{2}}$ such that y_ℓ takes the value $i \cdot d$ on S_i $(i = 0, 1, \frac{1}{2})$. We have from Lemma 3 that:

$$x = \sum_{m \in S_1} \mathbf{i}_{-m} + \frac{1}{2} \sum_{m \in S_{\frac{1}{2}}} \mathbf{i}_{-m}.$$

From $x_0 = 1$ we obtain the conditions:

$$1 = x(0) = |S_1| + \frac{1}{2}|S_{\frac{1}{2}}|.$$

This means that either S_1 has only one element and $S_{\frac{1}{2}} = \emptyset$ or $S_1 = \emptyset$ and $S_{\frac{1}{2}}$ has two elements. The first case corresponds to the case (i) where the x_ℓ's are d^{th} roots of unity. In the second case, if $S_{\frac{1}{2}} = \{m_1, m_2\}$ we obtain the following solution of the E–system:

$$x_\ell = \frac{1}{2}\left(\exp_{m_1}(\ell) + \exp_{m_2}(\ell)\right), \quad (0 \le \ell \le d-1), \tag{63}$$

which corresponds to $z = -\frac{1}{2}$.

We can now check that these solutions satisfy the system (Σ). Since $z = -\frac{1}{2}$ and $E = \frac{1}{2}$, we have that $E^{(\ell)} = x_\ell/2$, we have that $V_{c,a+b} = W_{a,b,c} = 0$, and $Z_{a,b,c}$ by Eq. 57 reduces to:

$$(u+1)z^2 x_{a+b+c} + \left((u-1)E^{(a+b+c)} + x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b}\right) z + x_a x_b x_c$$
$$= x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b} = x_{a+b+c} + 2x_a x_b x_c.$$

which can be checked to be satisfied by the values x_ℓ given in Eq. (63).

The rest of proof (the induction on n) follows by theorem 4. $\square$

Remark 6. The values for the trace parameter z in Theorems 5 and 6, $z = -\frac{1}{u+1}$ and $z = -1$, in order that tr on $\mathrm{Y}_{d,n}(u)$ passes to the quotient $\mathrm{YTL}_{d,n}(u)$ are the same as the values in Eq. 6 for ζ of the Ocneanu trace τ on $\mathrm{H}_n(u)$, so that τ passes to the quotient $\mathrm{TL}_n(u)$ (recall Section 1.2).

5. The Jones polynomial from $\mathrm{YTL}_{d,n}(u)$

The 2–variable Jones or Homflypt polynomial, $P(\lambda, u)$, can be defined through the Ocneanu trace on $\mathrm{H}_n(u)$ [14]. Indeed, for any braid $\alpha \in \cup_\infty B_n$ we have:

$$P(\lambda, u)(\widehat{\alpha}) = \left(-\frac{1-\lambda u}{\sqrt{\lambda}(1-u)}\right)^{n-1} \left(\sqrt{\lambda}\right)^{\varepsilon(\alpha)} \tau(\pi(\alpha)),$$

where: $\lambda = \frac{1-u+\zeta}{u\zeta}$, π is the natural epimorphism of $\mathbb{C}(u)B_n$ onto $\mathrm{H}_n(u)$ that sends the braid generator σ_i to v_i and $\varepsilon(\alpha)$ is the algebraic sum of the exponents of the σ_i's in α. Further, the Jones polynomial, $V(u)$, related to the algebras $\mathrm{TL}_n(u)$, can be redefined through the Homflypt polynomial, by specializing ζ to $-\frac{1}{u+1}$, see [14]. This is the non–trivial value for ζ, for which the Ocneanu trace τ passes to the quotient $\mathrm{TL}_n(u)$. Namely:

$$V(u)(\widehat{\alpha}) = \left(-\frac{1+u}{\sqrt{u}}\right)^{n-1} \left(\sqrt{u}\right)^{\varepsilon(\alpha)} \tau(\pi(\alpha)) = P(u,u)(\widehat{\alpha}).$$

As mentioned in Section 1.5, given a solution of the E–system parametrized by a subset S of $\mathbb{Z}/d\mathbb{Z}$, one can obtain an invariant for framed knots and links [11]:

$$\Gamma_{d,S}(w,u)(\widehat{\alpha}) = \left(-\frac{1-wu}{\sqrt{w}(1-u)E}\right)^{n-1} \left(\sqrt{w}\right)^{\varepsilon(\alpha)} \mathrm{tr}(\gamma(\alpha)), \tag{64}$$

where: $w = \frac{z+(1-u)E}{uz}$, γ the natural epimorphism of the framed braid group algebra $\mathbb{C}(u)\mathcal{F}_n$ onto the algebra $\mathrm{Y}_{d,n}(u)$, and $\alpha \in \cup_\infty \mathcal{F}_n$. Note that if the input braids α have all framings zero, then $\Gamma_{d,s}(w,u)$ restrict to invariants of classical knots and links, denoted $\Delta_{d,s}(w,u)$. In [1] it is shown that for generic values of the parameters u, z the invariants $\Delta_{d,S}(w,u)$ do not coincide with the Homflypt polynomial except in the trivial cases $u = 1$ or $E = 1$. More precisely, for $E = 1$ an algebra homomorphism can be defined, $h : \mathrm{Y}_{d,n}(u) \longrightarrow \mathrm{H}_n(u)$, and the composition $\tau \circ h$ is a Markov trace on $\mathrm{Y}_{d,n}(u)$ which takes the same values as the specialized trace tr, whereby the x_i's are specialized to the d^{th} roots of unity. For details see [1, §3]. Yet, as computational data [4] indicate, they may still be topologically equivalent to the Homflypt polynomial.

Recalling now the conditions of Theorem 6 for the trace tr to pass to the quotient $\mathrm{YTL}_{d,n}(u)$, we note that in both cases the x_i's are solutions of the E–system, as required by [11], in order to proceed with defining link invariants. We do not take into consideration case (i) for $z = -1$ and case (ii), where $z = -\frac{1}{2}$, since crucial braiding information is lost and therefore they are of no topological interest. For example, the trace tr for these two values of z gives the same result for all even (resp. odd) powers of the g_i's, as it becomes clear from the following formulas from [11], for $m \in \mathbb{Z}^{>0}$:

$$\mathrm{tr}(g_i^m) = \left(\frac{u^m-1}{u+1}\right) z + \left(\frac{u^m-1}{u+1}\right) E + 1 \qquad \text{if } m \text{ is even}$$

and

$$\mathrm{tr}(g_i^m) = \left(\frac{u^m-1}{u+1}\right) z + \left(\frac{u^m-1}{u+1}\right) E - E \qquad \text{if } m \text{ is odd},$$

since, for $z = -1$ and $z = -\frac{1}{2}$ we find from Eq. 47 $E = 1$ and $E = \frac{1}{2}$ respectively. The only remaining case of interest is case (i) of Theorem 6, where the x_ℓ's are the d^{th} roots of unity and $z = -\frac{1}{u+1}$. This implies that $E = 1$ and $w = u$ in Eq. 64. So, by [1] and [14], the invariant $\Delta_{d,s}(u, u)$ coincides with the Jones polynomial.

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