THE YOKONUMA–TEMPERLEY–LIEB ALGEBRA

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Abstract. In this paper we introduce the Yokonuma–Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra over a two-sided ideal generated by an expression analogous to the one of the classical Temperley-Lieb algebra. The main theorem provides necessary and sufficient conditions for the Markov trace defined on the Yokonuma–Hecke algebra to pass through to the quotient algebra, leading to a sequence of knot invariants which coincide with the Jones polynomial.

Introduction

The Temperley–Lieb algebra appeared originally in Statistical Mechanics and is important in several areas of Mathematics. In his seminal work V.F.R. Jones [15] constructed a Markov trace on the Temperley–Lieb algebra, leading to unexpected applications in knot theory as well as to a fertile interaction between Knot theory and Representation theory. In algebraic terms, the Temperley–Lieb algebra can be defined as a quotient of the Iwahori–Hecke algebra.

In [8] the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ (defined originally in [20]) has been defined as a quotient of the modular framed braid group $\mathcal{F}_{d,n}$, which comprises framed braids with framings modulo d, over a quadratic relation (Eq. 13) involving the framing generators t_i by means of certain weighted idempotents e_i (Eq. 9). Setting $d = 1$, the algebra $\mathrm{Y}_{1,n}(u)$ coincides with the Iwahori–Hecke algebra $\mathrm{H}_n(u)$. The Yokonuma–Hecke algebras have been studied in [20, 8, 10, 19, 3]. Further, in [8] the second author found an inductive linear basis for the algebras $\mathrm{Y}_{d,n}(u)$ and constructed a unique Markov trace tr on these algebras depending on parameters $z, x_1, \ldots, x_{d-1}$. Aiming to extracting framed link invariants from tr, as it turned out in [11], tr does not re-scale directly according to the framed braid equivalence, leading to conditions that have to be imposed on the trace parameters $x_1, \ldots, x_{d-1}$; namely, they had to satisfy a non-linear system of equations, the E–*system* (Eq. 16). The x_i's being d^{th} roots of unity is one obvious solution. Gérardin found in [11, Appendix] the full set of solutions of the E–system. Given now any solution of the E–system, 2–variable isotopy invariants for framed, classical and singular links were constructed in [11, 12, 13] respectively, which are studied further in [1, 4].

In this paper we define a Temperley–Lieb analogue of the Yokonuma–Hecke algebra, *the Yokonuma–Temperley–Lieb algebra* $\mathrm{YTL}_{d,n}(u)$, as a quotient of the Yokonuma–Hecke algebra over a two-sided ideal I (Eq. 21 and Definition 2), analogous to the classical case. For $d = 1$ the algebra $\mathrm{YTL}_{1,n}(u)$ coincides with the Temperley–Lieb algebra. We first show that I is a principal ideal (Corollary 1) and we give a presentation for $\mathrm{YTL}_{d,n}(u)$ with non-invertible generators, analogous to the classical case (Proposition 2). We then give a spanning set $\Sigma_{d,n}$ for $\mathrm{YTL}_{d,n}(u)$,

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where each word in $\Sigma_{d,n}$ contains the highest and lowest index braiding generator exactly once (Proposition 4). Moreover, any word in $\Sigma_{d,n}$ inherits the splitting property from $\mathrm{Y}_{d,n}(u)$, that is, it splits into the framing part and the braiding part. We also present the results of Chlouveraki and Pouchin [2] on the dimension (Proposition 5) and a linear basis for $\mathrm{YTL}_{d,n}(u)$ (Theorem 3). From the spanning set $\Sigma_{d,n}$, they extracted an explicit basis for $\mathrm{YTL}_{d,n}(u)$ by describing a set of linear dependence relations among the framing parts for each fixed element in the braiding part. Finally, using the dimension results of [2] we find a basis for $\mathrm{YTL}_{2,3}(u)$ different than the basis in [2]

Next, we seek conditions such that the trace tr, defined on the algebras $\mathrm{Y}_{d,n}(u)$, passes to the quotient algebras $\mathrm{YTL}_{d,n}(u)$. More precisely, we compute first the values of the trace parameter z that annihilate the generator of the defining ideal I, which are the roots of a quadratic equation (Eq. 51). Then we annihilate the traces of all elements of $\mathrm{Y}_{d,n}(u)$ that lie in I and so we end up with a system (Σ) of quadratic equations in z (Eqs. 61a–61d). If we demand that (Σ) has both roots of Eq. 51 as common solutions, which is essential for discussing link invariants, we end up with necessary conditions for the trace tr to pass to the quotient algebras $\mathrm{YTL}_{d,n}(u)$ (Theorem 4). More precisely, Theorem 4 states that the trace tr passes to the quotient algebra $\mathrm{YTL}_{d,n}(u)$ if the trace parameters are d^{th} roots of unity $x_1, \ldots, x_{d-1}$ and $z = -\frac{1}{u+1}$ and $z = -1$. Note that these two values for z are precisely the ones that Jones computed such that the Ocneanu trace on $\mathrm{H}_n(u)$ passes to the quotient, the Temperley–Lieb algebra $\mathrm{TL}_n(u)$. If we also let (Σ) to have one common solution for z we obtain the necessary and sufficient conditions for the trace tr to pass through to the quotient algebras $\mathrm{YTL}_{d,n}(u)$ (Theorem 5). More precisely, Theorem 5 states that the trace tr passes to the quotient algebras $\mathrm{YTL}_{d,n}(u)$ if and only if either the conditions of Theorem 4 are satisfied or the trace parameters $x_1, \ldots, x_{d-1}$ comprise a solution of the E–system (other than d^{th} roots of unity) and $z = -\frac{1}{2}$. This is our main result.

In [1] it is shown that if the trace parameters $x_1, \ldots, x_{d-1}$ are d^{th} roots of unity, then the classical link invariants derived from the algebra $\mathrm{Y}_{d,n}(u)$ coincide with the 2–variable Jones or HOMLYPT polynomial. Using Theorem 5 and the results in [1], we obtain from the invariants for framed and classical links in [11, 12] related to $\mathrm{Y}_{d,n}(u)$ 1–variable framed and classical link invariants through the algebras $\mathrm{YTL}_{d,n}(u)$ (Definition 4). As we show, these invariants coincide with the Jones polynomial for the case of classical links and they are framed analogues of the Jones polynomial for the case of framed links (Corollary 2).

The paper is organized as follows: In Section 1 we recall the definition and basic properties of the classical Temperley–Lieb algebra and the Yokonuma–Hecke algebra. In Section 2 we define the Yokonuma–Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra over a two-sided ideal (Eq. 21 and Definition 2), which we show that is a principal ideal (Corollary 1). Finally, we give a presentation for $\mathrm{YTL}_{d,n}(u)$ with non-invertible generators (Proposition 2). In Section 3 we present a spanning set for $\mathrm{YTL}_{d,n}(u)$ and the results of Chlouveraki and Pouchin [2] on the dimension and a linear basis for $\mathrm{YTL}_{d,n}(u)$. Then we give a basis for $\mathrm{YTL}_{2,3}(u)$. Section 4 focuses on the necessary and sufficient conditions under which the trace tr on $\mathrm{Y}_{d,n}(u)$ passes to the quotient algebra $\mathrm{YTL}_{d,n}(u)$ (Theorems 4 and 5). Finally, in Section 5 we discuss the invariants for classical and framed links that can be constructed through the trace tr and we recover the Jones polynomial (Corollary 2).

1. Preliminaries

1.1. *Notations.* Throughout the paper we shall fix the following notation. By the term algebra we mean an associative unital (with unity 1) algebra over the field $K := \mathbb{C}(u)$, where u is an indeterminate. The following two possitive integers are also fixed: d and n.

As usual we denote by B_n the braid group on n strands, that is the group generated by the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i is the positive crossing between the i^{th} and the $(i+1)^{st}$ strand, satisfying the well-known braid relations: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_i\sigma_j$ for $|i-j| > 1$.

We denote S_n the symmetric group on n symbols. Let s_i be the elementary transposition $(i, i+1)$. We denote by l the length function on S_n with respect to the s_i's.

Denote by $C_d = \langle t \,|\, t^d = 1\rangle$ the cyclic group of order d. Let $t_i = (1, \ldots, t, 1, \ldots, 1) \in C_d^n$, where t is in the i^{th} position.

Finally, we denote $C_{d,n} := C_d^n \rtimes S_n$, where the action is defined by permutation on the indices of the t_i's, namely: $s_i t_j = t_{s_i(j)} s_i$.

1.2. *The Temperley–Lieb algebra.* The Temperley–Lieb algebra, over $\mathbb{C}$, defined by generators $1, f_1, \ldots, f_{n-1}$ subject to the following relations:

$$\begin{array}{rcl} f_i^2 & = & f_i \\ f_i f_j f_i & = & \tau f_i, \quad |i-j| = 1 \\ f_i f_j & = & f_j f_i, \quad |i-j| > 1 \end{array}$$

where τ is a non–zero complex number (see [6],[14],[15]). The generators f_i are non-invertible; one can define the Temperley–Lieb algebra with the following invertible generators (see [14]):

$$h_i := (u+1)f_i - 1 \tag{1}$$

where u is defined via the relation $\tau^{-1} = 2 + u + u^{-1}$. The Temperley algebra $\mathrm{TL}_n(u)$, over K, is defined by generators $h_1, \ldots, h_{n-1}$ under the relations:

$$h_i h_j h_i = h_j h_i h_j, \quad |i-j| = 1 \tag{2}$$

$$h_i h_j = h_j h_i, \quad |i-j| > 1 \tag{3}$$

$$h_i^2 = (u-1)h_i + u \tag{4}$$

$$h_i h_j h_i + h_j h_i + h_i h_j + h_i + h_j + 1 = 0, \quad |i-j| = 1. \tag{5}$$

Note that relations (5) are symmetric with respect to the indices i, j, so relations (2) follow from relations (5). Relations (2)–(4) are the well–known defining relations of the Iwahori–Hecke algebra $\mathrm{H}_n(u)$. Therefore, $\mathrm{TL}_n(u)$ can be considered as a quotient of the $\mathrm{H}_n(u)$ over the two–sided ideal generated by relations (5). It turns out that the set:

$$\left\{(h_{j_1}h_{j_1-1}\ldots h_{j_1-k_1})\,(h_{j_2}h_{j_2-1}\ldots h_{j_2-k_2})\ldots\left(h_{j_p}h_{j_p-1}\ldots h_{j_p-k_p}\right)\right\}$$

where $1 \le j_1 < j_2 < \ldots < j_p \le n-1$ and $1 \le j_1 - k_1 < j_2 - k_2 < \ldots < j_p - k_p$, furnishes a linear basis for $\mathrm{TL}_n(u)$ and the dimension of $\mathrm{TL}_n(u)$ is equal to the n^{th} Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$ [14, 15]. Recall finally, that in [5], Ocneanu constructed a unique Markov trace on the algebras $\mathrm{H}_n(u)$:

Theorem 1 (Ocneanu). *For any $\zeta \in K^\times$ there exists a linear trace τ on $\cup_{n=1}^\infty \mathrm{H}_n(u)$ uniquely defined by the inductive rules:*

(1) $\tau(ab) = \tau(ba), \quad a, b \in \mathrm{H}_n(u)$
(2) $\tau(1) = 1$

(3) $\tau(ag_n) = \zeta\, \tau(a), \quad a \in \mathrm{H}_n(u)$.

Jones' methods for redefining his Markov trace on the Temperley–Lieb algebra as factoring of the Ocneanu trace on the Iwahori–Hecke algebra [14] tells us that the least requirement is that the Ocneanu trace respects the defining relations (5). This requirement implies:

$$\zeta = -\frac{1}{u+1} \quad \text{and} \quad \zeta = -1. \tag{6}$$

The Ocneanu trace is used in [14] for constructing the HOMFLYPT polynomial invariant for classical knots and links. Then, by specializing ζ to $-\frac{1}{u+1}$ the Jones polynomial was recovered.

1.3. *The Yokonuma–Hecke algebra.* The group $\mathbb{Z}^n$ is generated by the "framing generators" $t_1, \ldots, t_n$, the standard multiplicative generators of $\mathbb{Z}^n$. In this notation an element $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ in the additive notation can be expressed as $t_1^{a_1} \ldots t_n^{a_n}$. The *framed braid group* on n strands is then defined as:

$$\mathcal{F}_n = \mathbb{Z}^n \rtimes B_n$$

where the action of B_n on $\mathbb{Z}^n$ is given by the permutation induced by a braid on the indices:

$$\sigma_i t_j = t_{s_i(j)} \sigma_i. \tag{7}$$

In particular, $\sigma_i t_i = t_{i+1} \sigma_i$ and $t_{i+1} \sigma_i = \sigma_i t_i$. A word w in $\mathcal{F}_n$ has thus the "splitting property", that is, it splits into the "framing" part and the "braiding" part:

$$w = t_1^{a_1} \ldots t_n^{a_n}\, \sigma$$

where $\sigma \in B_n$ and $a_i \in \mathbb{Z}$. So w is a classical braid with an integer attached to each strand. Topologically, an element of $\mathbb{Z}^n$ is identified with a framed identity braid on n strands, while a classical braid in B_n is viewed as a framed braid with all framings 0. The multiplication in $\mathcal{F}_n$ is defined by placing one braid on top of the other and collecting the total framing of each strand to the top.

For a fixed positive integer d, the *d-modular framed braid group* on n strands, $\mathcal{F}_{d,n}$, is defined as the quotient of $\mathcal{F}_n$ over the *modular relations*:

$$t_i^d = 1 \quad (i = 1, \ldots, n). \tag{8}$$

Thus, $\mathcal{F}_{d,n} = C_d^n \rtimes B_n$, where C_d^n is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^n$ but with multiplicative notation. Framed braids in $\mathcal{F}_{d,n}$ have framings modulo d.

Passing now to the group algebra $\mathbb{C}\mathcal{F}_{d,n}$, we have the following elements $e_i \in \mathbb{C}C_d^n$ (see [10] for diagrammatic interpretations), which are idempotents (cf. [10, Lemma 4]):

$$e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}, \qquad i = 1, \ldots, n-1. \tag{9}$$

The definition of the idempotent e_i can be generalized in the following way. For any indices i, j and any $m \in \mathbb{Z}/d\mathbb{Z}$, we define the following elements in $\mathrm{Y}_{d,n}(u)$:

$$e_{i,j} := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_j^{-s}, \tag{10}$$

and:

$$e_i^{(m)} := \frac{1}{d} \sum_{s=0}^{d-1} t_i^{m+s} t_{i+1}^{-s}. \tag{11}$$

(notice that $e_i = e_{i,i+1} = e_i^{(0)}$). The following lemma collects some of the relations among the e_i's, the t_i's and the g_i's. These relations will be used in the paper.

Lemma 1. *For the idempotents e_i and for $1 \leq i,\ j \leq n-1$ the following relations hold:*

$$\begin{array}{rcl} t_j e_i & = & e_i t_j \\ e_{i+1} g_i & = & g_i e_{i,i+2} \\ e_i g_j & = & g_j e_i, \quad \textit{for } j \neq i-1, i+1 \\ e_j g_i g_j & = & g_i g_j e_i \quad \textit{for } |i-j| = 1 \\ e_i e_{i+1} & = & e_i e_{i,i+2} \\ e_i e_{i+1} & = & e_{i,i+2} e_{i+1}. \end{array}$$

Proof. All relations are immediate consequences of the definitions. The proofs for the first four relations can be found, for example, in [13, Lemma 2.1]. For the fifth relation we have:

$$\begin{aligned} e_i e_{i+1} &= \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}\, \frac{1}{d}\sum_{m=0}^{d-1} t_{i+1}^m t_{i+2}^{-m} \\ &= \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{m=0}^{d-1} t_i^s t_{i+1}^{m-s} t_{i+2}^{-m}. \end{aligned} \tag{12}$$

Setting now $k = m - s$ we obtain:

$$\begin{aligned} (12) &= \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} t_i^s t_{i+1}^{k} t_{i+2}^{-k-s} \\ &= \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_{i+2}^{-s}\, \frac{1}{d}\sum_{k=0}^{d-1} t_{i+1}^k t_{i+2}^{-k} \\ &= e_{i,i+2} e_{i+1}. \end{aligned}$$

The sixth relation is proved in an analogous way. □

The *Yokonuma–Hecke algebra* $\mathrm{Y}_{d,n}(u)$ is defined [8, 10] as the quotient of the group algebra $\mathbb{C}\mathcal{F}_{d,n}$ over the two-sided ideal generated by the elements:

$$\sigma_i^2 - 1 - (u-1)\, e_i - (u-1)\, e_i\, \sigma_i, \quad \text{for all } i,$$

which give rise to the following quadratic relations in $\mathrm{Y}_{d,n}(u)$:

$$g_i^2 = 1 + (u-1)\, e_i + (u-1)\, e_i\, g_i \tag{13}$$

where g_i corresponds to σ_i (see [10] for diagrammatic interpretations). Since the quadratic relations do not change the framing we have $\mathbb{C}C_d^n \subset \mathrm{Y}_{d,n}(u)$ and we keep the same notation for the elements of $\mathbb{C}C_d^n$ and for the elements e_i in $\mathrm{Y}_{d,n}(u)$. The elements g_i are invertible:

$$g_i^{-1} = g_i + (u^{-1}-1)\, e_i + (u^{-1}-1)\, e_i\, g_i.$$

For $d = 1$ we have $t_j = 1$ and $e_i = 1$, and in this case the quadratic relations (13) become $g_i^2 = (u-1)g_i + u$, which are the quadratic relations of the Iwahori–Hecke algebra $\mathrm{H}_n(u)$. So, $\mathrm{Y}_{1,n}(u)$ coincides with the algebra $\mathrm{H}_n(u)$. Further, there is an obvious epimorphism of the Yokonuma-Hecke algebra $\mathrm{Y}_{d,n}(u)$ onto the algebra $\mathrm{H}_n(u)$ via the map:

$$\begin{array}{rcl} g_i & \mapsto & h_i \\ t_j & \mapsto & 1. \end{array} \tag{14}$$

We can alternatively define the algebra $\mathrm{Y}_{d,n}(u)$ as a u–deformation of the algebra $\mathbb{C}C_{d,n}$. More precisely, let $w \in S_n$ and let $w = s_{i_1} \ldots s_{i_k}$ be a reduced expression for w. Since the generators g_i of $\mathrm{Y}_{d,n}(u)$ satisfy the same braiding relations as the generators of S_n, then together with the well-known theorem of Matsumoto [16], it follows that $g_w := g_{i_1} \ldots g_{i_k}$ is well defined. Notice that the defining generators g_i correspond to g_{s_i}. We have the following multiplication rule in $\mathrm{Y}_{d,n}(u)$ (see Proposition 2.4[7]):

$$g_{s_i} g_w = \begin{cases} g_{s_i w} & \text{for } l(s_i w) > l(w) \\ g_{s_i w} + (u-1) e_i g_{s_i w} + (u-1) e_i g_w & \text{for } l(s_i w) < l(w) \end{cases} \tag{15}$$

We also correspond g_{t_i} to t_i and we define: $g_{t_i w} = g_{t_i} g_w = t_i g_w$. Using the above multiplication formulas the second author proved in [8] that $\mathrm{Y}_{d,n}(u)$ has the following standard basis:

$$\{t_1^{a_1} \cdots t_n^{a_n} g_w \mid a_i \in \mathbb{Z}/d\mathbb{Z},\, w \in S_n\}$$

Further, we have an inductive basis of the Yokonuma–Hecke algebra, which is used in the proof of the main theorem.

Proposition 1 ([8] Proposition 8)**.** *Every element in $\mathrm{Y}_{d,n+1}(u)$ is a unique linear combination of words, each of one of the following types:*

$$\mathfrak{m}_n g_n g_{n-1} \ldots g_i t_i^k \quad \text{or} \quad \mathfrak{m}_n t_{n+1}^k,$$

where $k \in \mathbb{Z}/d\mathbb{Z}$ and $\mathfrak{m}_n$ is a word in the inductive basis of $\mathrm{Y}_{d,n}(u)$.

1.4. *A Markov trace on* $\mathrm{Y}_{d,n}(u)$**.** Using the above basis, the second author constructed in [8] a linear Markov trace on the algebra $\mathrm{Y}_{d,n}(u)$. Namely:

Theorem 2 ([8] Theorem 12)**.** *Let d a positive integer. For indeterminates z, x_1, $\ldots, x_{d-1}$ there exists a unique linear Markov trace* tr*:*

$$\mathrm{tr} : \cup_{n=1}^{\infty} \mathrm{Y}_{d,n}(u) \longrightarrow K[z, x_1, \ldots, x_{d-1}]$$

defined inductively on n by the following rules:

$$\begin{array}{rcll} \mathrm{tr}(ab) & = & \mathrm{tr}(ba) & \\ \mathrm{tr}(1) & = & 1 & \\ \mathrm{tr}(a g_n) & = & z\,\mathrm{tr}(a) & (\textit{Markov property}) \\ \mathrm{tr}(a t_{n+1}^s) & = & x_s \mathrm{tr}(a) & (s = 1, \ldots, d-1) \end{array}$$

where $a, b \in \mathrm{Y}_{d,n}(u)$.

By direct computation, $\mathrm{tr}(e_i)$ takes the same value for all i. We denote this value by E, that is:

$$E := \mathrm{tr}(e_i) = \frac{1}{d} \sum_{s=0}^{d-1} x_s x_{d-s},$$

where $x_0 := 1$. For all $0 \leq m \leq d-1$, we also define:

$$E^{(m)} := \mathrm{tr}(e_i^{(m)}) = \frac{1}{d} \sum_{s=0}^{d-1} x_{m+s} x_{d-s},$$

where $e_i^{(m)}$ is defined in (11). Notice that $E = E^{(0)}$.

1.5. *The* E*–system.* In order for an invariant for framed knots and links to be constructed through the trace on $\mathrm{Y}_{d,n}(u)$, tr should be normalized and rescaled properly. In [11] it is proved that such a rescaling is possible if the trace parameters x_i are solutions of a non-linear system of equations, the so–called E–system.

Definition 1. We say that the set of complex numbers $\{x_0, x_1, \ldots, x_{d-1}\}$ (where x_0 is always equal to 1) satisfies the E*–condition* if $x_1, \ldots, x_{d-1}$ satisfy the following E*–system* of non–linear equations in $\mathbb{C}$:

$$E^{(m)} = x_m E \qquad (1 \leq m \leq d-1)$$

or equivalently:

$$\sum_{s=0}^{d-1} x_{m+s} x_{d-s} = x_m \sum_{s=0}^{d-1} x_s x_{d-s} \qquad (1 \leq m \leq d-1). \tag{16}$$

In [11, Appendix] it is proved that the solutions of the E–system are the functions $x_{_S}$, from $\mathbb{Z}/d\mathbb{Z}$ to $\mathbb{C}$, parametrized by the non–empty subsets S of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ as follows:

$$x_{_S} = \frac{1}{|S|} \sum_{s \in S} \exp_s \tag{17}$$

where $\exp_s(k) = \cos \frac{2\pi s k}{d} + i \sin \frac{2\pi s k}{d}$ $(k \in \mathbb{Z}/d\mathbb{Z})$.

Remark 1. It is worth noting that the solution of the E–system can be interpreted as a generalization of the Ramanujan's sum. Indeed, by taking the subset P of $\mathbb{Z}/d\mathbb{Z}$ consisting of the numbers coprimes to d, then the solution parametrized by P is, up to the factor $|P|$, the Ramanujan's sum $c_d(k)$ (see [17]).

Equivalently, $x_{_S}$ can be seen as an element in $\mathbb{C}C_d$, namely:

$$x_{_S} = \sum_{k=0}^{d-1} x_k t^k \tag{18}$$

where $x_k = \frac{1}{|S|} \sum_{s \in S} \chi_s t^k$, $k = 0, \ldots, d-1$, and χ_s is the character of C_d defined as $\chi_s : t^m \mapsto \exp(sm)$. So, the coefficient x_k of t^k in (18) corresponds to $x_{_S}(k)$ in (17).

Recall now that on the group algebra $\mathbb{C}G$ of the finite group G, we have two products, one of them is the multiplication by coordinates, also called the multiplications of the values, which is defined as:

$$\left(\sum_{g \in G} a_g g\right) \cdot \left(\sum_{g \in G} b_g g\right) = \sum_{g \in G} a_g b_g g.$$

and the other product is the convolution product:

$$\left(\sum_{g \in G} a_g g\right) * \left(\sum_{h \in G} b_h h\right) = \sum_{g \in G} \sum_{h \in G} a_g b_h g h = \sum_{g \in G} \left(\sum_{h \in G} a_h b_{g h^{-1}}\right) g. \tag{19}$$

By taking $G = C_d$ and writing an arbitrary element x in $\mathbb{C}C_d$ as $x = \sum_{0 \leq k \leq d-1} a_k t^k$, we have the following lemma:

Lemma 2. *In $\mathbb{C}C_d$ we have:*

$$x * x = d \sum_{0 \leq \ell \leq d-1} E^{(\ell)} t^\ell$$

and

$$x * x * x = d^2 \sum_{0 \leq \ell \leq d-1} \mathrm{tr}(e_1^\ell e_2) t^\ell .$$

Proof. The expression for $x * x$ follows immediately by direct computation. For the second expression we have that:

$$\begin{aligned} x * x * x &= d \sum_{0 \leq \ell \leq d-1} E^{(\ell)} t^\ell * x \\ &= d \sum_{0 \leq \ell \leq d-1} E^{(\ell)} t^\ell * \sum_{0 \leq k \leq d-1} a_k t^k \\ &= d \sum_{0 \leq \ell, k \leq d-1} E^{(\ell)} a_k t^{\ell+k} \\ &= d \sum_{0 \leq \ell, k, s \leq d-1} a_s a_{\ell-s} a_k t^{\ell+k} \\ &= d \sum_{0 \leq \ell, k, s \leq d-1} a_s a_{\ell-s-k} a_k t^\ell \\ &= d^2 \mathrm{tr}(e_1^{(\ell)} e_2). \end{aligned}$$

□

For each $a \in \mathbb{Z}/d\mathbb{Z}$ the character χ_a defines, with respect to the convolution product, an element $\mathbf{i}_a$ of $\mathbb{C}C_d$,

$$\mathbf{i}_a := \sum_{0 \leq s \leq d-1} \chi_a(s) t^s .$$

One can verify that

$$\mathbf{i}_a * \mathbf{i}_b = \begin{cases} d\, \mathbf{i}_a & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

that is, $\mathbf{i}_a/d$ is an idempotent element. On the other hand, regarding $\delta_a := t^a$ as element in $\mathbb{C}C_d$, it is clear that,

$$\delta_a \cdot \delta_b = \begin{cases} \delta_a & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} .$$

The connection between the two products on $\mathbb{C}C_d$ is given by the *Fourier transform*. More precisely, the Fourier transform is the linear automorphism on $\mathbb{C}C_d$, defined as:

$$x := \sum_{0 \leq r \leq d-1} a_r t^r \mapsto \widehat{x} := (x * \mathbf{i}_s)(0) = \sum_{0 \leq \ell \leq d-1} a_\ell \chi_s(d - \ell) \tag{20}$$

With the above notation we have:

Lemma 3. *The following hold in $\mathbb{C}C_d$:*

$$\widehat{x * y} = \widehat{x} \cdot \widehat{y}, \qquad \widehat{x \cdot y} = d^{-1} \widehat{x} * \widehat{y},$$

$$\widehat{\delta}_a = \mathbf{i}_{-a}, \qquad \widehat{\mathbf{i}}_a = d\delta_a, \qquad \widehat{\widehat{x}}(u) = dx(-u).$$

Proof. The proof is just a straightforward computation (see [18]). □

2. The Yokonuma–Temperley–Lieb Algebra

In this section we define the Temperley–Lieb analogue, in the case of framing, as quotient of $\mathrm{Y}_{d,n}(u)$ over an appropriate two–sided ideal.

2.1. *The Yokonuma–Temperley–Lieb algebra.* The Hecke algebra, $\mathrm{H}_n(u)$, can be considered as a u–deformation of the $\mathbb{C}S_n$, while $\mathrm{TL}_n(u)$ is the quotient of $\mathrm{H}_n(u)$ over the two–sided ideal:

$$J = \langle h_{i,j}\,;\,\text{for all}\, i,j\, \text{such that}\,\ |i-j|=1\rangle$$

where $h_{i,j}$'s are the Steinberg elements $h_{i,j} := 1 + h_i + h_{i+1} + h_i h_{i+1} + h_{i+1} h_i + h_i h_{i+1} h_i$. It is well-known that that J is a principal ideal. Indeed,

$$J = \langle h_{1,2}\rangle.$$

Notice now that $h_{i,j}$ can be rewritten as

$$h_{i,j} = \sum_{\alpha \in W_{i,j}} h_\alpha$$

where $W_{i,j}$ is the subgroup of S_n generated by s_i and s_j (clearly, $W_{i,j}$ is isomorphic to S_3). On the other hand $\mathrm{Y}_{d,n}(u)$ can be regarded as a u–deformation of $\mathbb{C}[C_d^n \rtimes S_n]$. The symmetric group S_n can be considered as a subgroup of $C_d^n \rtimes S_n$, therefore the subgroups $W_{i,j}$ of S_n can be also regarded as subgroups of $C_d^n \rtimes S_n$. Thus, in analogy to the ideal J of $\mathrm{H}_n(u)$, it is natural to consider the following ideal I of $\mathrm{Y}_{d,n}(u)$:

$$I := \langle g_{i,j}\,;\,\text{for all}\, i,j\, \text{such that}\,\ |i-j|=1\rangle \tag{21}$$

where

$$g_{i,j} := \sum_{\alpha \in W_{i,j}} g_\alpha = 1 + g_i + g_j + g_i g_j + g_j g_i + g_i g_j g_i. \tag{22}$$

We then define:

Definition 2. For $n \geq 3$, the *Yokonuma–Temperley–Lieb* algebra, $\mathrm{YTL}_{d,n}(u)$, is defined as the quotient:

$$\mathrm{YTL}_{d,n}(u) = \frac{\mathrm{Y}_{d,n}(u)}{I}.$$

In other words, the algebra $\mathrm{YTL}_{d,n}(u)$ can be presented by the generators $1, g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$ (by abuse of notation), subject to the following relations:

$$g_i g_j = g_j g_i, \quad |i-j|>1 \tag{23}$$
$$g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i \tag{24}$$
$$g_i^2 = 1 + (u-1)e_i + (u-1)e_i g_i \tag{25}$$
$$t_i t_j = t_j t_i, \quad \text{for all } i,j \tag{26}$$
$$t_i^d = 1, \quad \text{for all } i \tag{27}$$
$$g_i t_i = t_{i+1} g_i \tag{28}$$
$$g_i t_{i+1} = t_i g_i \tag{29}$$
$$g_i t_j = t_j g_i, \quad \text{for } j \neq i, \text{ and } j \neq i+1 \tag{30}$$
$$g_i g_{i+1} g_i + g_i g_{i+1} + g_{i+1} g_i + g_i + g_{i+1} + 1 = 0 \tag{31}$$

We shall refer to relations (31) as *the Steinberg relations.*

Notice that relations (23)–(30) are the defining relations of the algebra $\mathrm{Y}_{d,n}(u)$. Note also that relations (31) are symmetric with respect to the indices i, $i+1$, i.e.:

$$g_i g_{i+1} g_i = -g_i g_{i+1} - g_{i+1} g_i - g_{i+1} - g_i - 1 = g_{i+1} g_i g_{i+1}.$$

so relations (24) follow from relations (31).

Remark 2. In analogy to the Yokonuma–Hecke algebra, $\mathrm{YTL}_{1,n}(u)$ coincides with the algebra $\mathrm{TL}_n(u)$. Further, the epimorphism (14) induces an epimorphism of the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$ onto the algebra $\mathrm{TL}_n(u)$. Clearly, by relations (28) and (29), any monomial in $\mathrm{YTL}_{d,n}(u)$ inherits the *splitting property* of $\mathrm{Y}_{d,n}(u)$, that is, it can be written in the form:

$$w = t_1^{a_1} \dots t_n^{a_n} g_{i_1} \dots g_{i_k}, \tag{32}$$

where: $a_1, \dots, a_n \in \mathbb{Z}/d\mathbb{Z}$.

We shall now prove that I is in fact a principal ideal.

Lemma 4. *The following hold in* $\mathrm{Y}_{d,n}(u)$ *for all* $i = 1, \dots, n-2$*:*

$$\begin{array}{lrcl}
(1) & g_i & = & (g_1 \dots g_{n-1})^{i-1}\, g_1\, (g_1 \dots g_{n-1})^{-(i-1)} \\
(2) & g_{i+1} & = & (g_1 \dots g_{n-1})^{i-1}\, g_2\, (g_1 \dots g_{n-1})^{-(i-1)} \\
(3) & g_i g_{i+1} & = & (g_1 \dots g_{n-1})^{i-1}\, g_1 g_2\, (g_1 \dots g_{n-1})^{-(i-1)} \\
(4) & g_{i+1} g_i & = & (g_1 \dots g_{n-1})^{i-1}\, g_2 g_1\, (g_1 \dots g_{n-1})^{-(i-1)} \\
(5) & g_i g_{i+1} g_i & = & (g_1 \dots g_{n-1})^{i-1}\, g_1 g_2 g_1\, (g_1 \dots g_{n-1})^{-(i-1)}
\end{array}$$

Proof. We will demonstrate the proof for the cases (1) and (5). The rest of the cases are proved in an analogous manner. For case (1) we have that the statement is true for $i = 2$. Indeed:

$$\begin{aligned}
(g_1 \dots g_{n-1})\, g_1\, (g_1 \dots g_{n-1})^{-1} &= g_1 g_2 g_1 g_3 \dots g_{n-1} (g_1 \dots g_{n-1})^{-1} \\
&= g_2 (g_1 g_2 \dots g_{n-1}) (g_1 \dots g_{n-1})^{-1} \\
&= g_2.
\end{aligned}$$

Suppose that the statement is true for $i = k$. We will show that the statement holds for $i = k+1$. We have:

$$\begin{aligned}
(g_1 \dots g_{n-1})^k g_1 (g_1 \dots g_{n-1})^{-k} &= (g_1 \dots g_{n-1})(g_1 \dots g_{n-1})^{k-1} g_1 (g_1 \dots g_{n-1})^{-(k-1)} (g_1 \dots g_{n-1})^{-1} \\
&= (g_1 \dots g_{n-1}) g_k (g_1 \dots g_{n-1})^{-1} \\
&= g_1 \dots g_{k-1} g_k g_{k+1} g_k g_{k+2} \dots g_{n-1} (g_1 \dots g_{n-1})^{-1} \\
&= g_1 \dots g_{k-1} g_{k+1} g_k g_{k+1} \dots g_{n-1} (g_1 \dots g_{n-1})^{-1} \\
&= g_{k+1} (g_1 \dots g_{n-1})(g_1 \dots g_{n-1})^{-1} \\
&= g_{k+1}.
\end{aligned}$$

For case (5) we have from (1):

$$\begin{aligned}
g_i g_{i+1} g_i &= (g_1 \dots g_{n-1})^{i-1} g_1 (g_1 \dots g_{n-1})^{-(i-1)} (g_1 \dots g_{n-1})^{i} g_1 (g_1 \dots g_{n-1})^{-i} \\
&\quad \cdot (g_1 \dots g_{n-1})^{i-1} g_1 (g_1 \dots g_{n-1})^{-(i-1)} \\
&= (g_1 \dots g_{n-1})^{i-1} g_1 (g_1 \dots g_{n-1})^{-(i-1)} (g_1 \dots g_{n-1})^{i-1} (g_1 \dots g_{n-1}) \\
&\quad \cdot g_1 (g_1 \dots g_{n-1})^{-1} (g_1 \dots g_{n-1})^{-(i-1)} (g_1 \dots g_{n-1})^{i-1} g_1 (g_1 \dots g_{n-1})^{-(i-1)} \\
&= (g_1 \dots g_{n-1})^{i-1} g_1 g_2 g_1 (g_1 \dots g_{n-1})^{-(i-1)}.
\end{aligned}$$

□

Corollary 1. $\mathrm{YTL}_{d,n}(u)$ *is the K–algebra generated by the set $\{1, t_1, \ldots, t_n, g_1, \ldots, g_{n-1}\}$ whose elements are subject to the defining relations of* $\mathrm{Y}_{d,n}(u)$ *and the relation:*

$$g_{1,2} = 0.$$

Proof. The result follows using the multiplication rule defined on $\mathrm{Y}_{d,n}(u)$ and Lemma 4. □

2.2. *A presentation with non-invertible generators.* In analogy with Eq. 1 one can obtain a presentation for the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}((u)$ with the non-invertible generators:

$$l_i := \frac{1}{u+1}(g_i + 1). \tag{33}$$

In particular we have:

Proposition 2. $\mathrm{YTL}_{d,n}(u)$ *can be viewed as the algebra generated by the elements:*

$$1, l_1, \ldots l_{n-1}, t_1, \ldots, t_n,$$

which satisfy the following defining relations:

$$\begin{aligned}
t_i^d &= 1, \quad \textit{for all } i && (34)\\
t_i t_j &= t_j t_i, \quad \textit{for all } i,j && (35)\\
l_i t_j &= t_j l_i, \quad \textit{for } j \neq i \textit{ and } j \neq i+1 && (36)\\
l_i t_i &= t_{i+1} l_i + \frac{1}{u+1}(t_i - t_{i+1}) && (37)\\
l_i t_{i+1} &= t_i l_i + \frac{1}{u+1}(t_{i+1} - t_i) && (38)\\
l_i^2 &= \frac{(u-1)e_i + 2}{u+1}\, l_i && (39)\\
l_i l_j &= l_j l_i, \quad |i-j| > 1 && (40)\\
l_i l_{i\pm 1} l_i &= \frac{(u-1)e_i + 1}{(u+1)^2}\, l_i && (41)
\end{aligned}$$

Proof. Obviously, $\mathrm{YTL}_{d,n}(u)$ is generated by the l_i's and the t_i's. It is a straightforward computation to see that relations (23)–(31) are transformed into the relations (34) – (41). However, we shall show here how it works for the quadratic relations (34) and the Steinberg relations (41). From Eq. 33 we obtain:

$$g_i = (u+1)l_i - 1. \tag{42}$$

We then have that:

$$g_i^2 = \left((u+1)^2 l_i - 1\right)^2$$

which is equivalent to:

$$1 + (u-1)e_i + (u-1)e_i g_i = (u+1)^2 l_i^2 - 2(u-1)l_i + 1$$

or equivalently:

$$(u-1)(u+1)e_i l_i = (u+1)^2 l_i^2 - 2(u+1)l_i$$

which leads to:

$$l_i^2 = \frac{(u-1)e_i + 2}{u+1} l_i.$$

which is Eq. 39.

For the Steinberg elements $g_{i,i\pm1}$ using Eq. 42 we have that:

$$g_{i,i\pm1} = g_i g_{i\pm1} g_i + g_{i\pm1} g_i + g_i g_{i\pm1} + g_{i\pm1} + g_i + 1 = (u+1)^3 l_i l_{i\pm1} l_i - (u+1)^2 l_i^2 + (u+1) l_i$$

From the Steinberg relation (31) and Eq. 39 we have that:

$$(u+1)^2 l_i l_{i\pm1} l_i = ((u-1)e_i + 1) l_i$$

or equivalently:

$$l_i l_{i\pm1} l_i = \frac{(u-1)e_i + 1}{(u+1)^2}\, l_i,$$

which is Eq. 41. □

Remark 3. Setting $d = 1$ in the presentation of $\mathrm{YTL}_{d,n}(u)$ in Proposition 2, one obtains the classical presentation of $\mathrm{TL}_n(u)$, as discussed in Subsection 1.2. Note also that, substituting in the braid relation (24) the g_i's using Eq. 42, we obtain the equation:

$$l_i l_{i+1} l_i - \frac{(u-1)e_i + 1}{(u+1)^2}\, l_i = l_{i+1} l_i l_{i+1} - \frac{(u-1)e_{i+1} + 1}{(u+1)^2}\, l_{i+1}$$

which becomes superfluous, since it can be deduced from Eq. 41. This was to be expected, since the braid relations (24) were also superfluous.

3. A spanning set for The Yokonuma–Temperley–Lieb algebra

In this section we discuss various properties of a word in $\mathrm{YTL}_{d,n}(u)$ and we present a spanning set for $\mathrm{YTL}_{d,n}(u)$ (Proposition 4). Furthermore, using the work of Chlouveraki and Pouchin in [2] we give their formula for the dimension of $\mathrm{YTL}_{d,n}(u)$ (Proposition 5) and we also discuss their results on the linear basis of $\mathrm{YTL}_{d,n}(u)$ (Theorem 3). We finally compute a basis for $\mathrm{YTL}_{2,3}(u)$ different than the one of Theorem 3.

3.1. We have the following definition:

Definition 3. *In* $\mathrm{YTL}_{d,n}(u)$ *we define a length function* l *as follows:*

$$l(t^a g_{i_1} \ldots g_{i_k}) := l'(s_{i_1} \ldots s_{i_k}),$$

where l' *is the usual length function of* S_n *and* $t^a := t_1^{a_1} \ldots t_n^{a_n} \in C_d^n$. *A word in* $\mathrm{YTL}_{d,n}(u)$ *of the form (32) shall be called reduced if it is of minimal length with respect to relations (23)–(25), (31).*

Proposition 3. *Each word in* $\mathrm{YTL}_{d,n}(u)$ *can be written as a sum of monomials, where the highest and lowest index of the generators* g_i *appear at most once.*

Proof. An analogous statement holds for the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ where only the highest index generators appear at most once [8, Proposition 8]. Since $\mathrm{YTL}_{d,n}(u)$ is a quotient of the algebra $\mathrm{Y}_{d,n}(u)$ the highest index property passes through to the algebra $\mathrm{YTL}_{d,n}(u)$. The idea is analogous to [15, Lemma 4.1.2] and it is based on induction on the length of reduced words, use of the braid relations and reduction of length using the quadratic relations (25). For the case of the lowest index generator g_i we use induction on the length of reduced words and the Steinberg relations (31). Indeed, clearly, the statement is true for all words of length ≤ 2, namely for words of the form t^a, $t^a g_1$ $t^a g_1 g_2$ and $t^a g_2 g_1$.

For words of length 3: Let $w = t^a g_1 g_2 g_1$. Applying relation (24) will violate the highest index property of the word, so we must use the Steinberg relation (31) and we have:

$$t^a g_1 g_2 g_1 \quad = \quad -t^a g_2 g_1 - t^a g_1 g_2 - t^a g_2 - t^a g_1 - t^a.$$

We assume that the lowest index generator appears at most once in all words of length $\leq r$, and we will show the lowest index property for words of length $r+1$. Let $w = t^a g_{i_1} g_{i_2} \dots g_{i_k}$ be a reduced word in $\mathrm{YTL}_{d,n}(u)$ of length $r+1$, and $l = \min\{i_1, \dots, i_k\}$.

Let first $w = t^a w_1 g_l w_2 g_l w_3$, and suppose that w_2 does not contain g_l. We then have two possibilities:

If w_2 does not contain g_{l+1}, then g_l commutes with all the g_i's in w_2 so the length of w can be reduced using the quadratic relations (25) for g_l^2 and we use the induction hypothesis:

$$\begin{aligned} w &= t^a w_1 g_l w_2 g_l w_3 \\ &= t^a w_1 w_2 g_l^2 w_3 \\ &= t^a w_1 w_2 (1 + (u-1)e_l + (u-1)e_l g_l)) w_3 \\ &= t^a w_1 w_2 w_3 + (u-1) t^a w_1 w_2 e_l w_3 + (u-1) t^a w_1 w_2 e_l g_l w_3. \end{aligned}$$

If w_2 does contain g_{l+1}, then, by the induction hypothesis w_2 has the form $w_2 = v_1 g_{l+1} v_2$, where in v_1, v_2 the lowest index generator is at least g_{l+2}, hence:

$$\begin{aligned} w &= t^a w_1 g_l v_1 g_{l+1} v_2 g_l w_3 \\ &= t^a w_1 v_1 g_l g_{l+1} g_l v_2 w_3 \\ &= t^a w_1 v_1 g_{l+1} g_l g_{l+1} v_2 w_3, \end{aligned}$$

and there is one less occurrence of g_l in w. In the case where $l+1 = m$, where $m = max\{i_1, \dots i_k\}$, we apply instead the Steinberg relation (31), so no contradiction is caused with respect to the highest index generator. Continuing in the same manner for all possible pairs of g_l in the word we reduce to having g_l at most once. □

The following proposition gives us a precise spanning set for $\mathrm{YTL}_{d,n}(u)$.

Proposition 4. *The following set of reduced words*

$$\Sigma_{d,n} \quad = \quad \left\{ t^a (g_{i_1} g_{i_1 - 1} \dots g_{i_1 - k_1})(g_{i_2} g_{i_2 - 1} \dots g_{i_2 - k_2}) \dots (g_{i_p} g_{i_p - 1} \dots g_{i_p - k_p}) \right\}, \qquad (43)$$

where

$$t^a = t_1^{a_1} \dots t_n^{a_n} \in C_d^n, \quad 1 \leq i_1 < i_2 < \dots < i_p \leq n-1,$$

and

$$1 \leq i_1 - k_1 < i_2 - k_2 < \dots < i_p - k_p,$$

spans the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$. *The highest index generator is* g_{i_p} *of the rightmost cycle and the lowest index generator is* $g_{i_1 - k_1}$ *of the leftmost cycle of a word in* $\Sigma_{d,n}$.

Proof. We will prove the statement by induction on the length of a word starting from the linear basis of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ [8, Proposition 8]. Namely,

$$\mathcal{B}_{\mathrm{Y}_{d,n}} \quad = \quad \left\{ t^a (g_{i_1} g_{i_1 - 1} \dots g_{i_1 - k_1})(g_{i_2} g_{i_2 - 1} \dots g_{i_2 - k_2}) \dots (g_{i_p} g_{i_p - 1} \dots g_{i_p - k_p}) \right\}, \qquad (44)$$

where:

$$a \in (\mathbb{Z}/d\mathbb{Z})^n, \quad 1 \leq i_1 < i_2 < \dots < i_p \leq n-1,$$

and $\mathcal{B}_{\mathrm{Y}_{d,n}}$ spans linearly the quotient $\mathrm{YTL}_{d,n}(u)$ since it is a quotient of $\mathrm{Y}_{d,n}(u)$. Note that in $\mathcal{B}_{\mathrm{Y}_{d,n}}$ there is no restriction on the indices $i_1 - k_1, \dots, i_p - k_p$. Starting now with a word in the set $\mathcal{B}_{\mathrm{Y}_{d,n}}$, we will show that it is a linear combination of words in the subset $\Sigma_{d,n}$. The statement holds trivially for words of length 0,1 and 2, since such words are in $\Sigma_{d,n}$. For length 3 consider the representative case of the word $t^a g_1 g_2 g_1$ which is not in $\Sigma_{d,n}$. Applying the Steinberg relation (31) a linear combination of words in $\Sigma_{d,n}$ is obtained (see Eq. 43). Suppose now that the statement holds for all words of length $\leq q$, namely, that any word in $\mathcal{B}_{\mathrm{Y}_{d,n}}$ of length q can be written as a linear combination of words in $\Sigma_{d,n}$. Let w be a word in $\mathcal{B}_{\mathrm{Y}_{d,n}}$ of length $q+1$ which is not contained in $\Sigma_{d,n}$. Then w must contain a pair of consecutive cycles:

$$(g_{i_1} g_{i_1-1} \dots g_k)(g_{i_2} g_{i_2-1} \dots g_l),$$

where $k \geq l$. It suffices to consider the situation where $i_2 = i_1 + 1$, otherwise the generators of higher index may pass temporarily to the left of the word. Next, we move the term g_k as far to the right as possible obtaining:

$$(g_{i_1} \dots g_{k+1})(g_{i_2} \dots g_{k+2} \underline{g_k g_{k+1} g_k} g_{k-1} \dots g_l).$$

We now apply the Steinberg relation (31) and we obtain five terms, all of length $< q+1$, and we apply the induction hypothesis. More precisely, we have the following five terms:

$(g_{i_1} \dots g_{k+1})(g_{i_2} \dots g_{k+2} \underline{g_{k+1} g_k} g_{k-1} \dots g_l)$,
$(g_{i_1} \dots g_{k+1})(g_{i_2} \dots g_{k+2} \underline{g_{k+1}} g_{k-1} \dots g_l)$,
$(g_{i_1} \dots g_{k+1})(g_{i_2} \dots g_{k+2} \underline{g_k} g_{k-1} \dots g_l)$,
$(g_{i_1} \dots g_{k+1})(g_{i_2} \dots g_{k+2} \underline{g_k g_{k+1}} g_{k-1} \dots g_l)$,
$(g_{i_1} \dots g_{k+1})(g_{i_2} \dots g_{k+2} g_{k-1} \dots g_l)$.

To see the exact position of the highest and lowest index generators in the words of $\Sigma_{d,n}$ one can observe that the position of the highest index generator g_i is already clear in the set $\mathcal{B}_{\mathrm{Y}_{d,n}}$ (cf. [8] [14]). To establish the position of the lowest index generator in the words of $\Sigma_{d,n}$ we shall analyze each of the five terms above. In the first term a cycle of smaller length is created and the difference between the lowest indices of the two cycles, $k+1$ and l, increases by one, so we need to apply the Steinberg relation once more and then use the induction hypothesis. In the second term the subword $(g_{k-1} \dots g_l)$ may pass to the left (since the generator g_k has disappeared), so we obtain the following word:

$$(g_{k-1} \dots g_l)(g_{i_1} \dots g_{k+1})(g_{i_2} \dots g_{k+1}). \tag{45}$$

This word contains two cycles with the same lowest index generators, hence we need to apply the Steinberg relation (31) and use the induction hypothesis as above. In the third term, g_k returns to its original position and the subword $(g_{k-1} \dots g_l)$ may pass to the left, obtaining a word in the set $\Sigma_{d,n}$, namely:

$$(g_{i_1} \dots g_{k+1} g_k g_{k-1} \dots g_l)(g_{i_2} \dots g_{k+2}). \tag{46}$$

The same holds for the forth term, which can be rewritten as:

$$(g_{i_1} \dots g_{k+1} g_k g_{k-1} \dots g_l)(g_{i_2} \dots g_{k+1}). \tag{47}$$

Finally, in the fifth term, the subword $(g_{k-1} \dots g_l)$ may pass to the far left, namely:

$$(g_{k-1} \dots g_l)(g_{i_1} \dots g_{k+1})(g_{i_2} \dots g_{k+2}), \tag{48}$$

which is a word in the set $\Sigma_{d,n}$. The fact that the lowest index generator g_i appears in the leftmost cycle of the monomial in $\Sigma_{d,n}$ is now clear from (45), (46), (47) and (48). Concluding, in each application of the Steinberg relation (31) the length of w is reduced by at least one, so, from the above and by the induction hypothesis the proof that $\Sigma_{d,n}$ is a spanning set is concluded. $\square$

M. Chlouveraki and G. Pouchin in [2] have computed the dimension for $\mathrm{YTL}_{d,n}(u)$ by using the representation theory of the Yokonuma–Hecke algebra [3]. More precisely, they proved the following result.

Proposition 5 (cf. Proposition 4 [2]). *The dimension of the Yokonuma–Temperley–Lieb algebra is:*

$$\dim(\mathrm{YTL}_{d,n}(u)) = dc_n + \frac{d(d-1)}{2}\sum_{k=1}^{n-1}\binom{n}{k}^2,$$

where c_n is the n^{th} Catalan number.

3.2. To find an explicit basis for $\mathrm{YTL}_{d,n}(u)$ Chlouveraki and Pouchin in [2] worked as follows: As mentioned in Remark 2 each word in $\mathrm{YTL}_{d,n}(u)$ inherits the splitting property. For each fixed element in the braiding part, they described a set of linear dependence relations among the framing parts (see [2, Proposition 5]). Using these relations they extracted from $\Sigma_{d,n}$ (recall Eq. 43) a smaller spanning set for $\mathrm{YTL}_{d,n}(u)$ and showed that the cardinality of this smaller spanning set is equal to the dimension of the algebra. Thus, it is a basis for $\mathrm{YTL}_{d,n}(u)$. Before describing this basis, we will need the following notations:

Let $\underline{i}$ and $\underline{k}$ be the following p–tuples:

$$\underline{i} = (i_1, \ldots, i_p) \quad \text{and} \quad \underline{k} = (k_1, \ldots, k_p)$$

and let $\mathcal{I}$ be the set of pairs $(\underline{i}, \underline{k})$ such that:

$$1 \leq i_1 < \ldots < i_p \leq n-1 \quad \text{and} \quad 1 \leq i_1 - k_1 < \ldots < i_p - k_p \leq n-1$$

We also denote by $g_{\underline{i},\underline{k}}$ the element:

$$g_{\underline{i},\underline{k}} := (g_{i_1}g_{i_1-1}\ldots g_{i_1-k_1})(g_{i_2}g_{i_2-1}\ldots g_{i_2-k_2})\ldots(g_{i_p}g_{i_p-1}\ldots g_{i_p-k_p})$$

Under these notations the set $\Sigma_{d,n}$ can be written as:

$$\Sigma_{d,n} = \{t_1^{r_1}\ldots t_n^{r_n}\, g_{\underline{i},\underline{k}} \,|\, r_1, \ldots, r_n \in \mathbb{Z}/d\mathbb{Z}, (\underline{i}, \underline{k}) \in \mathcal{I}\}.$$

The *degree of a word* $w = t_1^{r_1}\ldots t_n^{r_n} g_{i_1}\ldots g_{i_m}$ in $\mathrm{Y}_{d,n}(u)$, denoted $deg(w)$, is defined to be the integer m. Set:

$$\Sigma_{d,n}^{<w} := \{s \in \Sigma_{d,n} \,|\, deg(s) < deg(w)\}.$$

The group algebra $K(\mathbb{Z}/d\mathbb{Z})^n$ is isomorphic to the subalgebra of $\mathrm{Y}_{d,n}(u)$ that is generated by the t_i's but not to the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's. Further, the group algebra $K(\mathbb{Z}/d\mathbb{Z})^n$ has a natural basis, $B_{d,n}$, given by monomials in $t_1, \ldots, t_n$, the following:

$$B_{d,n} = \{t_1^{r_1}\ldots t_n^{r_n} \,|\, r_1, \ldots, r_n \in \mathbb{Z}/d\mathbb{Z}\}.$$

Thus, any element of $K(\mathbb{Z}/d\mathbb{Z})^n$ can be written as a linear combination of words in $B_{d,n}$. There is a surjective algebra morphism from $K(\mathbb{Z}/d\mathbb{Z})^n$ to the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's. We will denote the image of an element $b \in B_{d,n}$ into the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's with $\bar{b}$. We then have the following theorem:

Theorem 3 (Chlouveraki and Pouchin, cf. [2], Theorem 2). *The following set is a linear basis for $\mathrm{YTL}_{d,n}(u)$:*

$$S_{d,n} = \{\bar{b}_{\underline{i},\underline{k}}\, g_{\underline{i},\underline{k}} \,|\, (\underline{i}, \underline{k}) \in \mathcal{I},\, b_{\underline{i},\underline{k}} \in \mathcal{B}_{d,n}(g_{\underline{i},\underline{k}})\},$$

where $\mathcal{B}_{d,n}(g_{\underline{i},\underline{k}})$ is a proper subset of $B_{d,n}$ such that:

$$\{b_{\underline{i},\underline{k}} + R(g_{\underline{i},\underline{k}}) \,|\, b_{\underline{i},\underline{k}} \in \mathcal{B}(g_{\underline{i},\underline{k}})\}$$

is a basis of the quotient space $K(\mathbb{Z}/d\mathbb{Z})^n/R(g_{\underline{i},\underline{k}})$*, and where* $R(w)$ *is the following ideal of* $K(\mathbb{Z}/d\mathbb{Z})^n$*:*

$$R(w) = \{\mathfrak{m} \in K(\mathbb{Z}/d\mathbb{Z})^n \,|\, \bar{\mathfrak{m}}\, w \in Span_{\mathbb{C}(u)}(\Sigma_{d,n}^{<w})\}.$$

3.3. For $d = 2$, $n = 3$ it is relatively easy to find a basis for $\mathrm{YTL}_{2,3}(u)$. We will give here a basis different than the one in Theorem 3. Before continuing, we need the following technical lemma that will be also used in the proof of Theorem 4.

Lemma 5 (cf. Lemma 7.5 [9])**.** *For the element* $g_{1,2}$ *we have in* $\mathrm{Y}_{d,n}(u)$ *(recall (10) for* $e_{1,3}$*):*

$$\begin{array}{lrcl}
(1) & g_1 g_{1,2} & = & [1+(u-1)e_1]g_{1,2} \\
(2) & g_2 g_{1,2} & = & [1+(u-1)e_2]g_{1,2} \\
(3) & g_1 g_2 g_{1,2} & = & [1+(u-1)e_1+(u-1)e_{1,3}+(u-1)^2 e_1 e_2]g_{1,2} \\
(4) & g_2 g_1 g_{1,2} & = & [1+(u-1)\,e_2+(u-1)e_{1,3}+(u-1)^2 e_1 e_2]g_{1,2} \\
(5) & g_1 g_2 g_1 g_{1,2} & = & [1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1 e_2]g_{1,2}
\end{array}$$

Analogous relations hold for multiplications with $g_{1,2}$ *from the right.*

Proof. The idea is to expand the left–hand side of each equation and then use Eq. 25 and Lemma 1. We will demonstrate the proof for the indicative cases (1) and (4). The other cases are proved similarly.

For case (1) we have:

$$\begin{aligned}
g_1 g_{1,2} &= g_1 + g_1^2 + g_1 g_2 + g_1^2 g_2 + g_1 g_2 g_1 + g_1^2 g_2 g_1 \\
&= g_1 + [1+(u-1)e_1+(u-1)e_1 g_1] \\
&\quad + g_1 g_2 + [g_2 + (u-1)e_1 g_2 + (u-1)e_1 g_1 g_2] \\
&\quad + g_1 g_2 g_1 + [g_2 g_1 + (u-1)e_1 g_2 g_1 + (u-1)e_1 g_1 g_2 g_1] \\
&= g_{1,2} + (u-1)e_1 g_{1,2}.
\end{aligned}$$

Case (2) is completely analogous. In order to prove case (4) we will use cases (1) and (2):

$$\begin{aligned}
g_2 g_1 g_{1,2} &= g_2\,(g_{1,2} + (u-1)e_1 g_{1,2}) \\
&= g_2 g_{1,2} + (u-1)e_{1,3} g_2 g_{1,2} \quad \text{(Lemma 1)} \\
&= [1+(u-1)e_2]\,g_{1,2} + (u-1)e_{1,3}(1+(u-1)e_2)g_{1,2} \\
&= [1+(u-1)e_2]\,g_{1,2} + (u-1)e_{1,3}g_{1,2} + (u-1)^2 e_{1,3}e_2 g_{1,2} \quad \text{(Lemma 1)} \\
&= \left[1+(u-1)\,e_2+(u-1)e_{1,3}+(u-1)^2 e_1 e_2\right] g_{1,2}.
\end{aligned}$$

□

To find a basis for $\mathrm{YTL}_{2,3}(u)$: From Proposition 5 we have that $\dim(\mathrm{YTL}_{2,3}(u)) = 28$. On the other hand the spanning set $\Sigma_{2,3}$ of $\mathrm{YTL}_{2,3}(u)$ of Proposition 4, contains 40 elements. Thus, any relation $w_1 g_{1,2} w_2 = 0$ with w_1, $w_2 \in \mathrm{Y}_{2,3}(u)$ reduces to having w_1, $w_2 \in \Sigma_{2,3}$. Further, if any of w_1, w_2 contain braiding generators, then by Lemma 5 (after pushing framing generators in w_2 to the right) these get absorbed by $g_{1,2}$. Thus, and since $e_{i,j} = \frac{1}{2}(1 + t_i t_j)$ for $d = 2$, it suffices to consider the following system of equations:

$$w_1\, g_{1,2} w_2 = 0 \qquad w_1, w_2 \in \mathcal{T}, \tag{49}$$

where $\mathcal{T} := \{1, t_1, t_2, t_3, t_1t_2, t_1t_3,\ t_2t_3, t_1t_2t_3\}$. For finding all possible linear dependencies in $\Sigma_{2,3}$, after substituting $g_1g_2g_1$ with $-1-g_1-g_2-g_1g_2-g_2g_1$ in Eq. 49, note that some of these 64 equations reduce trivially to $g_{1,2} = 0$; for example if $w_2 = 1$ or $w_2 = t_1t_2t_3$ (since it commutes with $g_{1,2}$). From the rest one can extract 12 linearly independent equations which, applied on the spanning set $\Sigma_{2,3}$ lead to the following basis for $\mathrm{YTL}_{2,3}(u)$:

$$\begin{aligned}\mathcal{S}_{2,3} = \{&1, t_1, t_2, t_1t_2, g_1, t_2g_1, t_3g_1, t_2t_3g_1, g_2, t_1g_2, t_3g_2, t_1t_3g_2,\\ &g_1g_2, t_1g_1g_2, t_2g_1g_2, t_3g_1g_2, t_1t_2g_1g_2, t_1t_3g_1g_2, t_2t_3g_1g_2, t_1t_2t_3g_1g_2,\\ &g_2g_1, t_1g_2g_1, t_2g_2g_1, t_3g_2g_1, t_1t_2g_2g_1, t_1t_3g_2g_1, t_2t_3g_2g_1, t_1t_2t_3g_2g_1\}.\end{aligned}$$

4. A Markov trace on $\mathrm{YTL}_{d,n}(u)$

The following section is dedicated to finding the necessary and sufficient conditions for the trace tr on $\mathrm{Y}_{d,n}(u)$ to pass to the quotient algebra $\mathrm{YTL}_{d,n}(u)$, in analogy to the classical case, where the Ocneanu trace on $\mathrm{H}_n(u)$ passes to the quotient algebra $\mathrm{TL}_n(u)$ under the condition for certain values of the trace parameter ζ.

4.1. It is clear by now that tr will pass to $\mathrm{YTL}_{d,n}(u)$ if it kills the generator of the principal ideal through which the quotient is defined, that is, if $\mathrm{tr}(g_{1,2}) = 0$. We have the following lemma:

Lemma 6. *For the element $g_{1,2}$ we have:*

$$\mathrm{tr}(g_{1,2}) = (u+1)z^2 + ((u-1)E + 3)\, z + 1. \tag{50}$$

Proof. The proof is a straightforward computation:

$$\begin{aligned}\mathrm{tr}(g_{1,2}) &= \mathrm{tr}(1) + \mathrm{tr}(g_1) + \mathrm{tr}(g_2) + \mathrm{tr}(g_1g_2) + \mathrm{tr}(g_2g_1) + \mathrm{tr}(g_1g_2g_1)\\ &= 1 + 2z + 2z^2 + z + (u-1)Ez + (u-1)z^2\\ &= (u+1)z^2 + ((u-1)E + 3)\, z + 1.\end{aligned}$$

□

Lemma 6, together with the equation:

$$\mathrm{tr}(g_{1,2}) = (u+1)z^2 + ((u-1)E + 3)\, z + 1 = 0 \tag{51}$$

gives us the following values for z:

$$z_{\pm} = \frac{-\left((u-1)E+3\right) \pm \sqrt{\left((u-1)E+3\right)^2 - 4(u+1)}}{2(u+1)}. \tag{52}$$

We shall do now the analysis for all conditions that must be imposed on the trace parameters in order that tr passes to $\mathrm{YTL}_{d,n}(u)$. Having in mind Corollary 1 and the linearity of tr, it follows that tr passes to $\mathrm{YTL}_{d,n}(u)$ if and only if the following equations are satisfied for all monomials $\mathfrak{m}$ in the inductive basis of $\mathrm{Y}_{d,n}(u)$. Namely:

$$\mathrm{tr}(\mathfrak{m}\, g_{1,2}) = 0. \tag{53}$$

Let us first consider the case $n = 3$. By Proposition 1 the elements in the inductive basis of $\mathrm{Y}_{d,3}(u)$ are of the following forms:

$$t_1^a t_2^b t_3^c, \quad t_1^a g_1 t_1^b t_3^c, \quad t_1^a t_2^b g_2 g_1 t_1^c, \quad t_1^a t_2^b g_2 t_2^c, \quad t_1^a g_1 t_1^b g_2 t_2^c, \quad t_1^a g_1 t_1^b g_2 g_1 t_1^c \tag{54}$$

Using Lemma 5 and the following notations:

$$
\begin{aligned}
Z_{a,b,c} &:= (u+1)z^2 x_{a+b+c} + \left((u-1)E^{(a+b+c)} + x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b}\right) z + x_a x_b x_c \\
V_{a,b+c} &:= (u+1)z^2 x_{a+b+c} + (u+1)zE^{(a+b+c)} + z\, x_a x_{b+c} + x_a E^{(b+c)} \\
V_{b,a+c} &:= (u+1)z^2 x_{a+b+c} + (u+1)zE^{(a+b+c)} + z\, x_b x_{a+c} + x_b E^{(a+c)} \\
V_{c,a+b} &:= (u+1)z^2 x_{a+b+c} + (u+1)zE^{(a+b+c)} + z\, x_c x_{a+b} + x_c E^{(a+b)} \\
W_{a,b,c} &:= (u+1)z^2 x_{a+b+c} + (u+2)zE^{(a+b+c)} + \operatorname{tr}\left(e_1^{(a+b+c)} e_2\right)
\end{aligned}
$$

we obtain by (53) and (54) the following equations, for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$:

$$Z_{a,b,c} = 0 \tag{55}$$

$$Z_{a,b,c} + (u-1)V_{c,a+b} = 0 \tag{56}$$

$$Z_{a,b,c} + (u-1)V_{a,b+c} = 0 \tag{57}$$

$$Z_{a,b,c} + (u-1)\left[V_{c,a+b} + V_{b,a+c} + W_{a,b,c}\right] = 0 \tag{58}$$

$$Z_{a,b,c} + (u-1)\left[V_{a,b+c} + V_{b,a+c} + W_{a,b,c}\right] = 0 \tag{59}$$

$$Z_{a,b,c} + (u-1)\left[V_{a,b+c} + V_{b,a+c} + V_{c,a+b} + W_{a,b,c}\right] = 0 \tag{60}$$

Equations 55–60 reduce to the following system of equations of $z, x_1, \ldots, x_{d-1}$ for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$:

$$
(\Sigma)\begin{cases}
Z_{a,b,c} = 0 & \text{(61a)}\\
V_{c,a+b} = 0 & \text{(61b)}\\
V_{a,b+c} = 0 & \text{(61c)}\\
V_{b,a+c} + W_{a,b,c} = 0 & \text{(61d)}
\end{cases}
$$

Notice that for $a = b = c = 0$ Eq. 55 becomes Eq. 51. If, now, we require both solutions in (52) to participate in the solutions of (Σ), then we are led to necessary conditions for tr to pass to $\mathrm{YTL}_{2,3}(u)$ (Section 4.2). If not then we are led to necessary and sufficient conditions for tr to pass to $\mathrm{YTL}_{2,3}(u)$ (Section 4.3).

4.2. Suppose that both solutions for z from Eq. 52 participate in the solution set of (Σ). We have the following proposition:

Proposition 6. *The trace* tr *defined on* $\mathrm{Y}_{d,3}(u)$ *passes to the quotient* $\mathrm{YTL}_{d,3}(u)$ *if the trace parameters* x_i *are* d^{th} *roots of unity* ($x_i = x_1^i$, $1 \leq i \leq d-1$) *and* $z = -\frac{1}{u+1}$ *or* $z = -1$.

Proof. Suppose that tr passes to $\mathrm{YTL}_{d,3}(u)$ and that (Σ) has both solutions for z from Eq. 52. This implies that there exist λ in $K(x_1 \ldots, x_{d-1})$ such that:

$$Z_{a,b,c} = \lambda Z_{0,0,0}$$

From this we deduce that:

$$
\begin{aligned}
\lambda &= x_{a+b+c} \\
x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b} &= 3x_{a+b+c} \\
E^{(a+b+c)} &= x_{a+b+c} E \qquad (62)\\
x_{a+b+c} &= x_a x_b x_c. \qquad (63)
\end{aligned}
$$

Since this holds for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$, by taking $b = c = 0$ in Eq. 62 we have that:

$$E^{(a)} = x_a E \tag{64}$$

which is exactly the E–system. Moreover, by taking $c = 0$ in Eq. 63 we obtain:

$$x_a x_b = x_{a+b} \tag{65}$$

This implies that the x_i's are d^{th} roots of unity which is equivalent to $E = 1$ [11, Appendix]. In order to conclude the proof it is enough to verify that these conditions for the x_i's satisfy also (61b)–(61d) of (Σ). Since the x_i's are solutions of the E–system, Eqs. 61b and 61c are immediately satisfied. We will finally check Eq. 61d. Using Eqs. 65 and 64 we have that:

$$u\left((u+1)z^2 + (u+2)z + 1\right) x_a x_b x_c = 0,$$

from which we deduce that $z = -\frac{1}{u+1}$ or $z = -1$, which are precisely the solutions (52) for $E = 1$. □

Using induction on n one can prove the general case of the necessary conditions for tr to pass to $\mathrm{YTL}_{d,n}(u)$. Indeed we have:

Theorem 4. *For $n \geq 3$, the trace* tr *defined on $\mathrm{Y}_{d,n}(u)$ passes to the quotient $\mathrm{YTL}_{d,n}(u)$ if the trace parameters x_i are d^{th} roots of unity ($x_i = x_1^i$, $1 \leq i \leq d-1$) and $z = -\frac{1}{u+1}$ or $z = -1$.*

Proof. By induction on n. In Proposition 6 we proved the case where $n = 3$. Assume that the statement holds for all $\mathrm{YTL}_{d,k}(u)$, where $k \leq n$, that is:

$$\mathrm{tr}(a_k\, g_{1,2}) = 0$$

for all $a_k \in \mathrm{Y}_{d,k}(u)$, $k \leq n$. We will show the statement for $k = n+1$. It suffices to prove that the trace vanishes on any element in the form $a_{n+1} g_{1,2}$, where a_{n+1} belongs to the inductive basis of $\mathrm{Y}_{d,n+1}(u)$ (recall Proposition 1), given the conditions of the Theorem. Namely:

$$\mathrm{tr}(a_{n+1}\, g_{1,2}) = 0.$$

Since a_{n+1} is in the inductive basis of $\mathrm{Y}_{d,n+1}(u)$, it is of one of the following forms:

$$a_{n+1} = a_n g_n \ldots g_i t_i^k \quad \text{or} \quad a_{n+1} = a_n t_{n+1}^k,$$

where a_n is in the inductive basis of $\mathrm{Y}_{d,n}(u)$. For the first case we have:

$$\mathrm{tr}(a_{n+1}\, g_{1,2}) = \mathrm{tr}(a_n g_n \ldots g_i t_i^k\, g_{1,2}) = z\, \mathrm{tr}(a_n g_{n-1} \ldots g_i t_i^k\, r_{1,2}) = z\, \mathrm{tr}(\tilde{a} g_{1,2}),$$

where $\tilde{a} := a_n g_{n-1} \ldots g_i t_i^k$. Notice now that $\widetilde{a}$ is a word in $\mathrm{Y}_{d,n}(u)$ and so, by the linearity of the trace, we have that $\mathrm{tr}(\tilde{a}\, g_{1,2})$ is a linear combination of traces of the form $\mathrm{tr}(a_n\, g_{1,2})$, where a_n is in the inductive basis of $\mathrm{Y}_{d,n}(u)$. Therefore, by the induction hypothesis, we deduce that:

$$\mathrm{tr}(\widetilde{a}\, g_{1,2}) = 0,$$

if the conditions of the Theorem are satisfied. Therefore the statement is proved. The second case is proved similarly. Hence, the proof is concluded. □

4.3. In the proofs of Proposition 6 and Theorem 4 it became apparent that the x_i's are d^{th} roots of unity if and only if the values of z_+ and z_- satisfy all equations of (Σ). Clearly, if we loosen this last condition, then other solutions for the x_i's may appear such that the trace tr passes to the quotient $\mathrm{YTL}_{d,n}(u)$. Indeed, we have the following:

Theorem 5. *The trace* tr *passes to the quotient $\mathrm{YTL}_{d,n}(u)$ if and only if the x_i's are solutions of the* E*–system and one of the two cases holds:*

(i) *For some $0 \leq m_1 \leq d-1$ the x_ℓ's are expressed as:*

$$x_\ell = \exp(\ell m_1) \quad (0 \leq \ell \leq d-1).$$

In this case the x_ℓ's are d^{th} roots of unity and $z = -\frac{1}{u+1}$ or $z = -1$.

(ii) *For some* $0 \leq m_1, m_2 \leq d-1$ *the* x_ℓ*'s are expressed as:*

$$x_\ell = \frac{1}{2}\left(\exp(\ell m_1) + \exp(\ell m_2)\right) \quad (0 \leq \ell \leq d-1).$$

In this case we have $z = -\frac{1}{2}$*.*

Note that case (i) captures Theorem 4.

Proof. Observe that the x_ℓ's expressed by (i) are indeed solutions of the system (Σ). We will now assume that our solutions are not of this form. This implies that $x_\ell \neq E^{(\ell)}$ for some $0 \leq \ell \leq d-1$, and this will allow us to have this quantity in denominators later.

We will use induction on n. We will first prove the case $n = 3$. Suppose that trace tr passes to the quotient algebra $\mathrm{YTL}_{d,3}(u)$. This means that (Σ) has solutions for z any one of those in Eq. 52, for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$. Subtracting Eq. 61a from Eq. 61b we obtain:

$$z = -\frac{x_a x_b x_c - x_c E^{(a+b)}}{x_a x_{b+c} + x_b x_{a+c} - 2E^{(a+b+c)}}. \tag{66}$$

For $b = c = 0$ in Eq. 66 we obtain: $z = -\frac{1}{2}$. On the other hand, subtracting Eqs. 61a and 61b from Eq. 61d we have:

$$z = \frac{x_a x_b x_c + x_c E^{(a+b)} - x_b E^{(a+c)} - \mathrm{tr}(e_1^{(a+b+c)} e_2)}{3E^{(a+b+c)} - x_a x_{b+c} - 2x_c x_{a+b}}. \tag{67}$$

We will now assume that that x_ℓ's are not roots of unity. This implies that, for all $0 \leq a \leq d-1$, $x_a - E^{(a)} \neq 0$. For $b = c = 0$ in Eq. 67 we obtain:

$$z = -\frac{x_a - \mathrm{tr}(e_1^{(a)} e_2)}{3(x_a - E^{(a)})}. \tag{68}$$

From Eqs. 66 and 68 we have that:

$$\frac{1}{2} = \frac{x_a - \mathrm{tr}(e_1^{(a)} e_2)}{3(x_a - E^{(a)})}$$

or equivalently:

$$3(x_a - E^{(a)}) = 2(x_a - \mathrm{tr}(e_1^{(a)} e_2)).$$

Using Lemma 2, this is equivalent to:

$$3x - \frac{3}{d} x * x = 2x - \frac{2}{d^2} x * x * x.$$

By taking the Fourier transform (see Lemma 3) we arrive at:

$$\frac{2}{d^2}\widehat{x}^3 - \frac{3}{d}\widehat{x}^2 + \widehat{x} = 0.$$

Assuming that $\widehat{x} = \sum_{0 \leq \ell \leq d-1} y_\ell t^\ell$ we have the following expression for the cofficients y_ℓ in the expansion of $\widehat{x}$:

$$y_\ell \left(\frac{2}{d^2} y_\ell^2 - \frac{3}{d} y_\ell + 1 \right) = 0.$$

So either $y_\ell = 0$ or $y_\ell = d$ or $y_\ell = \frac{1}{2}d$. So if we take a partition of the set $\{\ell : 0 \leq \ell \leq d-1\}$ into sets S_0, S_1, $S_{\frac{1}{2}}$ such that y_ℓ takes the value $i \cdot d$ on S_i $(i = 0, 1, \frac{1}{2})$. We have from Lemma 3 that:

$$x = \sum_{m \in S_1} \mathbf{i}_{-m} + \frac{1}{2} \sum_{m \in S_{\frac{1}{2}}} \mathbf{i}_{-m}.$$

From $x_0 = 1$ we obtain the conditions:

$$1 = x(0) = |S_1| + \frac{1}{2}|S_{\frac{1}{2}}|.$$

This means that either S_1 has only one element and $S_{\frac{1}{2}} = \emptyset$ or $S_1 = \emptyset$ and $S_{\frac{1}{2}}$ has two elements. The first case corresponds to the case (i) where the x_ℓ's are d^{th} roots of unity. In the second case, if $S_{\frac{1}{2}} = \{m_1, m_2\}$ we obtain the following solution of the E–system:

$$x_\ell = \frac{1}{2}\left(\exp(\ell m_1) + \exp(\ell m_2)\right), \quad (0 \leq \ell \leq d-1)$$

which corresponds to $z = -\frac{1}{2}$.
The rest of proof (the induction on n) is analogous to the one of Theorem 4. ☐

Remark 4. The values for the trace parameter z in Theorems 4 and 5, $z = -\frac{1}{u+1}$ and $z = -1$, in order that tr on $\mathrm{Y}_{d,n}(u)$ passes to the quotient $\mathrm{YTL}_{d,n}(u)$ are the same as the values in Eq. 6 for ζ of the Ocneanu trace τ on $\mathrm{H}_n(u)$, so that τ passes to the quotient $\mathrm{TL}_n(u)$ (recall Section 1.2).

5. Knot Invariants from $\mathrm{YTL}_{d,n}(u)$

The 2–variable Jones or HOMFLYPT polynomial, $P(\lambda, u)$, can be defined through the Ocneanu trace on $\mathrm{H}_n(u)$ [14]. Indeed, for any braid $\alpha \in \cup_\infty B_n$ we have:

$$P(\lambda, u)(\hat{\alpha}) = \left(-\frac{1-\lambda u}{\sqrt{\lambda}(1-u)}\right)^{n-1} \left(\sqrt{\lambda}\right)^{\varepsilon(\alpha)} \tau(\pi(\alpha)),$$

where: $\lambda = \frac{1-u+\zeta}{u\zeta}$, π is the natural epimorphism of $\mathbb{C}B_n$ onto $\mathrm{H}_n(u)$ that sends the braid generator σ_i to h_i and $\varepsilon(\alpha)$ is the algebraic sum of the exponents of the σ_i's in α. Further, the Jones polynomial, $V(u)$, related to the algebras $\mathrm{TL}_n(u)$, can be redefined through the HOMFLYPT polynomial, related the algebras $\mathrm{H}_n(u)$, by specializing ζ to $-\frac{1}{u+1}$ [14]. This is the non–trivial value for which the Ocneanu trace τ passes to the quotient $\mathrm{TL}_n(u)$. Namely:

$$V(u)(\hat{\alpha}) = \left(-\frac{1+u}{\sqrt{u}}\right)^{n-1} \left(\sqrt{u}\right)^{\varepsilon(\alpha)} \tau(\pi(\alpha)) = P(u,u)(\hat{\alpha}).$$

5.1. In [11] it is proved that the trace tr can be re-scaled according to the braid equivalence corresponding to isotopic framed links if and only if the x_i's furnish a solution of the E–system. Then, by further normalizing an invariant for framed knots and links can be obtained [11]:

$$\Gamma_{d,S}(w,u)(\hat{\alpha}) = \left(-\frac{1-wu}{\sqrt{w}(1-u)E}\right)^{n-1} \left(\sqrt{w}\right)^{\varepsilon(\alpha)} \mathrm{tr}(\gamma(\alpha)), \tag{69}$$

where: S is a subset of $\mathbb{Z}/d\mathbb{Z}$ which parametrizes a solution of the E–system, $w = \frac{z+(1-u)E}{uz}$, γ the natural epimorphism of the framed braid group algebra $\mathbb{C}\mathcal{F}_n$ onto the algebra $\mathrm{Y}_{d,n}(u)$, and $\alpha \in \cup_\infty \mathcal{F}_n$. Note that for every $d \in \mathbb{N}$ we obtain $2^d - 1$ invariants for framed links.

Further, in [12] the second and the fourth authors represented the classical braid group B_n in the algebra $\mathrm{Y}_{d,n}(u)$ by regarding the framing generators t_i as formal elements. So, $\Gamma_{d,S}$ can be seen as an invariant of classical links. Namely:

$$\Delta_{d,S}(w,u)(\hat{\alpha}) = \left(-\frac{1-wu}{\sqrt{w}(1-u)E}\right)^{n-1} \left(\sqrt{w}\right)^{\varepsilon(\alpha)} \mathrm{tr}(\delta(\alpha)), \tag{70}$$

where: S, w as above, δ the natural homomorphism of the classical braid group algebra $\mathbb{C}B_n$ to the algebra $\mathrm{Y}_{d,n}(u)$ and $\alpha \in \cup_\infty B_n$. Further, in [13] the invariant $\Delta_{d,S}(w,u)$ was extended to an invariant for singular links.

In [1] it is shown that for generic values of the parameters u, z the invariants $\Delta_{d,S}(w,u)$ do not coincide with the HOMFLYPT polynomial except in the trivial cases $u = 1$ or $E = 1$. Yet, computational data [4] indicate that these invariants do not distinguish more or less knot pairs than the HOMFLYPT polynomial, so they may still be topologically equivalent to the HOMFLYPT polynomial.

5.2. We shall now define framed and classical link invariants related to the algebra $\mathrm{YTL}_{d,n}(u)$. In Theorem 5 we showed that the trace tr passes to the quotient $\mathrm{YTL}_{d,n}(u)$ if and only if one of the following cases holds:

(i) For some $0 \le m_1 \le d-1$ we have $x_\ell = \exp(\ell m_1)$ $(0 \le \ell \le d-1)$. In this case the x_i's are d^{th} roots of unity and $z = -\frac{1}{u+1}$ or $z = -1$.

(ii) For some $0 \le m_1, m_2 \le d-1$ the x_ℓ are expressed as $x_\ell = \frac{1}{2}\left(\exp(\ell m_1) + \exp(\ell m_2)\right)$ $(0 \le \ell \le d-1)$. In this case we have that $z = -\frac{1}{2}$.

We note that in both cases the x_i's are solutions of the E–system, as required by [11], in order to proceed with defining link invariants. We do not take into consideration the case where: $z = -1$ (and the x_i's are d^{th} roots of unity) and the case where $z = -\frac{1}{2}$ (and $x_\ell = \frac{1}{2}\left(\exp(\ell m_1) + \exp(\ell m_2)\right)$) since crucial braiding information is lost and therefore they are of no topological interest. Indeed, the trace tr, for these two values of z gives the same result for all even (resp. odd) powers of the g_i's, as it becomes clear from the following formulas from [11], for $m \in \mathbb{Z}^{>0}$:

$$\mathrm{tr}(g_i^m) = \left(\frac{u^m - 1}{u+1}\right) z + \left(\frac{u^m - 1}{u+1}\right) E + 1 \qquad \text{if } m \text{ is even} \tag{71}$$

and

$$\mathrm{tr}(g_i^m) = \left(\frac{u^m - 1}{u+1}\right) z + \left(\frac{u^m - 1}{u+1}\right) E - E \qquad \text{if } m \text{ is odd.} \tag{72}$$

Notice that, substituting in Eq. 51 $z = -1$ implies $E = 1$, while substituting $z = -\frac{1}{2}$ implies $E = \frac{1}{2}$.

The only remaining case of interest is case (i) where the x_i's are roots of unity and $z = -\frac{1}{u+1}$. This implies that $E = 1$ and $w = u$ in both Eqs. 69 and 70. We give the following definition:

Definition 4. *For x_i's d^{th} roots of unity ($x_i = x_1^i$, $1 \le i \le d-1$) and $z = -\frac{1}{u+1}$, we obtain from $\Gamma_{d,S}$ the following polynomial for $\alpha \in \cup_\infty \mathcal{F}_n$:*

$$(i) \quad \mathcal{V}_{d,S}(u)(\hat{\alpha}) \quad = \quad \left(-\tfrac{1+u}{\sqrt{u}}\right)^{n-1} (\sqrt{u})^{\varepsilon(\alpha)}\, \mathrm{tr}(\gamma(\alpha)) = \Gamma_{d,S}(u,u).$$

Further, from $\Delta_{d,S}$, we obtain the following polynomial for $\alpha \in \cup_\infty B_n$:

$$(ii) \quad V_{d,S}(u)(\hat{\alpha}) \quad = \quad \left(-\tfrac{1+u}{\sqrt{u}}\right)^{n-1} (\sqrt{u})^{\varepsilon(\alpha)}\, \mathrm{tr}(\delta(\alpha)) = \Delta_{d,S}(u,u).$$

Both polynomilas lie in $K(z, x_1, \ldots, x_{d-1})$.

By Theorem 5 and the results of [11] and [12], the polynomials $\mathcal{V}_{d,S}(u)$ and $V(u)$ are invariants of framed links and classical links respectively.

We know from [11, Remark 5] that the invariant $\Gamma_{d,S}(w,u)$ is not very interesting for framed links when the x_i's are d^{th} roots of unity because basic pairs of framed links are not distinguished.

For classical links, as mentioned earlier, we know from [1, Corollary 1] that the invariants $\Delta_{d,S}(w,u)$ coincide with the HOMFLYPT polynomial (case $E=1$). More precisely, for $E=1$ an algebra homomorphism can be defined, $h : \mathrm{Y}_{d,n}(u) \longrightarrow \mathrm{H}_n(u)$, and the composition $\tau \circ h$ is a Markov trace on $\mathrm{Y}_{d,n}(u)$ which takes the same values as the specialized trace tr, whereby the x_i's are specialized to d^{th} roots of unity ($x_i = x_1^i$, $1 \leq m \leq d-1$). For details see [1, §3]. The above discussion leads to the following corollary:

Corollary 2. *The invariants $V_{d,S}(u)$ coincide with the Jones polynomial. The invariants $\mathcal{V}_{d,S}(u)$ are analogues of the Jones polynomial in the framed category.*

References

[1] M. Chlouveraki, S. Lambropoulou, The Yokonuma–Hecke algebras and the HOMLFYPT polynomial, to appear in *J. Knot Theory Ramfications.* See also arXiv:1204.1871.

[2] M. Chlouveraki, G. Pouchin, Determination of the representations and a basis for the Yokonuma-Temperley-Lieb algebra, arXiv:1311.5626 [math.RT].

[3] M. Chlouveraki, L. Poulan, Representation theory of the Yokonuma–Hecke algebra. See arXiv:1302.6225.

[4] S. Chmutov, S.Jablan, J. Juyumaya, K. Karvounis, S. Lambropoulou, Computations on the Yokonuma–Hecke algebras, work in progress. See `http://www.math.ntua.gr/~sofia/yokonuma/index.html`.

[5] P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millett and A. Ocneanu, A new polynomial invariant of knots and links, Bull. AMS **12** (1985), 183-312.

[6] F. M. Goodman, P. de la Harpe, V.F.R. Jones, Coxeter Graphs and Towers of Algebras, Springer-Verlag, 1989.

[7] J. Juyumaya, Sur les noveaux générateurs de l'algèbre de Hecke $\mathcal{H}(G,U,1)$, *J Algebra* **204**, 40–68 (1998).

[8] J. Juyumaya, Markov Trace on the Yokonuma-Hecke Algebra, *J. Knot Theory Ramfications* **13** (2004), 25–39.

[9] J. Juyumaya, A partition Temperley–Lieb algebra. See arXiv:1304.5158.

[10] J. Juyumaya and S. Lambropoulou, p–adic Framed Braids, *Topology and its Applications* **154** (2007),1804–1826.

[11] J. Juyumaya and S. Lambropoulou, p–adic Framed Braids II, *Advances in Mathematics*, **234** (2013), 149–191.

[12] J. Juyumaya, S. Lambropoulou, An adelic extension of the Jones polynomial, in: M. Banagl, D. Vogel (Eds.) *The Mathematics of Knots, Theory and Application*, in: Contributions in Mathematical and Computational Sciences, **1**, Springer, 2010, pp. 125–142. See also arXiv:0909.2545.

[13] J. Juyumaya, S. Lambropoulou, An invariant for singular knots, *J. Knot Theory Ramifications* **18**(6) (2009), 825–840.

[14] V.F.R. Jones, Hecke Algebra representations of braid groups and link polynomials, *Annals of Mathematics* **126** (1987), 335–388.

[15] V.F.R. Jones, Index for Subfactors, *Inventiones Mathimaticae*, **72** (1983), 1–25.

[16] H. Matsumoto, Générateurs et relations des groupes de Weyl généralisés, *C. R. Acad. Sci. Paris*, **258**, 3419–3422 (1964).

[17] S. Ramanujan, On Certain Trigonometric Sums and their Applications in the Theory of Numbers, *Transactions of the Cambridge Philosophical Society*, **22**, No 13 (1918), 297–276.

[18] A. Terra, Fourier Analysis of Finite Groups and Applications, *London Math. Soc. student text* **43**, 1999.

[19] N. Thiem, Unipotent Hecke algebras, *Journal of Algebra*, **284** (2005), 559–577.

[20] T. Yokonuma, Sur la structure des anneux de Hecke d'un group de Chevalley fin, *C.R. Acad. Sc. Paris*, **264** (1967), 344–347.

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