AUTOMORPHISMS OF CURVES AND WEIERSTRASS SEMIGROUPS FOR KATZ-GABBER COVERS

SOTIRIS KARANIKOLOPOULOS AND ARISTIDES KONTOGEORGIS

ABSTRACT. We study p-group Galois covers $X \to \mathbb{P}^1$ with only one fully ramified point. These covers are important because of the Katz-Gabber compactification of Galois actions on complete local rings. The sequence of ramification jumps is related to the Weierstrass semigroup of the global cover at the stabilized point. We determine explicitly the jumps of the ramification filtrations in terms of pole numbers. We give applications for curves with zero p–rank: we focus on maximal curves and curves that admit a big action. Moreover the Galois module structure of polydifferentials is studied and an application to the tangent space of the deformation functor of curves with automorphisms is given.

1. INTRODUCTION

Let X be a projective nonsingular algebraic curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 3$. We will denote by F the function field of the curve X. Let G be a subgroup of the automorphism group $\mathrm{Aut}(X)$ of X and let $G(P)$ be the subgroup of automorphisms stabilizing a point P on X. The study of the group $G(P)$ is much more difficult in positive characteristic than in characteristic zero. In characteristic zero it is known that $G(P)$ is always a cyclic group, while when $p > 0$ and p divides $|G(P)|$ the group $G(P)$ does not have to be cyclic any more and admits the following ramification filtration:

$$G(P) = G_0(P) \supset G_1(P) \supset G_2(P) \supset \dots,$$

Recall that the groups $G_i(P)$ are defined as

$$G_i(P) = \{\sigma \in G(P) : v_P(\sigma(t) - t) \geq i + 1\},$$

for a local uniformizer t at P and v_P is the corresponding valuation. Notice that $G_1(P)$ is the p-part of $G(P)$. A natural question to answer is the determination of the jumps of the ramification filtration, i.e. of the numbers such that $G_i(P) \gneqq G_{i+1}(P)$. This a deep question related to the structure of $G_1(P)$ and of the curve in question. For instance if $G_1(P)$ is abelian then the Hasse-Arf theorem [36, Theorem p.76] puts very strong divisibility relations among the jumps. Let us fix the notation for the jumps of the ramification filtration:

$$G_0(P) = G_1(P) = G_{b_1} > G_{b_2} > \cdots > G_{b_\mu} > \{\mathrm{id}\}.$$

This means that $G_{b_\nu} \supsetneq G_{b_\nu+1}$ for every $1 \leq \nu \leq \mu$ and that there are μ jumps.

One the other hand, the Weierstrass semigroup at P consists of all elements of the function field of the curve that have a unique pole at P. More precisely we can consider the flag of vector spaces

$$k = L(0) = L(P) = \cdots = L((i-1)P) < L(iP) \leq \cdots \leq L((2g-1)P),$$

where

$$L(iP) := \{f \in F : \mathrm{div}(f) + iP \geq 0\} \cup \{0\}.$$

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We will write $\ell(D) = \dim_k L(D)$. An integer i will be called a pole number if there is a function $f \in F^*$ so that $(f)_\infty = iP$ or equivalently $\ell\big((i-1)P\big) + 1 = \ell\big(iP\big)$. The set of pole numbers at P form a semigroup $H(P)$ which is called the Weierstrass semigroup at P. It is known that there are exactly g pole numbers that are smaller or equal to $2g-1$ and that every integer $i \geq 2g$ is in the Weierstrass semigroup, see [41, I.6.7]. It is also known that there is a close connection to the group $G(P)$ and the Weierstrass semigroup at P. I. Morisson and H. Pinkham [29] studied this connection in characteristic zero for Galois Weierstrass points: a g_d^1 linear systems arising from the divisor dP, where d is the first non zero element in $H(P)$, $(f)_\infty = dP$ such that the cover that arises from $f : X \to \mathbb{P}^1(\mathbb{C})$ is Galois. This article can be seen as a natural generalization of some results in their article in positive characteristic. Notice that the first non zero element in $H(P)$ is not enough to grasp the group structure. We have to go up to the first pole number in $H(P)$ that is not divisible by p to do so. And of course the stabilizer $G(P)$ and its p-part $G_1(P)$ does not have to be a cyclic group anymore.

The starting point of our work is the definition by the second author [23, Lemmata 2.1,2.2] of a faithful action of the p-part $G_1(P)$ of the decomposition group $G(P)$ on the spaces $L(iP)$

Proposition 1. *If $g \geq 2$ and $p \neq 2, 3$ then there is at least one pole number $m_r \leq 2g-1$ not divisible by the characteristic p. Let $1 < m_r \leq 2g-1$ be the smallest pole number not divisible by the characteristic. There is a faithful representation*

$$\rho : G_1(P) \to \mathrm{GL}\big(L(m_r P)\big) \tag{1}$$

and proved the following

Proposition 2. *Let X be a curve acted on by the group G. For every fixed point P on X we consider the corresponding faithful representation defined in proposition 1:*

$$\rho : G_1(P) \to \mathrm{GL}_{\ell(m_r P)}(k).$$

Let

$$m_r > m_{r-1} > \cdots > m_0 = 0$$

be the pole numbers at P that are $\leq m_r$. If $G_i(P) > G_{i+1}(P)$, for $i \geq 1$, then $i = m_r - m_k$, for some pole number m_k.

The notion of the ramification filtration can be defined also for more general discrete valued rings see [36]. For the case of spectra $\mathscr{O}$ of local rings of the form $k[[t]]$ acted on by a group G_1, where k is an algebraically closed field of characteristic $p > 0$, we can pass from the local case to the global one.

More precisely, the Katz-Gabber compactification theorem asserts that there is a Galois cover $X \to \mathbb{P}^1$ ramified only at one point P of X with Galois group $G = \mathrm{Gal}(X/\mathbb{P}^1) = G_1$ such that $G_1(P) = G_1$ and the action of $G_1(P)$ on the completed local ring $\hat{\mathscr{O}}_{X,P}$ coincides with the original action of G_1 on $\mathscr{O}$. In this article we will focus on the so called Katz-Gabber covers for p-groups.

By considering the Katz-Gabber compactification to an action on the local ring $k[[t]]$, we have the advantage to attach global invariants, like genus, p-rank, differentials etc, in the local case. Also finite subgroups of the automorphism group $\mathrm{Aut}k[[t]]$, which is a difficult object to understand (and is a crucial object in understanding the deformation theory of deformations of curves with automorphisms, see [3]) become subgroups of $\mathrm{GL}(V)$ for a finite dimensional vector space V.

This article has the following aims:

(1) Proposition 2 gives us all the *possible* jumps of the representation filtration. In this article we will characterize exactly the lower ramification jumps, or equivalently in view of proposition 2 we will compute the pole numbers m_k for $m_r - m_k$ to be a ramification jump. We remark that we have not made any assumption for $G_1(P)$ to be an abelian group.

(2) Generalize the results of Pinkham and Morrison for the positive characteristic case.
(3) Study the Galois module structure of spaces of polydifferentials for Katz-Gabber covers, apply this computation to the open problem of computing the tangent space of the deformation functor of curves with automorphisms. Also a proposition concerning the p-rank representation of these curves is proved.
(4) We give a necessary and sufficient condition in order for a curve to admit a Katz-Gabber cover. We will prove that Katz-Gabber covers arise in a natural way as Galois covers of curves with zero p-rank. Then we will apply this to two such families, namely to maximal curves and to curves equipped with a "big action". For curves with a big-action we also show that the module of holomorphic differentials is an indecomposable $G_1(P)$-module.

Let us now sketch the methods and the results of our article: We will denote by

$$0 = m_0 < m_1 < \cdots < m_{r-1} < m_r$$

the elements of the Weierstrass semigroup at P up to m_r, the first pole number not divisible by the characteristic. Recall that the set of generators of this semigroup is the minimal set of elements such that they can generate the semigroup by their linear combinations with coefficients in $\mathbb{Z}_+$, i.e. we consider the minimal generators of the underlying numerical semigroup. We will see in theorem 3 that these generators contain essential information for the ramification filtration; their prime to p parts form the set of jumps of the ramification filtration.

In order to compute the generators of this Weierstrass semigroup we define in eq. (4) a new filtration of $G_1(P)$ the *representation filtration*:

$$G_1(P) = \ker\rho_0 \supseteq \ker\rho_1 \supseteq \ker\rho_2 \supseteq \cdots \supseteq \ker\rho_r = \{1\}. \tag{2}$$

This filtration leads to a successive sequence of elementary abelian extensions of the field $F^{G_1(P)}$:

$$F^{G_1(P)} = F^{\ker\rho_0} \subseteq F^{\ker\rho_1} \subseteq F^{\ker\rho_2} \subseteq \cdots \subseteq F^{\ker\rho_r} = F. \tag{3}$$

The above sequence of groups jumps at say n certain integers, we call them the jumps of the representation filtration,

$$c_1 < c_2 < \cdots < c_{n-1} < c_n,$$

and $c_n + 1 = r$. This last equality $c_n = r - 1$ comes from the faithful representation of proposition 1. These representation jumps give rise to generators of the Weierstrass semigroup, proposition 15. Since the sequence of the groups $\ker\rho_{c_i}$ jumps, the corresponding sequence of fields will also jump and moreover

$$F^{\ker\rho_{c_{i+1}}} = F^{\ker\rho_{c_i}}(f_{c_i+1}).$$

So in every extension we add an extra function f_{c_i+1} which in turn adds a new generator in the previous semigroup, lemma 22. In section 2 we will see the relation of the semigroups in a Galois extension of fields. More precisely define $Q_i = F^{\ker\rho_{c_i}} \cap P$ for $1 \leq i \leq n+1$ to be the *unique* ramification points of the tower defined in eq. (3); using the relation of the semigroups in Galois extension of function fields we will see that the semigroup of $F^{\ker\rho_{c_2}}$ at Q_2 is $\Sigma_2 := \left|\frac{\ker\rho_{c_1}}{\ker\rho_{c_2}}\right|\mathbb{Z}_+ + \lambda_1\mathbb{Z}_+$ with $(\lambda_1, p) = 1$. Notice that $\lambda_1 = 1$ if and only if $F^{\ker\rho_{c_2}}$ is rational. We proceed in this way and we have that

$$\Sigma_{i+1} = \left|\frac{\ker\rho_{c_i}}{\ker\rho_{c_{i+1}}}\right|\Sigma_i + \lambda_i\mathbb{Z}_+, \text{ for all } 1 \leq i \leq n,$$

i.e. the semigroup of a field at Q_{i+1} in the sequence given in eq. (3) is the semigroup of the previous field at Q_i multiplied by the order of their Galois group, plus and extra element

λ_i prime to p and all their $\mathbb{Z}_+$ linear combinations. Denote by $p^{h_i} = |\ker \rho_{c_{i+1}}|$, for all $1 \leq i \leq n$, and $p^{h_0} = G_1(P)$. We will see in proposition 15 that the elements

$$p^{h_1}\lambda_1 < p^{h_2}\lambda_2 < \cdots < p^{h_{n-1}}\lambda_{n-1} < \lambda_n = \frac{m_{c_n+1}}{|\ker \rho_{c_{n+1}}|} = m_r,$$

are inside the set of generators of the Weierstrass semigroup at P, and that if we add the element p^{h_0} then, by proposition 24:

$$\langle p^{h_0}, p^{h_1}\lambda_1, \ldots, p^{h_{n-1}}\lambda_{n-1}, \lambda_n \rangle_{\mathbb{Z}_+} = H(P).$$

The relation of the representation with the ramification filtration is given in terms of the following:

Theorem 3. *Assume that $X \to X/G_1(P) = \mathbb{P}^1$ is a Katz-Gabber cover. Then*

(1) $|G_{b_i}| = |\ker \rho_{c_i}| = p^{h_{i-1}}$ *for all* $2 \leq i \leq n$ *and* $p^{h_0} = |G_1(P)|$.
(2) *For every jump of the representation filtration c_i, $1 \leq i \leq n$ there exists a generator of $H(P)$ of the form $m_{c_i+1} = p^{h_i}\lambda_i$, where $(\lambda_i, p) = 1$.*
(3) *The jumps of the ramification filtration are the integers λ_i for $1 \leq i \leq n$, i.e. $\lambda_i = b_i$ for every such i, while the number of ramification and representation jumps coincide, i.e. $\mu = n$.*
(4) *Concerning the minimal set of generators of the Weierstrass semigroup at P, $H(P)$ we have the following two cases:*
    (a) *If $G_1(P) > G_2(P)$, then the extension $F/F^{G_2(P)}$ is also Katz-Gabber, and the Weierstrass semigroup $H(P)$ is minimally generated by m_{c_i+1}, with $1 \leq i \leq n$. Moreover $|G_2(P)| = m_{c_1+1} = m_1$.*
    (b) *If $G_1(P) = G_2(P)$ then we need m_{c_i+1}, $1 \leq i \leq n$ together with $p^{h_0} = |G_1(P)|$ in order to generate $H(P)$. In this case the element p^{h_0} does not correspond to a jump of the representation filtration and is a generator of $H(P)$.*

In both cases the semigroup $H(P)$ is symmetric.

Proof. 1 is corollary 28; 2 is proposition 15; 3 is theorem 27; 4a is corollary 30; 4b is proposition 31, while the assertion about the symmetric Weierstrass semigroup comes from corollary 41. □

Remark 4. For Katz-Gabber covers the field $F^{G_2(P)}$ is always rational this is [18, Theorem 11.78 (iii)].

Remark 5 (Upper ramification jumps)**.** The reader should notice that computing the jumps of the lower ramification filtration we gain information on the jumps of the *upper* ramification filtration through the Herbrand's formula, see [36, section IV]. As an application of this we get that, for p–groups, upper and lower ramification jumps are connected with the following formula:

$$b_i = \sum_{j=1}^{i} (u_j - u_{j-1}) p^{h_0 - h_{j-1}}, \text{ for every } 1 \leq i \leq n,$$

where $u_1, \ldots, u_n$ are the upper jumps of $G_1(P)$ and here $b_0 = u_0 = 0$.
Thus computing the lower jumps we also compute explicitly the upper jumps too.

1.1. **Applications.** Our motivation for studying actions of Katz-Gabber covers was the deformation theory of curves with automorphisms. J. Bertin and A. Mézard in [3] proved a local global principle that can be used to show that the "difficult part" of the study of the deformation functor of curves with automorphisms resides in the local deformation functors. This is a vast object of study to describe it here, the reader is advised to look at [3] for more information. Local actions can be compactified to Katz-Gabber covers, and at least the dimension of the tangent space of the deformation functor is reflected into the space of 2-holomorphic differentials $H^0(X, \Omega_X^{\otimes 2})$ of the corresponding Katz-Gabber cover. Indeed, the second author in [25] related the dimension $\dim H^0(X, \Omega_X^{\otimes 2})_G$ to the dimension of the tangent space of the deformation functor of curves with automorphisms.

Since we have to compute coinvariants of a $k[G]$-module we are lead to another open problem in positive characteristic:

Problem 6. Describe the Galois module structure of spaces $H^0(X, \Omega_X^{\otimes m})$, for a positive integer $m \geq 1$.

This problem is open and only some special cases can be found in the literature. However in the case of Katz-Gabber covers we will describe in sections 4, 5, both the spaces $H^0(X, \Omega_X^{\otimes m})$ and their G-action. Then we will show how this information will lead us to the computation of the space of coinvariants.

Regardless to the deformation theory of curves with automorphisms Katz-Gabber covers appear also to curves with zero p-rank; two such families are curves that admit "big actions" and maximal curves:

• The case of curves X with zero p-rank. More precisely in such a case every p group of automorphisms G of the curve X can be realized as the stabilizer of a *unique* place, see for example [18, paragraph 11.13]. Thus we can suppose that $G = G_1(P)$ for some P. This means that the Galois cover $X \longrightarrow X/G_1(P)$ is wildly ramified at the *unique* point P. The case with zero p–rank curves correspond to curves with "huge" automorphisms and among those curves the curves with most automorphisms occur exactly when $X/G_1(P)$ is rational (otherwise it is known that $|G_1(P)|$ is less than or equal to the genus of the curve, see [18, Theorem 11.78 (i)]). In this way, if $X/G_1(P)$ is rational, then we are exactly in the case of Katz–Gabber covers and our results can be applied. A useful criterion for this to happen is $|G_1(P)|$ to be a pole number at the point P, see corollary 9. From the other hand every curve X that admits a Katz–Gabber cover must also has zero p–rank, see theorem 33.

• C. Lehr, M. Matignon [27] defined the notion of *big actions* for groups acting on curves and big actions were studied further by M. Rocher and M. Matignon [28],[32]. All big actions are included in this set up (as we expected to, since they are certain Katz–Gabber p covers of the projective line). Notice also that for these curves we have $G_1(P) > G_2(P)$, and $F^{G_2(P)}$ is always rational, see proposition 34. Thus these curves provide us with examples that $F^{G_1(P)}$ cannot be generated by some function that gives rise to a generator of the Weierstrass semigroup, although $|G_1(P)|$ will always be a pole number since $|G_1(P)| > 2g$. We give a full description for them at corollary 35.

• Let X be a projective, geometrically irreducible, non-singular algebraic curve defined over $\mathbb{F}_{q^2}$, where q is a p–power. Such a curve is called maximal if the number of $\mathbb{F}_{q^2}$ rational points attains the Hasse–Weil upper bound

$$|X(\mathbb{F}_{q^2})| \leq q^2 + 1 + 2gq.$$

These curves have many applications to error correcting codes, see [15]. For a survey article see [8]; some other sources could be [7], [12], [13], [39], [17], [10], [6], as well as the book [18]. All these families of maximal curves with $|G_1(P)|$ a pole number can be also described. We show in theorem 36 that this condition for maximal curves over $\bar{\mathbb{F}}_{q^2}$ is equivalent to $q \leq |G_1(P)|$. Notice that this last condition is true for all the "generic" families of maximal curves that we know: the Hermitian, the (generalized) Giulietti-Korchmáros curve [13] ([39] and [17]), the Garcia–Stichtenoth curve [9], since for them we have $|G_1(P)| = q^3$, while for all maximal curves is true that $q, q+1 \in H(P)$ for a $\mathbb{F}_{q^2}$ rational point P. Finally, when $m_r = q+1$ then the linear series $|m_r P|$ that naturally arise from propositions 1 and 2 is called the Frobenius linear series and it is an invariant of the curve in a rational point, see [18] and remark 38. Although these curves are naturally defined over $\mathbb{F}_{q^2}$, here we view them over some algebraic closure $\bar{\mathbb{F}}_{q^2}$. We give a full description for them at corollary 37.

Notice that these two families are connected via the theory of global Ray class fields [26], [2], and through the identification of "many rational points" with "many automorphisms", see [28]. This is another reason why we believe that Katz-Gabber covers is the right tool to use in order to study them.

Finally, since we compute explicitly Weierstrass semigroups $H(P)$ for maximal curves satisfying the condition $q \leq |G_1(P)|$, we should mention the many connections that these semigroups have with the construction of AG (Algebraic geometric) codes [19]. All the semigroups that appear here are telescopic and thus symmetric, see remark 43; for some interesting geometric properties concerning this class of numerical semigroups, the reader can look at [1, p. 142] and at the references therein.

2. Decomposition Groups $G_i(P)$

2.1. Jumps in the ramification filtration and divisibility of the Weierstrass semigroup.

We begin our study by relating the semigroups in Galois covers. Consider a Galois cover $\pi : X \to Y = X/G$ of algebraic curves, and let P be a fully ramified point of X. How are the Weierstrass semigroup sequences of P, and $\pi(P)$ related?

Lemma 7. *Let $F(X)$, $F(Y) = F(X)^G$ denote the function fields of the curves X and Y respectively. The morphisms*

$$N_G : F(X) \to F(Y) \text{ and } \pi^* : F(Y) \to F(X),$$

sending $f \in F(X)$ to $N_G(f) = \prod_{\sigma \in G} \sigma f$ and $g \in F(Y)$ to $\pi^* g \in F(X)$ respectively, induce injections

$$N_G : H(P) \to H(Q) \text{ and } \pi^* : H(Q) \xrightarrow{\times |G|} H(P),$$

where $Q := \pi(P)$.

Proof. For every element $f \in F(X)$ such that $(f)_\infty = mP$, the element $N_G(f)$ is a G-invariant element, so it is in $F(Y)$. Moreover, the pole order of $N_G(f)$ seen as a function on $F(X)$ is $|G| \cdot m$. But since P is fully ramified the valuation of $N_G(f)$ expressed in terms of the local uniformizer at $\pi(P)$ is just $-m$.

On the other hand side an element $g \in F(Y)$ seen as an element of $F(X)$ by considering the pullback $\pi^*(g)$ has for the same reason valuation at P multiplied by the order of G. □

Remark 8. The condition of fully ramification is necessary in the above lemma. Indeed, if a point $Q \in Y$ has more than one elements in $\pi^{-1}(Q)$ then the pullback of g, such that $(g)_\infty = mQ$, is supported on $\pi^{-1}(Q)$ and gives no information for the Weierstrass semigroup at any of the points $P \in \pi^{-1}(Q)$.

Corollary 9. $|G_1(P)| \in H(P)$ *if and only if* $g_{X/G} = 0$.

Another immediate consequence of lemma 7 is the following

Corollary 10. *If an element f such that $(f)_\infty = aP$ is invariant under the action of a subgroup $H < G_1(P)$, then $|H|$ divides a.*

Proof. Since f is invariant it is the pullback of a function $g \in F(X/H)$. The result now follows from lemma 7. □

Definition 11. For each $0 \leq i \leq r$ we consider the representations

$$\rho_i : G_1(P) \to \mathrm{GL}(L(m_i P)).$$

We form the decreasing sequence of groups:

$$G_1(P) = \ker\rho_0 \supseteq \ker\rho_1 \supseteq \ker\rho_2 \supseteq \cdots \supseteq \ker\rho_r = \{1\}. \tag{4}$$

We will cal this sequence of groups "the representation" filtration.

Let $\sigma \in \ker\rho_i$. Then $\rho_{i+1}(\sigma)$ has the following form

$$\rho_{i+1}(\sigma) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ a_{i+1,1}(\sigma) & a_{i+1,2}(\sigma) & \cdots & a_{i+1,i}(\sigma) & 1 \end{pmatrix}.$$

Observe also that all functions $a_{i+1,\nu} : \ker\rho_i \to k$ are group homomorphisms into the additive group of the field k. Notice that

$$\ker\rho_{i+1} = \ker\rho_i \cap \bigcap_{\nu=1}^{i} \ker a_{i+1,\nu}.$$

Definition 12. From now on, $f_i \in F$, where $0 \leq i \leq r$ will denote the functions that give rise to the pole numbers m_i of proposition 2, that is $(f_i)_\infty = m_i P$.

Remark 13. For the case of Katz-Gabber covers we will prove in proposition 25 that for σ in $\ker \rho_i \setminus \ker \rho_{i+1}$ we have $\sigma(f_{i+1}) = f_{i+1} + c(\sigma)$ with $c(\sigma) \in k^*$ so that $a_{i+1,2}(\sigma) = \cdots = a_{i+1,i}(\sigma) = 0$.

3. Enumerating jumps

We call an index i a jump of the representation filtration if and only if $\ker \rho_i \gneq \ker \rho_{i+1}$. Let us also fix the notation for the representation jumps:

$$G_1(P) = \ker \rho_0 = \cdots = \ker \rho_{c_1} > \cdots > \ker \rho_{c_{n-1}} > \ker \rho_{c_n} = \ker \rho_{r-1} > \{\mathrm{id}\}.$$

Remark 14. Every element $\sigma \in \ker\rho_i$ fixes by definition all f_ν corresponding to m_ν for $\nu \leq i$. A non negative integer i is a jump whenever the function f_{i+1} is not $\ker \rho_i$ invariant.

Proposition 15. *If $\ker\rho_{c_i} \supsetneq \ker\rho_{c_i+1}$, i.e., when c_i is a representation jump then m_{c_i+1} is a generator of the Weierstrass semigroup at P.*

Proof. Suppose that $\ker \rho_i \supsetneq \ker \rho_{i+1}$. If m_{i+1} is in the semigroup $\langle m_1, \ldots, m_i \rangle_{\mathbb{Z}_+}$ generated by all $m_1, \ldots, m_i$ then

$$m_{i+1} = \sum_{j \leq i} \nu_j m_j, \text{ where } \nu_j \in \mathbb{Z}_+, \tag{5}$$

and

$$f_{i+1} = \prod_{j \leq i} f_j^{\nu_j} + \text{some terms that give rise to smaller than } m_{i+1} \text{ pole numbers.} \tag{6}$$

Since every element $\sigma \in \ker\rho_i$ fixes by definition the right hand side of the last equation, this implies that $\ker \rho_{i+1} = \ker \rho_i$, a contradiction. □

Remark 16.
- The reader should notice that when m_{i+1} is not a generator of the semigroup then, in general, the expression given in eq. (5) may not be unique. Although this fact does not affect the proof of proposition 15, we will actually see that in our case we can always write such a pole number in a *unique* way, by choosing

$$0 \leq \nu_j < \frac{p^{h_{j-1}}}{p^{h_j}} = \left| \frac{G_{b_j}}{G_{b_{j+1}}} \right|, \text{ for } 1 \leq j \leq i;$$

this comes from the fact that all the semigroups for us are *telescopic*, remark 43, coupled with corollary 28 and well known properties for such numerical semigroups, see for example [19, Lemma 5.34].

- Fixing a point P and a pole number m_i at this point, a function that has a unique pole at this point of order m_i is not unique. If f_i, f_i' are two functions such that $(f_i)_\infty = (f_i')_\infty = m_i P$, then the difference $f_i - f_i'$ is a function in $L(m_i P)$ and has pole order $|v_P(f_i - f_i')| \leq m_i$. By examining the Laurent expansion of f_i, f_i' we see that there is constant $c \in k^*$ such that:
$$f_i' = cf_i + g,$$
where g is a function in $L(m_{i-1}P)$.
- In eq. (6) the f_j's in the product give rise to generators m_j's of the Weierstrass semigroup. It may be the case that in the sum could appear a term $f_{j^*}^{\nu_{j^*}}$, where f_{j^*} gives rise to a generator $m_j < m_{j^*}$, for all m_j's that appear in the product. However the essential is to notice that for this case also $m_{i+1} > m_{j^*}\nu_{j^*}$.

Remark 17. The fields F, $F^{\ker \rho_{c_i}}$, $i = 1, \ldots, n$ given in eq. (3) and in definition 11 are generated by the elements f_{c_i+1} that we introduced in each step, i.e.
$$F^{\ker \rho_{c_{i+1}}} = F^{\ker \rho_{c_i}}(f_{c_i+1}) = F^{G_1(P)}(f_{c_1+1}, \ldots, f_{c_i+1}).$$
Moreover $F^{G_1(P)} = k(f_{i_0})$ for some index i_0 and $F = k(f_{i_0}, f_{c_1+1}, \ldots, f_{c_n+1} = f_r)$. We form the field $F^{\ker \rho_{c_i}}$ by successive extensions of the rational function field $F^{G_1(P)}$. At every jump c_i of the representation filtration we add an extra element f_{c_i+1} to the field $F^{\ker \rho_{c_i}}$ and we have $F^{\ker \rho_{c_{i+1}}} = F^{\ker \rho_{c_i}}(f_{c_i+1})$.

Remark 18. Notice that the additive polynomial of the Galois extension $F^{\ker \rho_{c_i}}/F^{\ker \rho_{c_{i-1}}}$ can be constructed explicitly using the theory of Moore determinants, [16, 1.3] together with proposition 25.

3.1. **Examples.** We will give examples of curves where m_{i_j+1} is a generator of the Weierstrass semigroup but $\ker \rho_{i_j} = \ker \rho_{i_j+1}$. In the first example $i_j = 0$.

Example 19. Consider the Artin Schreier extension of the rational function field given by the equation
$$y^p - y = f(x)$$
where $f(x)$ is a polynomial that has a unique pole at P and $\deg f(x) = m_r$, $(p, m_r) = 1$. Suppose that $m_r > p$. It is well known that the Weierstrass semi group at P is given by $\langle p, m_r \rangle_{\mathbb{Z}_+}$ [40, p. 618]. Notice that $|G| = |G_1(P)| = |\ker \rho_0| = p$, with $m_1 = p$ a generator of the Weierstrass semigroup *but* $\ker \rho_0 = \ker \rho_1$ since $|\ker \rho_0|$ divides m_1, so f_1 is a $\ker \rho_0$-invariant element and 0 is not a representation jump. Notice that here $m_r = -v_P(y) = -v_\infty(f(x))$ is the unique ramification jump of $G_1(P)$.

Next we will give an example, namely the Giulietti-Korchmáros curve (see [13]), where m_{i_j+1} is a Weierstrass generator at P with $i_j \neq 0$ such that $\ker \rho_{i_j} = \ker \rho_{i_j+1}$.

Example 20 (The GK–curve)**.** The Weierstrass semigroup at the unique ramified point is generated by $\langle m_1, m_2, m_3 \rangle_{\mathbb{Z}_+}$, with $m_2 = q^3 = |G_1(P)|$ and $F^{G_1(P)} = k(f_2)$. We compute the representation filtration and the picture is the following
$$G_1(P) = \ker \rho_0 \supsetneq \ker \rho_1 = \ker \rho_2 \supsetneq \{\mathrm{id}\}.$$
That is m_2 is a generator *but* 1 is not a representation jump (notice also that $|\ker \rho_2| = q$). Here $F^{\ker \rho_2} = k(f_1, f_2) = F^{G_1(P)}(f_1)$, see [6]. Moreover there are two ramification jumps for this case, [6, Proposition 4.2]: $m_r = -v_P(f_3)$ and $\frac{m_1}{|\ker \rho_2|}$.

Of course there are examples where it is not possible to generate $F^{G_1(P)}$ as a monomial on some f_i's, where each f_i corresponds to a generator m_i of the Weierstrass semigroup at P.

Example 21. Consider an Artin-Schreier cover of $\mathbb{P}^1 \to \mathbb{P}^1$ with equation $y^p - y = x$ and then a Katz-Gabber cover $X \to \mathbb{P}^1$ with equation $z^p - z = y^m$, $(m,p) = 1$, totally ramified at P. The semigroup of X is then $m\mathbb{Z}_+ + p\mathbb{Z}_+$, while F is generated by the functions x, y. The ramification jumps are given by 1 and $m = -v_P(z)$.

Lemma 22. *Define $Q_i := F^{\ker \rho_{c_i}} \cap P$ for $1 \le i \le n+1$. Then the semigroup $H(Q_{i+1})$ is generated by elements of the semigroup $H(Q_i)$ multiplied by $[\ker \rho_{c_i} : \ker \rho_{c_{i+1}}]$ and an extra prime to p generator that corresponds to the representation jump of proposition 15 and equals to*

$$-v_{Q_{i+1}}(f_{c_i+1}) = \frac{m_{c_i+1}}{|\ker \rho_{c_{i+1}}|}, \text{ for all } 1 \le i \le n.$$

Proof. From lemma 7 in every step of the representation tower we have

$$\frac{|\ker \rho_{c_i}|}{|\ker \rho_{c_{i+1}}|} H(Q_i) \subset H(Q_{i+1}).$$

We would like to apply proposition 15 for the extension $F^{\ker \rho_{c_{i+1}}}/F^{G_1(P)}$. For this reason we first show that the group $\ker \rho_{c_{i+1}}$ is a normal subgroup of $G_1(P)$. Indeed, for $\sigma \in \ker \rho_{c_{i+1}}$ and $\tau \in G_1(P)$ we have that

$$\tau^{-1}\sigma\tau(\tau^{-1} f_{c_j+1}) = \tau^{-1} f_{c_j+1}, \text{ for all } j \le i.$$

This means that $\tau^{-1}\sigma\tau$ fixes $\tau^{-1} f_{c_j+1}$. But since f_{c_j+1} corresponds to m_{c_j+1} then $\tau^{-1} f_{c_j+1}$ also corresponds to m_{c_j+1}, since

$$v_P(\tau^{-1} f_{c_j+1}) = v_{\tau^{-1}P}(f_{c_j+1}) = v_P(f_{c_j+1}).$$

Therefore $\tau^{-1}\sigma\tau$ fixes the generators of $F^{\ker \rho_{c_{i+1}}}$. Notice now that the field extension $F^{\ker \rho_{c_{i+1}}}/F^{G_1(P)}$ is also Katz-Gabber and their representation filtration is obtained from the quotients of the representation filtration of $F/F^{G_1(P)}$ by the group $\ker \rho_{c_{i+1}}$. Therefore, according to proposition 15, $H(Q_{i+1})$ can have only one extra generator compared to $H(Q_i)$ that is coming from the generator of the extension $F^{\ker \rho_{c_{i+1}}}/F^{\ker \rho_{c_i}}$ which is f_{c_i+1}. That is

$$-v_{Q_{i+1}}(f_{c_i+1}) = \frac{m_{c_i+1}}{|\ker \rho_{c_{i+1}}|}.$$

In order to finish the proof note that every Weierstrass semigroup $H(Q_{i+1})$, for every $1 \le i \le n$ must have a prime to p generator from proposition 1, while $-v_{Q_{i+1}}(f_{c_i+1})$ is the only such generator of $H(Q_{i+1})$ which is not a multiple of the characteristic. $\square$

We have the following picture of fields, groups, places and semigroups

$$\begin{tikzcd}
F \arrow[d, dash] & \{1\} \arrow[d, dash] & P \arrow[d, dash] & H(P) \arrow[d, dash] \\
F^{\ker \rho_{c_{i+1}}} \arrow[d, dash] & \ker \rho_{c_{i+1}} \arrow[d, dash] & Q_{i+1} \arrow[d, dash] & H(Q_{i+1}) = \left\langle \frac{|\ker \rho_{c_i}|}{|\ker \rho_{c_{i+1}}|} H(Q_i), \frac{m_{c_{i+1}}}{|\ker \rho_{c_{i+1}}|} \right\rangle \arrow[d, dash] \\
F^{\ker \rho_{c_i}} \arrow[d, dash] & \ker \rho_{c_i} \arrow[d, dash] & Q_i \arrow[d, dash] & H(Q_i) \arrow[d, dash] \\
F^{G_1(P)} & G_1(P) & Q_1 & \mathbb{Z}_+
\end{tikzcd}$$

According to proposition 15 since $\{c_1, \ldots, c_n\}$ are the jumps of the representation filtration the elements $\{m_{c_1+1}, \ldots, m_{c_n+1} = m_r\}$ are generators of the Weierstrass semigroup

$H(P)$. But it is not true that every generator of $H(P)$ comes this way as we already saw in the the examples of this section and as the following lemma indicates:

Lemma 23. *Let m_{i_0} be generator of the Weierstrass semigroup at Q_i for some index $1 \leq i \leq n+1$ and for some non negative integer i_0 which does not correspond to a representation jump, i.e. $m_{i_0} \neq m_{c_\nu+1}$ for all $1 \leq \nu \leq n$. Then, the corresponding element $f_{i_0} \in F$ is a $G_1(P)$-invariant element. The number of representation jumps are either equal to the number of the generators of the Weierstrass semigroup or equal to the number of the Weierstrass semigroup generators minus one and $|G_1(P)| = m_{i_0}$.*

Proof. If there is a generator of $H(Q_i)$ that does not correspond to the jump of the representation filtration, then this generator is a multiple of a generator of $H(Q_{i-1})$ by lemma 22. This means that the function f_{i_0} that corresponds to the generator is an element invariant under the Galois group of the extension $F^{\ker \rho_{c_\nu}}/F^{\ker \rho_{c_{\nu+1}}}$ for all $1 \leq \nu \leq n$. Using this argument inductively we arrive to the conclusion that the function f_{i_0} is $G_1(P)$ invariant and thus, by corollary 10, $|G_1(P)|$ divides m_{i_0} with m_{i_0} is a generator at $H(Q_i)$, i.e. $m_{i_0} = |G_1(P)|$. For this last assertion, the reader should notice that $F^{\ker \rho_{c_i}}/F^{G_1(P)}$ is Galois (see the proof of lemma 22) and thus $|G_1(P)| \in H(Q_i)$. Finally, if such an f_{i_0} exists it is unique since $F^{G_1(P)}$ is rational from our hypothesis. This completes the proof. $\square$

We sum up all the information concerning the Weierstrass semigroups of the field tower arising from the representation filtration in the next

Proposition 24. *The Weierstrass semigroups of the fields $F^{\ker \rho_{c_i}}$ at $Q_i = P \cap F^{\ker \rho_{c_i}}$ for every $1 \leq i \leq n$ and $\ker \rho_{c_1} = G_1(P)$ are given by*

$$H(Q_{i+1}) = \left\langle \frac{m_{c_j+1}}{|\ker \rho_{c_{i+1}}|}, \left| \frac{G_1(P)}{\ker \rho_{c_{i+1}}} \right| \right\rangle_{\mathbb{Z}_+},$$

where j runs through the indices $1 \leq j \leq i$. For the Weierstrass semigroup at P we get

$$H(P) = \langle m_{c_j+1}, |G_1(P)| \rangle_{\mathbb{Z}_+}, \text{ where } 1 \leq j \leq n, \text{ while } H(Q_1) = \mathbb{Z}_+.$$

Proposition 25. *Assume, that $\sigma \in \ker\rho_{c_i} - \ker\rho_{c_{i+1}}$. Then,*

$$\begin{aligned} \sigma(f_\nu) &= f_\nu && \text{for all } \nu \leq c_i \\ \sigma(f_{c_i+1}) &= f_{c_i+1} + c(\sigma), && \text{where } c(\sigma) \in k^*. \end{aligned} \tag{7}$$

Proof. In general $\sigma(f_{c_i+1}) = \alpha \cdot f_{c_i+1} + c(\sigma)$, where $c(\sigma) \in k[f_1, \ldots f_{c_i}]$, and $\alpha \in k^*$; since σ has order a power of p we see that $\alpha = 1$. But if $c(\sigma)$ is not constant then it has a root $Q \neq Q_i$. We will prove that Q is then a ramified point and this will lead to a contradiction since only one place can ramify, and this is Q_i.

Consider the ring $A := \mathscr{O}(X - Q_i)$, where $\mathscr{O}$ denotes the structure sheaf of a nonsingular projective model of our curve X that corresponds to the function field $F^{\ker \rho_{c_i}}$. The ring A is by definition

$$A = \bigcup_{\nu=0}^{\infty} L(\nu Q_i) = k[f_1, \ldots, f_{c_i}],$$

where the elements $f_1, \ldots, f_{c_i}$ are subject to several relations coming from the function field of the curve. Observe that when ν becomes greater than or equal to $\frac{m_{c_{i-1}+1}}{|\ker \rho_{c_i}|}$ (i.e. is greater than all the generators of the Weierstrass semigroup at Q_i) the algebra generated by $f_1, \ldots, f_{c_i}$ as elements of the vector space $L(\nu Q_i)$ is the ring A. Keep in mind that the vector space $L(\nu Q_i)$ is inside the function field of the curve so there is a well defined notion of multiplication on elements of $L(\nu Q_i)$. Every place $Q \neq Q_i$ of the function field $F^{\ker \rho_{c_i}}$ corresponds to a unique maximal ideal of the ring A.

Notice also that the automorphism group acts on A. We will prove that the ideal Q is left invariant under the action of σ. Let Q be a root of $c(\sigma)$ and denote by Q the corresponding ideal of A. It is finitely generated, so $Q = \langle g_j \rangle$ where g_j are polynomial expressions in f_i, where $1 \leq i \leq c_i$. We will prove that

$$\sigma(g_j) \in Q \text{ for all } j.$$

Indeed, write

$$g_j = \sum f_1^{\nu_1} \cdots f_{c_i}^{\nu_{c_i}}.$$

Then

$$\sigma(g_j) = \sum f_1^{\nu_1} \cdots (f_{c_i} + c(\sigma))^{\nu_i} =$$

$$\sum f_1^{\nu_1} \cdots f_{c_i}^{\nu_{c_i}} + \sum \sum_{\mu=1}^{\nu_i} f_1^{\nu_1} \cdots f_{c_i-1}^{\nu_{c_i-1}} \binom{\nu_{c_i}}{\mu} c(\sigma)^{\mu} f_{c_i}^{\nu_{c_i}-\mu}.$$

But Q is a root of $c(\sigma)$ and this is equivalent to $c(\sigma) \in Q$ so the second summand of the last equation is an element in Q.

We would like also to point out how we can construct the curve $X - Q_i$. If ν is big enough then the projective map Φ corresponding to the linear series $|\nu Q_i|$ is an embedding [14, Theorem 4.3.15]. The image $\Phi(X)$ is then a nonsingular curve; removing the point $\Phi(Q_i)$ we obtain the affine non-singular curve with coordinate ring A. Notice that, by construction, X is the projective closure of that curve with Q_i being the point at infinity, while the function fields for both curves are just $F^{\ker \rho_{c_i}}$. $\square$

In what follows we will use the following:

Lemma 26. *Let $f \in F$ such that $p \nmid v_P(f)$. If $\sigma \in G_i \setminus G_{i+1}$, then $\sigma(f) = f + f'$ with $f' \neq 0$ and $i = -v_P(f) + v_P(f')$.*

Proof. This is [18, Lemma 11.83] $\square$

Theorem 27. *Let P the totally ramified place of the Katz–Gabber cover. Denote with $Q_i = P \cap F^{\ker \rho_{c_i}}$, with $1 \leq i \leq n+1$.*

1. *the groups $\ker \rho_{c_i} / \ker \rho_{c_{i+1}}$, for each $1 \leq i \leq n$, have exactly one lower ramification jump that equals to $-v_{Q_{i+1}}(f_{c_i+1})$.*
2. *these jumps from part (1) are equal to the ramifications jumps of the groups $G_{b_i}/G_{b_{i+1}}$, for $1 \leq i \leq \mu$, thus $\mu = n$ and they exhaust all the ramification jumps of $G_1(P)$.*

Proof. By proposition 25 and lemma 26 in order to prove that the jump for $\ker \rho_{c_i} / \ker \rho_{c_{i+1}}$ is indeed $-v_{Q_{i+1}}(f_{c_i+1})$, we have to use that $\gcd(v_{Q_{i+1}}(f_{c_i+1}), p) = 1$, something that comes from lemma 22.

This jump is also unique by lemma 26, thus each extension $F^{\ker \rho_{c_{i+1}}} / F^{\ker \rho_{c_i}}$ is an elementary abelian extension. The group $\ker \rho_{c_n}$ is elementary abelian with jump at m_r, since this group is a subgroup of $G_1(P)$ and this is the maximum jump that we can have from proposition 2, we thus obtain $m_r = b_\lambda$.

For the next step notice that $\ker \rho_{c_n} / \ker \rho_{c_{n-1}}$ has unique jump at $-v_{Q_n}(f_{c_{n-1}+1})$ from the first part. This jump equals to the first jump of $\ker \rho_{c_{n-1}}$ (while the second is m_r).

We continue like this, using the fact that every ramification jump of a subgroup of $G_1(P)$ is a ramification jump of $G_1(P)$ as well [36, Proposition IV.2 p. 62], and get that all integers $-v_{Q_{i+1}}(f_{c_i+1})$ are indeed jumps of $G_1(P)$.

Are there more jumps of the ramification filtration? By construction $\ker \rho_{c_1} = G_1(P)$ and $\ker \rho_{c_1}$ has at least n (this number equals the number of representation jumps) lower ramification jumps from part (1). If the number of the ramification jumps was strictly greater than n, then some of the Galois groups $\ker \rho_{c_i} / \ker \rho_{c_{i+1}}$ should had more than one lower ramification jumps, something impossible from the computations done above. $\square$

Corollary 28. *The number of jumps of the ramification filtration equals the number of jumps of the representation filtration and moreover the orders of these groups are equal:*

$$|G_{b_i}| = |\ker \rho_{c_i}| \text{ for all } 1 \leq i \leq \mu = n.$$

Proof. We will prove first that $\ker \rho_{r-1} \subset G_{b_\lambda}$. But $b_\lambda = m_r$, thus for an element $\sigma \in \ker \rho_{r-1}$ we have $\sigma(f_r) = f_r + c(\sigma)$, with $c(\sigma) \in k^*$ so

$$v_P(\sigma(t) - t) = m_r + 1 = b_\lambda + 1 \Rightarrow \sigma \in G_{b_\lambda}.$$

Now we will prove that $\ker \rho_{r-1} \supset G_{b_\lambda}$. Notice that every element in G_{b_λ} satisfies $v_P(\sigma(t) - t) = b_\lambda + 1 = m_r + 1$. Let c_{i_0} be maximal such that $G_{b_\lambda} \subset \ker \rho_{c_{i_0}}$. Then by construction, there is an element $\sigma' \in G_{b_\lambda}$ that does not belong at $\ker \rho_{c_{i_0+1}}$, that is

$$\sigma'(f_j) = f_j \text{ for all } j \leq c_{i_0} \text{ and } \sigma'(f_{c_{i_0}+1}) = f_{c_{i_0}+1} + \sigma'(c), \text{ for some } \sigma'(c) \in k^*.$$

The class $\sigma' \ker \rho_{c_{i_0+1}} \in \frac{\ker \rho_{c_{i_0}}}{\ker \rho_{c_{i_0+1}}}$ jumps at $-v_{Q_{i_0+1}}(f_{c_{i_0}+1})$ and at b_λ on the hand other side. Notice, that since G_{b_λ} is elementary abelian with unique jump, lower and upper ramification filtrations coincide. So $m_r = b_\lambda = -v_{Q_{i_0+1}}(f_{c_{i_0}+1})$. Thus $i_0 = r$ and $c_{i_0} = c_n = r-1$ (for the last equality the reader should not forget our notational convention that we make through this paper, that is $c_n = r - 1$). This proves that $\ker \rho_{r-1} = G_{b_\lambda}$, i.e. the last groups in both filtrations coincide.

We now consider the Katz-Gabber extension of the rational function field given by:

$$F^{G_{b_\lambda}} = X/\ker \rho_{c_n} = F^{G_1(P)}(f_{c_1+1}, \ldots, f_{c_{n-1}+1}).$$

This extension, has ramification filtration

$$\frac{G_1(P)}{G_{b_\lambda}} \geq \cdots \geq \frac{G_i}{G_{b_\lambda}} \geq \cdots \geq \frac{G_{b_\lambda - 1}}{G_{b_\lambda}} > \{1\}.$$

Indeed, since we take the quotient by a subgroup that is a group in the ramification filtration the lower indices behave well, [36, Corollary on page 64], and has representation filtration $\ker \rho_{c_1} / \ker \rho_{r-1}$. Using the previous argument we see that the last groups in both filtrations are equal and proceed inductively using theorem 27. $\square$

We will now focus on the case where the first jump equals one:

Corollary 29. *The condition $G_1(P) > G_2(P)$ is equivalent to $F^{\ker \rho_{c_2}}$ being rational.*

Proof. Let $[G_1(P) : \ker \rho_{c_2}] =: q$. The group $G_1(P)/\ker \rho_{c_2}$ is elementary abelian of order q with a unique jump, say at υ. The Riemann–Hurwitz theorem implies:

$$2g_{F^{\ker \rho_{c_2}}} - 2 = -2q + (\upsilon + 1)(q - 1)$$

and $\upsilon = 1$ if and only if $g_{F^{\ker \rho_{c_2}}} = 0$. $\square$

Corollary 30. *Suppose that $G_1(P) > G_2(P)$. Let i_0 be the index such that $-v_P(f_{i_0}) = m_{i_0} = |G_1(P)|$ and $k(f_{i_0}) = F^{G_1(P)}$ as it is given in remark 17.*

1. *The element f_{i_0} is not needed for the generation of $F^{\ker \rho_{c_j}}$ for every $j > 1$.*
2. *Concerning the structure of the Weierstrass semigroups $H(Q_{i+1})$ given in proposition 24 we have*

$$H(Q_{i+1}) = \left\langle \frac{m_{c_j+1}}{|\ker \rho_{c_{i+1}}|} : 1 \leq j \leq i \right\rangle_{\mathbb{Z}_+},$$

while

$$H(P) = \langle m_{c_j+1} : 1 \leq j \leq n \rangle_{\mathbb{Z}_+}.$$

That is $\left\langle \frac{m_{c_j+1}}{|\ker \rho_{c_{i+1}}|} \right\rangle_{\mathbb{Z}_+} \ni \left| \frac{G_1(P)}{\ker \rho_{c_{i+1}}} \right|$. More precisely, $|G_2(P)| = m_1$, i.e. the order of the second lower ramification group equals to the first pole number and

$$m_r = m_{r-1} + 1.$$

Proof. From corollary 29 we have $F^{G_1(P)}(f_{c_1+1}) = F^{\ker \rho_{c_2}}$ is rational. The element f_{i_0} is a rational function on f_{c_1+1}, this proves the first assertion. Moreover in this case, we can normalize the Artin-Schreier generator f_{c_1+1} for the elementary abelian extension with unique ramification jump, and apply [41, Proposition 3.7.10] such that

$$f_{i_0} = f_{c_1+1}^q - f_{c_1+1},$$

where q equals to $[G_1(P) : \ker \rho_{c_2}]$.

For the second assertion, from corollary 9, $|G_1(P)|$ can result as a pole number from $|\ker \rho_{c_2}|$ since $|\ker \rho_{c_2}|$ divides $|G_1(P)|$. Moreover, from corollary 28 we have that $|G_2(P)| = |\ker \rho_{c_2}|$, while $|\ker \rho_{c_2}| = m_{c_1+1}$ and thus

$$\left|\frac{G_1(P)}{\ker \rho_{c_{i+1}}}\right| \in \left\langle \frac{m_{c_1+1}}{|\ker \rho_{c_{i+1}}|} \right\rangle_{\mathbb{Z}_+}, \text{ for every } 1 \leq i \leq n.$$

Notice that in this case $m_{c_1+1} = m_1$ and that the first mom zero pole number is always a generator.

Finally the last assertion about m_r, comes directly from proposition 2. $\square$

We would like now to discuss the case where $|G_1(P)|$ is a generator of the semigroup. It turns out that this happens if and only if 1 is *not* a ramification jump, i.e. $G_1(P) = G_2(P)$. We have seen that the generators of the semigroup $H(P)$ are of two types:

(1) they are induced by jumps of the representation filtration
(2) $|G_1(P)|$.

Proposition 31. *The number $|G_1(P)|$ is a generator of the Weierstrass semigroup at P if and only if $G_1(P) = G_2(P)$.*

Proof. If $G_1(P) > G_2(P)$, $F^{G_2(P)}$ is rational, $|G_2(P)|$ equals to the first pole number from corollary 30 and since $|G_2(P)|$ divides $|G_1(P)|$, $|G_1(P)|$ cannot be a generator.

For the other direction, assume that $|G_1(P)|$ is not a generator then we will prove that $G_1(P) > G_2(P)$. By our hypothesis, there is a semigroup $H(Q_i)$ where $|G_1(P)|/|\ker \rho_{c_i}|$ is not a generator for some $c_i < r$. Let ν_0 be the first index such that $|G_1(P)|/|\ker \rho_{c_i}|$ is a generator for $i \leq \nu_0$ and $|G_1(P)|/|\ker \rho_{c_{\nu_0+1}}|$ is not a generator for $H(Q_{\nu_0+1})$. We have the following generating sets for the semigroups:

$$H(Q_{\nu_0}) = \left\langle \left|\frac{G_1(P)}{\ker \rho_{c_{\nu_0}}}\right|, \frac{m_{c_j+1}}{|\ker \rho_{c_{\nu_0}}|} : 1 \leq j < \nu_0 \right\rangle_{\mathbb{Z}_+},$$

$$H(Q_{\nu_0+1}) = \left\langle \frac{m_{c_j+1}}{|\ker \rho_{c_{\nu_0+1}}|} : 1 \leq j \leq \nu_0 \right\rangle_{\mathbb{Z}_+},$$

i.e. both semigroups have the same number of generators. According to lemma 22 the semigroup $H(Q_{\nu_0+1})$ is generated by elements of the semigroup $H(Q_{\nu_0})$ multiplied by $[\ker \rho_{c_{\nu_0}} : \ker \rho_{c_{\nu_0+1}}]$ and an extra prime to p generator $\frac{m_{c_{\nu_0}+1}}{|\ker \rho_{c_{\nu_0+1}}|}$, i.e.:

$$H(Q_{\nu_0+1}) = [\ker \rho_{c_{\nu_0}} : \ker \rho_{c_{\nu_0+1}}] \cdot H(Q_{\nu_0}) + \mathbb{Z}_+ \frac{m_{c_{\nu_0}+1}}{|\ker \rho_{c_{\nu_0+1}}|}.$$

In order to finish the proof we need the following

Lemma 32. *Assume that S is a numerical semigroup and E is the semigroup such that $E = p^k S + N\mathbb{Z}_+$, where $(N, p) = 1$. Suppose further that the semigroups, S, E have the same cardinality of minimal generators. Then N is a generator of the semigroup S.*

Proof. This is proposition A.0.15 in the PhD thesis of H. Smith [37]. Notice that the result is proved only for $p^k = p$ but the same proof can be used for the more general case of higher values of k. $\square$

We will now complete the proof of proposition 31 by applying lemma 32. The prime to p generator $N = \frac{m_{c_{\nu_0}+1}}{|\ker \rho_{c_{\nu_0+1}}|}$ should be a generator of $H(Q_{\nu_0})$ but it can not be any of the $\frac{m_{c_j+1}}{|\ker \rho_{c_{\nu_0}}|} : 1 \leq j < \nu_0$ since it is greater of all of them, so the only remaining case is $N = \frac{|G_1(P)|}{|\ker \rho_{c_{\nu_0}}|}$, but since N is prime to p we have $|G_1(P)| = |\ker \rho_{c_{\nu_0}}|$, $N = 1$ and thus $\nu_0 = 1$ and $H(Q_1) = H(Q_2) = \mathbb{Z}_+$, something that contradicts the non-rationality of the field $F^{G_2(P)}$. □

Already from the introduction we saw that Katz–Gabber covers are related to zero p-rank curves. In the next theorem we examine further this connection.

Theorem 33. *The following conditions are equivalent*

1. *the curve X has zero p-rank, and $|\mathscr{A}|$ is a pole number at the point P that stabilizes, where $\mathscr{A}$ is a p-group of automorphisms of X.*
2. *the cover $X \to X/G_1(P)$ is Katz-Gabber and $\mathscr{A} = G_1(P)$.*

Proof. $1 \Rightarrow 2$. By [18, Lemma 11.129] every element of order p fixes exactly one point. This means that $\mathscr{A}$ can be realized as the stabilizer of a point $P \in X$ and that for the cover $X \to X/G_1(P)$, P is the unique totally ramified point. By corollary 9, $|\mathscr{A}| = |G_1(P)|$ is a pole number at P if and only if $X/G_1(P)$ is a rational curve.

$2 \Rightarrow 1$. Use the Deuring-Shafarevich formula [30, eq. (1.1)], or [30, Theorem 2i]. □

For maximal curves and curves equipped with a big action it is known that they have zero p-rank, see for example [11, Corollary 2.5] and [27, first lines of the proof of Proposition 2.5] respectively.

3.2. **Big actions, maximal curves.** Another case that forces $G_1(P)$ not to be a generator is when $|G_1(P)| \geq 2g$. In this case, since $2g$ is the conductor of the semigroup, the element $|G_1(P)|$ can be written as a sum of the generators smaller than the conductor. We would like to notice this situation is related to the theory of *big actions* as this is defined in the work of C. Lehr, M. Matignon [27]. Curves having a big action were studied further by M. Rocher and M. Matignon [28], [32]. Recall that a curve X together with a subgroup G of the automorphism group of X is called a big-action if G is a p-group and

$$\frac{|G|}{g} > \frac{2p}{p-1}.$$

All big actions have the following property [27]:

Proposition 34. *Assume that (X, G) is a big action. There is a unique point P of X such that $G_1(P) = G$, the group $G_2(P)$ is not trivial and strictly contained in $G_1(P)$ and the quotient $X/G_2(P) \cong \mathbb{P}^1$. Moreover, the group G is an extension of groups*

$$0 \to G_2(P) \to G = G_1(P) \xrightarrow{\pi} (\mathbb{Z}/p\mathbb{Z})^v \to 0.$$

Corollary 35. *If (X, G) is a big action, then*

1. *the jumps of $G_1(P)$ are given by theorem 27*
2. *the structure of $H(P)$ is given by corollary 30.*

We now focus on maximal curves. Theorem 33 can be used together with the following

Theorem 36. *Let X be a maximal curve defined over $\bar{\mathbb{F}}_{q^2}$, where q is a p-power. The integer $|G_1(P)|$ is a pole number if and only if $q \leq |G_1(P)|$ at some point P.*

Proof. By [18, Proposition 10.6 (XII)] q is a pole number for every point P. Thus if $q \leq |G_1(P)|$ then q divides $|G_1(P)|$ and $|G_1(P)|$ is a pole number.

For the opposite direction notice that $q = p^s$ for some s and that this s is the rank of nilpotency of the Cartier operator, see [11]. This means that if $|G_1(P)|$ is a pole number then it cannot be less that q, this is a consequence of the minimality of the rank of nilpotency of the Cartier operator. Indeed, according to [42, Corollary 2.7], the rank of the Cartier operator is greater than or equal to the number of gaps that are divisible by p^s; if we were in the case where $|G_1(P)| < p^s$ and $|G_1(P)|$ was a pole number, then $|G_1(P)|$ would divide p^s and thus the rank should then be strictly less than s, a contradiction. □

We thus get a corollary analogue to corollary 35 for the case of maximal curves

Corollary 37. *If X is maximal curve over $\bar{\mathbb{F}}_{q^2}$ and $q \leq |G_1(P)|$, then*

(1) *the jumps of $G_1(P)$ are given by theorem 27*
(2) *the structure of $H(P)$ is given by proposition 24.*

Notice that in the next section, corollary 41, we will show that $H(P)$ is also symmetric, and more precisely a telescopic numerical semigroup.

Remark 38. Under the hypothesis of corollary 37, we will have for m_r, the first pole number at P not divisible by the characteristic, that $m_r = q + 1$ whenever the maximal curve is not $\bar{\mathbb{F}}_{q^2}$ isomorphic with the curve

$$y^q + y = x^m, \ \text{ where } m \mid q + 1.$$

Notice that in any other case $q+1$ is a generator of the Weierstrass semigroup at P, according to [7, Theorem 2.3]. This exceptional generator is called the degree of the Frobenius linear series of the curve; and these are the cases for which this linear series coincide with $|m_r P|$, where m_r is the first not divisible by the characteristic pole number. It is an invariant of the curve at a rational point. For some deep connections with the arithmetic structure of the curve regarding this number the reader can look at [18]. It is also interesting to notice that in this case all the orders of the Frobenius linear series at P are exactly the possible ramification jumps given in proposition 2, while the projective map Φ arising from $|m_r P|$ is an embedding, [18, Theorem 10.7].

The Hasse-Arf theorem for abelian groups gives certain divisibility conditions for the jumps of the ramification filtration. Using theorem 3 restricted on the case of an abelian group $G_1(P)$, these divisibility conditions can be interpreted in terms of the Weierstrass semigroup at P:

Corollary 39 (Hasse–Arf theorem)**.** *Assume that a Katz-Gabber cover has abelian Galois p–group $G_1(P)$. Then the generators of the Weierstrass semigroup that result from the jumps of the representation filtration satisfy:*

$$\frac{m_{c_{i+1}+1}}{|G_{b_{i+2}}|} \equiv \frac{m_{c_i+1}}{|G_{b_{i+1}}|} \bmod p^{\sum_{j=1}^{i} r_j}$$

or

$$m_{c_i+1} \equiv \left|\frac{G_{b_{i+1}}}{G_{b_{i+2}}}\right| m_{c_{i+1}+1} \bmod \left|\frac{G_{b_1}}{G_{b_{i+1}}}\right|,$$

where $p^{r_i} = [G_{b_i} : G_{b_{i+1}}]$ for all $1 \leq i \leq n-1$.

Proof. We will use an equivalent form of Hasse–Arf theorem, see [33]; namely, every two subsequent ramification jumps b_{i+1}, b_i must satisfy:

$$b_{i+1} \equiv b_i \bmod p^{\sum_{j=1}^{i} r_j}, \text{ where } p^{r_i} := [G_{b_i} : G_{b_{i+1}}], \text{ for every } 1 \leq i \leq n-1.$$

Now replace b_i with $\frac{m_{c_i+1}}{|G_{b_{i+1}}|}$ for every $1 \leq i \leq n$ in order to derive the desired result. □

4. A Basis for Holomorphic polydifferentials

In what follows X is always a Katz-Gabber cover with Galois group a p-group. We can construct a basis for the m-holomorphic polydifferentials of X as follows:

Let f_{i_0} be the function generating the rational field $F^{G_1(P)} = k(f_{i_0})$. The function f_{i_0} can be selected so that it has a simple unique pole at infinity which is the restriction of the place P to $k(f_{i_0})$. Let $p^{h_0} = |G_1(P)|$. We observe first that

$$\operatorname{div}(df_{i_0}^{\otimes m}) = \left(-2mp^{h_0} + m \sum_{i=1}^{n} (b_i - b_{i-1})(p^{h_{i-1}} - 1) \right) P, \tag{8}$$

where

$$b_0 = -1,\ p^{h_0} = |G_1(P)|,\ p^{h_i} = |\ker \rho_{c_{i+1}}| = |G_{b_{i+1}}|, \text{ for } i \geq 1.$$

The right hand side of eq. (8) equals to $m(2g_X - 2)P$ by Riemann-Hurwitz formula.

Proposition 40. *For every pole number μ we select a function f_μ such that $(f_\mu)_\infty = \mu P$. The set $\{f_\mu df_{i_0}, \deg \operatorname{div}(f_\mu) \leq 2g_X - 2\}$ is a basis for the space of holomorphic differentials for X. The set $\{f_\mu df_{i_0}^{\otimes m} : \deg \operatorname{div}(f_i) \leq m(2g_X - 2)\}$ is a basis for the space of holomorphic m-polydifferentials of X.*

Proof. All m-holomorphic differentials are of the form $g df_{i_0}^{\otimes m}$. Therefore the condition for being holomorphic is translated to the condition $g \in L(m(2g_X - 2)P)$. Therefore the linear independent elements $f_i df_{i_0}^{\otimes m}$ with $\deg \operatorname{div} f_i = m_i \leq m(2g_X - 2)$ are holomorphic. In order to see that all the holomorphic differentials are of this form, we must count them:

Case $m = 1$. Notice that $\ell((2g_X - 2)P) = g_X$ and from the other hand $\ell((2g_X - 1)P) = g_X$ from the Weierstrass gap theorem [41]. This means that in the interval $[0, 2g_X - 2]$ there are exactly g_X pole numbers, equivalently $2g_X - 1$ is a gap.

Case $m > 1$. Similarly, observe using the Riemann-Roch theorem, that the space of m-holomorphic differentials has dimension

$$\dim L(mW) = m(2g_X - 2) + 1 - g_X = (2m - 1)g_X - 2m + 1.$$

On the other hand the number of f_i such that $\deg \operatorname{div}(f_i) \leq m(2g_X - 2)$ can be computed as follows:

In the interval $[0, 2g_X - 1]$ there are g_X such elements and every number greater than $2g_X$ is a pole number using again the Riemann-Roch theorem. So in the interval $(2g_X - 1, m(2g_X - 2)]$ there are $m(2g_X - 2) - (2g_X - 1) = 2mg_X - 2m - 2g_X + 1$ elements. In total there are $2mg_X - 2m - 2g_X + 1 + g_X = (2m - 1)g_X - 2m + 1$ and this coincides with the dimension of the space of m-holomorphic differentials. $\square$

Corollary 41. *The Weierstrass semigroup at P is symmetric, i.e. $2g_X - 1$ is a gap.*

We have proved in proposition 24 that the elements m_{c_i+1} for $1 \leq i \leq n$ together with the element p^{h_0} generate the Weierstrass semigroup. A numerical semigroup Σ that is not of the form $a\mathbb{Z}_+$ has a minimal element $\kappa(\Sigma)$ called *the conductor* such that all integers $n \geq \kappa(\Sigma)$ are in the semigroup.

Since the semigroup is symmetric we see that $\kappa(H(P)) = 2g_X$, recall that $2g_X - 1$ is a gap in this case and that Riemann-Roch theorem implies that all integers $\geq 2g_X$ are in $H(P)$.

Remark 42. Another way to show that the whole Weierstrass semigroup $H(P)$ is generated exactly by $\Lambda_i := p^{h_i}\lambda_i$ $1 \leq i \leq n$ and $p^{h_0} = |G_1(P)|$, is by using results of A.Brauer [5],[31]. This can be done as follows:

Set $d_{-1} = 0$ and $d_i = \gcd(p^{h_0}, \Lambda_1, \ldots, \Lambda_i) = p^{h_i}$. For the last assertion notice there cannot exist a higher power of p than p^{h_i} dividing all the Λ_i and p^{h_0}, since $\gcd(\lambda_i, p) = 1$.

Let $S = \left\langle p^{h_0}, \Lambda_1, \ldots, \Lambda_n \right\rangle_{\mathbb{Z}_+}$. Recall that for λ_i which is a generator of $H(Q_{i+1})$ we get that

$$\frac{\Lambda_i}{d_i} = \lambda_i \in H(Q_i), \text{ for all } 1 \leq i \leq n,$$

by lemma 7. Then by the theorem of A.Brauer [5], [31] we have that the conductor $\kappa(S)$ equals

$$\kappa(S) = \sum_{k=0}^{n} \left(\frac{d_{k-1}}{d_k} - 1 \right) \Lambda_k + 1 = -p^{h_0} + \sum_{k=1}^{n} \left(p^{h_{k-1}} - p^{h_k} \right) \lambda_k + 1. \tag{9}$$

Since $2g_X - 1$ is a gap for $H(P)$ the semigroup $S = H(P)$ if and only if $\kappa(S) = 2g_X$. This can be checked by using the Riemann-Hurwitz formula for the cover $X \to X/G_1(P)$:
(10)
$$2g_X = 2 - 2p^{h_0} + (p^{h_0} - 1)(\lambda_1 + 1) + (p^{h_1} - 1)(\lambda_2 - \lambda_1) + \cdots + (p^{h_{n-1}} - 1)(\lambda_n - \lambda_{n-1})$$

and by observing that the right hand side of eq. (10) equals $\kappa(S)$ given by eq. (9).

If $\lambda_1 = 1$ then the elements $\Lambda_1, \ldots, \Lambda_n$ generate the whole Weierstrass semigroup since the same argument can be used on the Katz-Gabber cover $F \to F^{G_2(P)}$.

Remark 43. Notice that the involved semigroups $H(Q_i)$ are telescopic, see [19, section 5.4] and [1], for all $1 \leq i \leq n+1$. Since every telescopic numerical semigroup is symmetric this gives us a proof for the fact that $H(Q_i)$ is symmetric for all $1 \leq i \leq n+1$, and not just only for the value $i = n+1$.

5. Galois module structure

The representation theory of p-groups in fields of characteristic p is much more difficult than the corresponding theory in characteristic zero. The notions of *irreducible* and *indecomposable* differ in the modular characteristic world. By the term Galois module structure of a certain G-module we mean analyzing the indecomposable factors together with their multiplicities. This is a difficult task because, unless G is a cyclic p-group, we lack of a classification for the indecomposable G-factors even for the simplest non cyclic case $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

For every any element $f_{c_i+1} \in \{f_{i_0},\ f_{c_1+1}, \ldots, f_{c_n+1}\}$ i.e. f_{c_i+1} corresponds to a generator of the semigroup at P we define a cocycle:

$$\begin{array}{rcl} d_i(\sigma) : G_1(P) & \to & k[f_{c_1+1}, \ldots, f_{c_{i-1}+1}], \\ \sigma & \mapsto & \sigma(f_{c_i+1}) - f_{c_i+1}. \end{array}$$

This cocycle defines an equivalence class in $H^1(G_1(P), k[f_{c_1+1}, \ldots, f_{c_{i-1}+1}])$. More general for every natural number n we consider the set

$$P_{<n} := \{\mu \in \mathbb{N} : \mu \text{ is a pole number}, \mu < n\},$$

and the corresponding vector space

$$V_n := \langle f_\mu : \mu \in P_{<n} \rangle_k = L\big((n-1)P\big).$$

Finally, for every element i in the Weierstrass semigroup $H(P)$ we will denote by g_i the monomial in $f_{c_1+1}, \ldots, f_{c_s+1}$ such that $(g_i)_\infty = iP$, i.e. $g_i = f_{c_1+1}^{a_1} \cdots f_{c_s+1}^{a_s}$, where $a_i \in \mathbb{Z}_+$ such that $i = a_1 m_{c_1+1} + \cdots + a_s m_{c_s+1}$. Notice that we can choose this g_i to be unique modulo multiplication by constants and functions that give rise to pole numbers less than i, see remark 16.

Using the element g_i we can define a cocycle:

$$\begin{array}{rcl} \delta_i : G_1(P) & \to & V_i \\ \sigma & \mapsto & \sigma(g_i) - g_i. \end{array}$$

Notice that for every element $a \in V_{i-1}$ the function $g_i + a$ has unique pole at P and $(g_i)_\infty = (g_i + a)_\infty = iP$ and the cocycle we form by $g_i + a$ is equivalent to δ_i since

$$\sigma(g_i + a) - (g_i + a) = \delta_i(\sigma) + \sigma(a) - a.$$

This means that every linear change of basis in V_i that respects the flag of subspaces V_ν $\nu < i$, induces the same class in cohomology.

Lemma 44. *Given a cocycle $d_i : G \to V_i$, where $k[f_{c_1+1}, \ldots, f_{c_{i-1}+1}]$ is a G-module, we can define an action of G on f_{c_i+1} by:*

$$\sigma(f_{c_i+1}) = f_{c_i+1} + d_i(\sigma).$$

Proof. We have to check that $(\tau\sigma)(f_{c_i+1}) = \tau\big(\sigma f_{c_i+1}\big)$, which is obvious from the cocycle condition. $\square$

Remark 45. The original action on the group $\mathrm{Aut}k[[t]]$ can be recovered from the information given in by the cocycle $d_n(\sigma) \in H^1(G_1(P), k[f_{c_1+1}, \ldots, f_{c_{n-1}+1}])$ by the following formula

$$\sigma(f_r) = \sigma(f_{c_n+1}) = f_{c_n+1} + d_n(\sigma) \Leftrightarrow \sigma(t) = t\,(1 + d_n(\sigma))^{-\frac{1}{m_{c_n+1}}} \tag{11}$$

where $d_n(\sigma)$ has a Laurent expansion in $k((t))$ with pole order less than $m_{c_n+1} = m_r$ see [23].

Remark 46. Notice that for an arbitrary monomial $f_{c_1+1}^{a_1} \cdots f_{c_s+1}^{a_s}$, with $s \leq r$ a positive integer, the cocycle is computed in terms of d_i by the formula

$$\sigma(f_{c_1+1}^{a_1} \cdots f_{c_s+1}^{a_s}) - f_{c_1+1}^{a_1} \cdots f_{c_s+1}^{a_s} = (f_{c_1+1} + d_1)^{a_1} \cdots (f_{c_s+1} + d_s)^{a_s} - f_{c_s+1}^{a_1} \cdots f_{c_s+1}^{a_s}$$
$$= \sum_{\substack{0 \leq \nu_i \leq a_i \\ (\nu_1, \ldots, \nu_s) \neq (a_1, \ldots, a_s)}} \binom{a_1}{\nu_1} \cdots \binom{a_s}{\nu_s} f_{c_1+1}^{\nu_1} \cdots f_{c_s+1}^{\nu_s} d_1^{a_1 - \nu_1} \cdots d_s^{a_s - \nu_s}.$$

Proposition 47. *Denote by $p^{h_0} = |G| = |G_1(P)|$. The module $\Omega_X^{\otimes m}$ is the direct sum of at most*

$$N := \left\lfloor \frac{m(2g-2)}{p^{h_0}} \right\rfloor = -2m + \left\lfloor m \frac{\sum_{i=1}^n (b_i - b_{i-1})(p^{h_{i-1}} - 1)}{p^{h_0}} \right\rfloor$$

direct indecomposable summands.

Proof. We have a representation of the group $G_1(P)$ in terms of lower diagonal matrices in $\Omega_X^{\otimes m} \cong L(m(2g_X - 2)P)$. For an element f in $L(m(2g_X - 2)P)$ we have the function $v_P : L(m(2g_X - 2)P) \to \mathbb{N}$ sending f to $-v_P(f)$ and $v_P(\sigma(f) - f) > v_P(f)$.

Assume that the space $L(m(2g_X - 2)P)$ admits a decomposition

$$L(m(2g_X - 2)P) = \bigoplus W_i$$

as a direct sum of G-modules W_i. We will prove that we can find a basis of elements $e_1, \ldots e_{\dim W_i}$ of W_i that have different valuations. Start from any basis of W_i. If there are two basis elements a, b of W_i such that $v_P(a) = v_P(b)$, then these are, locally at P, of the form

$$a = a_1 \frac{1}{t^v} + \text{higher order terms}, b = b_1 \frac{1}{t^v} + \text{ higher order terms}.$$

Therefore there is an element λ such that $a - \lambda b \neq 0$ has different valuation than a, b, ($\lambda = a_1/b_1$). We replace the element b by the element $a - \lambda b$. Proceeding this way we form the desired basis elements with different valuations. Now,

$$\sigma(e_i) = e_i + b_i(\sigma), \text{ with } b_i(\sigma) = 0 \text{ or } |v_P(b_i(\sigma))| < |v_P(e_i)|$$

and this proves that every direct summand W_i has an upper triangular representation matrix, so it contains at least one invariant element.

Therefore, the number of indecomposable summands is smaller than the number of $G_1(P)$-invariant elements. The space of invariant elements has a basis of elements of the form $f_{i_0}^j$ such that $-v_P(f_{i_0}^j) \leq m(2g-2)$, and the result follows. □

Corollary 48. *If $|G_1(P)| > m(2g-2)$ then the module $H^0(X, \Omega^{\otimes m})$ is indecomposable. In particular the space of holomorphic differentials $H^0(X, \Omega)$ is indecomposable for a curve X that admits a big action.*

Proof. If $|G_1(P)| > m(2g-2)$ then the only $G_1(P)$ invariant elements belonging to $L(2m(g-1))$ are the constants, thus this space includes a unique copy of the one dimensional irreducible representation, so is indecomposable. The assertion for curves admitting big action comes directly now from their definition. □

Remark 49. Let G be a p-group. The second author [25], observed that the tangent space of the global deformation functor $H^1(G, \mathscr{T}_X)$ can be computed in terms of coinvariants of 2-holomorphic differentials by:

$$H^1(G, \mathscr{T}_X) = \Omega_X^{\otimes 2} \otimes_{K[G]} K, \tag{12}$$

where $\Omega_X^{\otimes 2} := \Omega_X(2)$. Once the structure 2-holomorphic differentials is established the computation of coinvariants is a problem of linear algebra. Providing a closed formula in terms of the actions like we did in [20, Corollary 4.3] is still quite complicated and requires more effort. Notice also that using this approach we can compute the dimension $H^1(G, \mathscr{T}_{K[[t]]})$ of the local deformation functor in the sense of Bertin and Mézard [3] using their local global principle:

$$\Omega_X^{\otimes 2} \otimes_{K[G]} K = H^1(G, \mathscr{T}_X) = H^1(X/G, \pi_*^G(\mathscr{T}_X)) \oplus H^1(G, \mathscr{T}_{K[[t]]}). \tag{13}$$

For the dimension of the space $H^1(X/G, \pi_*^G(\mathscr{T}_X))$ we have an explicit formula, namely

$$\dim_K H^1(X/G, \pi_*^G(\mathscr{T}_X)) = 3g_{X/G} - 3 + \left\lceil \frac{\delta}{p^n} \right\rceil, \tag{14}$$

where δ is the local contribution to the different at the unique ramification point [24, eq. (38)]. In our case the constant δ can be computed as follows:

$$\delta := \sum_{i=0}^{\infty} (|G_i(P)| - 1) = (p^{h_0}-1)(\lambda_1+1) + (p^{h_1}-1)(\lambda_2-\lambda_1) + \cdots + (p^{h_{n-1}}-1)(\lambda_n - \lambda_{n-1})$$

An alternative approach for $m \geq 2$ is to use the p-rank representation. The divisor $D = df_{i_0}$ is a $G_1(P)$-invariant effective canonical divisor on X, see also [22, Lemma 3.4]. The space $H_{(m-1)D} = H^0(X, \Omega_X^{\otimes m}) = H^0(X, \Omega_X((m-1)D)$. There is a decomposition

$$H_{(m-1)D} = H^{\mathrm{s}}_{(m-1)D} \oplus H^{\mathrm{n}}_{(m-1)D}$$

where $H^{\mathrm{s}}_{(m-1)D}$ and $H^{\mathrm{n}}_{(m-1)D}$ are the spaces of semisimple and nilpotent differentials with respect to the Cartier operator in $H_{(m-1)D}$, see [22], [34], [43], [30], [38]. Since $(m-1)D$ is $G_1(P)$-invariant the above decomposition is a decomposition of $k[G_1(P)]$-modules. While little seems to be known about the $k[G_1(P)]$-module $H^{\mathrm{n}}_{(m-1)D}$, the $k[G_1(P)]$-module $H^{\mathrm{s}}_{(m-1)D}$ has been studied by many authors ([30], [21], [4], [38]) and is called the *p-rank representation*. As $G_1(P)$ is a p-group the only irreducible $k[G_1(P)]$-module is the trivial representation k and has projective cover $k[G_1(P)]$ [35, 15.6].

Proposition 50. *For the case of holomorphic differentials the p-rank of the Jacobian which equals to the dimension $\dim_k \Omega_X(0)^s$ is zero. For $m > 2$, the p-rank representation of $G_1(P)$ with respect to $(m-1)D = (m-1)df_{i_0}$ is zero. Therefore our space of holomorphic differentials consists only of nilpotent elements with respect to the Cartier operator for every $m \geq 1$.*

Proof. The $m = 1$ case follows from theorem 33.

For the $m > 2$ case we observe that since $(m-1)D \neq 0$ and its support contains all the ramification points. We have by [30, Theorem 1], [38, 4.5] that $H_{(m-1)}D^{\mathrm{s}}$ is a free $k[G_1(P)]$-module of rank $(m-1)D_{\mathrm{red}} - 1 = 0$. Observe that from [30, Theorem 1] the p-rank of the projective line is zero and the reduced divisor of $(m-1)D$ is just the unique ramified point and has cardinality one. □

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FREIE UNIVERSITÄT BERLIN, INSTITUT FÜR MATHEMATIK, ARNIMALLEE 3, 14195 BERLIN, GERMANY
E-mail address: `skaran@zedat.fu-berlin.de`

UNIVERSITY OF ATHENS, DEPARTMENT OF MATHEMATICS, PANEPISTIMIOUPOLIS, 15784 ATHENS, GREECE
E-mail address: `kontogar@math.uoa.gr`