GROUP ACTIONS ON CYCLIC COVERS OF THE PROJECTIVE LINE

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Abstract. We use tools from combinatorial group theory in order to study actions of three types on groups acting on a curve, namely the automorphism group of a compact Riemann surface, the mapping class group acting on a surface (which now is allowed to have some points removed) and the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ in the case of cyclic covers of the projective line.

1. Introduction

There is a variety of groups that can act on a Riemann surface/algebraic curve over $\mathbb{C}$; the automorphism group, the mapping class group (here we might allow punctures) and if the curve is defined over $\bar{\mathbb{Q}}$ the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is also acting on the curve. Understanding the above groups is a difficult problem and these actions provide information on both the curve and the group itself. For all the groups mentioned above, the action can often be understood in terms of linear representations, by allowing the group to act on vector spaces and modules related to the curve itself, as the (co)homology groups and section of holomorphic differentials.

For a compact Riemann surface X the automorphism group $\mathrm{Aut}(X)$, consists of all invertible maps $X \to X$ in the category of Riemann surfaces.

A compact Riemann surface minus a finite number of punctures, can be also seen as a conected, orientable topological surface and the mapping class group $\mathrm{Mod}(X)$ can be considered acting on X. The mapping class group is the quotient

$$\mathrm{Mod}(X) = \mathrm{Homeo}^+(X)/\mathrm{Homeo}^0(X),$$

where $\mathrm{Homeo}^+(X)$ is the group of orientation preserving homeomorphisms of X and $\mathrm{Homeo}^0(X)$ is the connected component of the identity in the compact-open topology.

These actions of the above mentioned three types of groups seem totally unrelated and come from different branches of Mathematics. Recent progress in the branch of "Arithmetic topology" provide us with a complete different picture. First the group $\mathrm{Aut}(X)$ can be seen as a subgroup of $\mathrm{Mod}(X)$ consisting of "rigid" automorphisms.

Y. Ihara in [8], [9], proposed a method to treat elements in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ as elements in the automorphism group of the profinite free group. This construction is similar to the realization of braids as automorphisms of the free group. This viewpoint of elements in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ as "profinite braids" allows us to give a series of Galois representations similar to classical Braid representations.

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In this article we will focus on curves which are cyclic ramified covers of the projective line. These curves form some of the few examples of Riemann surfaces, where explicit computations can be made. The first author considered the automorphism group of these in [12] and also considered the field of moduli versus field of definition in [1], [13].

A ramified cover of the projective curve reduces to a topological cover, when the branch points are removed. By covering map theory these covers correspond to certain subgroups of the fundamental group of the projective line with branch points removed, which is the free group.

Theorem 1. *Consider a set Σ of s points in the projective line and let α be the winding number with respect to these points (see eq. (12) for a precise definition). Set $X_s = \mathbb{P}^1 - \Sigma$ and fix generators for the fundamental group of X_s, that is $\pi(X_s, x_0) = \langle x_1, \ldots, x_{s-1} \rangle$. In table 1 we give the computation of generators for the open curves involved in this article. The curves on the third column correspond to the quotients of the universal covering space of X_s by the groups of the first column.*

Group	Generators	Curve	Galois group	Homology
F_{s-1}	$x_1, \ldots, x_{s-1}$	$\tilde{X}_s$	F_{s-1}	F_{s-1}/F'_{s-1}
F'_{s-1}	$[x_i, x_j], i \neq j$	Y	F_{s-1}/F'_{s-1}	F'_{s-1}/F''_{s-1}
R_0	$x_1^i x_j x_1^{-i+1}, {}_{2 \leq j \leq s-1}^{i \in \mathbb{Z}}$	C_s	$\mathbb{Z}$	R_0/R'_0
R_n	$x_1^i x_j x_1^{-i+1}, {}_{2 \leq j \leq s-1}^{0 \leq i \leq n-2}$ $x_1^{n-1} x_j, 1 \leq j \leq s-1$	Y_n	$\mathbb{Z}/n\mathbb{Z}$	R_n/R'_n

TABLE 1. Generators and homology

The homology groups for the cyclic covers C_s (resp. Y_n) can be seen as Galois modules over $\mathbb{Z}$ (resp. $\mathbb{Z}/n\mathbb{Z}$) as follows:

$$\begin{aligned} H_1(C_s, \mathbb{Z}) &= R_0/R'_0 = \mathbb{Z}[\mathbb{Z}]^{s-2} = \mathbb{Z}[t, t^{-1}]^{s-2} \\ H_1(Y_n, \mathbb{Z}) &= R_n/R'_n = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]^{s-2} \bigoplus \mathbb{Z}. \end{aligned} \tag{1}$$

The action of the braid group on $H_1(C_s, \mathbb{Z})$ give rise to the Burau representation.

Similar to the above we have that $H_1(C_s, \mathbb{Z}_\ell) = \mathbb{Z}_\ell[\mathbb{Z}]^{s-2}$ but in order to have an action of the absolute Galois group, a larger space is required, namelly the completed group algebra $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]^{s-2}$. In this way the pro-$\ell$ Burau representation can be defined:

$$\rho_{\mathrm{Burau}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_{s-2}(\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]).$$

Let g be a generator of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. The complete curve $\bar{Y}_n$ has homology

$$H_1(\bar{Y}_n, \mathbb{Z}) = J_{\mathbb{Z}/n\mathbb{Z}}^{s-2},$$

where $J_{\mathbb{Z}/n\mathbb{Z}} = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]/\langle \sum_{i=0}^{n=1} g^i \rangle$, is the co-augmentation module of $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$.

The later space when tensored with $\mathbb{C}$ gives a decomposition

$$H_1(\bar{Y}_n, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\nu=1}^{n-1} V_\nu,$$

where each V_ν is the $s-2$-dimensional eigenspace corresponding to eigenvalue $e^{\frac{2\pi i \nu}{n}}$. Each space V_ν gives rise to a representation of the braid group B_s, which is the reduction of the Burau representation at $t \mapsto e^{\frac{2\pi i \nu}{n}}$.

If $n = \ell^k$ then a similar reduction process can be applied to the pro-ℓ Burau representation. We consider the $\ell^k - 1$ non-trivial roots $\zeta_1, \ldots, \zeta_{\ell^k-1}$ of unity in the algebraically closed field $\bar{\mathbb{Q}}_\ell$. We have

$$\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]^{s-2} \otimes_{\mathbb{Z}_\ell} \bar{\mathbb{Q}}_\ell = \bigoplus_{\nu=1}^{\ell^k-1} V_\nu,$$

which after reducing $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]] \to \mathbb{Z}_\ell[\mathbb{Z}_\ell/\ell^k\mathbb{Z}_\ell] = \mathbb{Z}_\ell[\mathbb{Z}/\ell^k\mathbb{Z}]$ sending $t \mapsto \zeta_\nu$ gives rise to the representation in V_ν. The modules V_ν in the above decomposition are only $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]$-modules and $\ker N$-modules, where $N : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^*$ is the pro-ℓ cyclotomic character.

Several parts of the above theorem are handled in different parts of the article. Namelly the computations in table 1 as well as the decompositions of homology groups given in eq. (1) are discussed in section 4. Cyclic covers with infinite Galois group lead to the Burau representation which is discussed in 4.2.

In section 4.3 we give a pro-ℓ version of the analogon of a Burau representation and we can give it in matrix form by the following:

Theorem 2. *For $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $1 \leq i \leq s-1$ we have that $\sigma(x_i) = w_i(\sigma) x_i^{\chi(\sigma)} w_i(\sigma)^{-1}$, where $\chi(\sigma)$ is the cyclotomic character $\chi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^*$. Let t be a topological generator of $\mathbb{Z}_\ell$ written multiplicative, that is $\mathbb{Z}_\ell \cong \langle t^\alpha, \alpha \in \mathbb{Z}_\ell \rangle$. Let us write*

$$w_i(\sigma) = B_i(\sigma) x_1^{a_{1,i}(\sigma)} \cdots x_{s-1}^{a_{s-1,i}(\sigma)}, \qquad a_{\nu,i}(\sigma) \in \mathbb{Z}_\ell,$$

where $B_i(\sigma) \in R_0/R_0'$ is expressed as

$$B_i(\sigma) = \beta_2^{b_{2,i}(\sigma)} \cdots \beta_{s-1}^{b_{s-1,i}(\sigma)} C,$$

with $b_{i,j}(\sigma) \in \mathbb{Z}_\ell$ and $C \in R_0'$. Let t be a topological generator of $\mathbb{Z}_\ell$. The matrix representation of ρ_{Burau} with respect to the basis $\beta_j = x_j x_i^{-1}$, $j = 2, \ldots, s-1$ has the following form:

$$\rho_{\mathrm{Burau}}(\sigma) = \frac{t^{\chi(\sigma)} - 1}{t-1} L(\sigma) + \left(1 - t^{\chi(\sigma)}\right) M(\sigma) D(\sigma) + \left(1 - t^{\chi(\sigma)}\right) K(\sigma),$$

where L, M, K are $(s-2) \times (s-2)$ matrices given by

$$L(\sigma) = \mathrm{diag}\left(t^{\sum_{\nu=1}^{s-1} a_{\nu,2}(\sigma)}, \ldots, t^{\sum_{\nu=1}^{s-1} a_{\nu,s-2}(\sigma)} \right)$$

$$M(\sigma) = \begin{pmatrix} \Gamma(a_{2,2}) \cdot t^{a_{1,2}(\sigma)} & \cdots & \Gamma(a_{s,s-1}) \cdot t^{a_{1,s-1}(\sigma)} \\ \Gamma(a_{3,2}) \cdot t^{a_{1,2}(\sigma)+a_{2,2}(\sigma)} & \cdots & \Gamma(a_{3,s-1}) \cdot t^{a_{1,s-1}(\sigma)+a_{2,3}(\sigma)} \\ \vdots & & \vdots \\ \Gamma(a_{s-2,2}) \cdot t^{a_{1,2}(\sigma)+\cdots+a_{s-1,2}(\sigma)} & \cdots & \Gamma(a_{s-2,s-1}) \cdot t^{a_{1,s-1}(\sigma)+\cdots+a_{s-1,s-1}(\sigma)} \end{pmatrix}$$

$$K(\sigma) = \begin{pmatrix} b_{2,2}(\sigma) & b_{2,3}(\sigma) & \cdots & b_{2,s-1}(\sigma) \\ b_{3,2}(\sigma) & b_{3,3}(\sigma) & \cdots & b_{3,s-1}(\sigma) \\ \vdots & \vdots & & \vdots \\ b_{s-1,2}(\sigma) & b_{s-1,3}(\sigma) & \cdots & b_{s-1,s-1}(\sigma) \end{pmatrix}.$$

In the above theorem the term

$$\Gamma(a) := (t^a - 1)/(t-1)$$

for $a \in \mathbb{Z}_\ell$, is defined in lemma 18.

We would like to point out that the space $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]^{s-2}$, contains information of all covers $\bar{Y}_{\ell^k}$ for all $k \in \mathbb{N}$, and equals to the étale homology of a curve $\tilde{Y}$, which appears as a $\mathbb{Z}_\ell$-cover of the projective line, minus the same set of points removed. Going back from the arithmetic to topology we can say that the classical discrete Burau representation can be recovered by all representations of finite cyclic covers $\bar{Y}_n$, since we can define the inverse limit of all mod n representations obtaining the B_s-module $\mathbb{Z}[[\hat{\mathbb{Z}}]]^{s-2}$. This B_s-module in turn contains $\mathbb{Z}[\mathbb{Z}]^{s-2}$ as a dense subset. The computations involving the closed curves $\bar{Y}_n$ are discussed in section 5.

In [17] C. McMullen considered unitary representations of the braid group acting on global sections of differentials of cyclic covers of the projective line. His result can be recovered by our homological computations by dualizing. This approach was also mentioned in this article [17, p. 914 after th. 5.5.]. We believe that the details of this computation are worth studying and are by no means trivial. Also the homology approach allows us to study the pro-ℓ analogon according to Ihara's point of view.

Finally in section 5.2.1 we see how the analogon of the homology intersection pairing can be interpreted as an intersection pairing using the Galois action on the Weil pairing for the Tate module. For a free $\mathbb{Z}$ (resp. $\mathbb{Z}_\ell$)-module of rank $2g$, endowed with a symplectic pairing $\langle \cdot, \cdot \rangle$ the symplectic group is defined as

$$\mathrm{Sp}(2g, \mathbb{Z}) = \{M \in \mathrm{GL}(2g, \mathbb{Z}) : \langle Mv_1, Mv_2 \rangle = \langle v_1, v_2 \rangle\}$$

and the generalized symplectic group is defined as

$$\mathrm{GSp}(2g, \mathbb{Z}) = \{M \in \mathrm{GL}(2g, \mathbb{Z}_\ell) : \langle Mv_1, Mv_2 \rangle = m\langle v_1, v_2 \rangle, \text{ for some } m \in \mathbb{Z}_\ell^*\}.$$

In the topological setting the pairing is the intersection pairing and we have the following representation

$$\rho : B_{s-1} \to \mathrm{Sp}(2g, \mathbb{Z})$$

We employ properties of the Weil pairing in order to show that we have a representation

$$\rho' : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}(2g, \mathbb{Z}_\ell)$$

as an arithmetic analogon of the braid representation ρ.

2. On Artin and Ihara representations

2.1. **Artin representation.** It is known that the braid group can be seen as an automorphism group of the free group F_{s-1} in terms of the Artin representation. More precisely the group B_{s-1} can be defined as the subgroup of $\mathrm{Aut}(F_{s-1})$ generated by the elements σ_i for $1 \leq i \leq s-2$, given by

$$\sigma_i(x_k) = \begin{cases} x_k & \text{if } k \neq i, i+1, \\ x_i x_{i+1} x_i^{-1} & \text{if } k = i, \\ x_i & \text{if } k = i+1. \end{cases}$$

Topologically, the disc with $s-1$ points removed is the same with the projective line with the $s-1$ points and infinity removed, that is they have the isomorphic fundamental groups. Indeed, the free group F_{s-1} is the fundamental group of X_s defined as

$$X_s = \mathbb{P}^1 - \{P_1, \ldots, P_{s-1}, \infty\}. \tag{2}$$

In this setting the group F_{s-1} is given as:

$$F_{s-1} = \langle x_1, \dots, x_s | x_1 x_2 \cdots x_s = 1 \rangle, \tag{3}$$

the elements x_i correspond to homotopy classes of loop circling once clockwise around each removed point P_i, and distinguish the homotopy class $y = x_s$ of the loop circling around infinity.

Remark 3. Notice that not only B_{s-1} acts on F_{s-1} but also B_s acts on F_{s-1}. Indeed, for the extra generator $\sigma_{s-1} \in B_s - B_{s-1}$ we define

$$\sigma_{s-1}(x_i) = x_i \qquad \text{for } 1 \leq i \leq s-2 \tag{4}$$

$$\sigma_{s-1}(x_{s-1}) = x_{s-1} x_s x_{s-1}^{-1} = x_{s-2}^{-1} x_{s-3}^{-1} \cdots x_1^{-1} x_{s-1}^{-1} \tag{5}$$

and using eq. (4), (5) we compute

$$\sigma_{s-1}(x_s) = \sigma_{s-1}(x_{s-1}^{-1} \cdots x_1^{-1}) = \sigma_{s-1}(x_{s-1})^{-1} \left(x_{s-2}^{-1} \cdots x_1^{-1} \right) = x_{s-1}.$$

2.2. **Ihara representation.** We will follow the notation of [11]. Y. Ihara, by considering the étale (pro-ℓ) fundamental group of the space $\mathbb{P}^1_{\bar{\mathbb{Q}}} - \{P_1, \dots, P_{s-1}, \infty\}$, with $P_i \in \mathbb{Q}$, introduced the monodromy representation

$$\mathrm{Ih}_S : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\mathfrak{F}_{s-1}),$$

where $\mathfrak{F}_{s-1}$ is the pro-ℓ completion of the free group F_{s-1}. Here the group $\mathfrak{F}_{s-1}$ admits a presentation, similar to eq. (3),

$$\mathfrak{F}_{s-1} = \langle x_1, \dots, x_s | x_1 x_2 \cdots x_s = 1 \rangle. \tag{6}$$

The image of the Ihara representation is inside the group

$$\tilde{P}(\mathfrak{F}_{s-1}) := \left\{ \sigma \in \mathrm{Aut}(\mathfrak{F}_{s-1}) | \sigma(x_i) \sim x_i^{N(\sigma)} (1 \leq i \leq s) \text{ for some } N(\sigma) \in \mathbb{Z}_\ell^* \right\},$$

where $\sim$ denotes the conjugation equivalence. This group is the arithmetic analogon of the Artin representation of ordinary (pure) braid groups inside $\mathrm{Aut}(F_{s-1})$. Notice that the exponent $\sigma(x_i) \sim x_i^{N(\sigma)}$ depends only on σ and not on x_i. Moreover the map

$$N : \tilde{P}(\mathfrak{F}_{s-1}) \to \mathbb{Z}_\ell^*$$

is a group homomorphism and $N \circ \mathrm{Ih}_S : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^*$ coincides with the cyclotomic character χ_ℓ.

Remark 4. As in remark 3 the relation $x_1 \cdots x_{s-1} x_s = 1$ implies that $\tilde{P}(\mathfrak{F}_{s-1})$ also acts on the free group $\mathfrak{F}_s$ since $x_s = (x_1 \cdots x_{s-1})^{-1}$.

In this setting an element $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ can be seen acting on the topological generators $x_1, \dots, x_{s-1}$ of the free group by

$$\sigma(x_i) = w_i(\sigma) x_i^{N(\sigma)} w_i(\sigma)^{-1}. \tag{7}$$

Moreover, by normalizing by an inner automorphism we might assume that $w_1(\sigma) = 1$.

2.3. **Similarities.** For understanding representations of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the theory of coverings of $\mathbb{P}^1_{\bar{\mathbb{Q}}} - \{0,1,\infty\}$ is enough, by Belyi's theorem, [3]. On the other hand the study of topological covers of $\mathbb{P}^1_{\bar{\mathbb{Q}}} - \{0,1,\infty\}$ is not very interesting; both groups B_2 and B_3 which can act on covers of $\mathbb{P}^1_{\bar{\mathbb{Q}}} - \{0,1,\infty\}$ are not a very interesting braid groups. In order to seek out similarities between the Artin and Ihara representation, we will study covers with more than three points removed. Notice that when the number s of points we remove is $s > 3$, then we expect that their configuration might also affect our study.

Moreover elements in the braid group are acting like elements in the mapping class group of the punctured disk i.e. on the projective line minus s-points. The braid group acts like the symmetric group on the set of removed points Σ and acts like a complicated homeomorphism on the complement D_{s-1} of the $s-1$ points.

Since we have assumed $\Sigma \subset \mathbb{Q}$ the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ does not permute Σ and corresponds to the notion of pure braids. But $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the complement $\mathbb{P}^1_{\bar{\mathbb{Q}}} - \Sigma$ in a quite mysterious way.

Knot theorists study braid groups representations, in order to provide invariants of knots (after Markov equivalence, see [22, III.6 p.54]) and number theorists study Galois representations in order to understand the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Both kind of representations are important and bring knot and number theory together whitin the theory of arithmetic topology.

3. On the fundamental group of cyclic covers

Let $\pi : Y \to \mathbb{P}^1$ be a ramified Galois cover of the projective line ramified above the set $\Sigma = \{P_1, \ldots, P_s\} \subset \mathbb{P}^1$. The open curve $Y_0 = Y - \pi^{-1}(\Sigma)$ is then a topological cover of $X_s = \mathbb{P}^1 - \Sigma$ and can be seen as a quotient of the universal covering space $\tilde{X}_s$ by the free subgroup $R_0 = \pi_1(Y_0, y_0)$ of the free group $\pi_1(X_s, x_0) = F_{s-1}$ (resp. pro-ℓ free group $\mathfrak{F}_{s-1}$), where $s = \#\Sigma$. We will employ the Reidemeister Schreier method, algorithm [5, chap. 2 sec. 8],[16, sec. 2.3 th. 2.7] in order to compute the group R_0.

3.1. **Schreier' s Lemma.** Let $F_{s-1} = \langle x_1, \cdots, x_{s-1} \rangle$ be the free group with basis $X = \{x_1, \cdots, x_{s-1}\}$ and let H be a subgroup of of F_{s-1}.

A (right) **Schreier Transversal** for H in F_{s-1} is a set $T = \{t_1 = 1, \cdots, t_n\}$ of reduced words, such that each right coset of H in F_{s-1} contains a unique word of T (called a representative of this class) and all initial segments of these words also lie in T. In particular, 1 lies in T (and represents the class H) and $Ht_i \neq Ht_j$, $\forall i \neq j$. For any $g \in F_{s-1}$ denote by $\overline{g}$ the element of T with the property $Hg = H\overline{g}$.

If $t_i \in T$ has the reduced decomposition $t_i = x_{i_1}^{e_1} \cdots x_{i_k}^{e_k}$ (with $i_j = 1, \ldots, s-1$, $e_j = \pm 1$ and $e_j = e_{j+1}$ if $x_{i_j} = x_{i_{j+1}}$) then for every word t_i in T we have that

$$t_i = x_{i_1}^{e_1} \cdots x_{i_k}^{e_k} \in T \Rightarrow 1, x_{i_1}^{e_1}, x_{i_1}^{e_1} x_{i_2}^{e_2}, \ldots, x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_k}^{e_k} \in T. \tag{8}$$

Lemma 5. *Let T be a right Schreier Transversal for H in F_{s-1} and set $\gamma(t,x) := tx\overline{tx}^{-1}$, $t \in T$, $x \in X$ and $tx \notin T$. Then H is freely generated by the set*

$$\{\gamma(t,x) | \gamma(t,x) \neq 1 \rangle\}. \tag{9}$$

3.2. **Automorphisms of Free groups acting on subgroups.** If $R_0 = \pi_1(Y_0, y_0)$ is a characteristic subgroup of $F_{s-1} = \pi_1(\mathbb{P}^1 - \Sigma)$ (resp. of $\mathfrak{F}_{s-1}$ in the pro-ℓ case) then it is immediate that the Artin (resp. Ihara) representation can be lifted to an action on R_0.

In particular, in the discrete case, since the cover $\pi : Y \to \mathbb{P}^1$ is Galois we have that $R_0 \lhd F_{s-1}$, representation is expressed in terms of conjugation we have a well defined action of the Braid group on R_0.

The same argument applies for the kernel of the norm map in the Ihara case, that is since the pro-ℓ completion of R_0 is a normal subgroup of $\mathfrak{F}_{s-1}$, every element in $\mathrm{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})$ with cyclotomic character 1 defines an action on the pro-ℓ completion of $\pi_1(Y_0, y_0)$.

This in accordance with a result of J. Birman and H. Hilden [4, th. 5], which in the case of cyclic coverings $\pi : C \to (\mathbb{P}^1 - \Sigma)$, relates the subgroup $\mathrm{Mod}_\pi(C)$ of the mapping class group of C consisted by the fiber preserving automorphisms, the Galois group $\mathrm{Gal}(C/\mathbb{P}^1)$ and the mapping class group $\mathrm{Mod}(\mathbb{P}^1 - \Sigma)$ of $\mathbb{P}^1 - \Sigma$ in terms of the quotient

$$\mathrm{Mod}_\pi(C)/\mathrm{Gal}(C/\mathbb{P}^1) = \mathrm{Mod}(\mathbb{P}^1 - \Sigma).$$

For example when Y is the covering coresponding to the commutator group F'_{s-1}, then $\mathrm{Gal}(Y/X_s) \cong F_{s-1}/F'_{s-1} = H_1(X, \mathbb{Z})$. Therefore, the latter space is acted on by the group of automorphisms, and the braid group B_s. In the same way the pro-ℓ completion $H_1(Y, \mathbb{Z}_\ell) = H_1(Y, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is acted on by the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

3.3. **Automorphisms of curves.** For the case of automorphisms of curves, where the Galois cover $\pi : Y \to \mathbb{P}^1$ has Galois group H, we consider the short exact sequence

$$1 \to R_0 \to F_{s-1} \to H \to 1.$$

We see that there is an action of H on R_0 modulo inner automorphisms of R_0 and in particular a well defined action of H on $R_0/R'_0 = H_1(Y_0, \mathbb{Z})$. Therefore the space $H_1(Y_0, \mathbb{Z})$ can be seen as a direct sum of indecomposable $\mathbb{Z}[H]$-modules.

Remark 6. A cyclic cover X given in eq. (21) might have a bigger automorphism group then the cyclic group of order n, if the roots $\{b_i, 1 \leq i \leq s\}$ form a special configuration. Notice also that if the number s of branched points satisfies $s > 2n$ then the automorphism group G fits in a short exact sequence

$$1 \to \mathbb{Z}/n\mathbb{Z} \to G \to H \to 1 \tag{10}$$

where H is a subgroup of $\mathrm{PGL}(2, \mathbb{C})$ [12, prop. 1]. The first author in [12] classified all such extensions.

Observe that the action of the mapping class group of homology is of topological nature and hence independent of the special configuration of the roots b_i. If these roots have a special configuration, then elements of the mapping class group become automorphisms of the curve. This phenomenon is briefly explained on page 895 of [17].

Similarly, the action of elements of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which keeps invariant the set of branch points $\{b_i : 1 \leq i \leq s\}$ on homology is the same for all cyclic covers. For certain configurations of the branch points, elements of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ can be seen as automorphism of the curve.

If the branch locus $\{b_i : 1 \leq i \leq s\}$ is invariant under the group H then $H_1(X, \mathbb{Z})$ is a $\mathbb{Z}[G]$ module, where G is an extension of H with kernel $\mathbb{Z}/n\mathbb{Z}$ given by eq. (10).

3.4. **Adding the missing punctures.** Let us now relate the group $R = \pi_1(Y, y_0)$ corresponding to the complete curve Y with the group R_0 corresponding to the open curve $Y_0 = Y - \pi^{-1}(\Sigma)$. We know that the group R_0 admits a presentation

$$R_0 = \langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_s | \gamma_1\gamma_2 \cdots \gamma_s \cdot [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1\rangle,$$

where g is the genus of Y. The completed curve Y has a fundamental group which admits a presentation of the form

$$\begin{aligned} R &= \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g | [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1\rangle \\ &= \frac{R_0}{\langle \gamma_1, \ldots, \gamma_s\rangle}. \end{aligned}$$

There is the following short exact sequence relating the two homology groups:

$$\begin{tikzcd} 0 \arrow[r] & \langle \gamma_1, \ldots, \gamma_s\rangle \arrow[r] & H_1(Y_0, \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H_1(Y, \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 0 \\ & & R_0/R_0' \arrow[r] & R/R' = R_0/R_0'\langle \gamma_1, \ldots, \gamma_s\rangle & \end{tikzcd} \tag{11}$$

Convention 7. Given $e_1, \ldots, e_t$ elements inside a group E we will denote by $\langle e_1, \ldots, e_t\rangle$ the closed normal group generated by these elements. In the case of usual groups the extra "closed" condition is automatically satisfied, since these groups have the discrete topology. So the "closed group" condition has a non-trivial meaning only in the pro-ℓ case.

The open curve is a quotient of the universal covering space $\tilde{X}_s$ by R_0 and the closed curve is given as the quotient of the hyperbolic space $\mathbb{H}/R$,

$$\begin{tikzcd} \tilde{X}_s \arrow[d, "R_0"'] & \mathbb{H} \arrow[d, "R"] \\ Y_0 \arrow[r, hook] & Y \end{tikzcd}$$

For the sake of simplicity we will write only the pro-ℓ case and the case of usual discrete groups can be treated in a similar way, one has to replace all $\mathbb{Z}_\ell$ with $\mathbb{Z}$ in the discrete case.

Suppose that the curve Y (resp. Y_0) is a Galois cover of $\mathbb{P}^1$ (resp. $\mathbb{P}^1 - \Sigma$) with Galois group H. The elements $\gamma_1, \ldots \gamma_s$ are fixed by some $1 \neq g \in H$. Indeed, every such element circles around a branch point, so it is fixed by an element of the Galois group H. The construction of the fundamental group as a quotient is in accordance to the fact that the action of the automorphism group of a compact Riemann surface on the homology is faithful [6, V.3 p.269], so the fixed elements have to be factored out. More precisely we can compute the group Γ as follows: The elements $x_1, \ldots, x_s$ correspond to small paths circling around each point in S. Therefore we can take $\gamma_i = x_i^{e_i}$, where e_i is the ramification index in the branched cover $Y \to \mathbb{P}^1$.

So if a group acts on Y_0, then this action can be extended to an action of $R_0/\langle \gamma_1, \ldots, \gamma_s\rangle$ if and only if the group keeps invariant $\langle \gamma_1, \ldots, \gamma_s\rangle$.

4. Examples- Curves with punctures

Definition 8. Consider the projection

$$0 \to I \to H_1(X_s, \mathbb{Z}) \xrightarrow{\alpha} \mathbb{Z} \to 0$$

and let C_s be the curve given as quotient Y/I, so that $\mathrm{Gal}(C_s/X_s) = \mathbb{Z}$. The map α is the winding number map which can be defined both on the fundamental group and on its abelianization by: $(1 \leq i_1, \ldots, i_t \leq s, \ell_{i_1}, \ldots, \ell_{i_t} \in \mathbb{Z})$

$$\alpha : \pi_1(X_s, x_0) \longrightarrow \mathbb{Z} \qquad x_{i_1}^{\ell_{i_1}} x_{i_2}^{\ell_{i_2}} \cdots x_{i_t}^{\ell_{i_t}} \mapsto \sum_{\mu=1}^{t} \ell_{i_\mu}. \tag{12}$$

The following map is a pro-ℓ version of the w-map defined in eq. (12). Let $\mathfrak{F}_{s-1}$ be the free pro-ℓ group in generators $x_1, \ldots, x_{s-1}$. Consider the map

$$\alpha : \mathfrak{F}_{s-1} \to \mathfrak{F}_{s-1}/\langle x_1 x_j^{-1}, j = 2, \ldots, s-1 \rangle \cong \mathfrak{F}_1 \cong \mathbb{Z}_\ell. \tag{13}$$

The map α is continuous so if v_n is a sequence of words in F_{s-1} converging to $v \in \mathfrak{F}_{s-1}$ then

$$\lim_n \alpha(v_n) = \alpha(v) \in \mathbb{Z}_\ell.$$

4.1. **On certain examples of cyclic curves of $\mathbb{P}^1$.** Consider the commutative diagram below on the left:

$$\begin{tikzcd}
\tilde{X}_s \arrow[rr, "F'_{s-1}"] \arrow[dr, "R_0"] \arrow[dd, "F_{s-1}"'] & & Y \arrow[dl, "I"] \arrow[ddll, bend left, "H_1(X_s,\mathbb{Z})"] \\
& C_s \arrow[dl, "\mathbb{Z}"] & \\
X_s & &
\end{tikzcd}$$

Then $H_1(C_s, \mathbb{Z}) = R_0/R_0'$, where $R_0 = \pi_1(C_s)$ is the free subgroup of F_{s-1} corresponding to C_s. Moreover $H_1(C_s, \mathbb{Z})$ is a free $\mathbb{Z}[\mathbb{Z}]$-module free of rank $s-2$ acted on also by B_{s-1} giving rise to the so called Burau representation:

$$\rho : B_{s-1} \to \mathrm{GL}(s-2, \mathbb{Z}[t, t^{-1}]).$$

Keep in mind that $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$. In what follows will give a proof of these facts using the Schreier's lemma.

Lemma 9. *The group R_0 (is an infinite rank group) is given by*

$$R_0 = \{x_1^i x_j x_1^{-i+1} : i \in \mathbb{Z}, j \in 2, \ldots, s-1\}. \tag{14}$$

Proof. Consider the epimorphisms

$$\begin{tikzcd}
F_{s-1} \arrow[r, "p'"] \arrow[rr, bend right, "\alpha"'] & F_{s-1}/F'_{s-1} \arrow[r, "p''"] & \mathbb{Z} = H_1(Y, X_s)/I.
\end{tikzcd}$$

Set $\alpha = p'' \circ p'$. Let y be an element in $\alpha^{-1}(1_{\mathbb{Z}})$. By the properties of the winding number we can take as $y = x_1$. Moreover $\alpha(x_j) = y$ for all $1 \leq j \leq s-1$, since the automorphism $x_i \leftrightarrow x_j$ is compatible with I and therefore introduces an automorphism of $\mathbb{Z}$, so $\alpha(x_j) = y^{\pm 1}$, and we rename the generators x_i to x_i^{-1} if necessary.

Let $T := \{y^i : i \in \mathbb{Z}\} \subset F_{s-1}$ be a set of representatives of classes in $F_{s-1}/R_0 \cong \mathbb{Z}$. For every $x \in F_{s-1}$ we will denote by $\bar{x}$ the representative in T. Moreover for all $i \in \mathbb{Z}$ and $1 \leq j \leq s-1$ we have $\overline{y^i x_j} = y^{i+1}$ and by the Schreier's lemma we see that

$$y^i x_j \left(\overline{y^i x_j}\right)^{-1} = y^i x_j y^{-i-1} = x_1^i x_j x_1^{-i-1} \qquad i \in \mathbb{Z}, j \in 2, \ldots, s-1.$$

$\square$

Remark 10. The action of $\mathbb{Z}[\mathbb{Z}]$ on R_0/R_0' is given by conjugation. This means that for $n \in \mathbb{Z}$ we have

$$\begin{aligned} \mathbb{Z}[\mathbb{Z}] \times R_0 &\longrightarrow R_0 \\ (t^n, r) &\longmapsto x_1^n r x_1^{-n} \end{aligned} \tag{15}$$

A generating set for $H_1(C_s, \mathbb{Z})$ is given by the $s-2$ elements $\beta_j := x_j x_1^{-1}$. Moreover the action is given by

$$\left(x_i x_1^{-1}\right)^{t^n} = x_1^n x_i x_1^{-n-1},$$

i.e. that $H_1(C_s, \mathbb{Z})$ is a free $\mathbb{Z}[\mathbb{Z}]$-module of rank $s-2$.

Observe that in R_0/R_0' we have

$$\begin{aligned} x_j(x_i x_1^{-1})x_j^{-1} &= (x_j x_1^{-1})x_1\beta_i x_1^{-1}(x_j x_1^{-1})^{-1} \\ &= \beta_j x_1 \beta_i x_1^{-1}\beta_j^{-1} = \beta_i^t, \end{aligned}$$

i.e. the cunjugation by any generator x_j has the same effect by the conjugation by x_1.

Let us now consider a finite cyclic cover Y_n of X_s which is covered by C_s, i.e. we have the diagram on the right bellow:

$$\begin{tikzcd}
\tilde{X}_s \arrow[dr, "R_0"] \arrow[rr] \arrow[ddd] & & Y \arrow[dl, "I"'] \arrow[dddll, bend left, "H_1(X_s,\mathbb{Z})"] \\
& C_s \arrow[d] \arrow[ddl, "\mathbb{Z}"'] & \\
& Y_n \arrow[dl, "\mathbb{Z}/n\mathbb{Z}"] & \\
X_s & &
\end{tikzcd}$$

Lemma 11. *The group $R_n = \pi_1(Y_n) \supset R_0$ is the kernel of the map α_n*

$$\begin{tikzcd} \pi_1(X) \arrow[r, "\alpha"] \arrow[rr, bend left, "\alpha_n"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/n\mathbb{Z}. \end{tikzcd}$$

Proof. This is clear from the explicit description of the group R_0 given in eq. (14). □

Lemma 12. *The group R_n is generated by*

$$R_n = \{x_1^i x_j x_1^{-i-1} : 0 \le i \le n-2, 2 \le j \le s-1\} \cup \{x_1^{n-1}x_j : 1 \le j \le s-1\}.$$

which is a free group on $r = (s-2)n+1$ generators.

Proof. In this case the transversal set equals $T = \{y^i : 0 \le i \le n-1\}$. Moreover

$$\overline{y^i x_j} = \begin{cases} y^{i+1} & \text{if } i < n-1 \\ 1 & \text{if } i = n-1. \end{cases}$$

The desired result follows. □

Proposition 13. *The $\mathbb{Z}$-module R_n/R_n' as $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-module is isomorphic to*

$$R_n/R_n' = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]^{s-2} \bigoplus \mathbb{Z}.$$

Proof. Set $\beta_j = x_j x_1^{-1}$ for $2 \le j \le s-1$. Then the action of $\mathbb{Z}/n\mathbb{Z} = \langle g \rangle$ on β_j is given by

$$\beta_j^{g^\ell} = x_1^\ell x_j x_1^{-\ell-1} \text{ for } 0 \le \ell \le n-1.$$

It is clear that for each fixed j, $2 \le j \le s-1$, the elements $\beta_j^{g^\ell}$ generate a copy of the group algebra $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$. By the explicit form of the basis generators given in lemma 12 we have the alternative basis given by

$$\{x_1^i x_j x_1^{-i-1} : 2 \le j \le s-1, 0 \le i \le n-1\} \cup \{x_1^n\}. \tag{16}$$

The result follows. □

Remark 14. The above computation is compatible with the Schreier index formula [5, cor. 8.5 p.66] which asserts that

$$r - 1 = n(s-2). \tag{17}$$

Remark 15. Observe that there is no natural reduction modulo n map from $H_1(C_s, \mathbb{Z})$ to $H_1(Y_n, \mathbb{Z})$ corresponding to the group reduction $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.

4.2. **The Burau Representation.** Consider the action of a generator σ_i of B_s seen as an automorphism of the free group, given for $1 \leq i, j \leq s-2$ as

$$\sigma_i(x_j) = \begin{cases} x_j & \text{if } j \neq i, i+1 \\ x_i & \text{if } j = i+1 \\ x_i x_{i+1} x_i^{-1} & \text{if } j = i \end{cases}$$

Therefore the conjugation action on the generators $\beta_j = x_j x_1^{-1}$ of R, seen as a $\mathbb{Z}[\mathbb{Z}]$-module, is given for $j \geq 2$ by:

$$\sigma_j(\beta_{j+1}) = \sigma_j(x_{j+1}x_1^{-1}) = x_j x_1^{-1} = \beta_j,$$

$$\begin{aligned} \sigma_j(\beta_j) = \sigma_j(x_j x_1^{-1}) &= x_j \cdot x_{j+1} \cdot x_j^{-1} \cdot x_1^{-1} = x_j x_1^{-1} \cdot x_1 x_{j+1} x_1^{-2} x_1^2 x_j^{-1} \cdot x_1^{-1} \\ &= \beta_j x_1 \beta_{j+1} x_1^{-1} x_1 \beta_j^{-1} x_1^{-1} = \beta_j \beta_{j+1}^t \beta_j^{-t} = \beta_j^{1-t} \beta_{j+1}^t. \end{aligned}$$

The notation for t above is in accordance with the group algebra notation $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$. Also in the special case where $j = 1$ we compute:

$$\sigma_1(\beta_2) = \sigma_1(x_2 x_1^{-1}) = x_1 \cdot x_1 x_2^{-1} x_1^{-1} = \beta_2^{-t},$$

and if $i > 2$

$$\sigma_1(\beta_i) = \sigma_1(x_i x_1^{-1}) = x_i \cdot x_1 x_2^{-1} x_1^{-1} = x_i x_1^{-1} \cdot x_1 x_1 x_2^{-1} x_1^{-1} = \beta_i \beta_2^{-t}$$

We now compute the action on the $\mathbb{Z}$-module R/R', so the β_i, β_j are commuting and we arrive at the matrix of the action with respect to the basis $\{\beta_2, \ldots, \beta_{s-1}\}$:

$$\sigma_j \mapsto \begin{pmatrix} \mathrm{Id} & & & \\ & 1-t & 1 & \\ & t & 0 & \\ & & & \mathrm{Id} \end{pmatrix}, \text{if} \quad j \neq 1 \text{ and } \sigma_1 \mapsto \begin{pmatrix} -t & -t & & -t \\ 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Lemma 16. *The action of t on β_i commutes with the action of the braid group.*

Proof. It is obvious that for σ_j $j \geq 2$ and $a \in R_0$ we have

$$\sigma_j(a^t) = \sigma_j(x_1 a x_1^{-1}) = x_1 \sigma_j(a) x_1^{-1} = (\sigma_j(a))^t.$$

For σ_1 we observe that

$$\begin{aligned} \sigma_j(a^t) = \sigma_1(x_1 a x_1^{-1}) &= x_1 x_2 x_1^{-1} \sigma_1(a) x_1 x_2^{-1} x_1^{-1} = x_1 \beta_2 \sigma_1(a) \beta_2^{-1} x_1^{-1} \\ &= x_1 \sigma_1(a) x_1^{-1} = (\sigma_1(a))^t, \end{aligned}$$

since $\sigma_1(a)$ is expressed as product of β_ν and the elements β_i commute modulo R_0'. □

4.3. **The profinite Burau representation.** If we attempt to let the absolute Galois group act on $H_1(C_s,\mathbb{Z}_\ell) = H_1(C_s,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \mathbb{Z}_\ell[\mathbb{Z}]^{s-2}$ we see that this action is not well defined. It turns out that instead of the ordinary group algebra $\mathbb{Z}_\ell[\mathbb{Z}]$ we need the completed group algbera $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]$. In this way we see the profinite Burau representation as a linear representation:

$$\rho_{\mathrm{Burau}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_{s-2}(\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]).$$

Remark 17. The $\mathbb{Z}_\ell$-algebra $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]$ is a ring defined as the inverse limit

$$\mathbb{Z}_\ell[[\mathbb{Z}_\ell]] = \lim_{\substack{\leftarrow \\ n}} \mathbb{Z}_\ell[\mathbb{Z}/\ell^n\mathbb{Z}]$$

of the ordinary group algebra, see [23, p.171]. It contains the $\mathbb{Z}$-algebra $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t,t^{-1}]$ which appears in the discrete topological Burau representation as a dense subalgebra.

Lemma 18. *Recall that we have written $\mathbb{Z}_\ell$ as a multiplicative group, i.e. $\mathbb{Z}_\ell \cong \langle t^\alpha, \alpha \in \mathbb{Z}_\ell\rangle$. Let $\alpha = \sum_{\nu=0}^{\infty} a_\nu \ell^\nu \in \mathbb{Z}_\ell$, $0 \leq a_\nu < \ell$ for all $0 \leq \nu$. Set*

$$A_n = \left(1 + t + t^2 + \ldots + t^{(\sum_{\nu=0}^{n} a_\nu \ell^\nu)-1}\right).$$

Then the sequence above converges and we will denote its limit by $(t^\alpha - 1)/(t-1)$, that is

$$\lim_{n\rightarrow\infty} \left(1 + t + t^2 + \ldots + t^{(\sum_{\nu=0}^{n} a_\nu \ell^\nu)-1}\right) = \frac{t^\alpha - 1}{t-1}.$$

Proof. The algebra $\mathbb{Z}_\ell[\mathbb{Z}/\ell^n\mathbb{Z}]$ is identified by the set of all expressions $\sum_{\nu=0}^{\ell^n-1} \beta_\nu t_n^\nu$, where t_n is a generator of the cyclic group $\mathbb{Z}/\ell^n\mathbb{Z}$ and $\beta_\nu \in \mathbb{Z}_\ell$. In the inverse limit defining the ring of ℓ-adic numbers the generator t_{n+1} of $\mathbb{Z}/\ell^{n+1}\mathbb{Z}$ is sent to the generator t_n of $\mathbb{Z}/\ell^n\mathbb{Z}$. The corresponding map in the group algebras (by identifying $t_n = t_{n+1} = t$) is given by sending

$$\mathbb{Z}_\ell[\mathbb{Z}/\ell^{n+1}\mathbb{Z}] \ni \sum_{\nu=0}^{\ell^{n+1}-1} \beta_\nu t^\nu \longmapsto \sum_{\nu=0}^{\ell^n-1} \beta_\nu t^\nu \in \mathbb{Z}_\ell[\mathbb{Z}/\ell^n\mathbb{Z}].$$

We compute now for $m < n$

$$A_n - A_m = \sum_{\nu=a_0+a_1\ell+\cdots+a_m\ell^m}^{a_0+a_1\ell+\cdots+a_n\ell^n} t^\nu = t^{a_0+a_1\ell+\cdots+a_m\ell^m} \sum_{\nu=0}^{a_{m+1}\ell^{m+1}+\cdots+a_n\ell^n} t^\nu$$

Therefore, the sequence is Cauchy and converges in the complete group algebra $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]$. □

Lemma 19. *We have for $\alpha \in \mathbb{N}$*

$$x_k^\alpha x_1^{-\alpha} = \beta_k \cdot \beta_k^t \cdot \beta_k^{t^2} \cdots \beta_k^{t^{\alpha-1}}. \tag{18}$$

For $\alpha \in \mathbb{Z}_\ell$ we have

$$x_k^\alpha x_1^{-\alpha} = \beta_k^{\frac{t^\alpha - 1}{t-1}}. \tag{19}$$

Proof. We will prove first the result for $\alpha = n \in \mathbb{Z}$. Indeed, for $\alpha = 1$ the result is trivial while by induction

$$x_k^n x_1^{-n} = x_k \beta_k \cdots \beta_k^{t^{n-2}} x_1^{-1} = x_k x_1^{-1} x_1 \beta_k \cdots \beta_k^{t^{n-2}} x_1^{-1} = \beta_k \cdot \beta_k^t \cdot \beta_k^{t^2} \cdots \beta_k^{t^{n-1}}$$

Now for $\alpha = \sum_{\nu=0}^{\infty} a_\nu \ell^\nu \in \mathbb{Z}_\ell$ we consider the sequence $c_n = \sum_{\nu=0}^{n} a_\nu \ell^\nu \to \alpha$. We have

$$x_k^{\alpha} x_1^{-\alpha} = \lim_n x_k^{c_n} x_1^{-c_n} = \lim_n \beta_k^{\frac{t^{c_n}-1}{t-1}} = \beta_k^{\frac{t^{\alpha}-1}{t-1}}.$$

□

Lemma 20. *For every $i \neq 1$, and $N \in \mathbb{Z}_\ell$ we have*

$$x_i^{-1} x_1^{-N} = x_1^{-N} x_i^{-1} \cdot \beta_i^{1-t^N}.$$

More generally for $a \in \mathbb{Z}_\ell^*$

$$x_i^{-a} x_1^{-N} = x_1^{-N} x_i^{-a} \cdot \beta_i^{\frac{t^a-1}{t-1}(1-t^N)}$$

Proof. We compute

$$\begin{aligned} x_i^{-1} x_1^{-N} &= x_1^{-N} x_i^{-1} \cdot x_i x_1^N x_i^{-1} x_1^{-N} \\ &= x_1^{-N} x_i^{-1} \cdot x_i x_1^{-1} x_1^N (x_i x_1^{-1})^{-1} x_1^{-N} \\ &= x_1^{-N} x_i^{-1} \cdot \beta_i \beta_i^{-t^N} \\ &= x_1^{-N} x_i^{-1} \cdot \beta_i^{1-t^N}. \end{aligned}$$

The second equality is proved the same way

$$\begin{aligned} x_i^{-a} x_1^{-N} &= x_1^{-N} x_i^{-a} \cdot x_i^a x_1^N x_i^{-a} x_1^{-N} \\ &= x_1^{-N} x_i^{-a} \cdot x_i^a x_1^{-a} x_1^N (x_i^a x_1^{-a})^{-1} x_1^{-N} \\ &= x_1^{-N} x_i^{-a} \cdot \beta_i^{\frac{t^a-1}{t-1}(1-t^N)}. \end{aligned}$$

□

Lemma 21. *For a given word $x_{s-1}^{-a_{s-1}} \cdots x_1^{-a_1}$ we have*

$$x_{s-1}^{-a_{s-1}} \cdots x_1^{-a_1} x_1^{-N} = x_1^{-N} x_{s-1}^{-a_{s-1}} \beta_{s-1}^{\frac{t^{a_{s-1}}-1}{t-1}(1-t^N)} \cdots x_2^{-a_2} \beta_2^{\frac{t^{a_2}-1}{t-1}(1-t^N)} x_1^{-a_1}.$$

Proof. We use lemma 20 inductively to have

$$\begin{aligned} x_{s-1}^{-a_{s-1}} \cdots x_1^{-a_1} x_1^{-N} &= x_{s-1}^{-a_{s-1}} \cdots x_3^{-a_3} x_1^{-N} x_2^{-a_2} \beta_2^{\frac{t^{a_2}-1}{t-1}(1-t^N)} x_1^{-a_1} \\ &= x_{s-1}^{-a_{s-1}} \cdots x_4^{-a_4} x_1^{-N} x_3^{-a_3} \beta_3^{\frac{t^{a_3}-1}{t-1}(1-t^N)} x_2^{-a_2} \beta_2^{\frac{t^{a_2}-1}{t-1}(1-t^N)} x_1^{-a_1} \\ &= \cdots \\ &= x_1^{-N} x_{s-1}^{-a_{s-1}} \beta_{s-1}^{\frac{t^{a_{s-1}}-1}{t-1}(1-t^N)} \cdots x_2^{-a_2} \beta_2^{\frac{t^{a_2}-1}{t-1}(1-t^N)} x_1^{-a_1}. \end{aligned}$$

□

For simplicity denote $N(\sigma)$ by N and $w_i(\sigma)$ by w. We will consider $w x_i^N w^{-1} x_1^{-N}$, where $w^{-1} = x_{s-1}^{-a_{s-1}} \cdots x_1^{-a_1}$. We have

$$w x_i^N w^{-1} x_1^{-N} = \beta_i^{t^{\sum_{\nu=1}^{s-1} a_\nu} \frac{t^N-1}{t-1}} \beta_{s-1}^{t^{\sum_{\nu=1}^{s-2} a_\nu} \frac{t^{s-1}-1}{t-1}(1-t^N)} \cdots \beta_2^{t^{a_1} \frac{t^{a_2}-1}{t-1}(1-t^N)}$$

In the abelianization we have to consider the coefficients of β_j, $s-1 \geq j \geq 2$, that is the coefficient of β_j modulo the commutator of R_0 is given by

$$(1-t^N) \frac{t^{a_j}-1}{t-1} t^{\sum_{\mu=1}^{j-1} a_\mu}.$$

An arbitrary element $w \in \mathfrak{F}_{s-1}$ can be written in a unique way as

$$w = B \cdot x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}, \qquad a_i \in \mathbb{Z}_\ell$$

where B is an element in the group R_0 generated by the elements β_i, $i = 2, \ldots, s-1$. Observe now that for every β_i, and $N \in \mathbb{Z}_\ell$ we have

$$\beta_i x_1^{-N} = x_1^{-N} x_1^N \beta_i x_1^{-N} = x_1^{-N} \beta_i^{t^N}.$$

By considering a sequence of words in β_i tending to B we see that

$$B x_1^{-N} = x_1^{-N} B^{t^N},$$

for every element B in the pro-ℓ completion of R_0.

This means that

$$w x_i^N w^{-1} x_1^{-N} = B(x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}) x_i^N (x_{s-1}^{-a_{s-1}} \cdots x_1^{-a_1}) B^{-1} x_1^{-N}$$

$$= B\left(x_1^{a_1} \cdots x_{s-1}^{a_{s-1}}\right) x_i^N x_1^{-N} \left(x_{s-1}^{-a_{s-1}} \beta_{s-1}^{(1-t^N)\frac{t^{a_{s-1}}-1}{t-1}} \cdots x_2^{-a_2} \beta_2^{(1-t^N)\frac{t^{a_2}-1}{t-1}} x_1^{-a_1} \right) B^{-t^N}$$

$$= B \beta_i^{t^{a_1+\cdots a_{s-1}}\frac{t^N-1}{t-1}} \beta_{s-1}^{(1-t^N)t^{a_1+\cdots+a_{s-2}}\frac{t^{a_{s-1}}-1}{t-1}} \cdots \beta_2^{(1-t^N)t^{a_1}\frac{t^{a_2}-1}{t-1}} B^{-t^N}$$

The above in R_0/R_0' evaluates to

$$w x_i^N w^{-1} x_1^{-N} = \beta_i^{t^{a_1+\cdots+a_{s-1}}\frac{t^N-1}{t-1}} \beta_{s-1}^{(1-t^N)t^{a_1+\cdots+a_{s-2}}\frac{t^{a_{s-1}}-1}{t-1}} \cdots \beta_2^{(1-t^N)t^{a_1}\frac{t^{a_2}-1}{t-1}} B^{-t^N+1}. \tag{20}$$

We now return to the study of the dependence of the matrix of ρ with respect to the action given by $\sigma(x_i) = w_i(\sigma) x_i^{N(\sigma)} w_i(\sigma)^{-1}$. Let us write each $w_i(\sigma)$ as

$$w_i(\sigma) = B_i(\sigma) x_1^{a_{1,i}(\sigma)} \cdots x_{s-1}^{a_{s-1,i}(\sigma)},$$

where $B_i(\sigma) \in R_0/R_0'$ is expressed as

$$B_i(\sigma) = \beta_2^{b_{2,i}(\sigma)} \cdots \beta_{s-1}^{b_{s-1,i}(\sigma)} C,$$

with $b_{i,j}(\sigma) \in \mathbb{Z}_\ell[[\mathbb{Z}_\ell]]$ and $C \in R_0'$. The matrix form of ρ_{Burau} as given in theorem 2 follows by eq. (20).

5. Examples - Complete curves

5.1. **The compactification of cyclic covers.** Consider the complex compact Riemann surface corresponding to the cyclic cover of the projective line given by:

$$y^n = \prod_{i=1}^{s} (x - b_i)^{d_i}, \qquad (d_i, n) = 1 \tag{21}$$

where $\sum_{i=1}^{s} d_i \equiv 0 \mod n$, so that there is no ramification at infinity, see [12, p. 667].

Riemann-Hurwitz theorem implies that

$$g = \frac{(n-1)(s-2)}{2}, \tag{22}$$

which is compatible with the computation of $r = 2g + s - 1$ given in eq. (17).

This curve can be uniformized as a quotient $\mathbb{H}/\Gamma$ of the hyperbolic space modulo a discrete free subgroup of genus g, which admits a presentation

$$\Gamma = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g | [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle.$$

On the other hand side, when we remove the s-branch points we obtain a topological cover of the space X_s defined in the previous section. This topological cover corresponds to the free subgroup of $R_n < F_{s-1}$ given by

$$R_n = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_s | \gamma_1\gamma_2 \cdots \gamma_s \cdot [a_1, b_1] \cdots [a_g, b_g] = 1\rangle.$$

Lemma 22. *The $\mathbb{Z}/n\mathbb{Z}$-invariant elements of R_n/R_n' are given by multiples of*

$$\{x_i^n : 1 \leq i \leq s-1\}.$$

Proof. Observe that an element in the group algebra $\mathbb{Z}[\langle g\rangle]$ is g-invariant if and only if it is of the form $\sum_{i=0}^{n-1} ag^i$ for some $a \in \mathbb{Z}$. Hence the invariant elements are multiples (powers in the multiplicative notation) by

$$\beta_j \beta_j^g \beta_j^{g^2} \cdots \beta_j^{g^{n-1}} = x_j^n x_1^{-n}.$$

Since x_1^n is invariant the result follows. □

The elements γ_i are lifts of the loops x_i around each hole in the projective line. Thus γ_i are $\mathbb{Z}/n\mathbb{Z}$-invariant. Set $\gamma_i = x_i^n$. The quotient $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]/\langle \sum_{i=0}^{n-1} g^i\rangle$ is the co-augmentation module, see [21, sec. 1].

Lemma 23. *We have*

$$x_k^n x_i x_k^{-n} x_1^{-1} = \beta_k \cdot \beta_k^g \cdot \beta_k^{g^2} \cdots \beta_k^{g^{n-1}} \cdot \beta_i^{g^n} \cdot \beta_k^{-g^n} \cdot \beta_k^{-g^{n-1}} \cdots \beta_k^{-g^2} \cdot \beta_k^{-g}$$

Moreover in the abelian group R/R' we have

$$x_k^n x_i x_k^{-n} x_1^{-1} = \beta_i^{g^n} \beta_k^{1-g^n}.$$

Proof. Write

$$\begin{aligned}
x_k^n x_i x_k^{-n} x_1^{-1} &= x_k^n x_1^{-n} \cdot x_1^n x_i x_1^{-1} x_1^{-n} x_1^{n+1} x_k^{-n} x_1^{-1} \\
&= \beta_k \cdot \beta_k^g \cdot \beta_k^{g^2} \cdots \beta_k^{g^{n-1}} \cdot x_1^n \beta_i x_1^{-n} x_1 \left(\beta_k \cdot \beta_k^g \cdot \beta_k^{g^2} \cdots \beta_k^{g^{n-1}}\right)^{-1} x_1^{-1} \\
&= \beta_k \cdot \beta_k^g \cdot \beta_k^{g^2} \cdots \beta_k^{g^{n-1}} \cdot \beta_i^{g^n} \cdot \beta_k^{-g^n} \cdot \beta_k^{-g^{n-1}} \cdots \beta_k^{-g^2} \cdot \beta_k^{-g}
\end{aligned}$$

□

Lemma 24. *The subgroup of R_n/R_n' generated by $\mathbb{Z}/n\mathbb{Z}$-invariant elements*

$$\{x_1^n, x_j^n x_1^{-n} : 2 \leq j \leq s-1\}$$

is invariant under the action of the braid group and under the action of the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Proof. We consider first the Braid action. By lemma 19 we have

$$\begin{aligned}
\sigma_1(x_1^n) &= (x_1 x_2 x_1^{-1})^n = x_1 \cdot x_2^n \cdot x_1^{-1} = x_1 \cdot x_2^n x_1^{-n} \cdot x_1^{n-1} \\
&= x_1 \cdot \beta_2 \cdot \beta_2^g \cdot \beta_2^{g^2} \cdots \beta_2^{g^{n-1}} \cdot x_1^{-1} \cdot x_1^n = \beta_2^g \cdot \beta_2^{g^2} \cdots \beta_2^{g^n} \cdot x_1^n \\
&= \beta_2 \cdot \beta_2^g \cdots \beta_2^{g^{n-1}} \cdot x_1^n = x_2^n x_1^{-n} \cdot x_1^n = x_2^n
\end{aligned}$$

$$\sigma_1(x_2^n) = x_1^n, \sigma_1(x_i^n) = x_i^n \ (i > 2).$$

For $j \geq 2$:
$$\begin{aligned}
\sigma_j(x_j^n x_1^{-n}) &= (x_j x_{j+1} x_j^{-1})^n x_1^{-n} = x_j \cdot x_{j+1}^n \cdot x_j^{-1} \cdot x_1^{-n} \\
&= x_j x_1^{-1} \cdot x_1 (x_{j+1}^n x_1^{-n}) x_1^{-1} \cdot x_1^n \cdot x_1 x_j^{-1} \cdot x_1^{-n} \\
&= x_{j+1}^n x_1^{-n}
\end{aligned}$$

$$\sigma_j(x_j^n) \quad = \quad \sigma_j(x_j^n x_1^{-n})\sigma_j(x_1^n) = x_{j+1}^n.$$

We will now consider the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Each element $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on x_i by

$$\sigma(x_i) = w_i(\sigma) x_i^{N(\sigma)} w_i(\sigma)^{-1},$$

Therefore, if for every word $w_i(\sigma)$ we set $\alpha(w_i(\sigma)) \in \mathbb{Z}_\ell$ to be the sum of all p-adic exponents that appear in $w(\sigma)$, we have

$$\begin{aligned}\sigma(x_i^n x_1^{-n}) &= \sigma(\beta_j \beta_j^g \cdots \beta_j^{g^{n-1}}) \\ &= (\sigma(\beta_j))^{1+g+\cdots+g^{n-1}}\end{aligned}$$

which is an element invariant under the action of $\mathbb{Z}/n\mathbb{Z}$. □

Consider now the space

$$H_1(\bar{Y}_n, \mathbb{Z}) = \frac{R_n}{R_n' \cdot \langle \gamma_1, \ldots, \gamma_s \rangle} = \frac{R_n}{R_n' \cdot \langle x_1^n, \ldots, x_s^n \rangle}.$$

Observe that $R_n / R_n' \cdot \langle x_1 \rangle = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]^{s-2}$. Since $\langle \gamma_1, \ldots, \gamma_s \rangle$ is both $\mathbb{Z}/n\mathbb{Z}$ and B_s stable we have a natural defined action of B_s on the quotient. We compute now the action of the braid group on $\beta_j^{g^i} = x_1^i x_j x_1^{-i-1}$. We can pick as a basis of the $\mathbb{Z}$-module $H_1(\bar{Y}_n, \mathbb{Z})$ the elements

$$\{\beta_j^{g^i} = x_1^i x_j x_1^{-1-i} : 2 \leq j \leq s-1, 0 \leq i \leq n-2\}$$

and equation (18) written additively implies that $\beta_j^{g^{n-1}} = -\sum_{\nu=0}^{n-2} \beta_j^{g^\nu}$, recall that all powers x_i^n are considered to be zero.

Let $J_{\mathbb{Z}/n\mathbb{Z}}$ be the co-augmentation module. Observe that $\beta_j^{t^\nu - 1} = [x_1^\nu, x_j]$. It is well known (see, [21, Prop. 1.2]) that $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] = J_{\mathbb{Z}/n\mathbb{Z}} \oplus \mathbb{Z}$. Therefore $H_1(\bar{Y}_n, \mathbb{Z}) = J_{\mathbb{Z}/n\mathbb{Z}}^{s-2}$. Notice that the above $\mathbb{Z}$-module has the correct rank $2g = (n-1)(s-2)$. The direct sum above is in the category of $\mathbb{Z}$-modules not in the category of B_s-modules. Also on the co-augmentation module $J_{\mathbb{Z}/n\mathbb{Z}}$ the generator of the $\mathbb{Z}/n\mathbb{Z}$ is represented by the matrix:

$$A := \begin{pmatrix} 0 & \cdots & 0 & -1 \\ 1 & \ddots & \vdots & \vdots \\ 0 & \ddots & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{23}$$

which is the companion matrix of the polynomial $x^{n-1} + \cdots + x + 1$. One way to represent $J_{\mathbb{Z}/n\mathbb{Z}}$ is in terms of the $\mathbb{Z}$-module $\mathbb{Z}[\zeta]$, where ζ is a primitive n-th root of unity, i.e.

$$\mathbb{Z}[\zeta] = \bigoplus_{\nu=0}^{n-1} \zeta^\nu \mathbb{Z},$$

and the $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-module structure is given by multiplication by ζ.

Since the $\mathbb{Z}/n\mathbb{Z}$-action and the braid action are commuting we have a decomposition (notice that 1 does not appear in the eigenspace decomposition below)

$$H_1(\bar{Y}_n, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\nu=1}^{n-1} V_\nu$$

where V_ν is the eigenspace of the ζ^ν-eigenvalue. Each V_ν is a B_s-module of dimension $s-2$. In order to compute the spaces V_ν we have to diagonalize the matrix given in eq. (23). Consider the Vandermonde matrix given by:

$$P = \begin{pmatrix} 1 & \zeta_1 & \zeta_1^2 & \cdots & \zeta_1^{n-2} \\ 1 & \zeta_2 & \zeta_2^2 & \cdots & \zeta_2^{n-2} \\ \vdots & \vdots & & \vdots & \\ 1 & \zeta_{n-1} & \zeta_{n-1}^2 & \cdots & \zeta_{n-1}^{n-2} \end{pmatrix},$$

where $\{\zeta_1, \ldots, \zeta_{n-1}\}$ are all n-th roots of unity different than 1. Observe that

$$P \cdot A = \mathrm{diag}(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}) \cdot P.$$

Thus the action of the braid group on the eigenspace V_ν of the eigenvalue ζ^ν can be computed by a base change as follows: Consider the initial base $1, \beta_j, \beta_j^t, \ldots, \beta_j^{t^{n-2}}$ for $2 \leq j \leq s-1$. The eigenspace of the ζ^ν eigenvalue has as basis the k-element of the $1 \times (n-2)$ matrix

$$\left(1, \beta_j, \beta_j^g, \ldots, \beta_j^{g^{n-2}}\right) \cdot P^{-1}$$

for all j such that $2 \leq j \leq s-1$. These elements are $\mathbb{C}$-linear combinations of the elements β_j and the action of the braid generators on them can be easily computed.

Since the action of $\mathrm{Gal}(\bar{Y}_n/\mathbb{P}^1) = \langle g \rangle$ commutes with the action of B_s (resp. $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$) each eigenspace is a B_s (resp. $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$) module. The action of the operator t on each V_n is essentially the action of g, which by definition of eigenspace, acts by multiplication by ζ_ν. Therefore, the matrix representation corresponding to each eigenspace V_n is the matrix of the Burau (resp. pro-ℓ Burau) evaluated at $t = \zeta_\nu$.

Similarly in the pro-ℓ case we have

$$\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]^{s-2} \otimes_{\mathbb{Z}_\ell} \bar{\mathbb{Q}}_\ell = \bigoplus_{\nu=1}^{\ell^k-1} V_\nu,$$

which after reducing $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]] \to \mathbb{Z}_\ell[\mathbb{Z}_\ell/\ell^k\mathbb{Z}_\ell] = \mathbb{Z}_\ell[\mathbb{Z}/\ell^k\mathbb{Z}]$ sending $t \mapsto \zeta_\nu$ gives rise to the representation in V_ν.

This decomposition is a $\mathbb{Z}_\ell$ module. The Galois module structure and the $\mathbb{Z}_\ell$ action do not commute in this case. Indeed, the equation (7) implies that $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the pro-ℓ generator by

$$\sigma t = t^{N(\sigma)} \sigma.$$

Therefore, the modules V_ν defined above are $\ker N$-modules.

5.2. **Relation to actions on holomorphic differentials.** Let S be a compact Riemann-surface of genus g. Consider the first homology group $H_1(S, \mathbb{Z})$ which is a free $\mathbb{Z}$-module of rank $2g$. Let $H^0(S, \Omega_S)$ be the space of holomorphic differentials which is a $\mathbb{C}$-vector space of dimension g. The function

$$\begin{array}{rcl} H_1(S, \mathbb{Z}) \times H^0(S, \Omega_S) & \to & \mathbb{R} \\ \gamma, \omega & \mapsto & \langle \gamma, \omega \rangle = \mathrm{Re} \displaystyle\int_\gamma \omega \end{array}$$

induces a duality $H_1(S,\mathbb{Z})\otimes\mathbb{R}$ to $H^0(S,\Omega_S)^*$, see [14, th. 5.6], [7, sec. 2.2 p. 224]. Therefore an action of a group element on $H_1(S,\mathbb{Z})$ gives rise to the contragredient action on holomorphic differentials, see also [6, p. 271].

C. Mc Mullen in [17, sec. 3] considered the Hodge decomposition of the DeRham cohomology as

$$H^1(X) = \mathrm{Hom}_{\mathbb{C}}(H_1(X,\mathbb{Z}),\mathbb{C}) = H^{1,0}(X)\oplus H^{0,1}(X) \cong \Omega(X)\oplus\bar{\Omega}(X).$$

Of course this decomposition takes place in the dual space of holomorphic differentials, and is based on the intersection form

$$\langle\alpha,\beta\rangle = i/2\int_X \alpha\wedge\bar{\beta}, \qquad i^2=-1. \tag{24}$$

In this article we use the group theory approach and we focus around the homology group $H_1(X,\mathbb{Z})$. Homology group is equipped with an intersection form and a canonical symplectic basis $a_1,\ldots,a_g,b_1,\ldots,b_g$ such that

$$\langle a_i,b_j\rangle=\delta_{ij}, \qquad \langle a_i,a_j\rangle=\langle b_i,b_j\rangle=0.$$

Every two homology classes γ,γ' can be written as $\mathbb{Z}$-linear combinations of the canonical basis

$$\gamma=\sum_{i=1}^g(\lambda_i a_i+\mu_i b_i) \qquad \gamma'=\sum_{i=1}^g(\lambda'_i a_i+\mu'_i b_i)$$

and the intersection is given by

$$\langle\gamma,\gamma'\rangle = (\lambda_1,\ldots,\lambda_g,\mu_1,\ldots,\mu_g)\begin{pmatrix}0 & \mathbb{I}_g\\ -\mathbb{I}_g & 0\end{pmatrix}(\lambda'_1,\ldots,\lambda'_g,\mu'_1,\ldots,\mu'_g)^t.$$

This gives rise to a representation

$$\rho: B_{s-1}\rightarrow \mathrm{Sp}(2g,\mathbb{Z}) \tag{25}$$

since $\langle\sigma(\gamma),\sigma(\gamma')\rangle=\langle\gamma,\gamma'\rangle$. Indeed, it is known[10, sec. 3.2.1] that the action of the braid group keeps the intersection multiplicity of two curves. The relation to the unitary representation on holomorphic differentials (and the signature computations) is given by using the diagonalization of

$$\begin{pmatrix}0 & \mathbb{I}_g\\ -\mathbb{I}_g & 0\end{pmatrix} = P\cdot\mathrm{diag}(\underbrace{i,\ldots,i}_{g},\underbrace{-i,\ldots,-i}_{g})\cdot P^{-1},$$

and the extra"i" put in front of eq. (24).

5.2.1. *Arithmetic intersection.* In order to define an analogous result in the case of absolute Galois group we have first to define an intersection form in $H_1(X,\mathbb{Z}_\ell)$, which can be define as the limit of the intersection forms in $H_1(X,\mathbb{Z}/\ell^n\mathbb{Z})$. For every $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $\gamma,\gamma'\in H_1(X,\mathbb{Z}_\ell)$ we have

$$\langle\sigma(\gamma),\sigma(\gamma')\rangle=\chi_\ell(\sigma)\langle\gamma,\gamma'\rangle,$$

where $\chi_\ell(\sigma)$ is the ℓ-cyclotomic character.

Indeed, consider the Jacobian variety $J(X)$ for the curve X. By construction of the Jacobian variety as a quotient of its tangent space at the identity element it is clear that $H_1(J(X),\mathbb{Z}) = H_1(X,\mathbb{Z})$ and after tensoring with $\mathbb{Z}_\ell$ the same equality holds for the pro-ℓ homology groups. Consider the following diagram

$$\begin{tikzcd} H_1(X,\mathbb{Z}) \times H_1(X,\mathbb{Z}) \arrow[r, "\langle\cdot,\cdot\rangle"] \arrow[d] & \mathbb{Z} \arrow[d] \\ T_\ell(J(X)) \times T_\ell(J(X)) \arrow[r, "e^\lambda"] & \mathbb{Z}_\ell(1) = \varprojlim \mu_{\ell^n} \subset \bar{\mathbb{Q}}, \end{tikzcd}$$

where the down horizontal array is given by the Weil pairing e^λ with respect to the canonical polarization λ, and the upper map is the homology intersection form. The arrows pointing down on the left are the obvious ones, while the down pointing arrow $\mathbb{Z} \to \varprojlim \mu_{\ell^n}$ is given by $\mathbb{Z} \ni m \mapsto (\ldots, e^{\frac{2\pi i m}{\ell^n}}, \ldots)$. The above diagram is known to commute with a negative sign, see [20, p. 237], [19, ex.13.3 p.58] that is

$$e^\lambda(a,a') = (\ldots, e^{-\frac{2\pi i\langle a,a'\rangle}{\ell^n}}, \ldots)$$

By selecting a primitive ℓ^n-root of unity for every n, say $e^{2\pi i/\ell^n}$ we can write $\mathbb{Z}_\ell(1)$ as an additive module, that is we can send

$$\mathbb{Z}_\ell(1) \ni \alpha = (\ldots, e^{2\pi i a_n/\ell^n}, \ldots) \mapsto (\ldots, a_n, \ldots) \in \mathbb{Z}_\ell.$$

It is known that the Weil pairing induces a symplectic pairing in $T_\ell(J(X)) \cong H_1(X,\mathbb{Z}_\ell)$, [18, prop. 16.6],[2], [15] so that

$$\langle \sigma a, \sigma a' \rangle = \chi_\ell(\sigma)\langle a, a' \rangle.$$

In this way we obtain a representation

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}(2g, \mathbb{Z}_\ell)$$

which is the arithmetic analogon of the representation given in eq. (25).

References

[1] Jannis A. Antoniadis and Aristides Kontogeorgis. On cyclic covers of the projective line. *Manuscripta Math.*, 121(1):105–130, 2006.

[2] Sara Arias-de Reyna, Cécile Armana, Valentijn Karemaker, Marusia Rebolledo, Lara Thomas, and Núria Vila. Large Galois images for Jacobian varieties of genus 3 curves. *Acta Arith.*, 174(4):339–366, 2016.

[3] G. V. Belyĭ. Galois extensions of a maximal cyclotomic field. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(2):267–276, 479, 1979.

[4] Joan S Birman and Hugh M Hilden. Isotopies of homeomorphisms of Riemann surfaces and a theorem about Artin's braid group. *Bull. Am. Math. Soc.*, 78(6):1002–1004, November 1972.

[5] Oleg Bogopolski. *Introduction to group theory*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. Translated, revised and expanded from the 2002 Russian original.

[6] Hershel M. Farkas and Irwin Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1980.

[7] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[8] Yasutaka Ihara. Profinite braid groups, Galois representations and complex multiplications. *Ann. of Math. (2)*, 123(1):43–106, 1986.

[9] Yasutaka Ihara. Arithmetic analogues of braid groups and Galois representations. In *Braids (Santa Cruz, CA, 1986)*, volume 78 of *Contemp. Math.*, pages 245–257. Amer. Math. Soc., Providence, RI, 1988.

[10] Christian Kassel and Vladimir Turaev. *Braid groups*, volume 247 of *Graduate Texts in Mathematics*. Springer, New York, 2008. With the graphical assistance of Olivier Dodane.

[11] Hisatoshi Kodani, Masanori Morishita, and Yuji Terashima. Arithmetic Topology in Ihara Theory. *Publ. Res. Inst. Math. Sci.*, 53(4):629–688, 2017.
[12] Aristides Kontogeorgis. The group of automorphisms of cyclic extensions of rational function fields. *J. Algebra*, 216(2):665–706, 1999.
[13] Aristides Kontogeorgis. Field of moduli versus field of definition for cyclic covers of the projective line. *J. Théor. Nombres Bordeaux*, 21(3):679–692, 2009.
[14] Serge Lang. *Introduction to algebraic and abelian functions*, volume 89 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition, 1982.
[15] Pedro Lemos. *Residual Representations of Abelian Varieties*. PhD thesis, University of Warwick, 2017. Available at http://wrap.warwick.ac.uk/94788/.
[16] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory*. Dover Publications, Inc., Mineola, NY, second edition, 2004. Presentations of groups in terms of generators and relations.
[17] Curtis T. McMullen. Braid groups and Hodge theory. *Math. Ann.*, 355(3):893–946, 2013.
[18] J. S. Milne. Abelian varieties. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 103–150. Springer, New York, 1986.
[19] James S. Milne. Abelian varieties (v2.00), 2008. Available at www.jmilne.org/math/.
[20] David Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.
[21] Jürgen Neukirch. *Class field theory*. Springer, Heidelberg, 2013. The Bonn lectures, edited and with a foreword by Alexander Schmidt, Translated from the 1967 German original by F. Lemmermeyer and W. Snyder, Language editor: A. Rosenschon.
[22] V. V. Prasolov and A. B. Sossinsky. *Knots, links, braids and 3-manifolds*, volume 154 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1997. An introduction to the new invariants in low-dimensional topology, Translated from the Russian manuscript by Sossinsky [Sosinskiĭ].
[23] Luis Ribes and Pavel Zalesskii. *Profinite groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2010.

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