INEQUALITY

Schur's Inequality

Comprehensive Study

Gaussian Curvature

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Contents

Acknowledgements

This handout is on " The Wonderful Schur's Inequality " is intended for inequalities and it has been authored by Aritra12. The handout basically covers up some of the important cases, results & some techniques on Schur's Inequality, I am also thankful to the several resources which I listed also forgot some to enlist also users on AoPS who posted problems and solution. No handout can be all perfect so if you see any problem or typos don't forget to gmail at gaussiancurvature360@gmail.com.

0.1 Structure

The topic is basically divided into 9 parts namely the

- Schur's Inequality
  Introduction about it , what is schur's Inequality & its proof.
- The $k = 1$ case
  One of the most often used case of the schur's Inequality or the third degree Schur's Inequality.
- The $k = 2$ case
  One of the most often used case of the schur's Inequality or the fourth degree Schur's Inequality.
- Revisiting the $k = 1$ case
  Revisiting it for some more forms & variants with some more examples.
- Revisiting the $k = 2$ case
  Revisiting it for some more forms & variants with some more examples.
- Generalised Schur-Vornicu Schur Inequality
  About it & some example problems.
- Extension
  Extension of schur's Inequality to five , six seven variables.
- Two beautiful Results
  two beautiful results which can be deduced from schur namely Euler's Inequality & Gerretsen's Inequality.

- Some related problems
  Title is self explanatory also hints to selected problems are provided.

Chapter 1

Schur's Inequality

1.1 Schur's Inequality

Theorem 1.1.1 (Schur's Inequality) — For non-negative real numbers a, b, c and for $k \in \mathbb{R}^+$,

$$a^k(a-b)(a-c)+b^k(b-a)(b-c)+c^k(c-a)(c-b) \geq 0$$

Proof.[1] (Without loss of generality) That $0 \leq x \leq y \leq z$. At a minimum, this lets us identify some positive terms. For example, we now see that the first summand is non negative, so we can look at what's left. The common factor is of these two terms is $z-y$, and when we take it out our sum becomes

$$a^k(a-b)(a-c)+(c-b)\left\{c^k(c-a)-b^k(b-a)\right\}$$

By grace of the Fates, we now see why the second term is also non negative both factors of $c^k(c-a)$ are at least as large as the factors of $b^k(b-a)$ Finally, we can read off the case of equality from the above statement using the observation that for the sum to be zero, both summands must be zero. There are some special forms of schur's inequality(basically for $k=1,2$) which have been shown below, they are usually used in solving this inequalities.

Case 1 $(k=1)$**.** When $k=1$ in schur's inequality,

$$a^3+b^3+c^3+3abc \geq a^2(b+c)+b^2(a+c)+c^2(a+b)$$

also this is true

$$(a+b+c)^2+\frac{9abc}{a+b+c} \geq 4(ab+bc+ca)$$

Problem 1.1.1. Let a,b,c be three real numbers. Prove that

$$a^6+b^6+c^6+a^2b^2c^2 \geq \frac{2}{3}\left(a^5(b+c)+b^5(c+a)+c^5(a+b)\right)$$

According to AM-GM inequality and Schur inequality, we deduce that

$$\begin{aligned}3\sum_{cyc}a^6+3a^2b^2c^2 &\geq 2\sum_{cyc}a^6+\sum_{cyc}a^4\left(b^2+c^2\right)\\ &=\sum_{cyc}\left(a^6+a^4b^2\right)+\sum_{cyc}\left(a^6+a^4c^2\right) \geq 2\sum_{cyc}a^5(b+c)\end{aligned}$$

[1] http://www-stat.wharton.upenn.edu/ steele/Publications/Books/CSMC/New%20Problems/SchursInequality/SchursInquality

Problem 1.1.2. $a, b,$ and c are non-negative reals such that $a+b+c=1$. Prove that

$$a^3+b^3+c^3+6abc \geq \frac{1}{4}$$

Recall of schur's inequality for $k=1$, we know that,

$$a^3+b^3+c^3+3abc \geq a^2(b+c)+b^2(a+c)+c^2(a+b)$$

Now back to the main problem, just multiply both sides by 4,

$$4(a^3+b^3+c^3)+24abc \geq 1$$

Note that $a+b+c=1$ so $(a+b+c)^3=1$ So we can also rewrite that,

$$4(a^3+b^3+c^3)+24abc \geq (a+b+c)^3=1$$

Hence we get this inequality,

$$a^3+b^3+c^3+6abc \geq a^2(b+c)+b^2(c+a)+c^2(a+b)$$

which obviously is true as we got in the first line.

Problem 1.1.3. Let a, b, c be positive reals such that $a+b \geq c; b+c \geq a$; and $c+a \geq b$, we have

$$2a^2(b+c)+2b^2(c+a)+2c^2(a+b) \geq a^3+b^3+c^3+9abc$$

After checking that equality holds for $(a,b,c)=(t,t,t)$ and $(2t,t,t)$, it is apparent that more than straight AM-GM will be required. To handle the condition, put $a=y+z, b=z+x, c=x+y$ with $x, y, z \geq 0$. Now, the left hand side becomes

$$4x^3+4y^3+4z^3+10x^2(y+z)+10y^2(z+x)+10z^2(x+y)+24xyz$$

while the right hand side becomes

$$2x^3+2y^3+2z^3+12x^2(y+z)+12y^2(z+x)+12z^2(x+y)+18xyz$$

The desired is seen to be equivalent to

$$x^3+y^3+z^3+3xyz \geq x^2(y+z)+y^2(z+x)+z^2(x+y),$$

which is Schur's inequality. Equality holds where $x=y=z$, which gives $(a,b,c)=(t,t,t)$, or when two of x, y, z are equal and the third is 0, which gives $(a,b,c) \in \{(2t,t,t),(t,2t,t),(t,t,2t)\}$

Problem 1.1.4. For $x,y,z \geq 0$, the following inequality holds:

$$5\left(\sum\sqrt{x+y}\right)\left(\sum\sqrt{(x+y)(y+z)}\right) \geq \left(\sum\sqrt{x+y}\right)^3 + 18\sqrt{(x+y)(y+z)(z+x)}$$

If $(x+y)(y+z)(z+x)=0$, then at least two of x,y,z vanish so that the inequality reduces to $10a\sqrt{a} \geq 8a\sqrt{a}$, for some $a \geq 0$, which is obviously true. Assume $(x+y)(y+z)(z+x) \neq 0$. Then $\sqrt{x+y}, \sqrt{y+z}, \sqrt{z+x}$ form the sides of a triangle. The segments of the sides from the vertices of the triangle to the points of tangency with the incircle split each of the sides into two pats, thus insuring the existence of $a,b,c>0$ such that $\sqrt{x+y}=a+b, \sqrt{y+z}=b+c, \sqrt{z+x}=c+a$. In terms of a,b,c the inequality becomes

$$5(a+b+c)\left[a^2+b^2+c^2+3(ab+bc+ca)\right] \geq 4(a+b+c)^3 + 9(a+b)(b+c)(c+a)$$

Let for ou ease $S=a^3+b^3+c^3, s=ab(a+b)+bc(b+c)+ca(c+a)$, and $p=abc$. The straightforward algebraic manipulation reduces the above inequality to

$$5S+5s+15s+45p \geq 4S+12s+24p+9s+18p$$

which is simplified to $S+3p \geq s$ that is again the k=1 case of schur's Inequality.

Problem 1.1.5 (Cezar Lupu). Let a,b,c be positive reals such that $a+b+c+abc=4$. Prove that

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\frac{\sqrt{2}}{2}\cdot(a+b+c)$$

Note that Schur's inequality for $k=1$ can be expressed as

$$\frac{9abc}{a+b+c}\geq 4(ab+bc+ca)-(a+b+c)^2$$

let us claim that so as to proof the other , somehow going to follow the method of contradiction,

Claim— $ab+bc+ca>a+b+c$

$$\frac{9abc}{a+b+c}\geq 4(ab+bc+ca)-(a+b+c)^2>(a+b+c)(4-(a+b+c))=abc(a+b+c)$$

Hence according to our claim $a+b+c<3$,but wait also note that on the other hand $abc<1$, which implies $a+b+c+abc=4<4$ which is a clear contradiction. So the other

$$ab+bc+ca<a+b+c$$

is true & now by Cauchy Schwarz

$$\left(\sum_{cyc}a\sqrt{b+c}\right)\left(\sum_{cyc}\frac{a}{\sqrt{b+c}}\right)\geq(a+b+c)^2$$

But, also by Cauchy,

$$\sqrt{(a+b+c)(a(b+c)+b(c+a)+c(a+b))}\geq\sum_{cyc}a\sqrt{b+c}$$

Hence its true by our proved equation before that

$$\sum_{cyc}\frac{a}{\sqrt{b+c}}\geq\frac{\sqrt{2}}{2}\cdot(a+b+c)\cdot\sqrt{\frac{a+b+c}{ab+bc+ca}}\geq\frac{\sqrt{2}}{2}\cdot(a+b+c)$$

Case 2 $(k=2)$. When $k=2$ in schur's inequality,

$$a^4+b^4+c^4+abc(a+b+c)\geq a^3b+a^3c+b^3a+b^3c+c^3a+c^3b$$

Problem 1.1.6 (srijonrick). Prove that

$$\left(\sum_{\text{cyc}}(a-x)^4\right) + 2\left(\sum_{\text{sym}}x^3y\right) + 4\left(\sum_{\text{cyc}}x^2y^2\right) + 8xyza \geqslant \left(\sum_{\text{cyc}}(a-x)^2(a^2-x^2)\right)$$

where $a = x+y+z$ and $x,y,z \in \mathbb{R}$.

First recall Schur's Inequality with $k=2$,

$d^4+b^4+c^4+dbc(d+b+c) \geqslant d^3b+d^3c+b^3d+b^3c+c^3d+c^3b = d^3(b+c)+b^3(d+c)+c^3(d+b)-(*)$

Now in $(*)$ if we take

$$d = x+y, b = y+z \text{ and } c = z+x$$

we get,
$(x+y)^4+(y+z)^4+(z+x)^4+(x+y)(y+z)(z+x)2(x+y+z) \geqslant (x+y)^3(x+y+2z)+(y+z)^3(2x+y+z)+(z+x)^3(x+2y+z)$
Now it's given that $a = x+y+z$, using that along with the above equation we get,

$$(a-x)^4+(a-y)^4+(a-z)^4+E \geqslant (a-x)^3(a+x)+(a-y)^3(a+y)+(a-z)^3(a+z)-(1)$$

where $E = 2(x+y)(y+z)(z+x)(x+y+z)$
If we expand E we get it as

$$2x^3y+2x^3z+4x^2y^2+4x^2z^2+8x^2yz+2xy^3+2xz^3+8xyz^2+8xy^2z+2yz^3+4y^2z^2+2y^3z.$$

Now recall notations

$$\sum_{\text{cyc}}x^2y^2 = x^2y^2+y^2z^2+z^2x^2 \text{ and } \sum_{\text{sym}}x^3y = x^3y+x^3z+y^3z+y^3x+z^3y+z^3x$$

Hence $E = 2\left(\sum_{\text{sym}}x^3y\right) + 4\left(\sum_{\text{cyc}}x^2y^2\right) + 8xyza-(2)$ and $(a-x)^3(a+x) = (a-x)^2(a^2+x^2)-(3)$
On combining $(1),(2)$ and (3) we get the desired inequality:

$$\left(\sum_{\text{cyc}}(a-x)^4\right) + 2\left(\sum_{\text{sym}}x^3y\right) + 4\left(\sum_{\text{cyc}}x^2y^2\right) + 8xyza \geqslant \left(\sum_{\text{cyc}}(a-x)^2(a^2-x^2)\right)$$

Note. This two cases are generally in use in schur's inequality but don't think that problem are only with this two cases, these are just the two important cases of Schur's Inequality.

Case 3 (Third Degree Schur Inequality). Note again though same as the before we will revisit case of $k=1$ also useful form

$$\left(a^2+b^2+c^2\right)(a+b+c)+9abc \geq 2(a+b+c)(ab+bc+ca)$$

Even that is a third degree Schur Inequality

Oh also pls don't think they are bound to be 1 or 2 also can be 3, now if we let also now take $k = 3$,

Problem 1.1.7 (Aritra12). Let p, q, r be non negative reals such that $pqr = 1$. Find the maximum value for the expression

$$\sum_{cyc} p[r^4 + q^4 - p^4 - p]$$

This is Schur's Inequality with $k = 3$, Note that Schur's Inequality with $k = 3$ gives

$$\sum_{cyc} p^3(p-q)(p-r) \geq 0$$

Which implies that,

$$\sum_{cyc} p^5 + \sum_{cyc} p^3 qr \geq \sum_{cyc} p^4 q + \sum_{cyc} p^4 r$$

since

$$pqr = 1 \text{ and } \sum_{cyc} p^4 q = \sum_{cyc} pr^4, \quad \sum_{cyc} p^4 r = \sum_{cyc} pq^4$$

It concludes that,

$$\sum_{cyc} p^5 + \sum_{cyc} p^2 \geq \sum_{cyc} pr^4 + \sum_{cyc} pq^4$$

$$\sum_{cyc} p[r^4 + q^4 - p^4 - p] \leq 0,$$

Hence the maximum value of the expression is 0 with equality holding iff $p = q = r = 1$.

Problem 1.1.8. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{\sqrt{(1b+c)(b^8+c^3)}} + \frac{b^2}{\sqrt{(c+a)(c^8+a^3)}} + \frac{c^2}{\sqrt{(a+b)(a^8+b^3)}} \geq \frac{3}{2}$$

(Pham [Kim Hung)

By Holder inequality, we deduce that

$$\left(\sum_{cyc} \frac{a^2}{\sqrt{(b+c)(b^3+c^3)}}\right)^2 \left(\sum_{cyc} a^2(b+c)(b^3+c^3)\right) \geq \left(\sum_{cyc} a^2\right)^3$$

So it is enough to prove that

$$4\left(\sum_{cyc} a^2\right)^3 \geq 9\sum_{cyc} a^2(b+c)\left(b^3+c^3\right)$$

$$\Leftrightarrow 4\sum_{cyc} a^6 + 3\sum_{cyc} a^4\left(b^2+c^2\right) + 24a^2b^2c^2 \geq 9abc\sum_{cyc} a^2(b+c).$$

According to the third degree-Schur inequality $\sum_{cyc} a^2(b+c) \leq \sum_{cyc} a^3 + 3abc$, so it's enough to prove that

$$4\sum_{cyc} a^6 + 3\sum_{cyc} a^4\left(b^2+c^2\right) \geq 9\sum_{cyc} a^4bc + 3a^2b^2c^2$$

which is obvious by AM-GM inequality because

$$2\sum_{cyc} a^6 = \sum_{cyc}\left(a^6+b^6\right) \geq \sum_{cyc} a^2b^2\left(a^2+b^2\right) = \sum_{cyc} a^4\left(b^2+c^2\right) \geq 2\sum_{cyc} a^4bc \geq 6a^2b^2c^2$$

We are done. Equality holds for $a=b=c$.
Time for some substitution,

Problem 1.1.9. Let a,b,c be positive real numbers with $abc=1$. Prove that

$$\sum_{cyc}(a+bc) \leq 3+\frac{a}{c}+\frac{b}{a}+\frac{c}{b}$$

Let's put

$$a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x}$$

and the inequality becomes:

$$\sum_{cyc}\left(\frac{x}{y}+\frac{y}{x}\right) \leq 3+\sum_{cyc}\frac{x^2}{yz}$$

$$\Longleftrightarrow \sum_{cyc}\left(x^2y+xy^2\right) \leq 3xyz+\sum_{cyc} x^3$$

which is exactly Schur's inequality. This substitution

$$a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x}$$

has very wide use in many problems also in contests like USAMO/IMO.

Problem 1.1.10 (Tournament of Towns 1997). Let a,b,c be positive numbers such that abc = Prove that

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \leq 1$$

We can rewrite the given inequality as following :

$$\frac{1}{a+b+(abc)^{1/3}}+\frac{1}{b+c+(abc)^{1/3}}+\frac{1}{c+a+(abc)^{1/3}}\leq\frac{1}{(abc)^{1/3}}$$

We make the substitution $a=x^3, b=y^3, c=z^3$ with $x,y,z>0$. Then, it becomes

$$\frac{1}{x^3+y^3+xyz}+\frac{1}{y^3+z^3+xyz}+\frac{1}{z^3+x^3+xyz}\leq\frac{1}{xyz}$$

which is equivalent to

$$xyz\sum_{\text{cyclic}}\left(x^3+y^3+xyz\right)\left(y^3+z^3+xyz\right)\leq\left(x^3+y^3+xyz\right)\left(y^3+z^3+xyz\right)\left(z^3+x^3+xyz\right)$$

or

$$\sum_{\text{sym}}x^6y^3\geq\sum_{\text{sym}}x^5y^2z^2$$

Which is nothing but Schur.

Problem 1.1.11. For the above example determine how this satisfies with schur's Inequality

$$\sum_{\text{sym}}x^6y^3\geq\sum_{\text{sym}}x^5y^2z^2$$

> **Corollary 1.1.2** — If
>
> $$f(x)=\sum_{k=0}^{\infty}a_kx^k,$$
>
> where $a_k\geq 0$ $(k\in N)$, for all $x,y,z\geq 0$, then
>
> $$(x-y)(x-z)f(x)+(y-z)(y-x)f(y)+(z-x)(z-y)f(z)\geq 0$$

Sketch of proof. Using Schur's Inequality,

$$(x-y)(x-z)\left(a_kx^k\right)+(y-z)(y-x)\left(a_ky^k\right)+(z-x)(z-y)\left(a_kz^k\right)\geq 0$$

List of some corollaries

> **Corollary 1.1.3** —
>
> $$\frac{(x-y)(x-z)}{1-x^k}+\frac{(y-z)(y-x)}{1-y^k}+\frac{(z-x)(z-y)}{1-z^k}\geq 0\,(x,y,z\in(0,1),k\in N^*)$$

Corollary 1.1.4 —

$$(x-y)(x-z)\ln(1-x)+(y-z)(y-x)\ln(1-y)+(z-x)(z-y)\ln(1-z)\leq 0(x,y,z\in(0,1))$$

Corollary 1.1.5 —

$$(x-y)(x-z)\ln\frac{1+x}{1-x}+(y-z)(y-x)\ln\frac{1+y}{1-y}+(z-x)(z-y)\ln\frac{1+z}{1-z}\geq 0(x,y,z\in(0,1))$$

Corollary 1.1.6 —

$$(x-y)(x-z)\ln\frac{x+1}{x-1}+(y-z)(y-x)\ln\frac{y+1}{y-1}+(z-x)(z-y)\ln\frac{z+1}{z-1}\geq 0(x,y,z>1)$$

This are some of the many important results of **schur's inequality**. Why not look over $k=2$ case or fourth degree Schur Inequality again with some more variants?

Case 4 (Fourth Degree Schur's Inequality). Also can be written as

$$(a+b+c)\left(a^3+b^3+c^3+3abc\right)\geq 2\left(a^2+b^2+c^2\right)(ab+bc+ca)$$

also

$$a^2+b^2+c^2+\frac{6abc(a+b+c)}{a^2+b^2+c^2+ab+bc+ca}\geq 2(ab+bc+ca)$$

Problem 1.1.12. Let a,b,c be non-negative real numbers with $a+b+c=2$. Prove that

$$a^4+b^4+c^4+abc\geq a^3+b^3+c^3$$

According to the fourth degree Schur inequality, we have

$$a^4+b^4+c^4+abc(a+b+c)\geq a^3(b+c)+b^3(c+a)+c^3(a+b)$$

or

$$2\left(a^4+b^4+c^4\right)+abc(a+b+c)\geq\left(a^3+b^3+c^3\right)(a+b+c)$$

Now putting our given condition of $a+b+c=2$ into the above inequality, we get the our result. Equality holds for $a=b=c=\frac{2}{3}$, or $a=b=1,c=0$ or permutations.

Problem 1.1.13 (Vasile Cirtoaje). Suppose that a,b,c are non-negative real numbers. Prove that

$$\frac{a^2}{2b^2-bc+2c^2}+\frac{b^2}{2c^2-ca+2a^2}+\frac{c^2}{2a^2-ab+2b^2}\geq 1$$

According to Cauchy-Schwarz inequality, we deduce that

$$\sum_{cyc}\frac{a^2}{2b^2-bc+2c^2}\geq\frac{\left(a^2+b^2+c^2\right)^2}{a^2\left(2b^2-bc+2c^2\right)+b^2\left(2c^2-ca+2a^2\right)+c^2\left(2a^2-ac+2b^2\right)}$$

It suffices to prove that

$$\left(\sum_{cyc}a^2\right)^2\geq\sum_{cyc}a^2\left(2b^2-bc+2c^2\right)\Leftrightarrow\sum_{cyc}a^4+abc\left(\sum_{cyc}a\right)\geq 2\sum_{cyc}a^2b^2$$

According to the `fourth degree Schur inequality`, we conclude that

$$\sum_{cyc}a^4+abc\left(\sum_{cyc}a\right)\geq\sum_{cyc}ab\left(a^2+b^2\right)\geq 2\sum_{cyc}a^2b^2$$

We are done. Equality holds for $a=b=c$ and $a=b,c=10$ or permutations.

Problem 1.1.14 (IMO 2000/2). Let a,b,c be positive numbers such that $abc=1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1$$

It is equivalent to the following homogeneous inequality

$$\left(a-(abc)^{1/3}+\frac{(abc)^{2/3}}{b}\right)\left(b-(abc)^{1/3}+\frac{(abc)^{2/3}}{c}\right)\left(c-(abc)^{1/3}+\frac{(abc)^{2/3}}{a}\right)\leq abc$$

After the substitution $a=x^3,b=y^3,c=z^3$ with $x,y,z>0$, it becomes

$$\left(x^3-xyz+\frac{(xyz)^2}{y^3}\right)\left(y^3-xyz+\frac{(xyz)^2}{z^3}\right)\left(z^3-xyz+\frac{(xyz)^2}{x^3}\right)\leq x^3y^3z^3$$

which simplifies to

$$\left(x^2y-y^2z+z^2x\right)\left(y^2z-z^2x+x^2y\right)\left(z^2x-x^2y+y^2z\right)\leq x^3y^3z^3$$

Or

$$3x^3y^3z^3+\sum_{\text{cyclic}}x^6y^3\geq\sum_{\text{cyclic}}x^4y^4z+\sum_{\text{cyclic}}x^5y^2z^2$$

or

$$3\left(x^2y\right)\left(y^2z\right)\left(z^2x\right)+\sum_{\text{cyclic}}\left(x^2y\right)^3 \geq \sum_{\text{sym}}\left(x^2y\right)^2\left(y^2z\right)$$

which is a special case of Schur's inequality.

Theorem 1.1.7 (Vornicu-Schur inequality) — Consider real numbers a,b,c,x,y,z such that $a \geq b \geq c$ and either $x \geq y \geq z$ or $z \geq y \geq x$. Let $k \in \mathbb{Z}_{>0}$ be a positive integer and let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a function from the reals to the nonnegative reals that is either convex or monotonic. Then

$$f(x)(a-b)^k(a-c)^k + f(y)(b-a)^k(b-c)^k + f(z)(c-a)^k(c-b)^k \geq 0.$$

The below is a special case of the above also the below can also be termed as Vornicu-Schur Inequality.

Theorem 1.1.8 (Generalised Schur's Inequality) — Let a,b,c be three reals with $a \geq b \geq c$ and $y \geq x$ and the other pairs , where x,y,z be three non-negative reals. Then, the inequality

$$x(a-b)(a-c)+y(b-c)(b-a)+z(c-a)(c-b) \geq 0$$

Problem 1.1.15. Prove that if $x,y,z \geq 0$ satisfy $xy+yz+zx+xyz=4$ then $x+y+z \geq xy+yz+zx$.

Let us write the given condition as

$$\frac{x}{2}\cdot\frac{y}{2}+\frac{y}{2}\cdot\frac{z}{2}+\frac{z}{2}\cdot\frac{x}{2}+2\frac{x}{2}\cdot\frac{y}{2}\cdot\frac{z}{2}=1$$

Hence there are positive real numbers a,b,c such that

$$x=\frac{2a}{b+c}, \quad y=\frac{2b}{c+a}, \quad z=\frac{2c}{a+b}$$

But now the solution is almost over, since the inequality

$$x+y+z \geq xy+yz+zx$$

is equivalent to

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{2ab}{(c+a)(c+b)}+\frac{2bc}{(a+b)(a+c)}+\frac{2ca}{(b+a)(b+c)}$$

After clearing denominators, the inequality becomes

$$a(a+b)(a+c)+b(b+a)(b+c)+c(c+a)(c+b) \geq 2ab(a+b)+2bc(b+c)+2ca(c+a)$$

After basic computations, it reduces to

$$a(a-b)(a-c)+b(b-a)(b-c)+c(c-a)(c-b)\geq 0$$

But this is Schur's inequality.

Problem 1.1.16 (Duong Du Lam). Given non-negative reals a,b,c which have at least two positive numbers prove that

$$\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2+\frac{1}{2}\geq\frac{5}{4}\cdot\frac{a^2+b^2+c^2}{ab+bc+ca}$$

Multiply both sides of a given inequality by $(a+b)(b+c)(c+a)$, we can rewrite it as

$$\sum\frac{a^2(a+b)(c+a)}{b+c}+\frac{(a+b)(b+c)(c+a)}{2}\geq\frac{5}{4}\cdot\frac{\left(a^2+b^2+c^2\right)(a+b)(b+c)(c+a)}{ab+bc+ca}$$

or

$$\sum\frac{a^2(a-b)(a-c)}{b+c}+2\sum a^3+abc+\frac{\sum ab(a+b)}{2}$$
$$\geq\frac{5}{4}\left(\sum a^2\right)\left(\sum a\right)-\frac{5}{4}\cdot\frac{abc\left(a^2+b^2+c^2\right)}{ab+bc+ca}$$

So now by Vornicu-Schur Inequality

$$\sum\frac{a^2(a-b)(a-c)}{b+c}\geq 0$$

similarly now by AM-GM $\frac{a^2+b^2+c^2}{ab+bc+ca}\geq 1$, so what remains to prove is

$$2\sum a^3+abc+\frac{\sum ab(a+b)}{2}\geq\frac{5}{4}\left(\sum a^2\right)\left(\sum a\right)-\frac{5abc}{4}$$

The rest part is left to the readers.

Problem 1.1.17. How

$$2\sum a^3+abc+\frac{\sum ab(a+b)}{2}\geq\frac{5}{4}\left(\sum a^2\right)\left(\sum a\right)-\frac{5abc}{4}$$

is it true in the above problem?

We see several substitutions work along and play an important role here , using proper substitution is quite powerful while solving this inequalities since a proper one can lead you to the form which you actually want. Some quite useful theorems which you might want want to look at or more precisely when you move upon the bound of three variables .

Theorem 1.1.9 (Extension to 5 variables) — Let $x \geq y \geq z \geq v \geq 0$ and $t > 0$ such that $x+v \geq y+z$, then

$$x^t(x-y)(x-z)(x-v)+y^t(y-x)(y-x)(y-v)+z^t(z-x)(z-y)(z-v)+v^t(v-x)(v-y)(v-z) \geq 0$$

Using the generalisation of Schur's Inequality

$$f(x)(x-y)(x-z)+f(y)(y-x)(y-z)+f(z)(z-x)(z-y) \geq 0.$$

We can write the following Schur's type inequality,

$$\begin{gathered} f(x)(x-y)(x-z)(x-v)+f(y)(y-x)(y-z)(y-v)+ \\ +f(z)(z-x)(z-y)(z-v)+f(v)(v-x)(v-y)(v-z) \geq 0 \end{gathered}$$

Theorem 1.1.10 — If the real numbers $x,y,z,v,w,t \in \mathbb{R}$ are such that $x \geq y \geq z \geq v \geq w \geq 0, t > 0$ and $x+v \geq y+z$, then it's true that

$$\begin{aligned} x^2(x-y)(x-z)(x-v)(x-w)+y'(y-x)(y-z)(y-v)(y-w)+ \\ +z'(z-x)(z-y)(z-v)(z-w)+v^t(v-x)(v-y)(v-z)(v-w)+ \\ +w'(w-x)(w-y)(w-z)(w-v) \geq 0 \end{aligned}$$

Theorem 1.1.11 (Extension to 7 variables) — If the real numbers $x,y,z,v,w,u,t \in \mathbb{R}$ are such that $x \geq y \geq z \geq v \geq w \geq u \geq 0, t > 0$ and $x+u \geq y+w \geq z+v$, then

$$\begin{aligned} x^t(x-y)(x-z)(x-v)(x-w)(x-u)+ \\ +y^t(y-x)(y-z)(y-v)(y-w)(y-u)+ \\ +z^t(z-x)(z-y)(z-v)(z-w)(z-u)+ \\ +v^t(v-x)(v-y)(v-z)(v-w)(v-u)+ \\ +w^t(w-x)(w-y)(w-z)(w-v)(w-u)+ \\ +u^t(u-x)(u-y)(u-z)(u-v)(u-w) \geq 0 \end{aligned}$$

Let us denote $a = x-y \geq 0, b = y-z \geq 0, c = z-v \geq 0, d = v-w \geq 0, e = w-u \geq 0$. The condition $x+u \geq y+w$, means $x-y \geq w-u$, i.e. $a \geq e$, and the condition $y+w \geq z+v$, means $y-z \geq v-w$, i.e. $b \geq d$. Now we rewrite the left side of theorem 1.1.7 using

$$\begin{aligned} F(x,y,z,v,w,u,t) = & x^t(x-y)(x-z)(x-v)(x-w)(x-u)+ \\ & +y^t(y-x)(y-z)(y-v)(y-w)(y-u)+ \\ & +z^t(z-x)(z-y)(z-v)(z-w)(z-u)+ \\ & +v'(v-x)(v-y)(v-z)(v-w)(v-u)+ \\ & +w^t(w-x)(w-y)(w-z)(w-v)(w-u)+ \\ & +u^t(u-x)(u-y)(u-z)(u-v)(u-w), \end{aligned}$$

and the variables $a,b,c,d,e : F(x,y,z,v,w,u,t) = x^t \cdot a(a+b)(a+b+c)(a+b+c+d)(a+b+c+d+e)+y^t \cdot (-a)b(b+c)(b+c+d)(b+c+d+e)+z^t \cdot [-(a+b)](-b)c(c+d)(c+d+e)+$

$v^t \cdot [-(a+b+c)][-(b+c)](-c)d(d+e) + w^t \cdot [-(a+b+c+d)][-(b+c+d)][-(c+d)](-d)e + u^t[-(a+b+c+d+e)][-(b+c+d+e)][-(c+d+e)][-(d+e)](-e) =$

$$\begin{aligned}&[x^t \cdot a(a+b)(a+b+c)(a+b+c+d) - u^t \cdot (b+c+d+e)(c+d+e)(d+e)e]\\ &-(b+c+d) \cdot [y^t \cdot ab(b+c)(b+c+d+e) - w^t \cdot (a+b+c+d)(c+d)de] +\\ &+c \cdot [z^t \cdot (a+b)b(c+d)(c+d+e) - v^t \cdot (a+b+c)(b+c)d(d+e)] \geq 0\end{aligned}$$

We will demonstrate successively:

$$\begin{aligned}x^t \cdot a(a+b)(a+b+c)(a+b+c+d) - u^t \cdot (b+c+d+e)(c+d+e)(d+e)e \geq\\ y^t \cdot ab(b+c)(b+c+d+e) - w^t \cdot (a+b+c+d)(c+d)de\end{aligned}$$

$$y^t \cdot ab(b+c)(b+c+d+e) - w^t \cdot (a+b+c+d)(c+d)de \geq 0$$
$$z^t \cdot (a+b)b(c+d)(c+d+e) - v^t \cdot (a+b+c)(b+c)d(d+e) \geq 0$$

In order to show

$$\begin{aligned}x^t \cdot a(a+b)(a+b+c)(a+b+c+d) - u^t \cdot (b+c+d+e)(c+d+e)(d+e)e \geq\\ y^t \cdot ab(b+c)(b+c+d+e) - w^t \cdot (a+b+c+d)(c+d)de\end{aligned}$$

we have

$$x^t \cdot a(a+b)(a+b+c)(a+b+c+d) - u^t \cdot (b+c+d+e)(c+d+e)(d+e)e \geq$$

$$\geq y^t \cdot ab(b+c)(b+c+d+e) - w^t \cdot (a+b+c+d)(c+d)de$$

equivalent to

$$\begin{aligned}&x^t \cdot a(a+b)(a+b+c)(a+b+c+d) + w^t \cdot (a+b+c+d)(c+d)de \geq\\ &\geq y^t \cdot ab(b+c)(b+c+d+e) + u^t \cdot (b+c+d+e)(c+d+e)(d+e)e\end{aligned}$$

But $a \geq e$ so $a+b+c+d \geq b+c+d+e$. This means that the above inequality is valid if we demonstrate

$$x^t \cdot a(a+b)(a+b+c) + w^t \cdot (c+d)de \geq y^t \cdot ab(b+c) + u^t \cdot (c+d+e)(d+e)e$$

The above inequality can be written in the form: $x^t \cdot \left[a^3 + a^2(b+b+c) + ab(b+c)\right] + w^t \cdot (c+d)de \geq y^t \cdot ab(b+c) + u^t \cdot \left[e^3 + e^2(c+d+d) + e(c+d)d\right]$. Because $a \geq e \geq 0$ and $b \geq d \geq 0$ and $x \geq y \geq w \geq u \geq 0$ we have $x^t \cdot \left[a^3 + a^2(b+b+c)\right] \geq u^t \cdot \left[e^3 + e^2(c+d+d)\right]$ $x^t \cdot ab(b+c) \geq y^t \cdot ab(b+c), w^t \cdot (c+d)de \geq u^t \cdot e(c+d)d$. Adding these inequalities we show that above is true. In order to prove

$$y^t \cdot ab(b+c)(b+c+d+e) - w^t \cdot (a+b+c+d)(c+d)de \geq 0$$

we have $y^t \geq w^t \geq 0, a \cdot (b+c+d+e) = a \cdot (b+c+d) + ae \geq ae + (b+c+d) \cdot e = (a+b+c+d) \cdot e \geq 0$ and $b(b+c) \geq d(d+c) = (c+d)d \geq 0$. Multiplying these inequalities term by term we obtain

$$y^t \cdot ab(b+c)(b+c+d+e) - w^t \cdot (a+b+c+d)(c+d)de \geq 0$$

Now we demonstrate

$$z^t \cdot (a+b)b(c+d)(c+d+e) - v^t \cdot (a+b+c)(b+c)d(d+e) \geq 0$$

: $z^t \geq v^t \geq 0$ and $(a+b)b\left[c^2+c(d+d+e)\right] \geq d(d+e)\left[c^2+c(a+b+b)\right]$, because from $a \geq e$ and $b \geq d$ we get $(a+b)bc^2 \geq d(d+e)c^2$ and $(a+b)bc(d+d+e) \geq d(d+e)c(a+b+b)$ equivalent to $(a+b)b(d+e)+(a+b)bd \geq d(d+e)(a+b)+d(d+e)b$. But $(a+b)b(d+e) \geq (a+b)d(d+e) = d(d+e)(a+b)$, and $(a+b)bd \geq (e+d)bd = d(d+e)b$. Now by addition term by term we obtain the required inequality

$$z^t \cdot (a+b)b(c+d)(c+d+e) - v^t \cdot (a+b+c)(b+c)d(d+e) \geq 0$$

To finish the proof we have $a+b+c+d+e \geq b+c+d \geq 0, c \geq 0$ and if we use the above three deduced inequalities we can deduce our first expression. Using the hypothesis of schur's Inequality we can observe that in the Schur type inequality extension upto 7 variables we realize the equality in the following cases: i, $x=y=z=v=w=u \geq 0$; ii, $x=y=z=v=w \geq 0, u=0$; iii, $x=y=z=v \geq 0, w=u=0$; iv, $x=y=z \geq 0$ and $v=w=u=0$. v,$x=y \geq 0$ and $z=v=w=u=0$. Two beautiful results which can be resulted from our wonderful schur inequality

Theorem 1.1.12 (Euler's Inequality) — It states that if the circumradius of a triangle is R and the inradius is r, then $R \geq 2r$. If this is expressed in terms of the sides a,b,c, we get

$$\frac{abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \geq \sqrt{\frac{(-a+b+c)(a-b+c)(a+b-c)}{a+b+c}}$$

which is equivalent to

$$abc \geq (-a+b+c)(a-b+c)(a+b-c)$$

Theorem 1.1.13 (Gerretsen's Inequality) — It states that if s,r,R denotes the semi-perimeter, inradius and circumradius of a triangle, then

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + r^2$$

1.2 Problems

Problem 1.2.1. Show that $\forall x, y, z \in R$ we have:

$$\sum x^6 + 3x^2y^2z^2 \geq 2\sum x^3y^3$$

Hints: 1

Problem 1.2.2. Prove for all non-negative real numbers a, b, c that

$$\sum_{cyc}(a-b)^3 \geq \sum_{cyc}(a+b-c)(b-a)(a-c)$$

Problem 1.2.3. Given $a, b, c \geq 0$ such that $(a+b)(b+c)(c+a) = 8$, prove that:

$$a^3 + b^3 + c^3 + abc + \frac{108}{(a+b+c)^3} \geq 8.$$

Hints: 2

Problem 1.2.4. Prove that for any three non negative reals a, b, c, we have

$$\left(b^2-c^2\right)^2 + \left(c^2-a^2\right)^2 + \left(a^2-b^2\right)^2 \geq 4(b-c)(c-a)(a-b)(a+b+c)$$

Hints: 3

Problem 1.2.5 (IMO 1983). · Prove that if a, b, c are sidelengths of a triangle, then

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Hints: 4

Problem 1.2.6. Prove that if a, b, c are sidelengths of a triangle, and $t \geq 1$ is a real, then

$$(t+1)\sum a^3b \geq \sum ab^3 + tabc(a+b+c)$$

Hints: 5

Problem 1.2.7 (Nguyen Huy Tan). Prove the following inequality with non-negative real numbers a,b,c such that $ab+bc+ca>0$

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(a+c)}{c^2+ca+a^2}+\frac{c(a+b)}{a^2+ab+b^2}\geq 2+\frac{3[(a-b)(b-c)(c-a)]^2}{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)}$$

Hints: 6

Problem 1.2.8. Let $x,y,z\in(0,\infty)$ such that $x+y+z=1$. Prove that:

$$\frac{x}{xy+1}+\frac{y}{yz+1}+\frac{z}{zx+1}\geq\frac{36xyz}{13xyz+1}$$

Problem 1.2.9 (Nguyen Huy Tung). For positive numbers a,b,c prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3\sqrt[3]{\frac{9(a^3+b^3+c^3)}{(a+b+c)^3}}$$

Hints: 7

1.3 Hints to selected problems

1. As $x^2+y^2 \geq 2xy$, $\sum x^6+3x^2y^2z^2 \geq \sum x^2y^2(x^2+y^2) \geq 2\sum x^3y^3$.

2. By schur's inequality, $4\left(a^3+b^3+c^3\right)+15abc \geq (a+b+c)^3$

3. Rewrite this inequality in the form $\sum(a+b)^2(a-b)(a-c) \geq 0$

4. Rewrite this as

$$\sum c(a+b-c)(a-b)(a-c) \geq 0.$$

Now, this follows, it is shown that

$$c(a+b-c), a(b+c-a), b(c+a-b)$$

are the side lengths of a triangle.

5. For $t=1$, it becomes

$$\sum b(b+a)(a-b)(a-c) \geq 0$$

what follows from

$$\sum b^2(a-b)(a-c) \geq 0$$

and

$$\sum ba(a-b)(a-c) \geq 0.$$

6. Note that

$$\sum \frac{a(b+c)}{b^2+bc+c^2}-2=\frac{1}{ab+bc+ca}\sum\left[\frac{a(b+c)(ab+bc+ca)}{b^2+bc+c^2}-a(b+c)\right]$$

$$=\sum \frac{ab(a-b)^2}{\left(b^2+bc+c^2\right)\left(c^2+ca+a^2\right)}$$

Try to induce an inequality from it and try to take out this expression

$$2abc[a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)] \geq 0$$

7. Basically

$$4(x+y+z)^3 \geq 27\left(x^2y+y^2z+z^2x+xyz\right)$$

Use our famous substitution for a,b,c as

$$x=\frac{a}{b}\ ,\ y=\frac{b}{c}$$

and the similar for z. Try to considerate out some proper form and then apply Vornicu-Schur Inequality.

1.4 References

- ☐ Secrets in Inequality by Pham kim Hung
- ☐ Olympiad Inequalities by Thomas J. Mildorf
- ☐ Vornicu-Schur Inequality by Nguyn Huy Tùng
- ☐ A Schur type inequality for seven variables
- ☐ A Schur type inequality for five variables

Further reading

Schur's Inequality has a very important role in the pqr method, for understanding you can check this two handouts & yes the below two are links.

- ☐ pqr method (phoenixfire , Aritra12)
- ☐ An Interesting Property of Quadratic Polynomials (Aritra12 , Do Xua Trong)