PC 2021 P2

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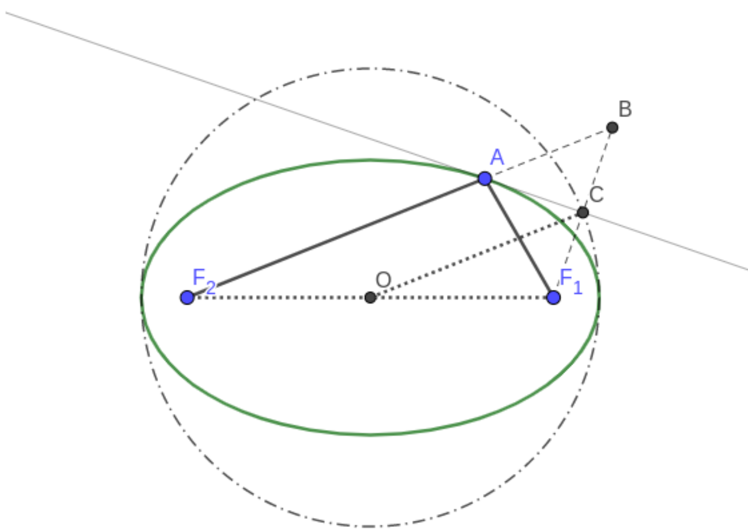
I make use of a few properties of an ellipse which I now present as lemmas. Assume that the ellipse has centre O, focii F_1 and F_2, and semi-minor axis a.

Lemma 1 Perpendicular from a focus onto a tangent meets the tangent on the auxiliary circle (which is the circle that is concentric with the ellipse and whose radius equals the semi-major axis of the ellipse)

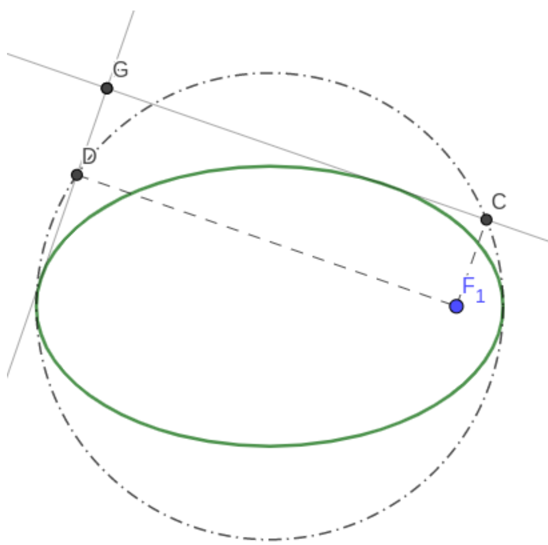


Proof: Refer to the figure. B is the reflection of F_1 in the tangent and C is the perpendicular drawn on the tangent from F_1. By the reflection property of the ellipse (which is pretty well known so I won't prove here), F_2, A and B are collinear. Also, because B is the reflection of F_1, $AF_1 = AB$. This means $F_2B = F_2A + AB = F_2A + AF_1 = 2a$.

Now, since C is the midpoint of F_1B and O is the midpoint of F_1F_2, the triangles F_1OC and F_1F_2C are similar, and scaled by a factor of 2 (this is also called the mid-point theorem in geometry). Thus $OC = \frac{1}{2}F_2B = a$, and therefore, C lies on the auxiliary circle. ■

Lemma 2 Perpendiculars drawn from a focus onto two perpendicular tangents are perpendicular to each other.

This isn't much of a property specific to an ellipse but rather of lines in general. We can prove this easily by seeing that F_1DGC must be a rectangle since three of it angles are already $90°$ (refer to the figure below) ■



Now on to the physics. I'll use the above figure as a reference. If it's not very clear from the figure, C and D are the feet of the perpendiculars drawn from the focus to the perpendicular tangents. The velocity at each point on the ellipse is along the tangent to the ellipse. Since v_2 has a larger magnitude, we must have that it must be along the tangent through C, as it must have a smaller perpendicular distance from the focus (because angular momentum is conserved about F_1). Using conservation of angular momentum about F_1, we get $\frac{F_1C}{F_1D} = \frac{|\vec{v}_1|}{|\vec{v}_2|} = \frac{1}{2}$

Now the problem reduces to a mathematical one :

> At what minimum distance must a point (F_1) be from the centre of a circle so that two mutually perpendicular segments can be drawn from that point, with the other end being on the circle, such that length of one is twice the other.

For an ellipse with eccentricity e, the focus is at a distance of ae from the centre. So knowing this minimum distance allows us to find e. To keep the equations short, I'll assume $a = 1$, since scaling the system would not change the value of e. Let the first segment drawn from the focus be of length r and at an angle θ with the x-axis. The second segment needs to be of a length $2r$ and at an angle $\frac{\pi}{2} + \theta$ (it doesn't matter whether we add or subtract θ, since θ will come out to be negative if it is in opposite direction).

Since the focus is $(e, 0)$, the two other end points of these segments would be $(e + r\cos\theta, r\sin\theta)$ and $(e - 2r\sin\theta, 2r\cos\theta)$

Both of these must lie on $x^2 + y^2 = 1$

Substituting the values and rearranging each equation gives

$$e^2 + r^2\cos^2\theta + 2er\cos\theta + r^2\sin^2\theta = 1 \implies \cos\theta = \frac{1 - e^2 - r^2}{2er}$$

and

$$e^2 + 4r^2\sin^2\theta + 4er\sin\theta + 4r^2\cos^2\theta = 1 \implies \sin\theta = \frac{e^2 + 4r^2 - 1}{4er}$$

Squaring and adding the above equations, we get after a few simplifications:

$$20r^4 - 16r^2 + 5(1 - e^2)^2 = 0 \tag{1}$$

This is a quadratic equation in r^2 and if it has a positive root then we can find a pair segments that satisfy the condition we want.

Applying the quadratic formula for (1), we get

$$r^2 = \frac{16 \pm \sqrt{\Delta}}{40}$$

, where $\Delta = 16^2 - 4 \times 20 \times 5(1 - e^2)^2 = 16 \times 25 \times ((\frac{4}{5})^2 - (1 - e^2)^2)$

This shows that if $\Delta \geq 0$, we will always have two positive roots. So the only condition we have is

$$\Delta \geq 0 \longleftrightarrow (\frac{4}{5})^2 \geq (1 - e^2)^2 \longleftrightarrow e^2 \geq \frac{1}{5} \longleftrightarrow \boxed{e \geq \frac{1}{\sqrt{5}}}$$