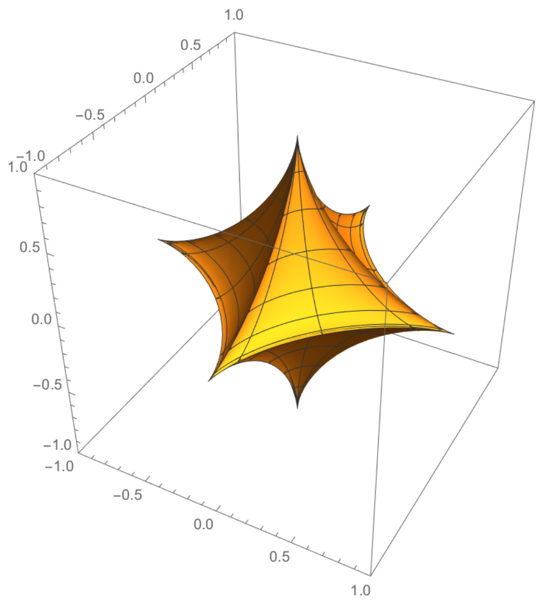


GAMO & GJMO

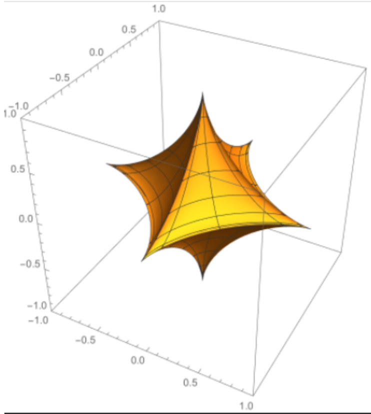
Solution and Report Booklet

Contents

Chapter 1

Problems

1.1 GJMO Problems



GAUSSIAN
1st USAJMO
2021

Day 1

J-1. Find the minimum possible value of the natural number x, such that:

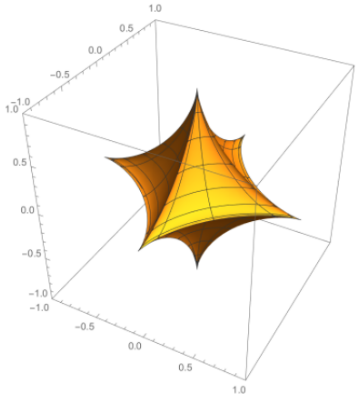
- $x > 2021$ and
- There is a positive integer y, co-prime with x, such that $x^2 - 4xy + 5y^2$ is a perfect square

J-2. In phoenix, a Galaxy far, far away, there are 2021 planets. Define a fire to be a path between two objects in phoenix. It is known that between every pair of planets either a single fire burns or no burning occurs. If we consider any subset of 2019 planets, the total number of fires burning between these planets is a constant. If there are $\mathscr{F}$ fires in phoenix, then find all possible values of $\mathscr{F}$.

J-3. Let ABC be a triangle with sides a, b, c and let r_a, r_b, r_c denote the radii of the excircles of triangle ABC. If R denotes the circumradius of triangle ABC then prove that

$$\frac{4[ABC]}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \leq R^2 \sum \frac{h_c}{r_a \cdot r_b}\left(\cos\frac{A}{2}\right)^4$$

where h denotes altitude , $[x]$ denotes area of x



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Day 2

J-4. On the board n positive integers are written, let them be $a_1, a_2, \ldots, a_n$. Let p, q be two prime numbers such that $p \neq q$. We are allowed to execute infinitely many times the following procedure: We pick two numbers a, b written on the board, we delete them and replace them with $pa - qb, pb - qa$. After 2021 applications of this procedure, let k be the product of all numbers on the board that time. If we know that $k^{(p-1)(q-1)} \not\equiv 1 \pmod{pq}$, then prove that there exists a $i \in \{1, 2, \ldots, n\}$, such that either $p|a_i$ or $q|a_i$.

J-5. In a $\triangle ABC$, let K be the intersection of the A-angle bisector and $\overline{BC}$. Let H be the orthocenter of $\triangle ABC$. If the line through K perpendicular to $\overline{AK}$ meets $\overline{AH}$ at P, and the line through H parallel to $\overline{AK}$ meets the A-tangent of (ABC) at Q, then prove that $\overline{PQ}$ is parallel to the A-symmedian.

Note: The A-symmedian is the reflection of the A-median over the A-angle bisector.

J-6. Let $S = \{1, 2, \ldots, n\}$, with $n \geq 3$ being a positive integer. Call a subset A of S *gaussian* if $|A| \geq 3$ and for all $a, b, c \in A$ with $a > b > c$,

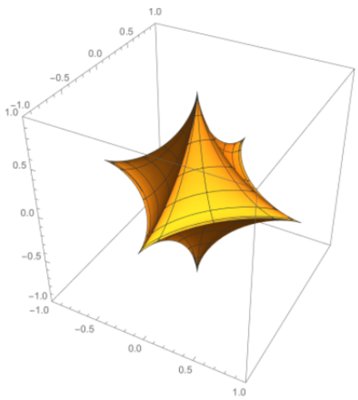
$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} < 5$$

holds true.

(i) Prove that $|A| \leq \left\lfloor \frac{n+2}{2} \right\rfloor$ for all gaussian subsets A of S.

(ii) If a gaussian subset of S contains exactly $\left\lfloor \frac{n+2}{2} \right\rfloor$ elements, then find all possible values of n.

1.2 GAMO Problems



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Day 1

A-1. Let $a, b > 1$ be any two distinct positive integers. You are given these two quantities:

$$\frac{a}{1} \text{ and } \frac{1}{a}$$

You are allowed to apply any of these three operations:

- You can add or subtract any positive integer onto/from the numerators of both the quantities (simultaneously).
- You can add or subtract any positive integer onto/from the denominators of both the quantities (simultaneously)
- You can reduce any quantity to it's lowest form (You need not reduce both quantities simultaneously).
- You can interchange the position of the two quantities.

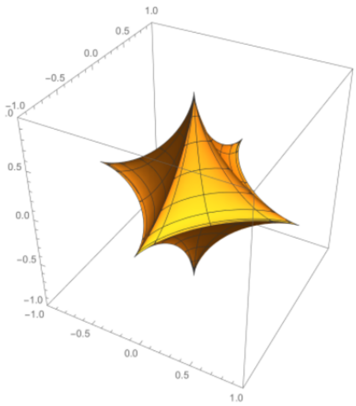
Note that the numerator is always non-negative and the denominator positive at any point. Determine whether it is possible to attain the following configuration of $\frac{b}{1}$ and $\frac{1}{b}$ from the current one by a finite (possibly empty) sequence of such operations.

A-2. Find all polynomials $P(x)$ with integer coefficients which satisfy the following conditions

- $P(n)$ is a positive integer for any positive integer n.
- $P(n)!$ divides $\prod_{k=1}^{n}\left(2^{P(k)+k-1} - 2^{k-1}\right)$ for all positive integers n.

A-3. Let ABC be a triangle with incenter I and Nagel Point N. Let N' be the reflection of N on BC. Let D be on the circumcircle of ABC such that $AD \perp BC$. Let the circle with diameter AI intersect the circumcircle of ABC at $S \neq A$. Let M be the midpoint of the arc BC not containing A and let AN intersect the circumcircle of ABC at X. Then MX, BC and the perpendicular from N' onto SD concur.

Note: The Nagel point of a triangle ABC is defined as the intersection point of the cevians joining the corresponding vertex to the point where the respective excircle touch the side opposite to that vertex



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Day 2

A-4. Let $n \geq 1$ be a positive integer, and let $\mathscr{S} \subset 0, 1, 2, \ldots, n$ such that

$$|\mathscr{S}| \geq \frac{n}{2} + 1.$$

Show that some power of 2 is either an element of $\mathscr{S}$ or the sum of two distinct elements of $\mathscr{S}$.

A-5. Let ABC be an acute, non-isosceles triangle, AD, BE, CF be its heights and (c) its circumcircle. FE cuts the circumcircle at points S, T, with point F being between points S, E. In addition, let P, Q be the midpoints of the major and the minor arc BC, respectively. Line DQ cuts (c) at R. The circumcircles of triangles RSF, TER, SFP and TEP cut again PR at points X, Y, Z and W, respectively. Suppose (ℓ) is the line passing through the circumcenters of triangles AXW, AYZ and $(\ell_B), (\ell_C)$ the parallel lines through B, C to (ℓ). If (ℓ_B) meets CF at U and (ℓ_C) meets BE at V, then prove that points U, V, F, E are concyclic.

A-6. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers x, y,

$$f(x^2 + f(y)) + f(yf(x)) = f(x)f(x+y) + f(y)$$

Chapter 2

GAMO Solution

2.1 Problem 1 proposed by EpicNumberTheory

Let $a > 1$ be any positive integer. You are given these two quantities:

$$\frac{a}{1} \text{ and } \frac{1}{a}$$

You are allowed to apply any of these three operations: You can add or subtract any positive integer onto/from the numerators of both the quantities (simultaneously) You can add or subtract any positive integer onto/from the denominators of both the quantities (simultaneously) You can reduce any quantity to it's lowest form (You need not reduce both quantities simultaneously) You can interchange the position of the two quantities Note that the numerator is always non-negative and the denominator positive at any point. Determine whether it is possible to attain the following configuration:

$$\frac{b}{1} \text{ and } \frac{1}{b}$$

from the current one by a finite (possibly empty) sequence of such operations.

The Answer to the problem is a very positive yes .

Claim 1: You can reach $\frac{n}{1}+\frac{1}{1}$ from $\frac{2}{1}+\frac{1}{2}$ where $n > 1$

Proof: $\frac{2}{1}+\frac{1}{2} \to \frac{4}{1}+\frac{3}{2} \to \frac{4}{2}+\frac{3}{3} \to \frac{2}{1}+\frac{1}{1}$ Now we would be proving that for any $n > 1$: $\frac{n}{1}+\frac{1}{1}$ can be obtained from $\frac{2}{1}+\frac{1}{1}$. $\frac{2}{1}+\frac{1}{1} \to \frac{k+1}{k}+\frac{1}{1} \to \frac{(k-1)+k+1}{k}+\frac{k}{1} \to \frac{2}{1}+\frac{k}{2} \to \frac{1}{1}+\frac{k-1}{1}$. Hence shown.

Claim 2: Let $d(n)$ be any divisor of n. Then you can obtain $\frac{d(n-1)+1}{1}+\frac{1}{1}$ from $\frac{n}{1}+\frac{1}{1}$.

Proof: $\frac{n}{1}+\frac{1}{1} \to \frac{n+k}{1+k}+\frac{1}{1}$. Now if $k+1 \mid n+k$ then $k+1 \mid n-1$. Choose such a k. Then $\frac{n+k}{1+k}+\frac{1}{1} \to \frac{d(n-1)+1}{1}+\frac{1}{1}$ where $d(n-1)$ is some divisor of $n-1$. But since we can choose $k+1$ to be any divisor of $n-1$, $d(n-1)$ can take all the divisor values of $n-1$. Hence proven.

Claim 3: $\frac{n}{1}+\frac{1}{1} \to \frac{n'}{1}+\frac{1}{1}$ for some $n' < n$.

Proof: We would show this by strong induction. Now for the base case we would show that this is true for 3: $\frac{3}{1}+\frac{1}{1} \to \frac{4}{1}+\frac{2}{1} \to \frac{4}{2}+\frac{2}{2} \to \frac{2}{1}+\frac{1}{1}$. Hence shown. Now there always exists a divisor $d(n-1)$ of $n-1$ such that $d(n-1) < n-1$ if $n-1 > 2$. So by Claim 3 this Claim is shown as well.

So if one obtains $\frac{l}{1}+\frac{1}{1}$ then one can obtain $\frac{2}{1}+\frac{1}{1}$

Claim 4: If you can reach $\frac{n^2}{1}+\frac{1}{1}$ then you can reach $\frac{n}{1}+\frac{1}{n}$

Proof: $\frac{n^2}{1}+\frac{1}{1} \to \frac{n^2}{n}+\frac{1}{n} \to \frac{n}{1}+\frac{1}{n}$

Now since obtaining $\frac{2}{1}+\frac{1}{1}$ finishes the problem as per Claim 1 and Claim 4 it suffices to show that it is possible to obtain $\frac{K}{1}+\frac{1}{1}$ from $\frac{c}{1}+\frac{1}{c}$ where c is any positive integer and K is some positive integer depending on c. But $\frac{c}{1}+\frac{1}{c} \to \frac{2c-1}{1}+\frac{1}{1}$. Hence we have shown that the answer to the problem is a very positive $\boxed{\text{yes}}$.

2.2 Problem 2 proposed by Aritra12, TLP.39, Orestis Lignos

Find all polynomials $P(x)$ with integer coefficients which satisfy the following conditions:

- $P(n)$ is a positive integer for any positive integer n.
- $P(n)!|\prod_{k=1}^{n}\left(2^{P(k)+k-1}-2^{k-1}\right)$ for all positive integers n.

First, we notice that since $P(k)+k-1>k-1$ for any positive integer k, we must have $\nu_2\left(2^{P(k)+k-1}-2^{k-1}\right)=k-1$ for all positive integers k. Thus, the 2-adic order of the dividend in the second condition must always be $2^{\frac{n(n-1)}{2}}$.

Thus, $\nu_2(P(n)!)\leq\frac{n(n-1)}{2}$. Since $\nu_2(P(n)!)=P(n)-s_2(P(n))\geq P(n)-\log_2(P(n)+1)$, where $s_2(P(n))$ is the sum of digits of $P(n)$ in its binary representation, we must have $0<P(n)\leq\frac{n^2-n}{2}+\log_2(P(n)+1)$ for all positive integers n.

As $\log_2(P(n)+1)<n$ for all sufficiently large n, the inequality is only possible when $P(x)$ is a linear polynomial.

This means that by finding the value of $P(n)$ for small n could lead to full set of solutions. In particular, only finding all possible values of $P(1)$ and $P(2)$ is enough.

When $n=1$, the second condition implies that $P(1)!|2^{P(1)}-1$. Since the dividend is odd, we must have $P(1)=1$.

When $n=2$, the second condition implies that $P(2)!|2\left(2^{P(2)}-1\right)$. Since the dividend is not divisible by 4, we must have $P(2)\leq 3$. Since $3!\nmid 2\left(2^3-1\right)$, we must have $P(2)\leq 2$.

If $P(2)=1$, then we have $\boxed{P(x)=1}$ which is clearly a solution.

If $P(2)=2$, then we have $P(x)=x$. It's not clear if this works, so we shall prove that this polynomial is indeed a solution.

What we need to prove is that

$$n!|2^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}\left(2^k-1\right)$$

In other word, we need to prove that $\nu_p(n!)\leq\nu_p\left(2^{\frac{n(n-1)}{2}}\right)+\sum_{k=1}^{n}\nu_p\left(2^k-1\right)$ for any prime p.

For $p = 2$, we have $\nu_2(n!) < n \implies \nu_2(n!) \leq n-1 \leq \frac{n(n-1)}{2} = \nu_2\left(2^{\frac{n(n-1)}{2}}\right)$. Thus, the inequality is true in this case.

For $p > 2$, since $ord_{p^k}(2) \leq \varphi(p^k) < p^k$ for any power p^k of p, at least $\lfloor \frac{n}{p^k} \rfloor$ terms in the summation will have values of at least k. Thus, $\sum_{k=1}^{n} \nu_p\left(2^k - 1\right) \geq \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor = \nu_p(n!)$. Thus, the inequality is also true in this case.

Hence, we can conclude that $\boxed{P(x) \equiv x}$ is the second and last solution.

2.3 Problem 3 proposed by Anonymous

Let ABC be a triangle with incenter I and Nagel Point N. Let N' be the reflection of N on BC. Let $D \in \odot(ABC)$ such that $AD \perp BC$. Let $\odot(AI) \cap \odot(ABC) = S$. Let M be the midpoint of $\widehat{BC}$ not containing A and let $AN \cap (ABC) = X$. Then MX, BC and the perpendicular from N' onto SD concur.

Note: Nagel point of a $\triangle ABC$ is defined as the intersection point of the cevians joining the corresponding vertex to the point where the respective excircle touch the side opposite to that vertex.

Reflect over the perpendicular bisector of BC. Let M_A be the midpoint of BC. N' goes to N^*, the reflection of N over BC. S goes to a point S', and D goes to A', the antipode of A on (ABC). X goes to T_A, the A-mixtilinear touchpoint.

Since $AN = 2 * IM$ and $\overline{AN} || \overline{IM}$, $\overline{AI} \cap \overline{NM}$ is both the reflection of A over I and the reflection of N over M. It is well known (enough) that $\overline{MT_A} \cap \overline{BC}$, which can be denoted as T, is such that $\measuredangle AIT = 90$ and $A - S - T$. It suffices to show that the line through I parallel to $\overline{AS'}$ bisects AT.

Let P be the midpoint of AT. Let L be the antipode of M on (ABC). $A(M, L; S', S) = -1 = (A, T; P, P_\infty) = I(A, T; P, P_\infty)$. Since $A - I - M, \overline{AL} \,||\, \overline{TI}, \overline{IP_\infty} || \overline{AS}$, $\overline{IP} \,||\, \overline{AS'}$, as desired.

2.4 Problem 4

Let $n \geq 1$ be a positive integer, and let $\mathscr{S} \subset 0, 1, 2, \ldots, n$ such that

$$|\mathscr{S}| \geq \frac{n}{2} + 1.$$

Show that some power of 2 is either an element of $\mathscr{S}$ or the sum of two distinct elements of $\mathscr{S}$.

We prove this by using induction on n. It is easy to check that the result is true for $n = 1, 2, 3,$ and 4. Let $n > 4$, and assume that the result holds for all postive integers $m < n$. Now choose $s > 2$ such that

$$2^x \leq n < 2^{s+1}$$

and let $r = n - 2^s \in \{0, 1, \ldots, 2^x - 1\}$. Let

$$S_1 = S \cap \{0, \ldots, 2^s - r - 1\}$$

and

$$S_2 = S \cap \{2^x - r, 2^s - r + 1, \ldots, 2^n + r\}$$

Then S is the disjoint union of S_1 and S_2, and $|S| = |S_1| + |S_2|$.

Suppose that the statement is false for the set S. Then $|S| \geq \frac{n}{2} + 1$, but no power of 2 belongs to S or is the sum of two distinct elements of S. It follows that $2^s \notin S_2$ and, for each $i = 1, 2, \ldots, r$, the set S_2 contains at most one of the two integers $2^s - i, 2^s + i$. Therefore, $|S_2| \leq r$.

If $n = 2^{s+1} - 1$, then $r = 2^x - 1$ and $S_1 \subseteq \{0\}$. Thus, $|S_1| \leq 1$. Now, it follows that

$$\frac{n}{2} + 1 \leq |S| \leq 1 + r = 2^s = \frac{n+1}{2}$$

which is impossible.

Similarly, if $2^s \leq n < 2^{s+1} - 1$, then $0 \leq r < 2^s - 1$ and $m = 2^x - r - 1 \geq 1$. Since the set S contains S_1, then no power of 2 belongs to S_1 or is the sum of two distinct elements of S_1. By the induction hypothesis, we have

$$|S_1| < \frac{m}{2} + 1 = \frac{2^s - r - 1}{2} + 1$$

and

$$\frac{n}{2} + 1 \leq |S| = |S_1| + |S_2| < \frac{2^x - r - 1}{2} + 1 + r = \frac{n+1}{2}$$

which is also impossible.

2.5 Problem 5 proposed by Orestis Lignos

Let ABC be an acute, non-isosceles triangle, AD, BE, CF be its heights and (c) its circumcircle. FE cuts the circumcircle at points S, T, with point F being between points S, E. In addition, let P, Q be the midpoints of the major and the minor arc BC, respectively. Line DQ cuts (c) at R. The circumcircles of triangles RSF, TER, SFP and TEP cut again PR at points X, Y, Z and W, respectively. Suppose (ℓ) is the line passing through the circumcenters of triangles AXW, AYZ and $(\ell_B), (\ell_C)$ the parallel lines through B, C to (ℓ). If (ℓ_B) meets CF at U and (ℓ_C) meets BE at V, then prove that points U, V, F, E are concyclic.

Let H be the orthocenter of triangle ABC and M be the midpoint of BC. The proof is based on the following crucial Claim:

Claim— $MH \parallel (\ell)$

Proof: Let, $N \equiv FE \cup BC$ and $N' \equiv PR \cup BC$. Note, that

$$\angle N'RD = 180^\circ - \angle PRQ = 90^\circ$$

and

$$\angle BRD = \angle BAQ = \angle QAC = \angle DRC,$$

hence DR and DN' are the two bisectors (internal and external) of angle $\angle BRC$, hence $(N', B, D, C) = -1$.

In addition, using the complete quadrilateral $AFDE.BC$, we deduce that $(N, B, D, C) = -1$. Therefore, we conclude that $N \equiv N'$, i.e. PR, FE, BC concur.

Thus,

$$NR \cdot NP = NB \cdot NC = NF \cdot NE,$$

which implies that $PRFE$ is cyclic.

Now, note that

$$\angle SXW = \angle RFE = \angle EPW = 180^\circ - \angle ETW,$$

which implies that $SXWT$ is cyclic. Let NA cut (c) at point K. Note, that

$$NK \cdot NA = NS \cdot NT = NX \cdot NW,$$

hence K belongs to the circumcircle of triangle AXW. In a similar manner, we deduce that K belongs to the circumcircle of triangle AYZ. Therefore, AK is the radical axis of the two circles, hence $AK \perp (\ell)$.

What remains, therefore, to be proved is $AK \perp HM$, which is well-known to be true. Indeed, considering circles (A, D, M) and (B, F, E, C), then:

• The line passing through their centers coincides with AM (the center of the first circle is the midpoint of AM while for the second it is point M).
• H belongs to their radical axis, since $HA \cdot HD = HF \cdot HC$
• N belongs to their radical axis, since $ND \cdot NM = NF \cdot NE$ (quadrilateral $FEMD$ is cyclic to the Euler circle).

Therefore, $NH \perp AM$, and since $AH \perp MN$, H is the orthocenter of triangle AMN, which implies that $MH \perp AN$.

Now, note that

$$NK \cdot NA = NB \cdot NC = NF \cdot NE,$$

hence points A, K, F, E are concyclic, therefore all points A, F, H, E, K are concyclic, hence

$$\angle HKA = \angle HFA = 90^\circ,$$

that is $HK \perp NA$.

So, $MH \perp NA$ and $HK \perp NA$, hence points K, H, M are collinear, which implies the desired ■

To the problem, since $MH \parallel (\ell)$, we deduce that $MH \parallel BU \parallel CV$, and since M is the midpoint of BC, H is the midpoint of UC and BV, hence we infer that $UVCB$ is a parallelogram. Therefore,

$$\angle UVE = 180^\circ - \angle UVB = 180^\circ - \angle VBC = 180^\circ - \angle EFC = \angle UFE,$$

which implies that $UFVE$ is cyclic, and the proof concludes.

2.6 Problem 6 proposed by EpicNumberTheory

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers x, y,

$$f(x^2 + f(y)) + f(yf(x)) = f(x)f(x+y) + f(y)$$

All functions satisfying the following F.E. are:

$$\boxed{\text{S1: } f(x) = x \ \forall x \in \mathbb{Z}}$$

$$\boxed{\text{S2: } f(x) = 0 \ \forall x \in \mathbb{Z}}$$

$$\boxed{\text{S3: } f(x) = 1 \ \forall x \in \mathbb{Z}}$$

$$\boxed{\text{S4: } f(x) = x \bmod 2 \ \forall x \in \mathbb{Z}}$$

which indeed work. Now we would show that these are the only solutions. Let $P(x,y)$ denote the assertion in the given problem. $P(0,0) \implies f(f(0)) = f(0)^2$ $P(x,0) \implies f(x^2 + f(0)) = f(x)^2$ $P(-x,0) \implies f(x^2 + f(0)) = f(-x)^2$ Comparing we get $f(x) = \pm - f(-x)$

Now we would be establish 'psuedo periodicity':
Now let t be such that $f(t) = f(-t)$ and $f(t) \neq 0$. $P(t,t) - P(-t,t) \implies f(2t) = f(0)$ $P(x,0) \implies f(x^2 + f(0) = f(x)^2$ So $P(x,2t) \implies f(x)^2 + f(2tf(x)) = f(x)f(x+2t) + f(0)$ $P(t,-t) - P(-t,-t) \implies f(-2t) = f(0) = f(2t)$. So if $f(t) = f(-t) \neq 0$ then $f(2t) = f(-2t) = f(0)$. $P(t,2t) - P(-t,2t) \implies f(3t) = f(t)$. And inductively we can obtain $f(t) = f(3t) = f(5t) = \dots$ and $f(2t) = f(4t) = f(6t) = \dots$ where $f(kt) = f(-kt)$ for any integer k. (So $f(2tf(x)) = f(2t) = f(0)$) So $P(x,2t) \implies f(x)^2 + f(2tf(x)) = f(x)f(x+2t) + f(0) \implies f(x)^2 = f(x)f(x+2t)$. So if $f(x) \neq 0$ then $f(x) = f(x+2t)$.

Case 1: There doesn't exist a z such that $f(z) = 0$.

So f is periodic with period $2t$. Let t be such that this period is minimum. (Sometimes we would be using it as minimum and sometimes not, it should be clear from the context) First of all if t is even then $t^2 \equiv 0 \bmod 2t$ Now we would have two cases depending on the parity of $f(t)$ and in both cases supposing t to be even because there always exists some even t.

Suppose for some t is even and $f(t)$ is odd. $P(t,t) \implies f(f(t)) + f(tf(t)) = f(t)f(0) + f(t)$ $P(2t,t) \implies f(f(t)) + f(tf(0)) = f(t)f(0) + f(t) \implies f(tf(t)) = f(tf(0))$. If $f(0)$ is even then $f(tf(t)) = f(0)$ which means that $f(t) = f(0)$ .Then consider $k = t/2$ (t is even). Note that $f(k) = -f(-k)$ (Because if $f(k) = f(-k)$ then we could never find minimal t such that $f(t) = f(-t)$ which means that t is zero.This means that their doesn't exist any non-zero t such that $f(t) = f(-t)$ Which would mean that f is odd). $P(k,k) - P(-k,k) \implies f(kf(k)) -$

$f(-kf(k)) = 2f(k)f(0)$ Now $f(kf(k)) \neq f(-f(k))$ because otherwise there would exist a zero. So we have: $f(kf(k)) = f(k)f(0)$. $P(k,k) \implies f(k^2+f(k))+f(kf(k)) = f(k)f(0)+f(k) \implies f(k^2+f(k)) = f(k)$ $P(-k,k)+P(k,-k) \implies f(k^2+f(k))+f(k^2+f(-k))+2f(-kf(k)) = 0$ $P(k,k)+P(-k,-k) \implies f(k^2+f(k)) = f(k^2+f(-k))+2f(kf(k)) = 0$. Comparing the above two we get: $f(-kf(k)) = -f(kf(k)) = f(kf(k)) \implies$ there exists a zero. Contradiction!

So $f(0)$ is odd. So $f(tf(0)) = f(t)$. $P(2t,t) \implies f(f(t))$ is even. Now choose any t' such that $f(t') = f(-t') \neq 0$ so $f(2t') = f(-2t')$ So $f(f(2t))$ is even which means $f(f(0))$ or $f(0)^2$ or $f(0)$ is even which is a contradiction.

Suppose for some t is even and $f(t)$ is even. $P(t,t) \implies f(f(t))+f(0) = f(t)f(0)+f(t)$ $P(2t,t) \implies f(f(t))+f(tf(0)) = f(t)f(0)+f(t)$ Suppose $f(0)$ is odd. (And so is $f(tf(0))$) But $P(2t,t)$ just means that $f(f(t))$ and $f(t)$ have the same parity i.e. even. But $P(t,t)$ leads to a contradiction.
Now $f(0)$ is even. $P(t,0) \implies f(0)^2 = f(t)^2 \implies f(t) \in \{f(0), -f(0)\}$ If $f(t) = f(0)$ then consider $k = t/2$ (t is even). Note that $f(k) = -f(-k)$ (Because if $f(k) = f(-k)$ then we could never find minimal t such that $f(t) = f(-t)$ which means that t is zero.This means that their doesn't exist any non-zero t such that $f(t) = f(-t)$ Which would mean that f is odd). $P(k,k)-P(-k,k) \implies f(kf(k))-f(-kf(k)) = 2f(k)f(0)$ Now $f(kf(k)) \neq f(-f(k))$ because otherwise there would exist a zero. So we have: $f(kf(k)) = f(k)f(0)$. $P(k,k) \implies f(k^2+f(k))+f(kf(k)) = f(k)f(0)+f(k) \implies f(k^2+f(k)) = f(k)$ $P(-k,k)+P(k,-k) \implies f(k^2+f(k))+f(k^2+f(-k))+2f(-kf(k)) = 0$ $P(k,k)+P(-k,-k) \implies f(k^2+f(k)) = f(k^2+f(-k))+2f(kf(k)) = 0$. Comparing the above two we get: $f(-kf(k)) = -f(kf(k)) = f(kf(k)) \implies$ there exists a zero. Contradiction!

So $f(t) = -f(0)$. So $f(-f(0)) = -f(0)^2 - 2f(0)$ (By $P(t,t)$) $P(t, f(0)) \implies f(f(0)^2) = -f(-f(0)^2)$ $P(f(0),0) \implies f(f(0)^2+f(0)) = f(0)^4$ $P(-f(0),0) \implies f(f(0)^2+f(0))+f(0) = f(-f(0))^2+f(0) = \implies f(-f(0)) = \pm f(0)^2$ So $\pm f(0)^2 - f(0)^2 - 2f(0) = 0 \implies f(0) \in \{0,1\} \implies f(0) = 1$ (Since there does not exist a z such that $f(z) = 0$)

Claim 1: If $f(0) = 1$ then $f(x) = 1$ $\forall x \in \mathbb{Z}$ Proof: Suppose $f(0) = 1$. Then $f(f(0)) = f(0)^2 \implies f(1) = 1$ $P(1,0) \implies f(1+f(0)) = f(1)^2 \implies f(2) = 1$. $P(1,2) \implies f(2) = f(3) \implies f(3) = 1$ and inductively (Use $P(1,k)$ repeatedly for induction) $f(x) = 1$ $\forall x \in \mathbb{N}$. By using $f(x)^2 = f(-x)^2$ we have that $f(v) = \pm 1$ for negative v. So let v such that $f(v) = 1$ and w such that $f(w) = -1$. $P(0,x) \implies f(f(x)) = f(x)$. So if $x \to w$ then $f(-1) = -1$. Choose $v > \mid w \mid$. $P(-1,v) \implies 0f(-v) = -f(v-1) = -1$. But v can be any positive integer. So $f(x) = -1$ $\forall x < 0 \in Z$. But $P(2020, -2021) \implies 1 = f(-1)$ which is a contradiction. So there doesn't exist any negative integer w such that $f(w) = -1$ so $f(x) = 1 \forall x \in \mathbb{Z}$

Now we don't need to consider t odd since we have completed the case where there is just the [i]existence[/i] of a an even t such that $f(t)$ is even too. So now we need to consider that there does not exist any such even t. But this cannot occur since always $f(2t) = f(-2t)$.

Case 2: There exists a z such that $f(z) = 0$. Comparing $P(z,0)$ and $P(-z,0)$ shows that $f(z)^2 = f(-z)^2 = 0$ $P(z,z) \implies f(z^2) = 0$ $P(-z,z) \implies f(0) = 0$.
Claim 2: If $f(0) = 0$ then $f(x) = x$. Proof: $P(1,0) \implies f(1) \in \{0,1\}$ Suppose $f(1) = 0$. Further suppose there exists a u such that $f(u) \neq 0$. $P(u,1) \implies f(u)+1 = f(u+1)$.

$P(1,u) \implies f(u+1)=f(u)$ which is wrong. (We used $f(f(u)+1)=f(f(u+1))=f(u+1)$) So there doesn't exist a u such that $f(u)\neq 0$ or we have that $f\equiv 0$.

Now we have that $f(1)=1$. Clearly $f(-1)\in\{-1,1\}$. Suppose $f(-1)=1$. $P(-1,1) \implies f(2)=0$. $P(1,2) \implies f(3)=1$. $P(1,2) \implies f(1)f(3)=0$. Which is clearly false. So we have $f(-1)=-1$. $P(-1,1) \implies f(2)=2$. $P(-1,2) \implies f(3)+f(-2)=1$. Suppose $P(1,-2) \implies f(3)=-1 \implies 1=2+f(-2)$. So $f(-2)=-2$ and so $f(3)=3$ and so inductively $f(x)=x\ \forall x\in\mathbb{N}$.Let w be a negative integer. So by comparing $P(w,0) \implies f(w)^2=w^2=f(-w)^2 \implies f(w)=\pm w$. Let w be such that $f(w)=w$ and v such that $f(v)=-v$ (v is also a negative integer here). $P(w,1) \implies w^2+w=wf(w+1) \implies f(w+1)=w+1$. So every negative integer $\geq w$ has $f(w)=w$. Now $w\leq -1$. So $f(-1)=-1$ (If there exists such w). $P(w,-1) \implies w^2-w=wf(w-1) \implies f(w-1)=w-1$ so we have $f(x)=x\in\mathbb{Z}$. Now we have the case that for all negative integers v we have $f(v)=-v$. So $P(0,w) \implies 0=w^2$ which is absurd.

Now suppose there doesn't exist any t such that $f(t)=f(-t)$ and $f(t)\neq 0$. So for any t such that $f(t)=f(-t)$ we have that $f(t)=0$. But then $f(t)=-f(-t)=0$. So f is odd too. But for the rest of the integers we have $f(x)=-f(-x)$. So we have $f(x)=-f(-x)$ for all non-zero integers. And so we will consider f odd now.

Now we assume $f(0)=0$.
A new assertion: $P(x,y)-P(-x,y) \implies f(yf(x))-f(-yf(x))=f(x)(f(x+y)+f(-x+y)) \implies 2f(yf(x))=f(x)(f(x+y)+f(-x+y))$ Let this be denoted by $Q(x,y)$.
Another assertion $Q(x,1) \implies 2f(x)=f(x)(f(x+1)-f(x-1))$ Suppose for some x, $f(x)\neq 0$. So $f(x+1)-f(x-1)=2 \implies 2=f(x+1)+f(1-x)$. Let this be denoted by $R(x)$ (Only true if $f(x)\neq 0$)
$f(0)=-f(0) \implies f(0)=0$. $P(0,x) \implies f(f(x))=f(x)$ $P(1,0) \implies f(1)=f(1)^2 \implies f(1)=0$ or 1.

Subcase 1: $f(1)=0$. $P(1,x) \implies f(f(x)+1)=f(x)=f(f(x))$. This holds for all integers x. But since f is odd we have that $f(-f(x))=-f(f(x))=-f(x)$. $P(x,-1) \implies f(x)^2+f(-f(x))=f(x)f(x-1) \implies f(x)(f(x)-1)=f(x)f(x-1)$. So shifting $x\to 2 \implies f(2)\in\{2,1\}$. But $f(f(2))=f(2)$. So if $f(2)=1$, then we would have a contradiction ($f(1)=1$). So $f(2)=0$. So by induction we can have that $f(x)\in\{0,1\}$. But $f(f(x))=x$ would imply that $f(x)=0$. Combining this with f odd we get that $f(x)=0\forall x\in\mathbb{Z}$.

Subcase 2: $f(1)=1$. $R(1) \implies f(2)=2$. $R(2) \implies f(3)=3$. $R(3) \implies f(4)=4$. $R(4) \implies f(5)=5$ and inductively $f(x)=x\ \forall x\in\mathbb{Z}$ (Since f is odd).

We are left with the case where $f(0)\neq 0$. We have to subcases.

One is that there exists a root h of f. $P(h,x) \implies f(h^2+f(x))=f(x)-f(0)$. This implies that $k-f(0)$ is inrange of f whenever k also is. Thus, $-nf(0)$ is in the range of f for any $n\geq -1$. Since $-nf(0)\neq f(0)$ for any $n\geq 0$, $nf(0)$ must also be in the range for $n\geq 0$. Thus, $nf(0)$ is in the range for any integer n.

Hence, $f(h^2+nf(0))=(n-1)f(0)$ for any integer n. Thus, $P(h+nf(0),0) \implies (2hn+n^2f(0))f(0)=f(h^2+(2hn+n^2f(0)+1)f(0))=f(h+nf(0))^2$. This means that $(2hn+n^2f(0))f(0)$ is always perfect square no matter what integer n is. This is impossible.

Else, 0 is not in the range of f. In this case, for any x,y such that $x\neq 0, y\neq 0, x+y\neq 0$, $P(x,y)$ and $P(-x,-y)$ implies that $f(x^2+f(y))-f(x^2-f(y))=2f(y)$. We denote this with $Q(x,y)$.

Since $f(x^2+f(0))=f(x)^2$, for any x such that $x^2+f(0)\neq 0$ and $f(x)+x^2+f(0)\neq 0$, $Q(f(x),x^2+f(0)) \implies f(2f(x)^2)=f(0)+2f(x)^2$.

Back to the original equation. For any $x,y\neq 0$, $P(x,y)$ and $P(-x,y)$ imply that $2f(yf(x))=f(x)(f(y+x)+f(y-x))$. Substituting $y\to 2f(x)$ imply that for any $x\neq 0$, either $f(x)|2f(0)$, $x^2+f(0)=0$, or $f(x)+x^2+f(0)=0$.

Thus, either f has finite range (which has been cleared) or there are infinitely many x such that $f(x)$ is a perfect square larger that $|2f(0)|$ and $-f(0)$. In this case, $f(x)|2f(0)$ and $f(x)+x^2+f(0)=0$ cannot be true, and $x^2+f(0)=0$ may be true for at most two cases. Thus, the statement 'for all x, either $f(x)|2f(0)$, $x^2+f(0)=0$, or $f(x)+x^2+f(0)=0$' cannot be true. Hence the last case has no solution.

Aliter: Let $P(x,y)$ denotes the equation.

$P(x,0) \implies f(x^2+f(0))=f(x)^2$. Thus, comparing $P(x,0)$ and $P(-x,0)$ gives $f(x)^2=f(-x)^2$ for all x. Moreover, for any k in the range of f, k^2 is also in the range.

If there is some $a\neq 0$ such that $f(a)\neq -f(-a)$, then we have $f(a)\neq 0$ and $f(a)=f(-a)$. Comparing $P(a,x)$ and $P(-a,x)$ gives $f(x+a)=f(x-a)$ for all x. Thus, f is periodic and bounded. In particular we also get that the range of f is finite.

If there's r such that $|f(r)|>1$, then $f(r)^{2^n}$ is in the range for all $n\in\mathbb{N}$ and so the range is infinite - a contradiction. Thus, $f(x)\in\{1,0,-1\}$ for all x.

If 0 is in the range, then there exists b such that $f(b)=0$. $P(b,y) \implies f(b^2+f(y))=f(y)-f(0)$. In particular, $k-f(0)$ is in the range whenever k is in the range. Since range of f is finite, this is impossible unless $f(0)=0$. $P(0,x)$ thus implies that $f(f(x))=f(x)$ for all x.

If 1 and -1 are also in the range, then $f(1)=1$ and $f(-1)=-1$. $P(-1,1)$ then implies that $f(2)=2$ whic his a contradiction.

If 1 is in the range but -1 is note, then $f(1)=1$, $f(-1)=1$ and $f(-x)=f(x)$ for all x. $P(-1,1)$ implies that $f(2)=0$. $P(1,y)$ implies that $f(1+f(y))=f(1+y)$ for all y. We can induct to prove that $f(x)=x \pmod 2$ for all $x>0$. Since $f(x)=f(-x)$, we have $\boxed{f(x)=x \pmod 2}$ which works.

If -1 is in the range but 1 is not, then $f(-1)=-1$ and $f(1)=-1$. $P(-1,0)$ then implies that $-1=1$ which is not true.

If neither -1 nor 1 is in the range, then $\boxed{f(x)\equiv 0}$ which works.

If 0 is not in the range, then $f(x) = \pm 1$ for all x. Thus, $f(x^2 + f(0)) = f(x)^2 = 1$ for all x.

If $f(0) = -1$, then $f(x^2 - 1) = 1$ for all x and so $f(0) = f(1^2 - 1) = 1$, a contradiction. So $f(0) = 1$. Thus, $f(x^2 + 1) = 1$ for all x. In particular, $f(1) = f(2) = 1$.

$P(1, y)$ implies that $f(1 + f(y)) = f(1 + y)$. Since $1 + f(y) \in \{0, 2\}$ for all y and $f(0) = f(2) = 1$, we have $f(1 + y) = 1$ for all y and so $\boxed{f(x) \equiv 1}$ which works.

Else, $f(x) = -f(-x)$ for all $x \neq 0$.

If $f(0) = 0$, then $P(0, x)$ implies that $f(f(x)) = f(x)$ for all x. Moreover, we have $f(x^2) = f(x)^2$. In particular, $f(1)^2 = f(1) \implies f(1) \in \{0, 1\}$.

If $f(1) = 0$, then $P(x, 1)$ implies that $f(x)^2 + f(x) = f(x)f(x+1)$. In particular, for any x, either $f(x) = 0$ or $f(x+1) = f(x) + 1$.

There are two cases here. The first is that $f(x) \equiv 0$ which is already include in the solution. Thesecond is that there's the largest negative c such that $f(c) \neq 0$. We can see that $c < -1$. Thus, $c + 1$ is still negative an so $f(c+1) = 0$. Hence, $f(c) = -1$ and so $f(-c) = 1$.

Since $f(-c) = 1 \neq 0$, we can induct to get that $f(c^2) = c^2 + c + 1$, but since $f(c^2) = f(c)^2 = 1$, we have $c \in \{0, -1\}$ which is not true.

If $f(1) = 1$, then $f(-1) = -1$ and $P(-1, x) \implies f(f(x) + 1) = 2f(x) - f(x-1)$ for all $x \neq 0$. We can then use induction to prove that $f(x) = x$ for all positive x and thus $\boxed{f(x) \equiv x}$ which is a solution.

We are left with the case where $f(0) \neq 0$. We have to subcases.

One is that there exists a root h of f. $P(h, x) \implies f(h^2 + f(x)) = f(x) - f(0)$. This implies that $k - f(0)$ is inrange of f whenever k also is. Thus, $-nf(0)$ is in the range of f for any $n \geq -1$. Since $-nf(0) \neq f(0)$ for any $n \geq 0$, $nf(0)$ must also be in the range for $n \geq 0$. Thus, $nf(0)$ is in the range for any integer n.

Hence, $f(h^2 + nf(0)) = (n-1)f(0)$ for any integer n. Thus, $P(h + nf(0), 0) \implies (2hn + n^2 f(0))f(0) = f(h^2 + (2hn + n^2 f(0) + 1)f(0)) = f(h + nf(0))^2$. This means that $(2hn + n^2 f(0))f(0)$ is always perfect square no matter what integer n is. This is impossible.

Else, 0 is not in the range of f. In this case, for any x, y such that $x \neq 0, y \neq 0, x + y \neq 0$, $P(x, y)$ and $P(-x, -y)$ implies that $f(x^2 + f(y)) - f(x^2 - f(y)) = 2f(y)$. We denote this with $Q(x, y)$.

Since $f(x^2 + f(0)) = f(x)^2$, for any x such that $x^2 + f(0) \neq 0$ and $f(x) + x^2 + f(0) \neq 0$, $Q(f(x), x^2 + f(0)) \implies f(2f(x)^2) = f(0) + 2f(x)^2$.

Back to the original equation. For any $x, y \neq 0$, $P(x, y)$ and $P(-x, y)$ imply that $2f(yf(x)) = f(x)(f(y+x) + f(y-x))$. Substituting $y \to 2f(x)$ imply that for any $x \neq 0$, either $f(x) | 2f(0)$, $x^2 + f(0) = 0$, or $f(x) + x^2 + f(0) = 0$.

Thus, either f has finite range (which has been cleared) or there are infinitely many x such that $f(x)$ is a perfect square larger that $|2f(0)|$ and $-f(0)$. In this case, $f(x) | 2f(0)$ and $f(x) + x^2 + f(0) = 0$ cannot be true, and $x^2 + f(0) = 0$ may be true for at most two cases.

Thus, the statement 'for all x, either $f(x)|2f(0)$, $x^2+f(0)=0$, or $f(x)+x^2+f(0)=0$' cannot be true. Hence the last case has no solution.

Chapter 3

GJMO Solution

3.1 Problem 1 proposed by Orestis Lignos

Find the minimum possible value of the natural number x, such that:

- $x > 2021$
- There is a positive integer y, co-prime with x, such that $x^2 - 4xy + 5y^2$ is a perfect square

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We will prove, that the minimum possible value of x is 2029.

Knowing that $x^2 - 4xy + 5y^2$ is a perfect square, we deduce that $(x-2y)^2 + y^2$ is a perfect square.

Note that $(x, y) = 1$, hence $(x-2y, y) = 1$ as well, therefore we have two cases to consider:

Case 1: $x - 2y = m^2 - n^2$ and $y = 2mn$ with $(m, n) = 1$ and $m, n \in \mathbb{Z}$. Then, $x = m^2 - n^2 + 4mn = (m+2n)^2 - 5n^2$.

We prove the following Claim:

Claim 1: $x = (m+2n)^2 - 5n^2 \geq 2029$.

Proof: Suppose otherwise. Then, $x \leq 2028$ and $x > 2021$, i.e. $x \in \{2022, 2023, 2024, 2025, 2026, 2027, 2028\}$

If now, $3 \mid x$, then

$$3 \mid (m+2n)^2 - 5n^2 \implies 3 \mid (m+2n)^2 + n^2 \implies 3 \mid m+2n \text{ and } 3 \mid n,$$

which is a contradiction since $(m+2n, n) = (m, n) = 1$.

Hence, we may exclude $2022, 2025$ and 2028 from the above list.

In addition, note that $(m+2n)^2 - 5n^2 \equiv 0, 1, 4 \pmod 5$, hence we may exclude 2023 and 2027.

The only possible value remaining is 2024. Suppose that $(m+2n)^2 - 5n^2 = 2024$.

Then, if both $m+2n$ and n are odd, $\pmod 8$ implies $1 - 5 \equiv 0 \pmod 8$, a contradiction.

Suppose now that $m + 2n = 2a$ and $n = 2b$. Then, $a^2 - 5b^2 = 506$.

Since 506 is even, a and b have the same parity.

If both are even, then $4 \mid (a^2 - 5b^2) = 506$, a contradiction.

If both are odd, then

$$506 = a^2 - 5b^2 \equiv 1 - 5 \equiv 4 \pmod 8,$$

a contradiction ■

To the problem, it's easy to see by some casework that taking $n = 6$ and $m = 35$ works, since $49^2 - 5 \cdot 6^2 = 2029$.

Case 2: $x-2y=2mn$ and $y=m^2-n^2$ with $(m,n)=1$ and $m,n\in\mathbb{Z}$. Then, $x=2(m^2-n^2+mn)$.

We make the following Claim:

Claim 2: $m^2-n^2+mn\geq 1015$.

Proof: Suppose otherwise. Then, since $x>2021$, we obtain $m^2-n^2+mn\in\{1011,1012,1013,1014\}$

If $3\mid(m^2-n^2+mn)$, then if $3\nmid m,n$ then

$$m^2-n^2+mn\equiv 1-1+mn \pmod 3\equiv mn \pmod 3,$$

hence $3\mid mn$, a contradiction.

Therefore, $3\mid m \text{ or } n\Rightarrow 3\mid m,n$, a contradiction since $(m,n)=1$.

Hence, $3\nmid(m^2-n^2+mn)$, so we may exclude 1011 and 1014.

Then, $m^2-n^2+mn\in\{1012,1013\}$.

Thus,

$$4(m^2-n^2+mn)\equiv(2m+n)^2-5n^2\equiv 0,1,4 \pmod 5,$$

therefore $m^2-n^2+mn\equiv 0,1,4 \pmod 5$, hence the two remaining values are as well excluded, since they are 2 or 3 $\pmod 5$ ■

Therefore, $x\geq 2030$, so its minimum value is certainly larger than the previously obtained 2029.

To conclude, $x_{\min}=2029$.

3.2 Problem 2 proposed by Phoenixfire

In phoenix, a Galaxy far, far away, there are 2021 planets and 1 sun. Define a fire to be a path between two objects in phoenix. It is known that between every pair of planets either a single fire burns or no burning occurs. If we consider any subset of 2019 planets, the total number of fires burning between these planets is a constant. If there are $\mathscr{F}_{(\mathscr{P})}$ fires in phoenix, then find all possible values of $\mathscr{F}_{(\mathscr{P})}$.

We consider the general case with n planets. Let K denote the constant number of fires burning in any subset of $n-2$ planets and let $f_{ij} \in \{0,1\}$ denote the number of fires burning between planet i and planet j. Finally, for $i = 1,2,\ldots,n$ let f_i denote the total number of fires burning to planet i.

Note that

$$\mathscr{F}_{(\mathscr{P})} \leq \binom{n}{2} = \frac{n(n-1)}{2}.$$

Clearly,

$$\sum_{i=1}^{n} f_i = \mathscr{F}_{(\mathscr{P})} \text{ and } \sum f_{ij} = \mathscr{F}_{(\mathscr{P})}$$

where the latter sum is over all 2-element subsets $\{i,j\}$ of the set $\{1,2,\ldots,n\}$. The number of fires burning to at least one of the planets with number i or j is equal to $f_i + f_j - f_{ij}$. Thus, for any 2-element subsets $\{i,j\} \subset \{1,2,\ldots,n\}$, we have

$$K = \mathscr{F}_{(\mathscr{P})} - f_i - f_j + f_{ij}.$$

Adding all these equations for every 2-element subset $\{i,j\}$ yields

$$\binom{n}{2}K = \binom{n}{2}\mathscr{F}_{(\mathscr{P})} - 2(n-1)\mathscr{F}_{(\mathscr{P})} + \mathscr{F}_{(\mathscr{P})}$$

which may be written as

$$n(n-1)K = (n-2)(n-3)\mathscr{F}_{(\mathscr{P})}.$$

Note that both $n(n-1)$ and $(n-2)(n-3)$ are divisible by 2, and that the only integer $k > 2$ which divides both $n(n-1)$ and $(n-2)(n-3)$ is 3, this latter case occurring if and only if n is divisible by 3. Since 3 does not divide 2021, in the situation of the given problem $\dfrac{n(n-1)}{2}$ and $\dfrac{(n-2)(n-3)}{2}$ are coprime. Hence, $\mathscr{F}_{(\mathscr{P})}$ is a multiple of $\dfrac{n(n-1)}{2}$.

As $\mathscr{F}_{(\mathscr{P})} \leq \frac{n(n-1)}{2}$ with equality when a fire burns between all the pairs of planets, the only possibilities are $\mathscr{F}_{(\mathscr{P})} = \dfrac{n(n-1)}{2}$ or $\mathscr{F}_{(\mathscr{P})} = 0$. Therefore, the total number of fires is

$$\mathscr{F}_{(\mathscr{P})} = \frac{2020 \cdot 2021}{2} \text{ or } \mathscr{F}_{(\mathscr{P})} = 0.$$

3.3 Problem 3 proposed by Aritra12

Let r_a, r_b, r_c denote the radius of the excircles of $\triangle ABC$ having sides a, b, c. If R, r are the usual notations for circumradius and inradius then prove that

$$\frac{4[ABC]}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \leq R^2 \sum \frac{h_c}{r_a \cdot r_b}\left(\cos\frac{A}{2}\right)^4$$

where h denotes altitude , $[x]$ denotes area of x

We know the basic inequalities mitronivic & euler's Inequality such as $3\sqrt{3}R \geq 2s$ and $R \geq 2r$, are true, by multiplying them we get

$$3\sqrt{3}R^2 \geq 4[ABC] \implies \text{squaring both sides} \implies 27R^4 \geq 16[ABC]^2$$

which means that

$$27R^4 \geq 16[ABC]^2$$

using the identity $[ABC] = \frac{abc}{4R}$ it can be written as

$$27 \cdot 4R[ABC] \geq \frac{64[ABC]^3}{R^3} \implies 27abcR^3 \geq 64[ABC]^3$$

so its true that

$$3\sqrt[3]{abc} \geq \frac{4[ABC]}{R}$$

By AM-GM we can say that ,

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq 3(abc)^{\frac{1}{3}}$$

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq 3(abc)^{\frac{1}{3}} \geq \frac{4[ABC]}{R}$$

So it's not wrong that

$$\frac{1}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \leq \frac{R}{4[ABC]}$$

$$\frac{4}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \leq \frac{4R}{4[ABC]}$$

Since $4R+r > 4R$

$$\frac{4}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \leq \frac{4R+r}{4[ABC]}$$

We will denote the above as $(*)$,

Now note that,

$$4R+r = \sum \frac{\cos^4\frac{A}{2}}{\frac{h_c}{r_a r_b}\cos^4\frac{A}{2}}$$

Since

$$2\sum \frac{r_b r_c}{h_a} = 2\sum r_a - 2(4R+r) \text{ or } \sum \frac{r_b r_c}{h_a} = 4R+r$$

and

$$\sum \frac{\cos^4 \frac{A}{2}}{\frac{h_c}{r_a r_b}\cos^4 \frac{A}{2}} \geq \frac{\left(\sum \cos^2 \frac{A}{2}\right)^2}{\sum \frac{h_c}{r_a r_b}\cos^4 \frac{A}{2}}$$
$$\implies \sum \frac{h_c}{r_a r_b}\cos^4 \frac{A}{2} \geq \frac{4R+r}{4R^2}$$
$$\implies 4R^2 \sum \frac{h_c}{r_a r_b}\cos^4 \frac{A}{2} \geq 4R+r$$

Now taking $(*)$ in action we know that

$$\frac{16[ABC]}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \leq 4R+r$$

$$\frac{16[ABC]}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \leq 4R+r \leq 4R^2 \sum \frac{h_c}{r_a r_b}\cos^4 \frac{A}{2}$$
$$\frac{16[ABC]}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \leq 4R^2 \sum \frac{h_c}{r_a r_b}\cos^4 \frac{A}{2}$$

that is

$$\frac{4[ABC]}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \leq R^2 \sum \frac{h_c}{r_a r_b}\cos^4 \frac{A}{2}$$

Proved .

3.4 Problem 4 proposed by Orestis Lignos

On the board n positive integers are written, let them be $a_1, a_2, \ldots, a_n$. Let p, q be two prime numbers such that $p \neq q$. We are allowed to execute infinitely many times the following procedure: We pick two numbers a, b written on the board, we delete them and replace them with $pa - qb, pb - qa$. After 2021 applications of this procedure, let k be the product of all numbers on the board that time. If we know that $k^{(p-1)(q-1)} \not\equiv 1 \pmod{pq}$, then prove that there exists a $i \in \{1, 2, \ldots, n\}$, such that either $p|a_i$ or $q|a_i$.

We make firstly the following Claim:

***Claim*—** Either p or q divides k.

Proof: Suppose not. Then, by Fermat's Little Theorem,

$$k^{p-1} \equiv 1 \pmod{p}$$

and

$$k^{q-1} \equiv 1 \pmod{q},$$

therefore

$$k^{(p-1)(q-1)} \equiv 1 \pmod{p}$$

and

$$k^{(p-1)(q-1)} \equiv 1 \pmod{q},$$

hence $k^{(p-1)(q-1)} \equiv 1 \pmod{pq}$, which is contradictive to the problem hypothesis ■

Suppose WLOG that $p \mid k$. Then, there exists a number on the board that time that is divisible by p. Let this number be m.

Then, $m = pa - qb$ or $m = pb - qa$ for some a, b that were on the board on the previous move.

In any case, since $p \neq q$ by the problem statement, we deduce that at least one number on the board is divisible by p in the previous move.

Going now continually backwards, we obtain that at least one number on the board initially is a multiple of p, as desired

3.5 Problem 5 proposed by i3435

In a $\triangle ABC$, let K be the intersection of the A-angle bisector and $\overline{BC}$. Let H be the orthocenter of $\triangle ABC$. If the line through K perpendicular to $\overline{AK}$ meets $\overline{AH}$ at P, and the line through H parallel to $\overline{AK}$ meets the A-tangent of (ABC) at Q, then prove that $\overline{PQ}$ is parallel to the A-symmedian

Note: The A-symmedian is the reflection of the A-median over the A-angle bisector).

Let V be the intersection of the tangent to (ABC) at A and $\overline{BC}$. $\measuredangle VAK = \measuredangle VAB + \measuredangle BAK = \measuredangle ACB + \measuredangle BAK$ and $\measuredangle KVA = \measuredangle BVA = \measuredangle BAV + \measuredangle VBA = \measuredangle CBA - \measuredangle ACB$. Since $2\measuredangle VAK = 2\measuredangle ACB + \measuredangle BAC = \measuredangle ACB - \measuredangle CBA = -\measuredangle KVA$, $VA = VK$. Letting T be the reflection of A over V, we see that $K - P - T$.

Let M be the midpoint of BC, let D be the foot from A to $\overline{BC}$, and let X be the midpoint of AH. Let Y be the intersection of (DXM)(which is the 9-point circle) and $\overline{AM}$. $\measuredangle AYX = \measuredangle MDA = 90$. Let H_A and A' be the reflections of H over D and M respectively. It is well known that H_A and A' lie on (ABC), and that A' is the antipode of A on (ABC). $\measuredangle DYA = \measuredangle DYM = \measuredangle DXM = \measuredangle H_A AA' = \measuredangle BAC - 2(90 - \measuredangle CBA) = \measuredangle BAC + 2\measuredangle CBA = \measuredangle CBA - \measuredangle ACB = \measuredangle DVA$, so V, A, Y, D are cyclic. Since $\measuredangle AYV = \measuredangle ADV = 90$, $T - X - Y$, so $\overline{VX} \perp \overline{AM}$. This means that $\overline{TH} \perp \overline{AM}$.

$\measuredangle TQH = \measuredangle TAK = \measuredangle AKV = 90 - \measuredangle HAK = \measuredangle TPH$, so Q, H, P, T are concyclic. $\measuredangle PQH = \measuredangle PTH = \measuredangle KAM$, thus $\overline{PQ}$ is parallel to the reflection of the A-median over the A-angle bisector, as desired.

3.6 Problem 6 proposed by Orestis Lignos

Let $S = \{1, 2, \ldots, n\}$, with $n \geq 3$ being a positive integer. Call a subset A of S *gaussian* if $|A| \geq 3$ and for all $a, b, c \in A$ with $a > b > c$,

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} < 5$$

holds true.

- Prove that $|A| \leq \left\lfloor \frac{n+2}{2} \right\rfloor$ for all gaussian subsets A of S.
- If a gaussian subset of S contains exactly $\left\lfloor \frac{n+2}{2} \right\rfloor$ elements, then find all possible values of n.

(i) Let $A = \{a_1, a_2, \ldots, a_k\}$ be a gaussian subset of S with $a_i < a_j$ when $i < j$. We contend that $a_x + a_y > a_z$ for all $x, y, z \leq k$.

Indeed, if $a_x \geq a_z$ or $a_y \geq a_z$, then the result is clear. Suppose henceforth that $a_z > a_y > a_x$.

Then, consider the function

$$f(x) = \frac{x^2}{a_y^2} + \frac{a_y^2}{a_x^2} + \frac{a_x^2}{x^2},$$

with $x \geq \sqrt{a_x a_y}$.

The above function has a positive derevative, since

$$f'(x) = \frac{2x}{a_y^2} - \frac{2a_x^2}{x^3} = \frac{2(x^4 - (a_x a_y)^2}{a_y^2 x^3} \geq 0,$$

therefore it is strictly increasing at $[\sqrt{a_x a_y}, +\infty)$.

Now, note that if $a_z > a_x + a_y$, then

$$a_z > a_x + a_y \geq 2\sqrt{a_x a_y} > \sqrt{a_x a_y},$$

implying that $f(a_z) > f(a_x + a_y)$.

From the problem's condition, though, we notice that $f(a_z) < 5$, therefore $f(a_x + a_y) < f(a_z) < 5$. Let $\frac{a_x}{a_y} = t$.

Then, $f(a_x + a_y) < 5$ easily rewrites as

$$(t+1)^2 + \frac{1}{t^2} + \frac{1}{(\frac{1}{t}+1)^2} < 5,$$

which in turn after manipulation rewrites as $(t^3 + 2t^2 - t - 1)^2 < 0$, which is clearly impossible.

Therefore, $a_x+a_y>a_z$, or equivalently $a_x+a_y\geq a_z+1$, for all $x,y,z\leq k$.

Thus,

$$a_1+a_2\geq a_k+1\geq a_2+k-1\Longrightarrow a_1\geq k-1,$$

therefore

$$n\geq a_k\geq a_1+k-1\geq 2k-2\Longrightarrow k\leq\left\lfloor\frac{n+2}{2}\right\rfloor,$$

which evidently implies the desired conclusion.

(ii) We initially prove two Claims that will be used later.

Claim 1: $\frac{d^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{d^2}>\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}$ for all $d>a>b>c>0$. Proof: After manipulation it rewrites as $(d^2-a^2)((ad)^2-(bc)^2)>0$, which trivially holds ■

Claim 2: $\frac{a^2}{b^2}+\frac{b^2}{d^2}+\frac{d^2}{a^2}>\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}$ for all $a>b>c>d>0$. Proof: After manipulation it rewrites as $(c^2-d^2)((ab)^2-(cd)^2)>0$, which trivially holds ■

Now, to the problem, we distinguish two cases.

Case 1: n is even. Then, let $n=2m$ and note that $|A|=m+1$, that is, $S=\{a_1,a_2,\dots,a_{m+1}\}$.

Note that, as in the first part, $a_1+a_2\geq a_{m+1}+1$, hence

$$a_1+a_2\geq a_{m+1}+1\geq a_2+m-1+1\Longrightarrow a_1\geq m,$$

therefore

$$2m\geq a_{m+1}\geq a_1+m\geq 2m,$$

thus equality must hold, implying that $A=\{m,m+1,\dots,2m\}$.

We prove the following Claim:

Claim 3: $\frac{(2m)^2}{(2m-1)^2}+\frac{(2m-1)^2}{m^2}+\frac{m^2}{(2m)^2}\geq\frac{i^2}{j^2}+\frac{j^2}{k^2}+\frac{k^2}{j^2}$ for all $i>j>k$ such that $i,j,k\in A$.

Proof: According to Claims 1 and 2, in order to maximize $\frac{i^2}{j^2}+\frac{j^2}{k^2}+\frac{k^2}{j^2}$, we need to take i as large as possible, and k as small as possible. Hence, let $i=2m,k=m$ and we just need to prove

$$\frac{(2m)^2}{(2m-1)^2}+\frac{(2m-1)^2}{m^2}+\frac{m^2}{(2m)^2}\geq\frac{(2m)^2}{j^2}+\frac{j^2}{m^2}+\frac{m^2}{(2m)^2},$$

for all $m+1\leq j\leq 2m-1$.

The above rewrites as

$$((2m-1)^2-j^2)([(2m-1)j]^2-(2m^2)^2)\geq 0,$$

which holds true, since the first parenthesis is nonnegative, since $2m-1\geq j$, while for the second one,

$$(2m-1)j\geq(2m-1)(m+1)=2m^2+m-1\geq 2m^2,$$

hence we are done ■

To the problem, in order for A to be a gaussian subset, by Claim 3 we just need to have

$$\frac{(2m)^2}{(2m-1)^2}+\frac{(2m-1)^2}{m^2}+\frac{m^2}{(2m)^2}<5,$$

which after solving gives $m \leq 11$, that is, $\boxed{n \in \{4,6,8,\dots,22\}}$.

Claim 2: n is odd. Then, let $n=2m+1$ and note that $|A|=m+1$, that is $A=\{a_1,a_2,\dots,a_{m+1}\}$.

As before, we prove that $a_1 \geq m$. Since $a_{m+1} \leq 2m+1$, we have

$$a_1 \leq a_{m+1}-m=m+1,$$

therefore $a_1 \in \{m,m+1\}$.

Subcase 1: $a_1=m+1$. We have equality, hence $A=\{m+1,\dots,2m+1\}$.

In a similar manner as before (using Claims 1 and 2), we deduce that

$$\frac{i^2}{j^2}+\frac{j^2}{k^2}+\frac{k^2}{j^2}$$

maximizes when $i=2m+1, j=2m, k=m+1$, which after solving implies that $m \leq 25$.

Hence, solutions in this case are all $\boxed{n \in \{5,7,\dots,51\}}$.

For brevity, call the above set M.

Subcase 2: $a_1=m$. Then, $2m+1 \geq a_{m+1} \geq 2m$, hence $a_{m+1} \in \{2m,2m+1\}$.

We have two cases to consider.

• $a_{m+1}=2m$. Then, $A=\{m,m+1,\dots,2m\}$, which is identical to Case 1, hence $m \geq 11$. The resulting solutions belong to set M.

• $a_{m+1}=2m+1$. Then, $2m-1 \leq a_m \leq 2m$, hence $a_m \in \{2m-1,2m\}$.

– In the first case, by the above logic we should have

$$\frac{(2m+1)^2}{(2m-1)^2}+\frac{(2m-1)^2}{m^2}+\frac{m^2}{(2m+1)^2}<5,$$

which implies $m \leq 7$, hence the solutions emerging are covered by set M.

– In the second case, by the above logic we should have

$$\frac{(2m+1)^2}{(2m)^2}+\frac{(2m)^2}{m^2}+\frac{m^2}{(2m+1)^2}<5,$$

which rewrites as $(2m^2+4m+1)^2<0$, hence no solutions in this case.

To conclude, the set of solutions is

$$\{4,6,\dots,22\} \cup \{5,7,\dots,51\} = \{4,5,\dots,23\} \cup \{25,27,\dots,49,51\}.$$

Aliter: For part (i), assume that the statement is false, then there is a guassian set with at least $\frac{n+3}{2}$ members, and thus, the ratio (r) between the maximum (M) and the minimum (m) elements of the set is at least $\frac{2n}{n-1}$.

Consider an element x of the gaussian set that is not m nor M. Let $\frac{x}{m} = y$ and $\frac{M}{x} = z$, then we must have $yz = r$ and $y^2 + z^2 < 5 - \frac{1}{r^2}$. After calculation, we get that $\frac{A-B}{2} < y < \frac{A+B}{2}$ where $A = \sqrt{5 - \frac{1}{r^2} + 2r}$ and $B = \sqrt{5 - \frac{1}{r^2} - 2r}$.

This means that the maximum and minimum elements of the gaussian set (not counting M and m) must be less than Bm apart, so the gaussian set has size less than $Bm + 3$.

Thus, $\frac{n+3}{2} < Bm + 3 \implies \frac{n-3}{2} < Bm \leq \sqrt{5 - \left(\frac{n-1}{2n}\right)^2 - \left(\frac{4n}{n-1}\right)} \times \frac{n-1}{2} < \frac{\sqrt{(n-5)(n-1)}}{2}$, which is not true - a contradiction.

(Note that we use the fact that $f(r) = 2r + \frac{1}{r^2}$ is increasing when $r \geq 1$.)

For (ii), we first prove three lemmas:

For any $a > b > c$, $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{a^2}{b^2} + \frac{b^2}{(c+1)^2} + \frac{(c+1)^2}{a^2}$

This is equivalent to $\frac{b^2}{c^2(c+1)^2}(2c+1) \geq \frac{2c+1}{a^2}$, which is true.

For any $a > b > c$, $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \leq \frac{(a+1)^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{(a+1)^2}$

This is equivalent to $\frac{c^2}{a^2(a+1)^2}(2a+1) \leq \frac{2a+1}{b^2}$, which is true.

If $a > b > c$, then $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{a^2}{(b+1)^2} + \frac{(b+1)^2}{(c+1)^2} + \frac{(c+1)^2}{a^2}$

This is true since $\frac{b}{c} > \frac{b+1}{c+1}$ and $\left(\frac{a^2}{b^2} - \frac{a^2}{(b+1)^2}\right) + \left(\frac{c^2}{a^2} - \frac{(c+1)^2}{a^2}\right) = \frac{a^2(2b+1)}{b^2(b+1)^2} - \frac{2c+1}{a^2} \geq 0$

From these lemmas, we can conclude that a set is gaussian if the inequality is satisfied when we choose a as the maximum element c as the minimum element, and b be the element such that $\min\{\frac{a}{b}, \frac{b}{c}\}$ is the smallest (which maximize the sum). We will consider two cases based on parity of n.

If $n \geq 4$ is even, then any gaussian set with $\frac{n+2}{2}$ member must have n as a member. (Else, it's a subset of $\{1, 2, ..., n-1\}$ and we can use (i) to get contradiction.)

In this case, among the $\frac{n}{2}$ left members, some pair of them are at least $\frac{n}{2} - 1$ apart, and so, by the two lemmas, we must have

$$\frac{n^2}{(n-1)^2} + \frac{(2n-2)^2}{n^2} + \frac{1}{4} < 5 \implies \frac{n^2}{(n-1)^2} > \frac{19 - \sqrt{105}}{8} \implies \frac{2}{n-1} + \frac{1}{(n-1)^2} > \frac{11 - \sqrt{105}}{8}$$

This is true when $n \leq 22$ as $\frac{2}{21} > \frac{11-\sqrt{105}}{8}$, but not true for $n \geq 24$ as $\frac{11-\sqrt{105}}{8} > \frac{1}{11} > \frac{2}{23} + \frac{1}{529}$.

To prove that any even number $4 \leq n \leq 22$ satisfies the condition, simply choose $\{\frac{n}{2}, ..., n-1, n\}$

as the gaussian set and note that $\frac{n}{n-1} \leq \frac{\frac{n}{2}+1}{\frac{n}{2}}$.

For odd number, $n=3$ doesn't satisfy the condition, and $5 \leq n \leq 23$ work by choosing the same gaussian set with previous case. For higher n, the gaussian set must have n as an element. Similar to the previous case, we must have

$$\frac{n^2}{(n-1)^2} + \frac{(2n-2)^2}{(n+1)^2} + \frac{(n+1)^2}{4n^2} < 5$$

or, equivalently

$$\frac{2}{n-1} + \frac{1}{(n-1)^2} + \frac{16}{(n+1)^2} + \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2} < \frac{16}{n+1}$$

This is false for $n \geq 53$ as $\frac{2}{n-1} + \frac{1}{4} + \frac{1}{2n} > \frac{16}{n+1}$, and for $25 \leq n \leq 51$, this is true as $\frac{16}{n+1} - \left(\frac{2}{n-1} + \frac{1}{4} + \frac{1}{2n}\right) > \frac{20}{n^2-1}$ and $\frac{1}{(n-1)^2} + \frac{16}{(n+1)^2} + \frac{1}{4n^2} < \frac{19}{n^2-1}$.

Again, simply take the set $\{\frac{n+1}{2}, ..., n-1, n\}$ as example and note that $\frac{n}{n-1} \leq \frac{\frac{n+3}{2}}{\frac{n+1}{2}}$.

Thus, the answer is $\{4, 6, ..., 22\} \cup \{5, 7, ..., 51\}$.

Chapter 4

Results

4.1 GAMO Results

We received a total of 24 submissions: 13 for GAMO and 11 for GJMO.

The GAMO leadearboard is presented below , we congratulate all the participants of GAMO specially the first, first runner up and second runner up that is

- ywq233
- k12byda5h
- korncrazy.

GAMO 2021 Leaderboard								
AoPS Username	Ranking Position	Day 1			Day 2			Total Score
		A-1	A-2	A-3	A-4	A-5	A-6	
ywq233	1	7	7	7	6	7	2	36
k12byda5h	2	7	7	7	6	7	0	34
korncrazy	3	7	7	7	6	7	0	34
L567	4	7	6	0	7	7	4	31
Pitagar	5	7	7	0	7	7	0	28
hyay	6	7	7	0	6	0	7	27
chirita.andrei	7	7	0	7	6	7	0	27
KaiDaMagical336	8	7	7	0	7	0	5	26
bluelinfish	9	7	6	0	7	0	0	20
DreamDream	10	7	0	0	7	0	0	14
LTH-0-	11	3	0	0	6	0	0	413
FAA2533	12	0	0	3	0	0	0	3
Ucchash	13	0	0	0	0	0	1	1
GAMO Results								

4.2 GJMO Results

The GJMO leadearboard is presented below , we congratulate all the participants of GJMO specially the first, first runner up and second runner up that is

- Pitagar
- bluelinfish
- L567.

GJMO 2021 Leaderboard								
AoPS Username	Ranking Position	Day 1			Day 2			Total Score
		J-1	J-2	J-3	J-4	J-5	J-6	
Pitagar	1	7	7	7	7	7	7	42
bluelinfish	2	7	7	7	7	0	7	35
L567	3	7	7	0	7	6	2	29
DreamDream	4	0	6	0	7	7	0	20
NTFEGAC	5	7	3	0	7	0	2	19
Anonymous	6	7	0	0	7	0	3	17
IMOTC	7	0	0	0	7	0	0	7
hemlock	8	0	0	0	7	0	0	7
UKR3IMO	9	0	0	0	7	0	0	7
ishan3.14	10	0	6	0	0	0	0	6
Souparna(NAoPS)	11	0	0	0	6	0	0	6
GJMO Results								

Note that NAoPS means not a member of AoPS.

Problem Proposers

USAMO IMO Team Members of Gaussian Curvature

Problem Selection Committee

Aritra12	Orestis Lignos	TLP.39
EpicNumberTheory	Phoenixfire	Anonymous

GAMO & GJMO Graders

Aritra12	Orestis Lignos	TLP.39
EpicNumberTheory	Phoenixfire	i3435
Anonymous		

Test Solvers

Test Solver Team of Gaussian Curvature

Participants(Sign Ups)

L567, bluelinfish3, Pitagar, Ninjasolver0201, IMOTC, weaving2, UKR3IMO9, hyay10, Ishan3.14, franzliszt, PlaneGod, Souparna, DrYouKnowWho, kattyames, sparkaks, jasperE3, Pluto04, CaptainLevi16, DebayuRMO, mufree, iman007, superagh, LTH-0-, Mop2018, shalomrav, ike.chen, JustinLee2017, Dreamdream, Paradoxes, Mattii, Wisphard, Sohil_Doshi, NTFEGAC, ucchash, FAA2533, B1002342, Interstigation, The_Musilm, Enthurelx, deepakies, Abrar_Sharia, Samariun_42, rafaello, third_one_is_jerk, ashraful7525, korncrazy, chirita.andrei, molecules_mal, Lorgenoob9696, Com10atorics, k12byda5h, hakN, MathLuis, RADHEY_123