in Section 4.2.3. Namely, for a given basic solution (and, hence, inverse), the dual values must be unique. Problem 5-31 addresses this point.

5.4 THE ASSIGNMENT MODEL

The classical assignment model deals with matching workers (with varying skills) to jobs. Presumably, skill variation affects the cost of completing a job. The goal is to determine the minimum cost assignment of workers to jobs. The general assignment model with n workers and n jobs is represented in Table 5.18. The element c_{ij} represents the cost of assigning worker i to job $j(i, j = 1, 2, \ldots, n)$. There is no loss of generality in assuming that the number of workers and the number of jobs are equal, because we can always add fictitious workers or fictitious jobs to satisfy this assumption.

The assignment model is a special case of the transportation model where workers represent sources and jobs represent destinations. The supply (demand) amount at each source (destination) exactly equals 1. The cost of "transporting" worker i to job j is c_{ij}. In effect, the assignment model can be solved directly as a regular transportation model (or as a regular LP). Nevertheless, the fact that all the supply and demand amounts equal 1 has led to the development of a simple solution algorithm called the **Hungarian method**. Although the new solution method appears totally unrelated to the transportation model, the algorithm is actually rooted in the simplex method, just as the transportation model is.

5.4.1 The Hungarian Method[8]

We will use two examples to present the mechanics of the new algorithm. The next section provides a simplex-based explanation of the procedure.

TABLE 5.18 Assignment Model

		Jobs				
		1	2	…	n	
Worker	1	c_{11}	c_{12}	…	c_{1n}	**1**
	2	c_{21}	c_{22}	…	c_{2n}	**1**
	⋮	⋮	⋮	⋮	⋮	⋮
	N	c_{n1}	c_{n2}	…	c_{nn}	**1**
		1	**1**	…	**1**	

[8]As with the transportation model, the classical Hungarian method, designed primarily for hand computations, is something of the past and is presented here for historical reasons. Today, the need for such computational shortcuts is not warranted, as the problem can be solved by highly efficient LP computer codes. Perhaps the benefit from studying these classical techniques is that they are based on a sophisticated theory that reduces the solution steps to simple rules suitable for hand computations.

Example 5.4-1

Joe Klyne's three children, John, Karen, and Terri, want to earn some money for personal expenses. Mr. Klyne has chosen three chores for his children: mowing the lawn, painting the garage door, and washing the family cars. To avoid anticipated sibling competition, he asks them to submit individual (secret) bids for what they feel is fair pay for each of the three chores. Table 5.19 summarizes the bids received. The children will abide by their father's decision regarding the assignment of chores.

The assignment problem will be solved by the Hungarian method.

Step 1. Determine p_i, the minimum cost element of row i in the original cost matrix, and subtract it from all the elements of row i, $i = 1, 2, 3$.

Step 2. For the matrix created in step 1, determine q_j, the minimum cost element of column j, and subtract it from all the elements of column j, $j = 1, 2, 3$.

Step 3. From the matrix in step 2, attempt to find a *feasible* assignment among all the resulting zero entries.

3a. If such an assignment can be found, it is optimal.

3b. Else, additional calculations are needed (as will be explained in Example 5.4-2).

Table 5.20 shows the application of the three steps to the current problem.

The cells with underscored zero entries in step 3 provide the (feasible) optimum solution: John gets the paint job, Karen gets to mow the lawn, and Terri gets to wash the family cars. The total cost to Mr. Klyne is $9 + 10 + 8 = \$27$. This amount also will always equal $(p_1 + p_2 + p_3) + (q_1 + q_2 + q_3) = (9 + 9 + 8) + (0 + 1 + 0) = \$27$. (A justification of this result is given in the next section.)

TABLE 5.19 Klyne's Assignment Problem

	Mow	Paint	Wash
John	\$15	\$10	\$9
Karen	\$9	\$15	\$10
Terri	\$10	\$12	\$8

TABLE 5.20 Application of the Hungarian Method to the Assignment Problem of Example 5.4-1

Step 1:

	Mow	Paint	Wash	Row min
John	15	10	9	$p_1 = 9$
Karen	9	15	10	$p_2 = 9$
Terri	10	12	8	$p_3 = 8$

⇒

Step 2:

	Mow	Paint	Wash
John	6	1	0
Karen	0	6	1
Terri	2	4	0
Column max	$q_1 = 0$	$q_2 = 1$	$q_3 = 0$

⇒

Step 3:

	Mow	Paint	Wash
John	6	$\underline{\mathbf{0}}$	0
Karen	$\underline{\mathbf{0}}$	5	1
Terri	2	3	$\underline{\mathbf{0}}$

As stated in step 3 of the Hungarian method, the zeros created by steps 1 and 2 may not yield a feasible solution directly. In this case, further steps are needed to find the optimal (feasible) assignment. The following example demonstrates this situation.

Example 5.4-2

Suppose that the situation discussed in Example 5.4-1 is extended to four children and four chores. Table 5.21 summarizes the cost elements of the problem.

The application of steps 1 and 2 to the matrix in Table 5.21 (using $p_1 = 1, p_2 = 7, p_3 = 4, p_4 = 5, q_1 = 0, q_2 = 0, q_3 = 3$, and $q_4 = 0$) yields the reduced matrix in Table 5.22 (verify!):

The locations of the zero entries do not allow assigning unique chores to all the children. For example, if we assign child 1 to chore 1, then column 1 will be eliminated, and child 3 will not have a zero entry in the remaining three columns. This obstacle can be accounted for by adding the following step to the procedure given in Example 5.4-1:

Step 3b. If no feasible zero-element assignments can be found,

(i) Draw the *minimum* number of horizontal and vertical lines in the last reduced matrix to cover *all* the zero entries.

(ii) Select the *smallest uncovered* entry, subtract it from every uncovered entry, and then add it to every entry at the intersection of two lines.

(iii) If no feasible assignment can be found among the resulting zero entries, repeat step 3a.

The application of step 3b to the last matrix produces the shaded cells in Table 5.23. The smallest unshaded entry (shown underscored) equals 1. This entry is added to the intersection cells and subtracted from the remaining shaded cells to produce the matrix in Table 5.24, and the optimal solution shown by underscored zeros.

TABLE 5.21 Assignment Model

		Chore 1	2	3	4
Child	1	\$1	\$4	\$6	\$3
	2	\$9	\$7	\$10	\$9
	3	\$4	\$5	\$11	\$7
	4	\$8	\$7	\$8	\$5

TABLE 5.22 Reduced Assignment Matrix

		Chore 1	2	3	4
Child	1	**0**	3	2	2
	2	2	**0**	**0**	2
	3	**0**	1	4	3
	4	3	2	**0**	**0**

TABLE 5.23 Application of Step 3b

		Chore 1	2	3	4
Child	1	0	3	2	2
	2	**2**	0	0	2
	3	0	<u>1</u>	4	3
	4	**3**	2	0	0

TABLE 5.24 Optimal Assignment

		Chore 1	2	3	4
Child	1	**<u>0</u>**	2	1	1
	2	3	0	**<u>0</u>**	2
	3	0	**<u>0</u>**	3	2
	4	4	2	0	**<u>0</u>**

AMPL Moment

File *amplEx5.4-2.txt* provides the AMPL model for the assignment model. The model is similar to that of the transportation model.

5.4.2 Simplex Explanation of the Hungarian Method

The assignment problem in which n workers are assigned to n jobs can be represented as an LP model in the following manner: Let c_{ij} be the cost of assigning worker i to job j, and define

$$x_{ij} = \begin{cases} 1, & \text{if worker } i \text{ is assigned to job } j \\ 0, & \text{otherwise} \end{cases}$$

Then the LP model is given as

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots, n$$

$$x_{ij} = 0 \text{ or } 1$$

The optimal solution of the preceding LP model remains unchanged if a constant is added to or subtracted from any row or column of the cost matrix (c_{ij}). To prove this point, let p_i and q_j be constants subtracted from row i and column j. Thus, the cost element c_{ij} is changed to

$$c'_{ij} = c_{ij} - p_i - q_j$$

Now

$$\sum_i \sum_j c'_{ij} x_{ij} = \sum_i \sum_j (c_{ij} - p_i - q_j)\, x_{ij} = \sum_i \sum_j c_{ij} x_{ij} - \sum_i p_i \Big(\sum_j x_{ij} \Big) - \sum_j q_j \Big(\sum_i x_{ij} \Big)$$

$$= \sum_i \sum_j c_{ij} x_{ij} - \sum_i p_i(1) - \sum_j q_j(1)$$

$$= \sum_i \sum_j c_{ij} x_{ij} - \text{constant}$$

Because the new objective function differs from the original by a constant, the optimum values of x_{ij} are the same in both cases. The development shows that steps 1 and 2 of the Hungarian method, which call for subtracting p_i from row i and then subtracting q_j from column j, produce an equivalent assignment model. In this regard, if a feasible solution can be found among the zero entries of the cost matrix created by steps 1 and 2, then it must be optimum (because the cost in the modified matrix cannot be less than zero).

If the created zero entries cannot yield a feasible solution (as Example 5.4-2 demonstrates), then step 2a (dealing with the covering of the zero entries) must be applied. The validity of this procedure is again rooted in the simplex method of linear programming and can be explained by duality theory (Chapter 4) and the complementary slackness theorem (Chapter 7). We will not present the details of the proof here because they are somewhat involved.

The reason $(p_1 + p_2 + \ldots + p_n) + (q_1 + q_2 + \ldots + q_n)$ gives the optimal objective value is that it represents the dual objective function of the assignment model. This result can be seen through comparison with the dual objective function of the transportation model given in Section 5.3.3. [See Bazaraa and Associates (2009) for the details.]

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