TABLE 37
Tableau Indicating Infeasibility for Bevco (Infeasible)

z	x_1	s_2	s_1	e_2	a_2	a_3	rhs	Basic Variable
1	$1 - 2M$	0	0	$-M$	0	$3 - 4M$	$30 + 6M$	$z = 6M + 30$
0	$\frac{1}{4}$	0	1	0	0	$-\frac{1}{4}$	$\frac{3}{2}$	$s_1 = \frac{3}{2}$
0	-2	0	0	-1	1	-3	6	$a_2 = 6$
0	1	1	0	0	0	1	10	$x_2 = 10$

Note that when the Big M method is used, it is difficult to determine how large M should be. Generally, M is chosen to be at least 100 times larger than the largest coefficient in the original objective function. The introduction of such large numbers into the problem can cause roundoff errors and other computational difficulties. For this reason, most computer codes solve LPs by using the two-phase simplex method (described in Section 4.13).

PROBLEMS

Group A

Use the Big M method to solve the following LPs:

1 $\min z = 4x_1 + 4x_2 + x_3$

$$\begin{aligned} \text{s.t.}\quad & x_1 + x_2 + x_3 \le 2 \\ & 2x_1 + x_2 \le 3 \\ & 2x_1 + x_2 + 3x_3 \ge 3 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

2 $\min z = 2x_1 + 3x_2$

$$\begin{aligned} \text{s.t.}\quad & 2x_1 + x_2 \ge 4 \\ & x_1 - x_2 \ge -1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

3 $\max z = 3x_1 + x_2$

$$\begin{aligned} \text{s.t.}\quad & x_1 + x_2 \ge 3 \\ & 2x_1 + x_2 \le 4 \\ & x_1 + x_2 = 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

4 $\min z = 3x_1$

$$\begin{aligned} \text{s.t.}\quad & 2x_1 + x_2 \ge 6 \\ & 3x_1 + 2x_2 = 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

5 $\min z = x_1 + x_2$

$$\begin{aligned} \text{s.t.}\quad & 2x_1 + x_2 + x_3 = 4 \\ & x_1 + x_2 + 2x_3 = 2 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

6 $\min z = x_1 + x_2$

$$\begin{aligned} \text{s.t.}\quad & x_1 + x_2 = 2 \\ & 2x_1 + 2x_2 = 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

4.13 The Two-Phase Simplex Method[†]

When a basic feasible solution is not readily available, the two-phase simplex method may be used as an alternative to the Big M method. In the two-phase simplex method, we add artificial variables to the same constraints as we did in the Big M method. Then we find a bfs to the original LP by solving the Phase I LP. In the Phase I LP, the objective function is to minimize the sum of all artificial variables. At the completion of Phase I, we reintroduce the original LP's objective function and determine the optimal solution to the original LP.

The following steps describe the two-phase simplex method. Note that steps 1–3 for the two-phase simplex are identical to steps 1–3 for the Big M method.

[†]This section covers topics that may be omitted with no loss of continuity.

Step 1 Modify the constraints so that the right-hand side of each constraint is nonnegative. This requires that each constraint with a negative right-hand side be multiplied through by -1.

Step 1′ Identify each constraint that is now (after step 1) an $=$ or $\geq$ constraint. In step 3, we will add an artificial variable to each constraint.

Step 2 Convert each inequality constraint to the standard form. If constraint i is a $\leq$ constraint, then add a slack variable s_i. If constraint i is a $\geq$ constraint, subtract an excess variable e_i.

Step 3 If (after step 1′) constraint i is a $\geq$ or $=$ constraint, add an artificial variable a_i. Also add the sign restriction $a_i \geq 0$.

Step 4 For now, ignore the original LP's objective function. Instead solve an LP whose objective function is min w' = (sum of all the artificial variables). This is called the **Phase I LP.** The act of solving the Phase I LP will force the artificial variables to be zero.

Because each $a_i \geq 0$, solving the Phase I LP will result in one of the following three cases:

Case 1 The optimal value of w' is greater than zero. In this case, the original LP has no feasible solution.

Case 2 The optimal value of w' is equal to zero, and no artificial variables are in the optimal Phase I basis. In this case, we drop all columns in the optimal Phase I tableau that correspond to the artificial variables. We now combine the original objective function with the constraints from the optimal Phase I tableau. This yields the **Phase II LP.** The optimal solution to the Phase II LP is the optimal solution to the original LP.

Case 3 The optimal value of w' is equal to zero and at least one artificial variable is in the optimal Phase I basis. In this case, we can find the optimal solution to the original LP if at the end of Phase I we drop from the optimal Phase I tableau all nonbasic artificial variables and any variable from the original problem that has a negative coefficient in row 0 of the optimal Phase I tableau.

Before solving examples illustrating Cases 1–3, we briefly discuss why $w' > 0$ corresponds to the original LP having no feasible solution and $w' = 0$ corresponds to the original LP having at least one feasible solution.

Phases I and II Feasible Solutions

Suppose the original LP is infeasible. Then the only way to obtain a feasible solution to the Phase I LP is to let at least one artificial variable be positive. In this situation, $w' > 0$ (Case 1) will result. On the other hand, if the original LP has a feasible solution, then this feasible solution (with all $a_i = 0$) is feasible in the Phase I LP and yields $w' = 0$. This means that if the original LP has a feasible solution, the optimal Phase I solution will have $w' = 0$. We now work through examples of Cases 1 and 2 of the two-phase simplex method.

EXAMPLE 5 **Two-Phase Simplex: Case 2**

First we use the two-phase simplex to solve the Bevco problem of Section 4.12. Recall that the Bevco problem was

$$\begin{aligned}
\min z &= 2x_1 + 3x_2 \\
\text{s.t.} \quad \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 &\leq 4 \\
x_1 + 3x_2 &\geq 20 \\
x_1 + x_2 &= 10 \\
x_1, x_2 &\geq 0
\end{aligned}$$

Solution As in the Big M method, steps 1–3 transform the constraints into

$$\begin{aligned} \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 \phantom{{}- e_2 + a_2 + a_3} &= 4 \\ x_1 + 3x_2 \phantom{{}+ s_1} - e_2 + a_2 \phantom{{}+ a_3} &= 20 \\ x_1 + x_2 \phantom{{}+ s_1 - e_2 + a_2} + a_3 &= 10 \end{aligned}$$

Step 4 yields the following Phase I LP:

$$\begin{aligned} \min w' &= a_2 + a_3 \\ \text{s.t.} \quad \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 \phantom{{}- e_2 + a_2 + a_3} &= 4 \\ x_1 + 3x_2 - e_2 + a_2 \phantom{{}+ a_3} &= 20 \\ x_1 + x_2 \phantom{{}- e_2 + a_2} + a_3 &= 10 \end{aligned}$$

This set of equations yields a starting bfs for Phase I ($s_1 = 4$, $a_2 = 20$, $a_3 = 10$).

Note, however, that the row 0 for this tableau ($w' - a_2 - a_3 = 0$) contains the basic variables a_2 and a_3. As in the Big M method, a_2 and a_3 must be eliminated from row 0 before we can solve Phase I. To eliminate a_2 and a_3 from row 0, simply add row 2 and row 3 to row 0:

$$\begin{aligned} &\text{Row 0:} && w' \phantom{{}+ 2x_1 + 4x_2 - e_2} - a_2 - a_3 = 0 \\ &+ \text{Row 2:} && x_1 + 3x_2 - e_2 + a_2 \phantom{{}- a_3} = 20 \\ &+ \text{Row 3:} && x_1 + x_2 \phantom{{}- e_2 + a_2} + a_3 = 10 \\ &= \text{New row 0:} && w' + 2x_1 + 4x_2 - e_2 \phantom{{}- a_2 - a_3} = 30 \end{aligned}$$

Combining the new row 0 with the Phase I constraints yields the initial Phase I tableau in Table 38. Because the Phase I problem is *always* a min problem (even if the original LP is a max problem), we enter x_2 into the basis. The ratio test indicates that x_2 will enter the basis in row 2, with a_2 exiting the basis. After performing the necessary EROs, we obtain the tableau in Table 39. Because $5 < 20$ and $5 < \frac{28}{5}$, x_1 enters the basis in row 3. Thus, a_3 will leave the basis. Because a_2 and a_3 will be nonbasic after the current pivot is completed, we already know that the next tableau will be optimal for Phase I. A glance at the tableau in Table 40 confirms this fact.

Because $w' = 0$, Phase I has been concluded. The basic feasible solution $s_1 = \frac{1}{4}$, $x_2 = 5$, $x_1 = 5$ has been found. No artificial variables are in the optimal Phase I basis, so the problem is an example of Case 2. We now drop the columns for the artificial variables a_2 and a_3 (we no longer need them) and reintroduce the original objective function.

$$\min z = 2x_1 + 3x_2 \qquad \text{or} \qquad z - 2x_1 - 3x_2 = 0$$

Because x_1 and x_2 are both in the optimal Phase I basis, they must be eliminated from the Phase II row 0. We add 3(row 2) + 2(row 3) of the optimal Phase I tableau to row 0.

$$\begin{aligned} &\text{Phase II row 0:} && z - 2x_1 - 3x_2 \phantom{{}- \tfrac{1}{2}e_2} = 0 \\ &+ 3(\text{row 2}): && 3x_2 - \tfrac{3}{2}e_2 = 15 \\ &+ 2(\text{row 3}): && 2x_1 \phantom{{}+ 3x_2} + e_2 = 10 \\ &= \text{New Phase II row 0:} && z \phantom{{}- 2x_1 - 3x_2} - \tfrac{1}{2}e_2 = 25 \end{aligned}$$

We now begin Phase II with the following set of equations:

$$\begin{aligned} \min z - \tfrac{1}{2}e_2 &= 25 \\ s_1 - \tfrac{1}{8}e_2 &= \tfrac{1}{4} \\ x_2 \quad - \tfrac{1}{2}e_2 &= 5 \\ x_1 \quad + \tfrac{1}{2}e_2 &= 5 \end{aligned}$$

TABLE 38
Initial Phase I Tableau for Bevco

w'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable	Ratio
1	2	4	0	-1	0	0	30	$w' = 30$	
0	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	$s_1 = 4$	16
0	1	3	0	-1	1	0	20	$a_2 = 20$	$\frac{20}{3}$*
0	1	1	0	0	0	1	10	$a_3 = 10$	10

TABLE 39
Phase I Tableau for Bevco after One Iteration

w'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable	Ratio
1	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{4}{3}$	0	$\frac{10}{3}$	$w' = \frac{10}{3}$	
0	$\frac{5}{12}$	0	1	$\frac{1}{12}$	$-\frac{1}{12}$	0	$\frac{7}{3}$	$s_1 = \frac{7}{3}$	$\frac{28}{5}$
0	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{20}{3}$	$x_2 = \frac{20}{3}$	20
0	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{10}{3}$	$a_3 = \frac{10}{3}$	5*

TABLE 40
Optimal Phase I Tableau for Bevco

w'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable
1	0	0	0	0	-1	-1	0	$w' = 0$
0	0	0	1	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{5}{8}$	$\frac{1}{4}$	$s_1 = \frac{1}{4}$
0	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5	$x_2 = 5$
0	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	5	$x_1 = 5$

This is optimal. Thus, in this problem, Phase II requires no pivots to find an optimal solution. If the Phase II row 0 does not indicate an optimal tableau, then simply continue with the simplex until an optimal row 0 is obtained. In summary, our optimal Phase II tableau shows that the optimal solution to the Bevco problem is $z = 25$, $x_1 = 5$, $x_2 = 5$, $s_1 = \frac{1}{4}$, and $e_2 = 0$. This agrees, of course, with the optimal solution found by the Big M method in Section 4.12.

EXAMPLE 6 Two-Phase Simplex: Case I

To illustrate Case 1, we now modify Bevco's problem so that 36 mg of vitamin C are required. From Section 4.12, we know that this problem is infeasible. This means that the optimal Phase I solution should have $w' > 0$ (Case 1). To show that this is true, we begin with the original problem:

$$\begin{aligned}
\min z &= 2x_1 + 3x_2 \\
\text{s.t.} \quad \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 &\leq 4 \\
x_1 + 3x_2 &\geq 36 \\
x_1 + x_2 &= 10 \\
x_1, x_2 &\geq 0
\end{aligned}$$

TABLE 41
Initial Phase I Tableau for Bevco (Infeasible)

w'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable	Ratio
1	2	4	0	-1	0	0	46	$w' = 46$	
0	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	$s_1 = 4$	16
0	1	3	0	-1	1	0	36	$a_2 = 36$	12
0	1	1	0	0	0	1	10	$a_3 = 0$	10*

TABLE 42
Tableau Indicating Infeasibility for Bevco (Infeasible)

w'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable
1	-2	0	0	-1	0	-4	6	$w' = 6$
0	$\frac{1}{4}$	0	1	0	0	$-\frac{1}{4}$	$\frac{3}{2}$	$s_1 = \frac{3}{2}$
0	-2	0	0	-1	1	-3	6	$a_2 = 6$
0	1	1	0	0	0	1	10	$x_2 = 10$

Solution After completing steps 1–4 of the two-phase simplex, we obtain the following Phase I problem:

$$
\begin{aligned}
\min w' &= a_2 + a_3 \\
\text{s.t.} \quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\
& x_1 + 3x_2 - e_2 + a_2 = 36 \\
& x_1 + x_2 + a_3 = 10
\end{aligned}
$$

From this set of equations, we see that the initial Phase I bfs is $s_1 = 4$, $a_2 = 36$, and $a_3 = 10$. Because the basic variables a_2 and a_3 occur in the Phase I objective function, they must be eliminated from the Phase I row 0. To do this, we add rows 2 and 3 to row 0:

$$
\begin{aligned}
&\text{Row 0:} && w' - a_2 - a_3 = 0 \\
+\ &\text{Row 2:} && x_1 + 3x_2 - e_2 + a_2 = 36 \\
+\ &\text{Row 3:} && x_1 + x_2 + a_3 = 10 \\
=\ &\text{New row 0:} && w' + 2x_1 + 4x_2 - e_2 = 46
\end{aligned}
$$

With the new row 0, the initial Phase I tableau is as shown in Table 41. Because $4 > 2$, we should enter x_2 into the basis. The ratio test indicates that x_2 should enter the basis in row 3, forcing a_3 to leave the basis. The resulting tableau is shown in Table 42. No variable in row 0 has a positive coefficient, so this is an optimal Phase I tableau, and since the optimal value of w' is $6 > 0$, the original LP must have no feasible solution. This is reasonable, because if the original LP had a feasible solution, it would have been feasible in the Phase I LP (after setting $a_2 = a_3 = 0$). This feasible solution would have yielded $w' = 0$. Because the simplex could not find a Phase I solution with $w' = 0$, the original LP must have no feasible solution.

REMARKS **1** As with the Big M method, the column for any artificial variable may be dropped from future tableaus as soon as the artificial variable leaves the basis. Thus, when we solved the Bevco problem, a_2's column could have been dropped after the first Phase I pivot, and a_3's column could have been dropped after the second Phase I pivot.

2 It can be shown that (barring ties for the entering variable and in the ratio test) the Big M method and Phase I of the two-phase method make the same sequence of pivots. Despite this equivalence, most computer codes utilize the two-phase method to find a bfs. This is because M, being a large positive number, may cause roundoff errors and other computational difficulties. The two-phase method does not introduce any large numbers into the objective function, so it avoids this problem.

EXAMPLE 7 Two-Phase Simplex: Case 3

Use the two-phase simplex method to solve the following LP:

$$
\begin{aligned}
\min z &= 40x_1 + 10x_2 + 7x_5 + 14x_6 \\
\text{s.t.} \quad & x_1 - x_2 + 2x_5 = 0 \\
& -2x_1 + x_2 - 2x_5 = 0 \\
& x_1 + x_3 + x_5 - x_6 = 3 \\
& 2x_2 + x_3 + x_4 + 2x_5 + x_6 = 4 \\
& \text{All } x_i \geq 0
\end{aligned}
$$

Solution We may use x_4 as a basic variable for the fourth constraint and use artificial variables a_1, a_2, and a_3 as basic variables for the first three constraints. Our Phase I objective is to minimize $w = a_1 + a_2 + a_3$. After adding the first three constraints to $w - a_1 - a_2 - a_3 = 0$, we obtain the initial Phase I tableau shown in Table 43.

Even though x_5 has the most positive coefficient in row 0, we choose to enter x_3 into the basis (as a basic variable in row 3). We see that this will immediately yield $w = 0$. Our final Phase I tableau is shown in Table 44.

Because $w = 0$, we now have an optimal Phase I tableau. Two artificial variables remain in the basis (a_1 and a_2) at a zero level. We may now drop the artificial variable a_3 from our first Phase II tableau. The only original variable with a negative coefficient in the optimal Phase I tableau is x_1, so we may drop x_1 from all future tableaus. This is because from the optimal Phase I tableau we find $w = x_1$. This implies that x_1 can never become positive during Phase II, so we may drop x_1 from all future tableaus. Because $z - 40x_1 - 10x_2 - 7x_5 - 14x_6 = 0$ contains no basic variables, our initial tableau for Phase II is as in Table 45.

TABLE 43

w	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	rhs	Basic Variable
1	0	0	1	0	1	−1	0	0	0	3	$w = 3$
0	1	−1	0	0	2	0	1	0	0	0	$a_1 = 0$
0	−2	1	0	0	−2	0	0	1	0	0	$a_2 = 0$
0	1	0	1	0	1	−1	0	0	1	3	$a_3 = 3$
0	0	2	1	1	2	1	0	0	0	4	$x_4 = 4$

TABLE 44

w	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	rhs	Basic Variable
1	−1	0	0	0	0	0	0	0	−1	0	$w = 0$
0	1	−1	0	0	2	0	1	0	0	0	$a_1 = 0$
0	−2	1	0	0	−2	0	0	1	0	0	$a_2 = 0$
0	1	0	1	0	1	−1	0	0	1	3	$x_3 = 3$
0	−1	2	0	1	1	2	0	0	−1	1	$x_4 = 1$

TABLE 45

z	x_2	x_3	x_4	x_5	x_6	a_1	a_2	rhs	Basic Variables
1	-10	0	0	-7	-14	0	0	0	$z = 0$
0	-1	0	0	2	0	1	0	0	$a_1 = 0$
0	1	0	0	-2	0	0	1	0	$a_2 = 0$
0	0	1	0	1	-1	0	0	3	$x_3 = 3$
0	2	0	1	1	2	0	0	1	$x_4 = 1$

TABLE 46

z	x_2	x_3	x_4	x_5	x_6	a_1	a_2	rhs	Basic Variables
1	4	0	7	0	0	0	0	7	$z = 7$
0	0	0	0	2	0	1	0	0	$a_1 = 0$
0	1	0	0	0	0	0	1	0	$a_2 = 0$
0	1	1	$\frac{1}{2}$	$\frac{3}{2}$	0	0	0	$\frac{7}{2}$	$x_3 = \frac{7}{2}$
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$x_4 = \frac{1}{2}$

We now enter x_6 into the basis in row 4 and obtain the optimal tableau shown in Table 46.

The optimal solution to our original LP is $z = 7$, $x_3 = 7/2$, $x_4 = \frac{1}{2}$, $x_2 = x_5 = x_6 = x_3 = 0$.

PROBLEMS

Group A

1 Use the two-phase simplex method to solve the Section 4.12 problems.

2 Explain why the Phase I LP will usually have alternative optimal solutions.

4.14 Unrestricted-in-Sign Variables

In solving LPs with the simplex algorithm, we used the ratio test to determine the row in which the entering variable became a basic variable. Recall that the ratio test depended on the fact that any feasible point required all variables to be nonnegative. Thus, if some variables are allowed to be unrestricted in sign (urs), the ratio test and therefore the simplex algorithm are no longer valid. In this section, we show how an LP with unrestricted-in-sign variables can be transformed into an LP in which all variables are required to be nonnegative.

For each urs variable x_i, we begin by defining two new variables x_i' and x_i''. Then substitute $x_i' - x_i''$ for x_i in each constraint and in the objective function. Also add the sign restrictions $x_i' \geq 0$ and $x_i'' \geq 0$. The effect of this substitution is to express x_i as the difference of the two nonnegative variables x_i' and x_i''. Because all variables are now required to be nonnegative, we can proceed with the simplex. As we will soon see, no basic feasible solution can have both $x_i' > 0$ and $x_i'' > 0$. This means that for any basic feasible solution, each urs variable x_i must fall into one of the following three cases: