EXAMPLE: Cobb-Douglas Demand Functions

In Chapter 4 we introduced the **Cobb-Douglas utility function**

$$u(x_1, x_2) = x_1^c x_2^d.$$

Since utility functions are only defined up to a monotonic transformation, it is convenient to take logs of this expression and work with

$$\ln u(x_1, x_2) = c \ln x_1 + d \ln x_2.$$

Let's find the demand functions for x_1 and x_2 for the Cobb-Douglas utility function. The problem we want to solve is

$$\max_{x_1, x_2} c \ln x_1 + d \ln x_2$$

$$\text{such that } p_1 x_1 + p_2 x_2 = m.$$

There are at least three ways to solve this problem. One way is just to write down the MRS condition and the budget constraint. Using the expression for the MRS derived in Chapter 4, we have

$$\frac{cx_2}{dx_1} = \frac{p_1}{p_2}$$
$$p_1 x_1 + p_2 x_2 = m.$$

These are two equations in two unknowns that can be solved for the optimal choice of x_1 and x_2. One way to solve them is to substitute the second into the first to get

$$\frac{c(m/p_2 - x_1 p_1/p_2)}{dx_1} = \frac{p_1}{p_2}.$$

Cross multiplying gives

$$c(m - x_1 p_1) = dp_1 x_1.$$

Rearranging this equation gives

$$cm = (c + d)p_1 x_1$$

or

$$x_1 = \frac{c}{c+d} \frac{m}{p_1}.$$

This is the demand function for x_1. To find the demand function for x_2, substitute into the budget constraint to get

$$\begin{aligned} x_2 &= \frac{m}{p_2} - \frac{p_1}{p_2} \frac{c}{c+d} \frac{m}{p_1} \\ &= \frac{d}{c+d} \frac{m}{p_2}. \end{aligned}$$

The second way is to substitute the budget constraint into the maximization problem at the beginning. If we do this, our problem becomes

$$\max_{x_1} c \ln x_1 + d \ln(m/p_2 - x_1 p_1/p_2).$$

The first-order condition for this problem is

$$\frac{c}{x_1} - d \frac{p_2}{m - p_1 x_1} \frac{p_1}{p_2} = 0.$$

A little algebra—which you should do!—gives us the solution

$$x_1 = \frac{c}{c+d} \frac{m}{p_1}.$$

Substitute this back into the budget constraint $x_2 = m/p_2 - x_1 p_1/p_2$ to get

$$x_2 = \frac{d}{c+d} \frac{m}{p_2}.$$

These are the demand functions for the two goods, which, happily, are the same as those derived earlier by the other method.

Now for Lagrange's method. Set up the Lagrangian

$$L = c \ln x_1 + d \ln x_2 - \lambda(p_1 x_1 + p_2 x_2 - m)$$

and differentiate to get the three first-order conditions

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{c}{x_1} - \lambda p_1 = 0 \\ \frac{\partial L}{\partial x_2} &= \frac{d}{x_2} - \lambda p_2 = 0 \\ \frac{\partial L}{\partial \lambda} &= p_1 x_1 + p_2 x_2 - m = 0. \end{aligned}$$

Now the trick is to solve them! The best way to proceed is to first solve for λ and then for x_1 and x_2. So we rearrange and cross multiply the first two equations to get

$$\begin{aligned} c &= \lambda p_1 x_1 \\ d &= \lambda p_2 x_2. \end{aligned}$$

These equations are just asking to be added together:

$$c + d = \lambda(p_1 x_1 + p_2 x_2) = \lambda m,$$

which gives us

$$\lambda = \frac{c+d}{m}.$$

Substitute this back into the first two equations and solve for x_1 and x_2 to get

$$\begin{aligned} x_1 &= \frac{c}{c+d} \frac{m}{p_1} \\ x_2 &= \frac{d}{c+d} \frac{m}{p_2}, \end{aligned}$$

just as before.