Decision Modelling (MSC 507)

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Decision modelling

- Decision-making involves the selection of a course of action from among two or more possible alternatives in order to arrive at a solution for a given problem.

- Decision Modelling helps to take decisions

- Modelling is the scientific approach to decision making usually involves the use of one or more mathematical models.

- Decision models are mathematical models of an actual situation that may be used to make better decisions for improving the existing situation.

Model building

Optimization models: helps to find values of the ***decision variables*** that ***optimize (maximize or minimize)*** an ***objective function*** among the set of all values for the decision variables that satisfy the given ***constraints***

- Objective Function(s)
- Decision Variables
- Constraints

Suppose you need to buy some cabinets for a room. There are two types of cabinets that you have liked in the market, say X and Y . Each unit of cabinet X costs you Rs 15000 and requires 6 square feet of floor space, and holds 8 cubic feet of files. On the other hand each unit of cabinet Y Costs Rs 12000, requires 8 square feet of floor space and holds 12 cubic feet of files. You have been given Rs 140,000 for this purchase, though you may not spend all. The office has room for no more than 72 square feet of cabinets.

How many of each model you must buy in order to maximize the storage volume?

Sign Restrictions

Can the decision variable only assume nonnegative values, or is the decision variable allowed to assume both positive and negative values?

If a decision variable x_i can only assume nonnegative values, then we add the **sign restriction** $x_i \geq 0$. If a variable x_i can assume both positive and negative (or zero) values, then we say that x_i is **unrestricted in sign** (often abbreviated **urs**).

Ex 1. Suppose you need to buy some cabinets for a room. There are two types of cabinets that you have liked in the market, say X and Y . Each unit of cabinet X costs you Rs 15000 and requires 6 square feet of floor space, and holds 8 cubic feet of files. On the other hand each unit of cabinet Y Costs Rs 12000, requires 8 square feet of floor space and holds 12 cubic feet of files. You have been given Rs 140,000 for this purchase, though you may not spend all. The office has room for no more than 72 square feet of cabinets.

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let us introduce two variables x and y for numbers of models X and Y cabinets purchased respectively. Naturally, $x \geq 0$, $y \geq 0$.

If we look at the cost restriction then, we must have,

$15000x + 12000y \leq 140000$

or equivalently

$15x + 12y \leq 140$.

Now, think of floor space restriction. Then,

$6x + 8y \leq 72$ or $3x + 4y \leq 36$

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the model that we need to solve is the following

max $(8x + 12y)$

subject to

$15x + 12y \leq 140$

$3x + 4y \leq 36$

$x, y \geq 0.$

We try and plot the graph of restrictions on x and y.

- The model’s ***objective Function*** is the function which we wish to optimize.
- The ***decision variables*** are the variables that influence the performance of the system and alter the value of objective function.
- ***Constraints*** are the restrictions on decision variables.
- An ***optimal solution*** to an optimization model is any point in the feasible region that optimizes the objective function.
- The ***feasible region*** contains the set of all points that satisfies all the constraints and sign restrictions of the problem.
- For a ***maximization*** problem, an optimal solution to an LP is a point in the feasible region with the ***largest objective function value***. Similarly, for a ***minimization*** problem, an optimal solution is a point in the feasible region with the ***smallest objective function value***.

Linear Programming Definitions

A **linear programming problem (LP)** consists of three parts:

1 A linear function (the **objective function**) of decision variables (say, $x_1, x_2, \ldots, x_n$) that is to be maximized or minimized.

2 A set of **constraints** (each of which must be a linear equality or linear inequality) that restrict the values that may be assumed by the decision variables.

3 The **sign restrictions,** which specify for each decision variable x_j either (1) variable x_j must be nonnegative—$x_j \geq 0$; or (2) variable x_j may be positive, zero, or negative—x_j is **unrestricted in sign (urs).**

The coefficient of a variable in the objective function is the variable's **objective function coefficient.** The coefficient of a variable in a constraint is a **technological coefficient.** The right-hand side of each constraint is called a **right-hand side (rhs).**

A *point* is simply a specification of the values of each decision variable. The **feasible region** of an LP consists of all points satisfying the LP's constraints and sign restrictions. Any point in the feasible region that has the largest z-value of all points in the feasible region (for a max problem) is an **optimal solution** to the LP. An LP may have no optimal solution, one optimal solution, or an infinite number of optimal solutions.

Ex 2. max $7x + 5y$

subject to

$x + y \geq 1$

$2x - y \leq 2$

$x, y \geq 0.$

Ex 3. An auto company manufactures cars and trucks. Each vehicle must be processed in the paint shop and body assembly shop. If the paint shop were only painting trucks, then 40 per day could be painted. If the paint shop were only painting cars, then 60 per day could be painted. If the body shop were only producing cars, then it could process 50 per day.

If the body shop were only producing trucks, then it could process 50 per day. Each truck contributes $300 to profit, and each car contributes $200 to profit. Use linear programming to determine a daily production schedule that will maximize the company's profits.

The company must decide how many cars and trucks should be produced daily. This leads us to define the following decision variables:

x_1 number of trucks produced daily

x_2 number of cars produced daily

The company's daily profit (in hundreds of dollars) is $3x_1 + 2x_2$, so the company's objective function may be written as max $z = 3x_1 + 2x_2$

The company's two constraints are the following:

Constraint 1 The fraction of the day during which the paint shop is busy is less than or equal to 1.

Constraint 2 The fraction of the day during which the body shop is busy is less than or equal to 1.

We have

$$\begin{aligned}
\text{Fraction of day paint shop works on trucks} &= \left(\frac{\text{fraction of day}}{\text{truck}}\right)\left(\frac{\text{trucks}}{\text{day}}\right) \\
&= \tfrac{1}{40}\, x_1 \\
\text{Fraction of day paint shop works on cars} &= \tfrac{1}{60}\, x_2 \\
\text{Fraction of day body shop works on trucks} &= \tfrac{1}{50}\, x_1 \\
\text{Fraction of day body shop works on cars} &= \tfrac{1}{50}\, x_2
\end{aligned}$$

Thus, Constraint 1 may be expressed by

$$\frac{1}{40}\, x_1 + \frac{1}{60}\, x_2 \le 1 \qquad \text{(Paint shop constraint)}$$

and Constraint 2 may be expressed by

$$\frac{1}{50}\, x_1 + \frac{1}{50}\, x_2 \le 1 \qquad \text{(Body shop constraint)}$$

Because $x_1 \geq 0$ and $x_2 \geq 0$ must hold, the relevant LP is

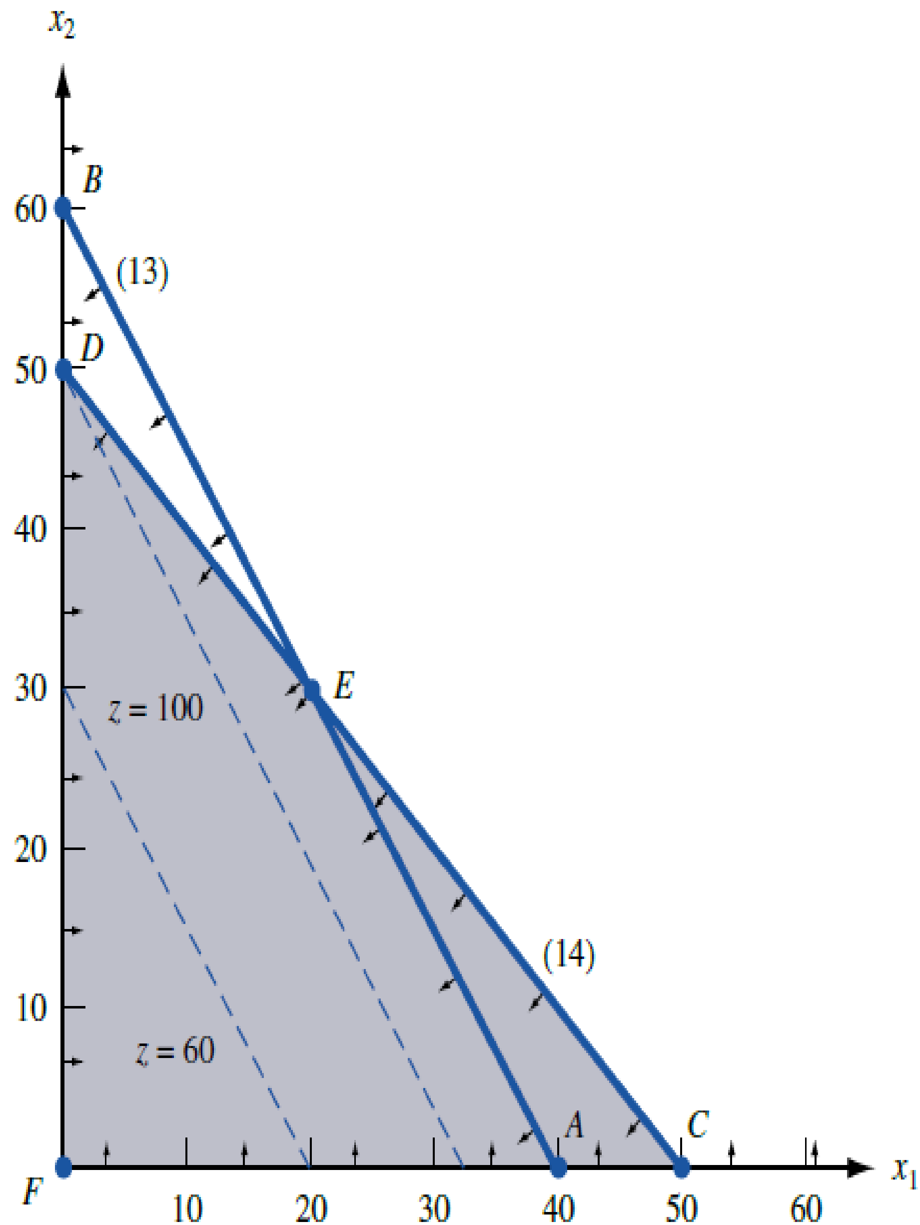
$$\max z = 3x_1 + 2x_2$$

$$\text{s.t.} \quad \frac{1}{40}x_1 + \frac{1}{60}x_2 \leq 1$$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

The feasible region for this LP is the shaded region in Figure bounded by *AEDF*.



From the discussion in the last two sections, we see that every LP with two variables must fall into one of the following four cases:

Case 1 The LP has a unique optimal solution.

Case 2 The LP has alternative or multiple optimal solutions: Two or more extreme points are optimal, and the LP will have an infinite number of optimal solutions.

Case 3 The LP is infeasible: The feasible region contains no points.

Case 4 The LP is unbounded: There are points in the feasible region with arbitrarily large z-values (max problem) or arbitrarily small z-values (min problem).

Ex 4. My diet requires that all the food I eat come from one of the four "basic food groups" (chocolate cake, ice cream, soda, and cheesecake). At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake.

Each brownie costs 50¢, each scoop of chocolate ice cream costs 20¢, each bottle of cola costs 30¢, and each piece of pineapple cheesecake costs 80¢. Each day, I must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in Table. Formulate a linear programming model that can be used to satisfy my daily nutritional requirements at minimum cost.

Nutritional Values for Diet

Type of Food	Calories	Chocolate (Ounces)	Sugar (Ounces)	Fat (Ounces)
Brownie	400	3	2	2
Chocolate ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Pineapple cheesecake (1 piece)	500	0	4	5

x_1 = number of brownies eaten daily

x_2 = number of scoops of chocolate ice cream eaten daily

x_3 = bottles of cola drunk daily

x_4 = pieces of pineapple cheesecake eaten daily

My objective is to minimize the cost of my diet. The total cost of any diet may be determined from the following relation: (total cost of diet) = (cost of brownies) + (cost of ice cream) + (cost of cola) + (cost of cheesecake). To evaluate the total cost of a diet, note that, for example,

$$\text{Cost of cola} = \left(\frac{\text{cost}}{\text{bottle of cola}}\right)\left(\begin{array}{c}\text{bottles of}\\ \text{cola drunk}\end{array}\right) = 30x_3$$

Applying this to the other three foods, we have (in cents)

$$\text{Total cost of diet} = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

Thus, the objective function is

$$\min z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

The decision variables must satisfy the following four constraints:

Constraint 1 Daily calorie intake must be at least 500 calories.

Constraint 2 Daily chocolate intake must be at least 6 oz.

Constraint 3 Daily sugar intake must be at least 10 oz.

Constraint 4 Daily fat intake must be at least 8 oz.

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Type of Food	Calories	Chocolate (Ounces)	Sugar (Ounces)	Fat (Ounces)
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To express Constraint 1 in terms of the decision variables, note that (daily calorie intake) = (calories in brownies) + (calories in chocolate ice cream) + (calories in cola) + (calories in pineapple cheesecake).

The calories in the brownies consumed can be determined from

$$\text{Calories in brownies} = \left(\frac{\text{calories}}{\text{brownie}}\right)\left(\begin{matrix}\text{brownies}\\ \text{eaten}\end{matrix}\right) = 400x_1$$

Applying similar reasoning to the other three foods shows that

$$\text{Daily calorie intake} = 400x_1 + 200x_2 + 150x_3 + 500x_4$$

Constraint 1 may be expressed by

$$400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \qquad \text{(Calorie constraint)}$$

Constraint 2 may be expressed by

$$3x_1 + 2x_2 \geq 6 \qquad \text{(Chocolate constraint)}$$

Constraint 3 may be expressed by

$$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \qquad \text{(Sugar constraint)}$$

Constraint 4 may be expressed by

$$2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 \qquad \text{(Fat constraint)}$$

Finally, the sign restrictions $x_i \geq 0$ $(i = 1, 2, 3, 4)$ must hold.

$$
\begin{aligned}
\min z = 50x_1 + 20x_2 + 30x_3 + 80x_4 \\
\text{s.t.} \quad & 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 && \text{(Calorie constraint)} \\
& 3x_1 + 2x_2 \geq 6 && \text{(Chocolate constraint)} \\
& 2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 && \text{(Sugar constraint)} \\
& 2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 && \text{(Fat constraint)} \\
& x_i \geq 0 \ (i = 1, 2, 3, 4) && \text{(Sign restrictions)}
\end{aligned}
$$

Ex5. A post office requires different numbers of full-time employees on different days of the week. The number of full-time employees required on each day is given in Table. Union rules state that each full-time employee must work five consecutive days and then receive two days off. For example, an employee who works Monday to Friday must be off on Saturday and Sunday. The post office wants to meet its daily requirements using only fulltime employees. Formulate an LP that the post office can use to minimize the number of full-time employees who must be hired.

Requirements for Post Office

Day	Number of Full-time Employees Required
1 = Monday	17
2 = Tuesday	13
3 = Wednesday	15
4 = Thursday	19
5 = Friday	14
6 = Saturday	16
7 = Sunday	11

$$x_i = \text{number of employees beginning work on day } i$$

For example, x_1 is the number of people beginning work on Monday (these people work Monday to Friday). With the variables properly defined, it is easy to determine the correct objective function and constraints. To determine the objective function, note that (number of full-time employees) = (number of employees who start work on Monday) + (number of employees who start work on Tuesday) $+\cdots+$ (number of employees who start work on Sunday). Because each employee begins work on exactly one day of the week, this expression does not double-count employees. Thus, when we correctly define the variables, the objective function is

$$\min z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

$$
\begin{array}{llll}
\text{s.t.} & x_1 + x_4 + x_5 + x_6 + x_7 \geq 17 & & \text{(Monday constraint)} \\
& x_1 + x_2 + x_5 + x_6 + x_7 \geq 13 & & \text{(Tuesday constraint)} \\
& x_1 + x_2 + x_3 + x_6 + x_7 \geq 15 & & \text{(Wednesday constraint)} \\
& x_1 + x_2 + x_3 + x_4 + x_7 \geq 19 & & \text{(Thursday constraint)} \\
& x_1 + x_2 + x_3 + x_4 + x_5 \geq 14 & & \text{(Friday constraint)} \\
& x_2 + x_3 + x_4 + x_5 + x_6 \geq 16 & & \text{(Saturday constraint)} \\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq 11 & & \text{(Sunday constraint)} \\
& x_i \geq 0 & (i = 1, 2, \ldots, 7) & \text{(Sign restrictions)}
\end{array}
$$