Problem:

Breadco Bakeries bakes two kinds of bread: French and sourdough. Each loaf of French bread can be sold for 36¢, and each loaf of sourdough bread for 30¢. A loaf of French bread requires 1 yeast packet and 6 oz of flour; sourdough requires 1 yeast packet and 5 oz of flour. At present, Breadco has 5 yeast packets and 10 oz of flour. Additional yeast packets can be purchased at 3¢ each, and additional flour at 4¢/oz. Formulate and solve an LP that can be used to maximize Breadco's profits (revenues costs).

$$
\begin{aligned}
x_1 &= \text{number of loaves of French bread baked}\\
x_2 &= \text{number of loaves of sourdough bread baked}\\
x_3 &= \text{number of yeast packets purchased}\\
x_4 &= \text{number of ounces of flour purchased}
\end{aligned}
$$

Then Breadco's objective is to maximize z = revenues − costs, where

$$\text{Revenues} = 36x_1 + 30x_2 \qquad \text{and} \qquad \text{Costs} = 3x_3 + 4x_4$$

Thus, Breadco's objective function is

$$\max z = 36x_1 + 30x_2 - 3x_3 - 4x_4$$

Constraint 1 may be written as

$$x_1 + x_2 \le 5 + x_3 \qquad \text{or} \qquad x_1 + x_2 - x_3 \le 5$$

and Constraint 2 may be written as

$$6x_1 + 5x_2 \le 10 + x_4 \qquad \text{or} \qquad 6x_1 + 5x_2 - x_4 \le 10$$

$$
\begin{aligned}
\max z &= 36x_1 + 30x_2 - 3x_3 - 4x_4\\
\text{s.t.} \quad & x_1 + x_2 - x_3 \le 5\\
& 6x_1 + 5x_2 - x_4 \le 10\\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
$$

Bevco manufactures an orange-flavored soft drink called Oranj by combining orange soda and orange juice. Each ounce of orange soda contains 0.5 oz of sugar and 1 mg of vitamin C. Each ounce of orange juice contains 0.25 oz of sugar and 3 mg of vitamin C. It costs Bevco 2¢ to produce an ounce of orange soda and 3¢ to produce an ounce of orange juice. Bevco's marketing department has decided that each 10-oz bottle of Oranj must contain at least 20 mg of vitamin C and at most 4 oz of sugar. Use linear programming to determine how Bevco can meet the marketing department's requirements at minimum cost.

Let, $x1$ number of ounces of orange soda in a bottle of Oranj

$x2$ number of ounces of orange juice in a bottle of Oranj

Then the appropriate LP is

$$
\begin{aligned}
\min z &= 2x_1 + 3x_2 \\
\text{s.t.} \quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \le 4 && \text{(Sugar constraint)} \\
& x_1 + 3x_2 \ge 20 && \text{(Vitamin C constraint)} \\
& x_1 + x_2 = 10 && \text{(10 oz in bottle of Oranj)} \\
& && x_1, x_2 \ge 0
\end{aligned}
$$

we obtain the following standard form:

$$
\begin{aligned}
\text{Row 0:} \quad & z - 2x_1 - 3x_2 = 0 \\
\text{Row 1:} \quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\
\text{Row 2:} \quad & x_1 + 3x_2 - e_2 = 20 \\
\text{Row 3:} \quad & x_1 + x_2 = 10
\end{aligned}
$$

All variables nonnegative

In a min problem, we can ensure that all the artificial variables will be zero by adding a term Ma_i to the objective function for each artificial variable a_i. (In a max problem, add a term $-Ma_i$ to the objective function.) Here M represents a "very large" positive number.

$$\min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$
$$z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0$$

Because a_2 and a_3 are in our starting bfs (that's why we introduced them), they must be eliminated from row 0. To eliminate a_2 and a_3 from row 0, simply replace row 0 by row 0 + M(row 2) + M(row 3). This yields

$$\begin{array}{llllllll}
\text{Row 0:} & z - & 2x_1 - & 3x_2 & & - Ma_2 & - Ma_3 & = 0 \\
M(\text{row 2}): & & Mx_1 + & 3Mx_2 & - Me_2 & + Ma_2 & & = 20M \\
M(\text{row 3}): & & Mx_1 + & Mx_2 & & & + Ma_3 & = 10M \\
\text{New row 0:} & z + & (2M-2)x_1 + & (4M-3)x_2 & - Me_2 & & & = 30M
\end{array}$$

Total variables= $\{x_1, x_2, e_2, s_1, a_2, a_3\}$

NBV=$\{x_1, x_2, e_2\}$ $BV=\{s_1, a_2, a_3\}$

New row 0: $z + (2M - 2)x_1 + (4M - 3)x_2 - Me_2 = 30M$

$$
\begin{array}{ll}
\text{Row 1:} & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\
\text{Row 2:} & x_1 + 3x_2 - e_2 + a_2 = 20 \\
\text{Row 3:} & x_1 + x_2 + a_3 = 10
\end{array}
$$

NBV=$\{x_1, x_2, e_2\}$ BV=$\{s_1, a_2, a_3\}$

Initial Tableau for Bevco

z	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable	Ratio
1	$2M - 2$	$4M - 3$	0	$-M$	0	0	$30M$	$z = 30M$	
0	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	$s_1 = 4$	16
0	1	3	0	-1	1	0	20	$a_2 = 20$	$\frac{20}{3}$*
0	1	1	0	0	0	1	10	$a_3 = 10$	10

We are solving a min problem, so the variable with the most positive coefficient in row 0 should enter the basis. Because $4M$ - 3 > $2M$ - 2, variable x_2 should enter the basis. The ratio test indicates that x_2 should enter the basis in row 2, which means the artificial variable a_2 will leave the basis.

NEW NBV=$\{x_1, a_2, e_2\}$ BV=$\{s_1, x_2, a_3\}$

$$\text{the new row 2 is} \qquad \tfrac{1}{3}x_1 + x_2 - \tfrac{1}{3}e_2 + \tfrac{1}{3}a_2 = \tfrac{20}{3}$$

We can now eliminate x_2 from row 0 by adding $-(4M - 3)$(new row 2) to row 0 or $(3 - 4M)$(new row 2) + row 0. Now

$$(3 - 4M)(\text{new row 2}) =$$

$$\frac{(3-4M)x_1}{3} + (3-4M)x_2 - \frac{(3-4M)e_2}{3} + \frac{(3-4M)a_2}{3} = \frac{20(3-4M)}{3}$$

$$\text{Row 0:} \qquad z + (2M-2)x_1 + (4M-3)x_2 - Me_2 = 30M$$

$$\text{New row 0: } z + \frac{(2M-3)x_1}{3} + \frac{(M-3)e_2}{3} + \frac{(3-4M)a_2}{3} = \frac{60+10M}{3}$$

First Tableau for Bevco

z	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable	Ratio
1	$\frac{2M-3}{3}$	0	0	$\frac{M-3}{3}$	$\frac{3-4M}{3}$	0	$\frac{60+10M}{3}$	$z = \frac{60+10M}{3}$	
0	$\frac{5}{12}$	0	1	$\frac{1}{12}$	$-\frac{1}{12}$	0	$\frac{7}{3}$	$s_1 = \frac{7}{3}$	$\frac{28}{5}$
0	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{20}{3}$	$x_2 = \frac{20}{3}$	20
0	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{10}{3}$	$a_3 = \frac{10}{3}$	5*

Because $\frac{2M-3}{3} > \frac{M-3}{3}$, we next enter x_1 into the basis. The ratio test indicates that x_1 should enter the basis in the third row of the current tableau. Then a_3 will leave the basis, and our next tableau will have $a_2 = a_3 = 0$. To enter x_1 into the basis in row 3, we first replace row 3 by $\frac{3}{2}$(row 3). Thus, new row 3 will be

Renewed NBV={e_2, a_2, a_3}

$BV=\{x_1, x_2, s_1\}$

$$x_1 + \frac{e_2}{2} - \frac{a_2}{2} + \frac{3a_3}{2} = 5$$

To eliminate x_1 from row 0, we replace row 0 by row 0 + $(3 - 2M)$(new row 3)/3.

$$\text{Row 0:} \quad z + \frac{(2M-3)x_1}{3} + \frac{(M-3)e_2}{3} + \frac{(3-4M)a_2}{3} = \frac{60+10M}{3}$$

$$\frac{(3-2M)(\text{new row 3})}{3}: \frac{(3-2M)x_1}{3} + \frac{(3-2M)e_2}{6} + \frac{(2M-3)a_2}{6} + \frac{(3-2M)a_3}{2} = \frac{15-10M}{3}$$

$$\text{New row 0:} \quad z - \frac{e_2}{2} + \frac{(1-2M)a_2}{2} + \frac{(3-2M)a_3}{2} = 25$$

Optimal Tableau for Bevco

z	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable
1	0	0	0	$-\frac{1}{2}$	$\frac{1-2M}{2}$	$\frac{3-2M}{2}$	25	$z = 25$
0	0	0	1	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{5}{8}$	$\frac{1}{4}$	$s_1 = \frac{1}{4}$
0	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5	$x_2 = 5$
0	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	5	$x_1 = 5$

Description of Big M Method

Step 1 Modify the constraints so that the right-hand side of each constraint is nonnegative. This requires that each constraint with a negative right-hand side be multiplied through by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed. For example, our method would transform the inequality $x_1 + x_2 \geq -1$ into $-x_1 - x_2 \leq 1$. It would also transform $x_1 - x_2 \leq -2$ into $-x_1 + x_2 \geq 2$.

Step 1′ Identify each constraint that is now (after step 1) an $=$ or $\geq$ constraint. In step 3, we will add an artificial variable to each of these constraints.

Step 2 Convert each inequality constraint to standard form. This means that if constraint i is a $\leq$ constraint, we add a slack variable s_i, and if constraint i is a $\geq$ constraint, we subtract an excess variable e_i.

Step 3 If (after step 1 has been completed) constraint i is a $\geq$ or $=$ constraint, add an artificial variable a_i. Also add the sign restriction $a_i \geq 0$.

Step 4 Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function. If the LP is a max problem, add (for each artificial variable) $-Ma_i$ to the objective function.

Step 5 Because each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. This ensures that we begin with a canonical form. In choosing the entering variable, remember that M is a very large positive number. For example, $4M - 2$ is more positive than $3M + 900$, and $-6M - 5$ is more negative than $-5M - 40$. Now solve the transformed problem by the simplex. If all artificial variables are equal to zero in the optimal solution, then we have found the optimal solution to the original problem. If any artificial variables are positive in the optimal solution, then the original problem is infeasible.

When an artificial variable leaves the basis, its column may be dropped from future tableaus because the purpose of an artificial variable is only to get a starting basic feasible solution. Once an artificial variable leaves the basis, we no longer need it. Despite this fact, we often maintain the artificial variables in all tableaus.

There is a possibility that when the LP (with the artificial variables) is solved, the final tableau may indicate that the LP is unbounded. If the final tableau indicates the LP is unbounded and all artificial variables in this tableau equal zero, then the original LP is unbounded. If the final tableau indicates that the LP is unbounded and at least one artificial variable is positive, then the original LP is infeasible.