Ex6. Rylon Corporation manufactures Brute and Chanelle perfumes. The raw material needed to manufacture each type of perfume can be purchased for \$3 per pound. Processing 1 lb of raw material requires 1 hour of laboratory time. Each pound of processed raw material yields 3 oz of Regular Brute Perfume and 4 oz of Regular Chanelle Perfume. Regular Brute can be sold for \$7/oz and Regular Chanelle for \$6/oz. Rylon also has the option of further processing Regular Brute and Regular Chanelle to produce Luxury Brute, sold at \$18/oz, and Luxury Chanelle, sold at \$14/oz. Each ounce of Regular Brute processed further requires an additional 3 hours of laboratory time and \$4 processing cost and yields 1 oz of Luxury Brute. Each ounce of Regular Chanelle processed further requires an additional 2 hours of laboratory time and \$4 processing cost and yields 1 oz of Luxury Chanelle. Each year, Rylon has 6,000 hours of laboratory time available and can purchase up to 4,000 lb of raw material. Formulate an LP that can be used to determine how Rylon can maximize profits. Assume that the cost of the laboratory hours is a fixed cost.

Rylon must determine how much raw material to purchase and how much of each type of perfume should be produced. We therefore define our decision variables to be

x_1 = number of ounces of Regular Brute sold annually

x_2 = number of ounces of Luxury Brute sold annually

x_3 = number of ounces of Regular Chanelle sold annually

x_4 = number of ounces of Luxury Chanelle sold annually

x_5 = number of pounds of raw material purchased annually

Rylon wants to maximize

$$\begin{aligned}
\text{Contribution to profit} &= \text{revenues from perfume sales} - \text{processing costs} \\
&\quad - \text{costs of purchasing raw material} \\
&= 7x_1 + 18x_2 + 6x_3 + 14x_4 - (4x_2 + 4x_4) - 3x_5 \\
&= 7x_1 + 14x_2 + 6x_3 + 10x_4 - 3x_5
\end{aligned}$$

Thus, Rylon's objective function may be written as

$$\max z = 7x_1 + 14x_2 + 6x_3 + 10x_4 - 3x_5$$

$$\begin{aligned}
\max z = 7x_1 + 14x_2 + 6x_3 + 10x_4 - 3x_5 & \\
\text{s.t.} \qquad\qquad x_5 &\le 4{,}000 \\
3x_2 + 2x_4 + x_5 &\le 6{,}000 \\
x_1 + x_2 - 3x_5 &= 0 \\
x_3 + x_4 - 4x_5 &= 0 \\
x_i \ge 0 \qquad (i = 1, 2, 3, 4, 5) &
\end{aligned}$$

Ex7. Sunco Oil manufactures three types of gasoline (gas 1, gas 2, and gas 3). Each type is produced by blending three types of crude oil (crude 1, crude 2, and crude 3). The sales price per barrel of gasoline and the purchase price per barrel of crude oil are given in Table. Sunco can purchase up to 5,000 barrels of each type of crude oil daily.

The three types of gasoline differ in their octane rating and sulfur content. The crude oil blended to form gas 1 must have an average octane rating of at least 10 and contain at most 1% sulfur. The crude oil blended to form gas 2 must have an average octane rating of at least 8 and contain at most 2% sulfur. The crude oil blended to form gas 3 must have an octane rating of at least 6 and contain at most 1% sulfur. The octane rating and the sulfur content of the three types of oil are given in Table . It costs $4 to transform one barrel of oil into one barrel of gasoline, and Sunco's refinery can produce up to 14,000 barrels of gasoline daily.

Sunco's customers require the following amounts of each gasoline: gas 1—3,000 barrels per day; gas 2—2,000 barrels per day; gas 3—1,000 barrels per day. The company considers it an obligation to meet these demands. Sunco also has the option of advertising to stimulate demand for its products. Each dollar spent daily in advertising a particular type of gas increases the daily demand for that type of gas by 10 barrels. For example, if Sunco decides to spend $20 daily in advertising gas 2, then the daily demand for gas 2 will increase by 20(10) 200 barrels. Formulate an LP that will enable Sunco to maximize daily profits (profits revenues - costs).

Sunco must make two types of decisions: first, how much money should be spent in advertising each type of gas, and second, how to blend each type of gasoline from the three types of crude oil available. For example, Sunco must decide how many barrels of crude 1 should be used to produce gas 1. We define the decision variables

a_i = dollars spent daily on advertising gas i $\quad (i = 1, 2, 3)$

x_{ij} = barrels of crude oil i used daily to produce gas j $\quad (i = 1, 2, 3; j = 1, 2, 3)$

For example, x_{21} is the number of barrels of crude 2 used each day to produce gas 1.

Gas and Crude Oil Prices for Blending

Gas	Sales Price per Barrel ($)	Crude	Purchase Price per Barrel ($)
1	70	1	45
2	60	2	35
3	50	3	25

Octane Ratings and Sulfur Requirements for Blending

Crude	Octane Rating	Sulfur Content (%)
1	12	0.5
2	6	2.0
3	8	3.0

Knowledge of these variables is sufficient to determine Sunco's objective function and constraints, but before we do this, we note that the definition of the decision variables implies that

$$
\begin{aligned}
x_{11} + x_{12} + x_{13} &= \text{barrels of crude 1 used daily} \\
x_{21} + x_{22} + x_{23} &= \text{barrels of crude 2 used daily} \\
x_{31} + x_{32} + x_{33} &= \text{barrels of crude 3 used daily}
\end{aligned}
$$

$$
\begin{aligned}
x_{11} + x_{21} + x_{31} &= \text{barrels of gas 1 produced daily} \\
x_{12} + x_{22} + x_{32} &= \text{barrels of gas 2 produced daily} \\
x_{13} + x_{23} + x_{33} &= \text{barrels of gas 3 produced daily}
\end{aligned}
$$

To simplify matters, let's assume that gasoline cannot be stored, so it must be sold on the day it is produced. This implies that for $i = 1, 2, 3$, the amount of gas i produced daily should equal the daily demand for gas i. Suppose that the amount of gas i produced daily exceeded the daily demand. Then we would have incurred unnecessary purchasing and production costs. On the other hand, if the amount of gas i produced daily is less than the daily demand for gas i, then we are failing to meet mandatory demands or incurring unnecessary advertising costs.

$$\text{Daily revenues from gas sales} = 70(x_{11} + x_{21} + x_{31}) + 60(x_{12} + x_{22} + x_{32}) + 50(x_{13} + x_{23} + x_{33})$$

$$\text{Daily cost of purchasing crude oil} = 45(x_{11} + x_{12} + x_{13}) + 35(x_{21} + x_{22} + x_{23}) + 25(x_{31} + x_{32} + x_{33})$$

$$\text{Daily advertising costs} = a_1 + a_2 + a_3$$

$$\text{Daily production costs} = 4(x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33})$$

Then,

$$\begin{aligned}
\text{Daily profit} &= \text{daily revenue from gas sales} \\
&\quad - \text{daily cost of purchasing crude oil} \\
&\quad - \text{daily advertising costs} - \text{daily production costs} \\
&= (70 - 45 - 4)x_{11} + (60 - 45 - 4)x_{12} + (50 - 45 - 4)x_{13} \\
&\quad + (70 - 35 - 4)x_{21} + (60 - 35 - 4)x_{22} + (50 - 35 - 4)x_{23} \\
&\quad + (70 - 25 - 4)x_{31} + (60 - 25 - 4)x_{32} \\
&\quad + (50 - 25 - 4)x_{33} - a_1 - a_2 - a_3
\end{aligned}$$

Thus, Sunco's goal is to maximize

$$\begin{aligned}
z = {}& 21x_{11} + 11x_{12} + x_{13} + 31x_{21} + 21x_{22} + 11x_{23} + 41x_{31} \\
& + 31x_{32} + 21x_{33} - a_1 - a_2 - a_3
\end{aligned}$$

Constraint 1 Gas 1 produced daily should equal its daily demand.

Constraint 2 Gas 2 produced daily should equal its daily demand.

Constraint 3 Gas 3 produced daily should equal its daily demand.

Constraint 4 At most 5,000 barrels of crude 1 can be purchased daily.

Constraint 5 At most 5,000 barrels of crude 2 can be purchased daily.

Constraint 6 At most 5,000 barrels of crude 3 can be purchased daily.

Constraint 7 Because of limited refinery capacity, at most 14,000 barrels of gasoline can be produced daily.

Constraint 8 Crude oil blended to make gas 1 must have an average octane level of at least 10.

Constraint 9 Crude oil blended to make gas 2 must have an average octane level of at least 8.

Constraint 10 Crude oil blended to make gas 3 must have an average octane level of at least 6.

Constraint 11 Crude oil blended to make gas 1 must contain at most 1% sulfur.

Constraint 12 Crude oil blended to make gas 2 must contain at most 2% sulfur.

Constraint 13 Crude oil blended to make gas 3 must contain at most 1% sulfur.

Objective Function and Constraints for Blending

x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	a_1	a_2	a_3	
21	11	1	31	21	11	41	31	21	−1	−1	−1	(max)
1	0	0	1	0	0	1	0	0	−10	0	0	= 3,000
0	1	0	0	1	0	0	1	0	0	−10	0	= 2,000
0	0	1	0	0	1	0	0	1	0	0	−10	= 1,000
1	1	1	0	0	0	0	0	0	0	0	0	≤ 5,000
0	0	0	1	1	1	0	0	0	0	0	0	≤ 5,000
0	0	0	0	0	0	1	1	1	0	0	0	≤ 5,000
1	1	1	1	1	1	1	1	1	0	0	0	≤ 14,000
2	0	0	−4	0	0	−2	0	0	0	0	0	≥ 0
0	4	0	0	−2	0	0	0	0	0	0	0	≥ 0
−0.005	0	0	0.01	0	0	0.02	0	0	0	0	0	≤ 0
0	−0.015	0	0	0	0	0	0.01	0	0	0	0	≤ 0
0	0	−0.005	0	0	0.01	0	0	0.02	0	0	0	≤ 0

Simplex Algorithm

- Convert an LP with m constraints into standard form. Assuming that the standard form contains n variables (labeled for convenience $x1, x2, \ldots, xn$), the standard form for such an LP is

$$
\begin{aligned}
&\max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
&\text{(or min)} \\
&\text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&\qquad\; a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\
&\qquad\; a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\
&\qquad\qquad x_i \geq 0 \quad (i = 1, 2, \ldots, n)
\end{aligned}
$$

the set of constraints for may be written as the system of equations $A\mathbf{x} = \mathbf{b}$.

If we define

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
\qquad
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Any basic solution to above set of equations in which all variables are nonnegative is a **basic feasible solution** (or **bfs**).

Slack variable: if constraint i of an LP is a $\leq$ constraint, then we convert it to an equality constraint by adding a slack variable s_i to the ith constraint and adding the sign restriction $s_i \geq 0$.

Excess Variable: To convert the ith $\geq$ constraint to an equality constraint, we define an **excess variable** (sometimes called a surplus variable) e_i. (e_i will always be the excess variable for the ith constraint.) We define ei to be the amount by which the ith constraint is oversatisfied.

$$
\begin{array}{llllll}
\min z = & 50x_1 + & 20x_2 + & 30x_3 + & 80x_4 & \\
\text{s.t.} & 400x_1 + & 200x_2 + & 150x_3 + & 500x_4 & \geq 500 \\
& 3x_1 + & 2x_2 & & & \geq 6 \\
& 2x_1 + & 2x_2 + & 4x_3 + & 4x_4 & \geq 10 \\
& 2x_1 + & 4x_2 + & x_3 + & 5x_4 & \geq 8 \\
& & & & x_1, x_2, x_3, x_4 & \geq 0
\end{array}
$$

The resulting LP is in standard form and may be written as

$$
\begin{array}{llllllllll}
\min z = & 50x_1 + & 20x_2 + & 30x_3 + & 80x_4 & & & & & \\
\text{s.t.} & 400x_1 + & 200x_2 + & 150x_3 + & 500x_4 & - e_1 & & & & = 500 \\
& 3x_1 + & 2x_2 & & & & - e_2 & & & = 6 \\
& 2x_1 + & 2x_2 + & 4x_3 + & 4x_4 & & & - e_3 & & = 10 \\
& 2x_1 + & 4x_2 + & x_3 + & 5x_4 & & & & - e_4 & = 8 \\
\end{array}
$$

$$x_i, e_i \geq 0 \qquad (i = 1, 2, 3, 4)$$

In summary, *if the ith constraint of an LP is a $\geq$ constraint, then it can be converted to an equality constraint by subtracting an excess variable e_i from the ith constraint and adding the sign restriction $e_i \geq 0$.*

$$
\begin{array}{ll}
\max z = 20x_1 + 15x_2 & \\
\text{s.t.} & x_1 \le 100 \\
 & x_2 \le 100 \\
 & 50x_1 + 35x_2 \le 6{,}000 \\
 & 20x_1 + 15x_2 \ge 2{,}000 \\
 & x_1, x_2 \ge 0
\end{array}
$$

Following the procedures described previously, we transform this LP into standard form by adding slack variables s_1, s_2, and s_3, respectively, to the first three constraints and subtracting an excess variable e_4 from the fourth constraint. Then we add the sign restrictions $s_1 \ge 0$, $s_2 \ge 0$, $s_3 \ge 0$, and $e_4 \ge 0$. This yields the following LP in standard form:

$$
\begin{array}{ll}
\max z = 20x_1 + 15x_2 & \\
\text{s.t.} & x_1 + s_1 = 100 \\
 & x_2 + s_2 = 100 \\
 & 50x_1 + 35x_2 + s_3 = 6{,}000 \\
 & 20x_1 + 15x_2 - e_4 = 2{,}000 \\
\multicolumn{2}{l}{x_i \ge 0 \quad (i = 1, 2); \qquad s_i \ge 0 \quad (i = 1, 2, 3); \qquad e_4 \ge 0}
\end{array}
$$

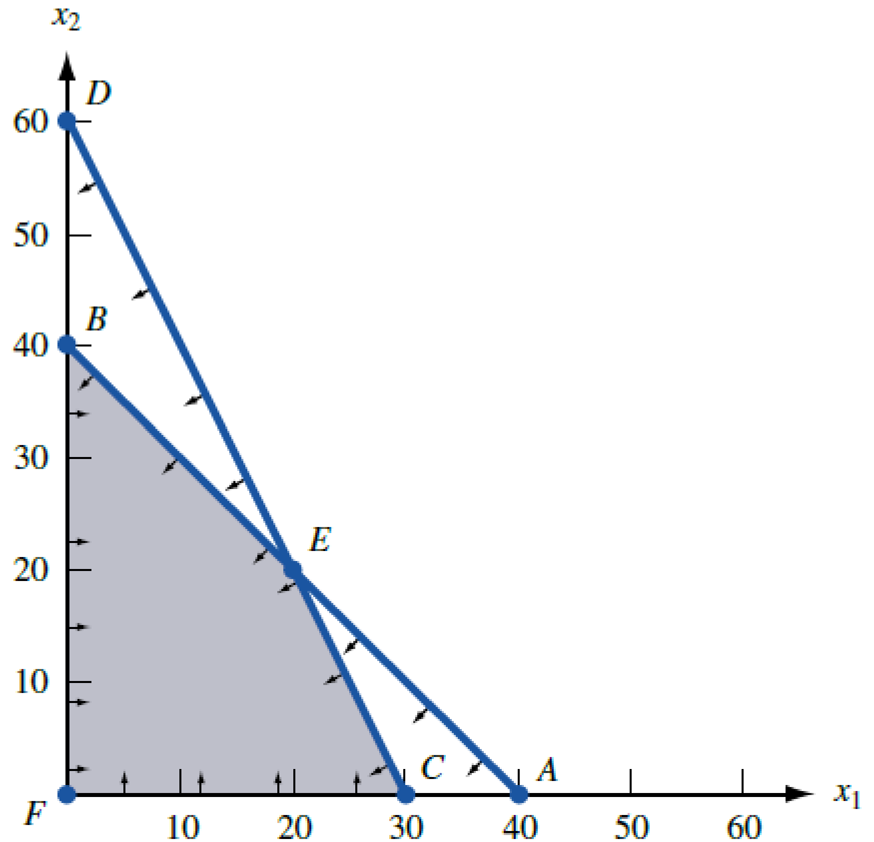
Of course, we could easily have labeled the excess variable for the fourth constraint e_1 (because it is the first excess variable). We chose to call it e_4 rather than e_1 to indicate that e_4 is the excess variable for the fourth constraint.

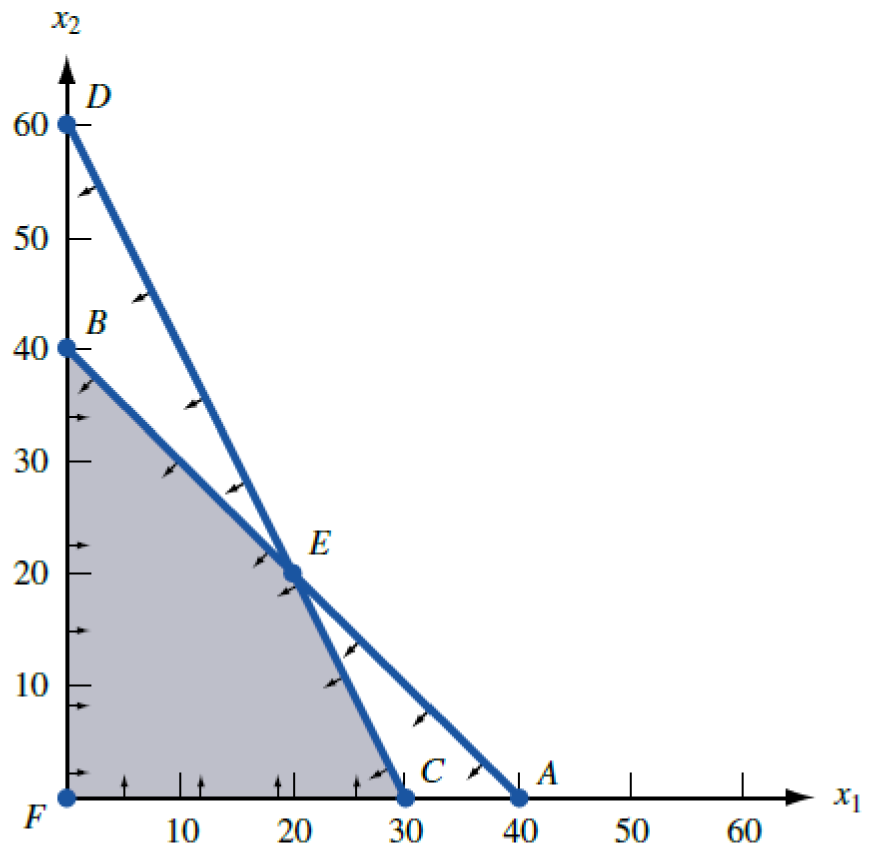
To find a basic solution to $A\mathbf{x} = \mathbf{b}$, we choose a set of $n-m$ variables (the **nonbasic variables,** or **NBV**) and set each of these variables equal to 0. Then we solve for the values of the remaining $n - (n - m) = m$ variables (the **basic variables,** or **BV**) that satisfy $A\mathbf{x} = \mathbf{b}$.

A point in the feasible region of an LP is an extreme point if and only if it is a basic feasible solution to the LP.

$$\begin{aligned} \max z &= 4x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 40 \\ & 2x_1 + x_2 \le 60 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \max z &= 4x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 + s_1 = 40 \\ & 2x_1 + x_2 + s_2 = 60 \\ & x_1, x_2, s_1, s_2 \ge 0 \end{aligned}$$





TABLE

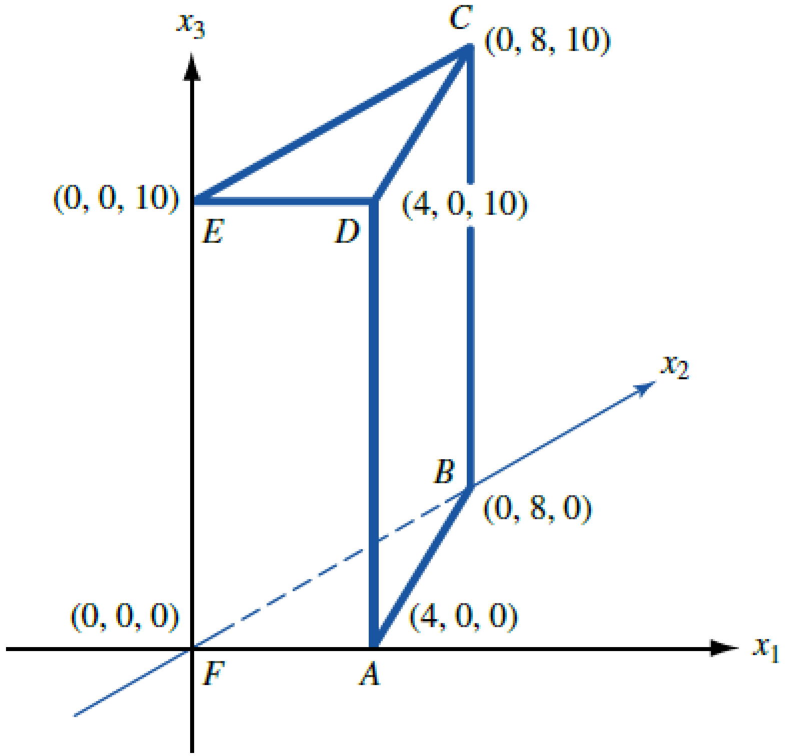
Correspondence between Basic Feasible Solutions and Corner Points

Basic Variables	Nonbasic Variables	Basic Feasible Solution	Corresponds to Corner Point
x_1, x_2	s_1, s_2	$s_1 = s_2 = 0, x_1 = x_2 = 20$	E
x_1, s_1	x_2, s_2	$x_2 = s_2 = 0, x_1 = 30, s_1 = 10$	C
x_1, s_2	x_2, s_1	$x_2 = s_1 = 0, x_1 = 40, s_2 = -20$	Not a bfs because $s_2 < 0$
x_2, s_1	x_1, s_2	$x_1 = s_2 = 0, s_1 = -20, x_2 = 60$	Not a bfs because $s_1 < 0$
x_2, s_2	x_1, s_1	$x_1 = s_1 = 0, x_2 = 40, s_2 = 20$	B
s_1, s_2	x_1, x_2	$x_1 = x_2 = 0, s_1 = 40, s_2 = 60$	F

Adjacent basic feasible solution

- For any **LP** with *m* constraints, two basic feasible solutions are said to be **adiacent** if their sets of basic variables have *m* 1 basic variables in common

$$
\begin{aligned}
\max z = {} & x_1 + 2x_2 + 2x_3 \\
\text{s.t.} \quad & 2x_1 + x_2 \le 8 \\
& x_3 \le 10 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$



Correspondence between bfs and Corner Points

Basic Variables	Basic Feasible Solution	Corresponds to Corner Point
x_1, x_3	$x_1 = 4, x_3 = 10, x_2 = s_1 = s_2 = 0$	D
s_1, s_2	$s_1 = 8, s_2 = 10, x_1 = x_2 = x_3 = 0$	F
s_1, x_3	$s_1 = 8, x_3 = 10, x_1 = x_2 = s_2 = 0$	E
x_2, x_3	$x_2 = 8, x_3 = 10, x_1 = s_1 = s_2 = 0$	C
x_2, s_2	$x_2 = 8, s_2 = 10, x_1 = x_3 = s_1 = 0$	B
x_1, s_2	$x_1 = 4, s_2 = 10, x_2 = x_3 = s_1 = 0$	A

The simplex algorithm proceeds as follows:

Step 1 Convert the LP to standard form (see Section 4.1).

Step 2 Obtain a bfs (if possible) from the standard form.

Step 3 Determine whether the current bfs is optimal.

Step 4 If the current bfs is not optimal, then determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value.

Step 5 Use EROs to find the new bfs with the better objective function value. Go back to step 3.

In performing the simplex algorithm, write the objective function

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$z - c_1x_1 - c_2x_2 - \cdots - c_nx_n = 0$$

We call this format the **row 0 version** of the objective function (row 0 for short).