To see this, suppose that we are at a basic feasible solution (call it bfs 1) that has $z = 20$. Fact 1 shows that our next pivot will take us to a bfs (call it bfs 2) and has $z > 20$. Because no future pivot can decrease z, we can never return to a bfs having $z = 20$. Thus, we can never return to bfs 1. Now recall that every LP has only a finite number of basic feasible solutions. Because we can never repeat a bfs, this argument shows that when we use the simplex algorithm to solve a nondegenerate LP, we are guaranteed to find the optimal solution in a finite number of iterations. For example, suppose we are solving a nondegenerate LP with 10 variables and 5 constraints. Such an LP has at most

$$\binom{10}{5} = 252$$

basic feasible solutions. We will never repeat a bfs, so we know that for this problem, the simplex is guaranteed to find an optimal solution after at most 252 pivots.

However, the simplex may fail for a degenerate LP.

DEFINITION ■ An LP is **degenerate** if it has at least one bfs in which a basic variable is equal to zero. ■

The following LP is degenerate:

$$\begin{aligned} \max z &= 5x_1 + 2x_2 \\ \text{s.t.} \quad x_1 + x_2 &\le 6 \\ x_1 - x_2 &\le 0 \\ x_1, x_2 &\ge 0 \end{aligned} \tag{16}$$

What happens when we use the simplex algorithm to solve (16)? After adding slack variables s_1 and s_2 to the two constraints, we obtain the initial tableau in Table 29. In this bfs, the basic variable $s_2 = 0$. Thus, (16) is a degenerate LP. Any bfs that has at least one basic variable equal to zero (or, equivalently, at least one constraint with a zero right-hand side) is a **degenerate bfs.** Because $-5 < -2$, we enter x_1 into the basis. The winning ratio is 0. This means that after x_1 enters the basis, x_1 will equal zero in the new bfs. After doing the pivot, we obtain the tableau in Table 30. Our new bfs has the same z-value as

TABLE 29
A Degenerate LP

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	−5	−2	0	0	0	$z = 0$	
0	1	1	1	0	6	$s_1 = 6$	6
0	1	−1	0	1	0	$s_2 = 0$	0*

TABLE 30
First Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	0	−7	0	5	0	$z = 0$	
0	0	2	1	−1	6	$s_1 = 6$	$\frac{6}{2} = 3$*
0	1	−1	0	1	0	$x_1 = 0$	None

TABLE 31
Optimal Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable
1	0	0	3.5	1.5	21	$z = 21$
0	0	1	0.5	-0.5	3	$x_2 = 3$
0	1	0	0.5	0.5	3	$x_1 = 3$

the old bfs. This is consistent with fact 2. In the new bfs, all variables have exactly the same values as they had before the pivot! Thus, our new bfs is also degenerate. Continuing with the simplex, we enter x_2 in row 1. The resulting tableau is shown in Table 31. This is an optimal tableau, so the optimal solution to (16) is $z = 21$, $x_2 = 3$, $x_1 = 3$, $s_1 = s_2 = 0$.

We can now explain why the simplex may have problems in solving a degenerate LP. Suppose we are solving a degenerate LP for which the optimal z-value is $z = 30$. If we begin with a bfs that has, say, $z = 20$, we know (look at the LP we just solved) that it is possible for a pivot to leave the value of z unchanged. This means that it is possible for a sequence of pivots like the following to occur:

$$
\begin{aligned}
\text{Initial bfs (bfs 1): } z &= 20\\
\text{After first pivot (bfs 2): } z &= 20\\
\text{After second pivot (bfs 3): } z &= 20\\
\text{After third pivot (bfs 4): } z &= 20\\
\text{After fourth pivot (bfs 1): } z &= 20
\end{aligned}
$$

In this situation, we encounter the same bfs twice. This occurrence is called **cycling.** If cycling occurs, then we will loop, or cycle, forever among a set of basic feasible solutions and never get to the optimal solution ($z = 30$, in our example). Cycling can indeed occur (see Problem 3 at the end of this section). Fortunately, the simplex algorithm can be modified to ensure that cycling will never occur [see Bland (1977) or Dantzig (1963) for details].† For a practical example of cycling, see Kotiah and Slater (1973).

If an LP has many degenerate basic feasible solutions (or a bfs with many basic variables equal to zero), then the simplex algorithm is often very inefficient. To see why, look at the feasible region for (16) in Figure 12, the shaded triangle *BCD*. The extreme points of the feasible region are *B*, *C*, and *D*. Following the procedure outlined in Section 4.2, let's look at the correspondence between the basic feasible solutions to (16) and the extreme points of its feasible region (see Table 32). Three sets of basic variables correspond to extreme point *C*. It can be shown that for an LP with n decision variables to be degenerate, $n + 1$ or more of the LP's constraints (including the sign restrictions $x_i \geq 0$ as constraints) must be binding at an extreme point.

In (16), the constraints $x_1 - x_2 \leq 0$, $x_1 \geq 0$, and $x_2 \geq 0$ are all binding at point *C*. Each extreme point at which three or more constraints are binding will correspond to more than one set of basic variables. For example, at point *C*, s_1 must be one of the basic variables, but the other basic variable may be x_2, x_1, or s_2.

†Bland showed that cycling can be avoided by applying the following rules (assume that slack and excess variables are numbered $x_{n+1}, x_{n+2}, \ldots$):
1 Choose as the entering variable (in a max problem) the variable with a negative coefficient in row 0 that has the smallest subscript.
2 If there is a tie in the ratio test, then break the tie by choosing the winner of the ratio test so that the variable leaving the basis has the smallest subscript.

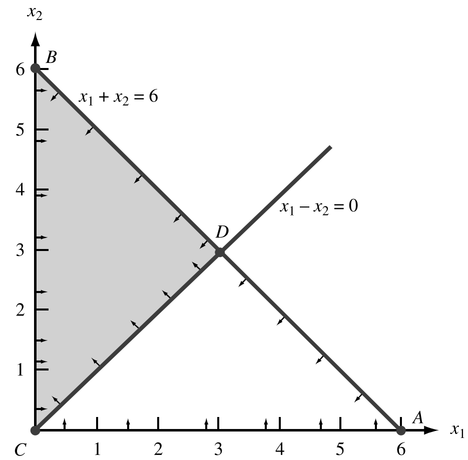

FIGURE 12
Feasible Region for the LP (16)

TABLE 32
Three Sets of Basic Variables Correspond to Corner Point *C*

Basic Variables	Basic Feasible Solution	Corresponds to Extreme Point
x_1, x_2	$x_1 = x_2 = 3, s_1 = s_2 = 0$	D
x_1, s_1	$x_1 = 0, s_1 = 6, x_2 = s_2 = 0$	C
x_1, s_2	$x_1 = 6, s_2 = -6, x_2 = s_1 = 0$	Infeasible
x_2, s_1	$x_2 = 0, s_1 = 6, x_1 = s_2 = 0$	C
x_2, s_2	$x_2 = 6, s_2 = 6, s_1 = x_1 = 0$	B
s_1, s_2	$s_1 = 6, s_2 = 0, x_1 = x_2 = 0$	C

We can now discuss why the simplex algorithm often is an inefficient method for solving degenerate LPs. Suppose an LP is degenerate. Then there may be many sets (maybe hundreds) of basic variables that correspond to some nonoptimal extreme point. The simplex algorithm might encounter all these sets of basic variables before it finds that it was at a nonoptimal extreme point. This problem was illustrated (on a small scale) in solving (16): The simplex took two pivots before it found that point C was suboptimal. Fortunately, some degenerate LPs have a special structure that enables us to solve them by methods other than the simplex (see, for example, the discussion of the assignment problem in Chapter 7).

PROBLEMS

Group A

1 Even if an LP's initial tableau is nondegenerate, later tableaus may exhibit degeneracy. Degenerate tableaus often occur in the tableau following a tie in the ratio test. To illustrate this, solve the following LP:

$$\begin{aligned} \max z &= 5x_1 + 3x_2 \\ \text{s.t.} \quad 4x_1 + 2x_2 &\le 12 \\ 4x_1 + x_2 &\le 10 \\ x_1 + x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Also graph the feasible region and show which extreme points correspond to more than one set of basic variables.

2 Find the optimal solution to the following LP:

$$\begin{aligned} \min z &= -x_1 - x_2 \\ \text{s.t.} \quad x_1 + x_2 &\le 1 \\ -x_1 + x_2 &\le 0 \\ x_1, x_2 &\ge 0 \end{aligned}$$