Transportation Problem

Powerco has three electric power plants that supply the needs of four cities. Each power plant can supply the following numbers of kilowatt-hours (kwh) of electricity: plant 1—35 million; plant 2—50 million; plant 3—40 million (see Table 1). The peak power demands in these cities, which occur at the same time (2 P.M.), are as follows (in kwh): city 1—45 million; city 2—20 million; city 3—30 million; city 4—30 million. The costs of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel. Formulate an LP to minimize the cost of meeting each city's peak power demand.

To formulate Powerco's problem as an LP, we begin by defining a variable for each decision that Powerco must make. Because Powerco must determine how much power is sent from each plant to each city, we define (for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$)

$$x_{ij} = \text{number of (million) kwh produced at plant } i \text{ and sent to city } j$$

In terms of these variables, the total cost of supplying the peak power demands to cities 1–4 may be written as

$$\begin{aligned} &8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} && \text{(Cost of shipping power from plant 1)} \\ + &9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} && \text{(Cost of shipping power from plant 2)} \\ + &14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} && \text{(Cost of shipping power from plant 3)} \end{aligned}$$

Shipping Costs, Supply, and Demand for Powerco

From	To: City 1	City 2	City 3	City 4	Supply (million kwh)
Plant 1	\$8	\$6	\$10	\$9	35
Plant 2	\$9	\$12	\$13	\$7	50
Plant 3	\$14	\$9	\$16	\$5	40
Demand (million kwh)	45	20	30	30	

constraint that ensures that the total quantity shipped from a plant does not exceed plant capacity is a **supply constraint.** The LP formulation of Powerco's problem contains the following three supply constraints:

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 35 \qquad \text{(Plant 1 supply constraint)}$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 50 \qquad \text{(Plant 2 supply constraint)}$$
$$x_{31} + x_{32} + x_{33} + x_{34} \leq 40 \qquad \text{(Plant 3 supply constraint)}$$

$$x_{11} + x_{21} + x_{31} \geq 45 \qquad \text{(City 1 demand constraint)}$$
$$x_{12} + x_{22} + x_{32} \geq 20 \qquad \text{(City 2 demand constraint)}$$
$$x_{13} + x_{23} + x_{33} \geq 30 \qquad \text{(City 3 demand constraint)}$$
$$x_{14} + x_{24} + x_{34} \geq 30 \qquad \text{(City 4 demand constraint)}$$

Combining the objective function, supply constraints, demand constraints, and sign restrictions yields the following LP formulation of Powerco's problem:

$$
\begin{aligned}
\min z = {} & 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} \\
& + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} + x_{14} \le 35 \qquad \text{(Supply constraints)} \\
& x_{21} + x_{22} + x_{23} + x_{24} \le 50 \\
& x_{31} + x_{32} + x_{33} + x_{34} \le 40 \\
& x_{11} + x_{21} + x_{31} \ge 45 \qquad \text{(Demand constraints)} \\
& x_{12} + x_{22} + x_{32} \ge 20 \\
& x_{13} + x_{23} + x_{33} \ge 30 \\
& x_{14} + x_{24} + x_{34} \ge 30 \\
& x_{ij} \ge 0 \quad (i = 1, 2, 3;\ j = 1, 2, 3, 4)
\end{aligned}
$$

General Description of a Transportation Problem

In general, a transportation problem is specified by the following information:

1 A set of m *supply points* from which a good is shipped. Supply point i can supply at most s_i units. In the Powerco example, $m = 3$, $s_1 = 35$, $s_2 = 50$, and $s_3 = 40$.

2 A set of n *demand points* to which the good is shipped. Demand point j must receive at least d_j units of the shipped good. In the Powerco example, $n = 4$, $d_1 = 45$, $d_2 = 20$, $d_3 = 30$, and $d_4 = 30$.

3 Each unit produced at supply point i and shipped to demand point j incurs a *variable cost* of c_{ij}. In the Powerco example, $c_{12} = 6$.

Let

$$x_{ij} = \text{number of units shipped from supply point } i \text{ to demand point } j$$

then the general formulation of a transportation problem is

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{j=n} x_{ij} \le s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)}$$

$$\sum_{i=1}^{i=m} x_{ij} \ge d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)}$$

$$x_{ij} \ge 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

If a problem has the constraints given in (1) and is a *maximization* problem, then it is still a transportation problem (see Problem 7 at the end of this section). If

$$\sum_{i=1}^{i=m} s_i = \sum_{j=1}^{j=n} d_j$$

then total supply equals total demand, and the problem is said to be a **balanced transportation problem.**

Two reservoirs are available to supply the water needs of three cities. Each reservoir can supply up to 50 million gallons of water per day. Each city would like to receive 40 million gallons per day. For each million gallons per day of unmet demand, there is a penalty. At city 1, the penalty is \$20; at city 2, the penalty is \$22; and at city 3, the penalty is \$23. The cost of transporting 1 million gallons of water from each reservoir to each city is shown in Table 4. Formulate a balanced transportation problem that can be used to minimize the sum of shortage and transport costs.

$$\text{Daily supply} = 50 + 50 = 100 \text{ million gallons per day}$$

$$\text{Daily demand} = 40 + 40 + 40 = 120 \text{ million gallons per day}$$

To balance the problem, we add a dummy (or shortage) *supply point* having a supply of $120 - 100 = 20$ million gallons per day. The cost of shipping 1 million gallons from the dummy supply point to a city is just the shortage cost per million gallons for that city. Table 5 shows the balanced transportation problem and its optimal solution. Reservoir 1 should send 20 million gallons per day to city 1 and 30 million gallons per day to city 2, whereas reservoir 2 should send 10 million gallons per day to city 2 and 40 million gallons per day to city 3. Twenty million gallons per day of city 1's demand will be unsatisfied.

Shipping Costs for Reservoir

	To		
From	City 1	City 2	City 3
Reservoir 1	\$7	\$8	\$10
Reservoir 2	\$9	\$7	\$8

when we solve a balanced transportation problem, we may omit from consideration any one of the problem's constraints and solve an LP having $m + n - 1$ constraints. We (arbitrarily) assume that the first supply constraint is omitted from consideration.

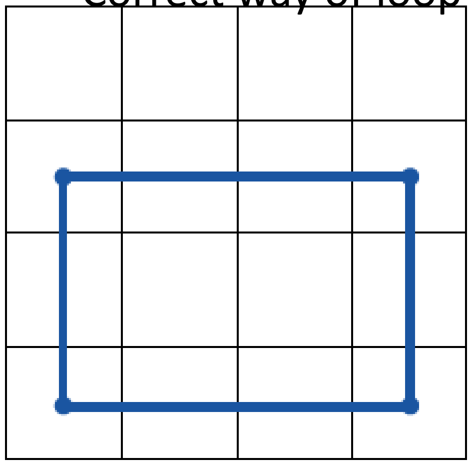
An ordered sequence of at least four different cells is called a **loop** if

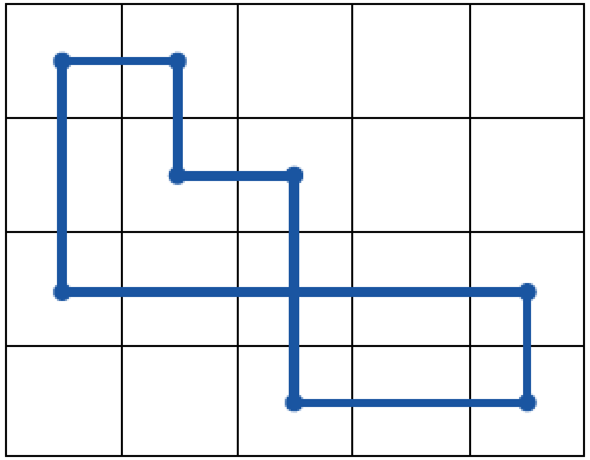
1 Any two consecutive cells lie in either the same row or same column

2 No three consecutive cells lie in the same row or column

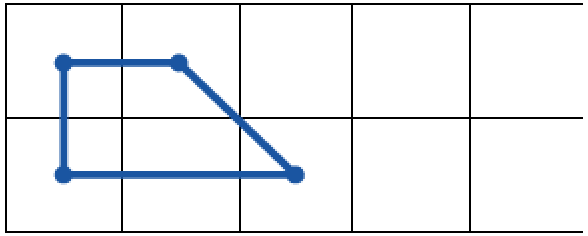
3 The last cell in the sequence has a row or column in common with the first cell in the sequence ■

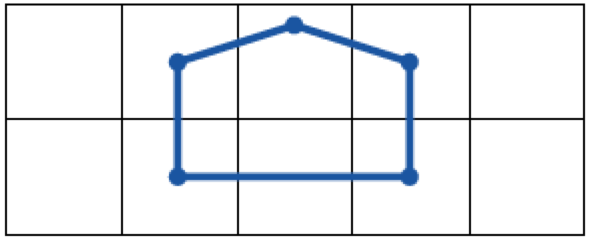
Correct way of loop





Wrong looping: Diagnols not allowed





In a balanced transportation problem with m supply points and n demand points, the cells corresponding to a set of $m + n - 1$ variables contain no loop if and only if the $m + n - 1$ variables yield a basic solution.

a set of $m + n - 1$ cells contains no loop if and only if the $m + n - 1$ columns corresponding to these cells are linearly independent.

Methods to find basic feasible solutions

1. northwest corner method
2. minimum-cost method
3. Vogel's method

Methods to find the Optimal solutions

1. Stepping Stone Method
2. Modified Distribution Method (MoDi)