Production Function

It is the mathematical function that illustrates the relationship between output and inputs.

The production function is an engineering relation that defines the ***maximum amount of output*** that can be produced with a given set of inputs.

Q or the output is also known as the TOTAL PRODUCT (TP).

Inputs to production are called **factors of production**.

Factors of production are often classified into broad categories such as land, labor, capital, and raw materials.

Capital goods are those inputs to production that are themselves produced goods. Basically capital goods are machines of one sort or another: tractors, buildings, computers, or whatever.

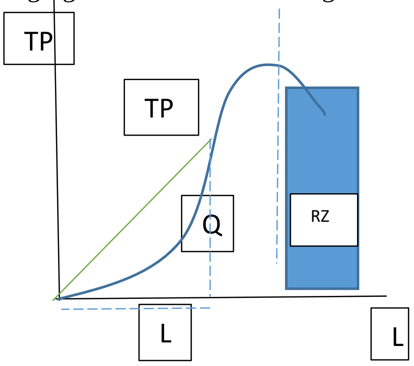
Q = F(X_1, X_2, X_3, $X_{4,....}$ X_n) where X_i is the i^{th} input

Generally, we take only two major inputs, labour and capital.

In the L-K space i.e. where there are only two inputs L and K, the production function or **Total product** is

Q = TP = F(K,L)

The more input is supplied, the more output, i.e. marginal value of the slope of the production function with respect to each input is positive. We don't consider the restricted zone RZ in adjoining figure 1 where TP is falling as we increase L.



Mathematically, $\frac{\delta TP}{\delta L} = \frac{\delta Q}{\delta L} > 0 \quad and \ \frac{\delta Q}{\delta K} > 0$

Two important parameters of the production function:

Average product: Total product divided by the quantity used of the input.

$AP_L = \frac{Q}{L}$, $AP_K = \frac{Q}{K}$

Average product on any point of the TP curve can be plotted using a straight line that connects the origin and the point of the TP curve since the triangle that is formed by that straight line has height equal to Q and base equal to quantity of input (see fig 1).

Marginal product: The change in total output attributable to the last unit of an input.

MP is also the slope of the production function with respect to any particular input, L or K, when the other input is held constant.

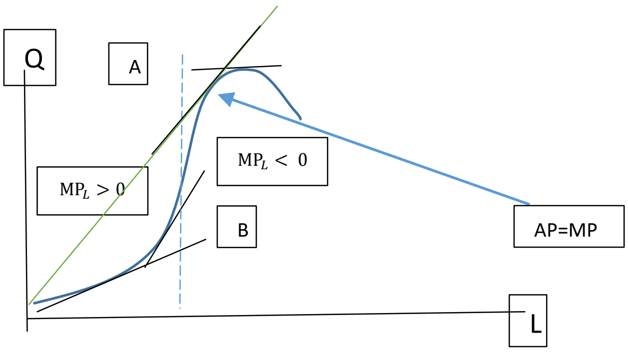
$MP_L(L,K) = \frac{\delta Q}{\delta L} = \frac{Q(L+\delta L, \ K) - Q(L,K)}{\delta L}$

$MP_K\ (L,K) = \frac{\delta}{\delta K}\left(\frac{\delta Q}{\delta K}\right) = \frac{Q(L,K+\delta K) - Q(L,K)}{\delta K}$

$\frac{\delta}{\delta L}\left(\frac{\delta Q}{\delta L}\right) > 0$ in zone A

Beyond that, it is the zone of diminishing MP

$\frac{\delta}{\delta L}\left(\frac{\delta Q}{\delta L}\right) < 0$ in zone B



I.e. in zone A, MP_L is rising and in zone B, MP_L is falling. Similar result shall apply if we plot TP with respect to input capital, K.

Note that the point where both MP and AP of labour are same is given by the green line through origin where the slop of tangent to TP curve is same as the slope of the straight line connecting origin to TP curve.

Diminishing returns operate with respect to each factor after a point. In above figure the point of is given by the dotted line. Note that this comes before the point where AP=MP. So, MP_L **starts falling before** AP_L (figure 2).

Due to diminishing returns to labor, the marginal product of labor declines after A, i.e. keeping capital fixed, as we increase labour, the additional output from an extra hour's labor decreases.

Due to diminishing returns to capital, the marginal product of capital declines, i.e. keeping labour fixed, as we increase capital, the additional output from an extra unit of capital decreases.

$$\frac{\delta}{\delta L}\left(\frac{\delta Q}{\delta L}\right) < 0$$

$$\frac{\delta}{\delta K}\left(\frac{\delta Q}{\delta K}\right) < 0$$

Phases of Marginal Returns: As the usage of an input increases, marginal product initially increases (increasing marginal returns), then begins to decline (decreasing marginal returns), and eventually becomes negative (negative marginal returns).

FIGURE 5–1 Increasing, Decreasing, and Negative Marginal Returns

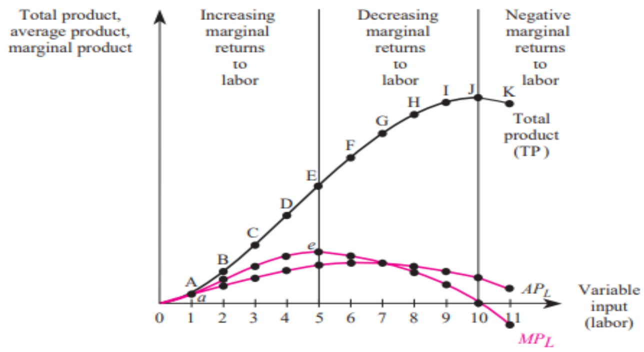
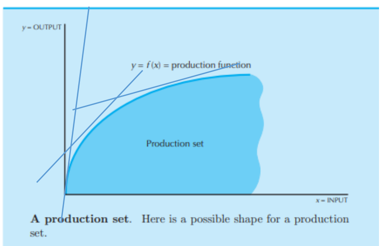


Figure 2, Source: Baye

A possible shape for a production function can also be:



A production set. Here is a possible shape for a production set.

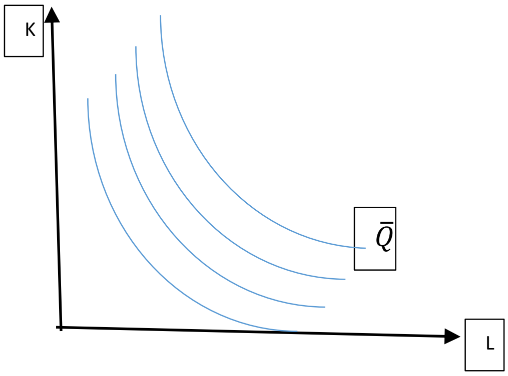
Here only the diminishing marginal returns is operating. The increasing returns is not present.

Isoquant

In the two-input case there is a convenient way to depict production relations on the L-K plane.

An isoquant is the set of all possible combinations of inputs L and K that are just sufficient to produce a given amount of output.

Isoquants are similar to indifference curves. Isoquants are labelled with the amount of output they can produce, not with a utility level (Varian) .



Isoquant is specified by a given output $\bar{Q}$.

So equation of an isoquant is $\bar{Q} = F(K, L)$

Totally differentiating we get,

$$d\bar{Q} = \frac{\partial F}{\partial L}\, dL + \frac{\partial F}{\partial K}\, dK = 0$$

$$\text{or, } dK/dL = -\frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}} = -\frac{MPl}{MPk}$$

So the slope of the isoquant is changing along the different points of an isoquant since both the marginal products are themselves functions of K and L.

The Technical Rate of Substitution: The technical rate of substitution measures the trade-off between two inputs in production. It measures the rate at which the firm will have to substitute one input for another in order to keep ***output constant***. It is the absolute value of the slope of the isoquant.

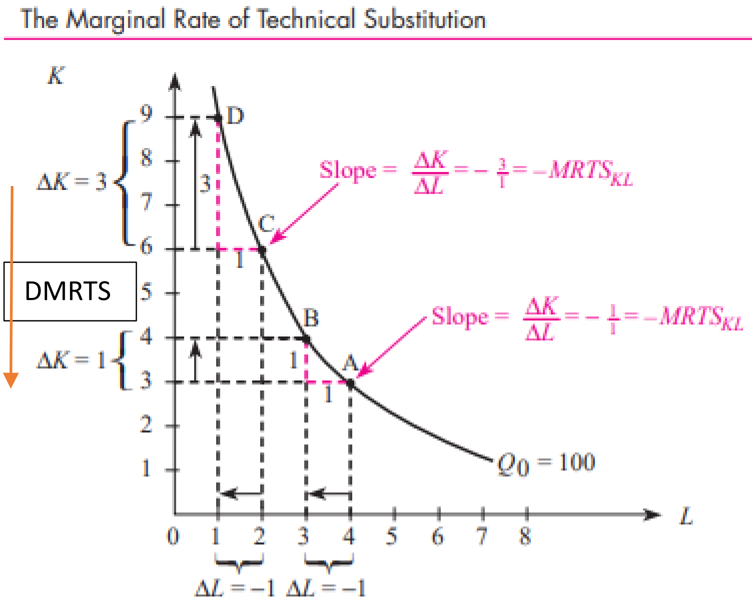
$$TRS = \left|\frac{dK}{dL}\right| = \frac{MPl}{MPk}$$

$$\text{Diminishing Technical rate of substitution: } \frac{\delta}{\delta L}(TRS) = \frac{\delta}{\delta L}\left(\frac{MPl}{MPk}\right) < 0$$

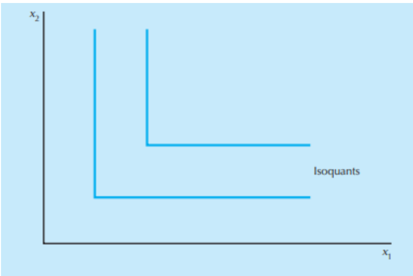
Similar to DMRS of consumer theory.

Absolute value of the slope of the isoquant decreases as we increase L.

As we increase the amount of factor 1, and adjust factor 2 so as to stay on the same isoquant, the technical rate of substitution declines. Roughly speaking, the assumption of diminishing TRS means that the slope of an isoquant must decrease in absolute value as we move along the isoquant in the direction of increasing x1, and it must increase as we move in the direction of increasing x2. This means that the isoquants will have the same sort of convex shape that well-behaved indifference curves have.

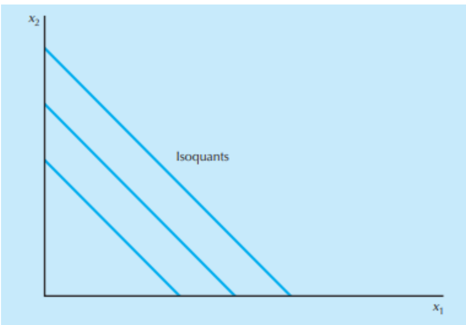


Different kinds of isoquants: Read textbook (Michael R. Baye)



Fixed proportions. Isoquants for the case of fixed proportions.

Q = {min (K, L)} perfect complements or Q = {min (aK, bL)}



Perfect substitutes. Isoquants for the case of perfect substitutes.

Q = K + L perfect substitutes or, Q = aK + bL

Returns to scale (from Varian):

Instead of increasing the amount of one input while holding the other input fixed, let's increase the amount of all inputs to the production function. This is different from Diminishing MP as there one input is held constant.

In other words, let's scale the amount of all inputs up by some constant factor: for example, use twice as much of both L and K. By how much output will increase?

When we will get twice as much output, this is called the case of constant returns to scale (CRS).

When we will get ***more than*** twice as much output, this is called the case of increasing returns to scale (IRS).

When we will get ***less than*** twice as much output, this is called the case of decreasing returns to scale (DRS).

Mathematically,

t^n Q = F (tK, tL)

when n =1, CRS

when n>1, IRS

when n <1, DRS

In terms of the production function, this means that two times as much of each input gives two times as much output. In the case of two inputs we can express this mathematically by 2f(L, K) = f(2L, 2K). In general, if we scale all of the inputs up by some amount t, constant returns to scale implies that we should get t times as much output: tf(L, K) = f(tL, tK)

How we derive inverse demand curve of each input? (From Varian)

The profit-maximizing input usage rule defines the demand for an input by a profit-maximizing firm.

The factor demand curves of a firm measure the relationship between the factor price and the profit-maximizing choice of that factor.

For this we use the profit ($\prod$) maximisation of the firm.

Let's consider the short-run unconstrained profit-maximization problem when K is fixed at some level $\bar{K}$.

Let f($\bar{K}$, L) be the production function for the firm, let p be the price of output at which the firm sells its product, and let w and r be the prices of the capital and labour.

Then the profit-maximization problem facing the firm can be written as

$$\max_L \ \prod \quad = \ p\, f(\bar{K}, L) \ - w\, L \ - r\, \bar{K}$$

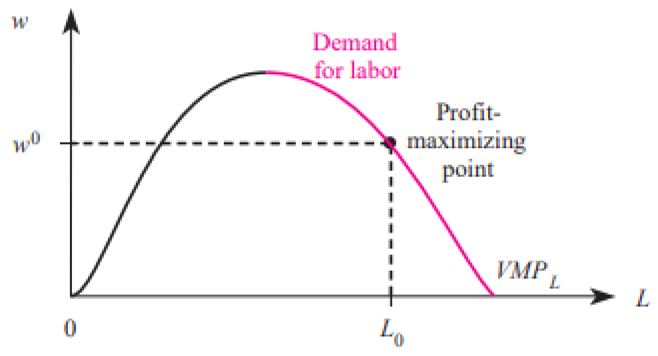
There is no constraint here, we just use the first order condition of maximisation i.e. partial derivative with respect to L is set equal to zero.

$$\frac{\delta \Pi}{\delta L} = p \quad MPl \ - w = 0$$

$$\text{Or, } p\,.\,MPl(L^*, \bar{K}\,) \quad = \quad w$$

This is the inverse demand curve of L^*. This is given by the falling portion of the MP curve of labour.

The Demand for Labor



In the long run, a similar profit maximisation can be done when both K and L are variable inputs, however the optimality condition remains same as the other factor of production is taken as constant.

The inverse demand curves are derived below:

$\max_{L,K} \; \prod \quad = \; p \; f(\overline{K}, L) \; - w \, L \; - rK$

FOC:

$\frac{\delta \prod}{\delta L} \; = \; p \quad MPl \; - w = 0$

Or, $\boxed{p \, . \, MPl(L^{*}, \overline{K}\,) \quad = \quad w}$

$\frac{\delta \prod}{\delta K} \; = \; p \quad MPk \; - r = 0$

Or, $\boxed{p \, . \, MPl(\overline{L}, K^{*}) \quad = \quad r}$

<u>Read from "Bayes" book why the second point in above figure is the profit maximising point.</u>

<u>Isocost</u>

Isocost line represents the combinations of inputs that will cost the producer the same amount of money.

<u>Optimal Input</u>

We use the constrained optimisation problems of

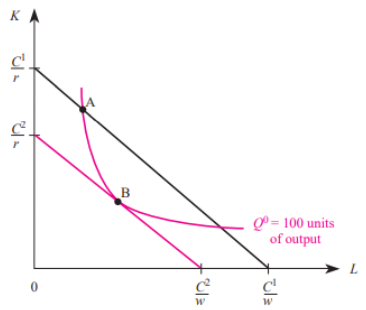
<u>Cost minimisation subject to a level of output</u>

<u>Output maximisation subject to a certain cost</u>

Either leads to same optimality condition

$$\boxed{\frac{MP_L}{MP_K} = \frac{w}{r}}$$

Input Mix *B* Minimizes the Cost of Producing 100 Units of Output



Cost minimisation

$\min_{L,K,\lambda}$ wL + rK

subject to $\bar{Q} = F(K, L)$

The Lagrangian can be written as :

$$\mathcal{L} = wL + rK + \lambda\,[\,\bar{Q} - F(K,L)\,]$$

FOCs

$\frac{\partial \mathcal{L}}{\partial L}$ = w - λ MP_L = 0 => w = λ MP_L…………..(1)

$\frac{\partial \mathcal{L}}{\partial K}$ = r - λ MP_K = 0 => r = λ MP_K…………..(2)

$\frac{\partial \mathcal{L}}{\partial \lambda}$ = $\bar{Q} - F(K,L)$ = 0 => $\bar{Q} = F(K,L)$ …………..(3)

Dividing (1) and (2),

$$\frac{MP_L}{MP_K} = \frac{w}{r}$$

Use this in the constraint, given by (3) and derive L and K in terms of w, r, and $\bar{Q}$

L^* = L (w,r, $\bar{Q}$)

K^* = K (w, r, $\bar{Q}$)

Do the output maximisation similarly.

Here, L^* = L (w,r, $\bar{C}$)

K^* = K (w, r, $\bar{C}$)

Read Cost Section from textbook Bayes and Prince