Mathematical Logic, End-Semester Exam

Chennai Mathematical Institute

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1. Suppose ϕ is a propositional logic formula and p is an atomic proposition occurring in ϕ. Prove that there are formulas ϕ_1, ϕ_2 in which p doesn't occur, such that ϕ is logically equivalent to $(\phi_1 \wedge p) \vee (\phi 2 \wedge \neg p)$. (4)

2. We could consider enriching first-order logic by the addition of a new quantifier. The formula $\exists! x\phi$ (read "there exists a unique x such that ϕ") is to be satisfied by a structure A and interpretation σ iff there is exactly one element e in the universe of discourse of A such that $(A, \sigma[x \mapsto e]) \models \phi$. Show that this apparent enrichment comes to naught, in the sense that we can find an ordinary formula logically equivalent to $\exists! x\phi$. (2)

3. Let $k \in \mathbb{N}$. Assume that FO$[k]$ contains only finitely many formulas up to logical equivalence, if the free variables are among the fixed set $\{x_1, x_2, \dots, x_m\}$. Suppose there is a property $\mathcal{P}$ satisfying the following condition: for any two structures A, B, if A has the property $\mathcal{P}$ and agrees on FO$[k]$ with B, then B also has the property $\mathcal{P}$. Prove that the property $\mathcal{P}$ is FO-definable. (4)

4. A set M of natural numbers is called a *spectrum* if there is a first-order language L and a sentence ϕ over L such that

$$M = \{n \mid \phi \text{ has a model containing exactly } n \text{ elements }\} .$$

Show that:

   (a) Every finite subset of $\{1, 2, 3, \dots\}$ is a spectrum. (2)

   (b) The set of squares greater than 0 is a spectrum. (6)

5. Suppose $G = (V, E)$ is a graph, whose vertices are coloured with either red, blue or green. The *interpreted graph* associated with G is the graph $H = (V', E')$ where $V' = \{(v_1, v_2, v_3) \in V \times V \times V \mid (v_1, v_2) \in E, (v_2, v_3) \in E, (v_1, v_3) \notin E, v_1 \text{ is red}, v_2 \text{ is blue}, v_3 \text{ is green}\}$ and the set of edges is $E' = \{((v_1, v_2, v_3), (u_1, u_2, u_3)) \in V' \times V' \mid (v_1, u_1) \in E, (v_2, u_2) \notin E, (v_3, u_3) \in E\}$. The problem is to write a first-order formula ϕ that can only use the equality symbol, the unary relation symbols red, blue, green and a binary relation symbol E. The formula ϕ should be true in G iff the interpreted graph H has a dominating set of size k, where $k \in \mathbb{N}$ is a constant. A dominating set in a graph is a set of vertices such that all other vertices are adjacent to at lest one vertex in the dominating set. (7)

6. In propositional logic, recall that $\mathrm{voc}(\phi)$ is the set of atomic propositions occurring in ϕ. suppose $\phi_1 \models \phi_2$ and $\mathrm{voc}(\phi_1) \cap \mathrm{voc}(\phi_2) = \emptyset$. Prove that there is a formula ϕ_i such that $\mathrm{voc}(\phi_i) = \mathrm{voc}(\phi_1) \cap \mathrm{voc}(\phi_2)$, $\phi_1 \models \phi_i$ and $\phi_i \models \phi_2$. (5)

7. Prove the *interpolation theorem*: if $\phi_1 \models \phi_2$, there is a formula ϕ_i such that $\mathrm{voc}(\phi_i) = \mathrm{voc}(\phi_1) \cap \mathrm{voc}(\phi_2)$, $\phi_1 \models \phi_i$ and $\phi_i \models \phi_2$. *Hint*: do an induction on $|\mathrm{voc}(\phi_1) \cap \mathrm{voc}(\phi_2)|$. If $p \in \mathrm{voc}(\phi_1) \cap \mathrm{voc}(\phi_2)$, Problem 1 shows that ϕ_1 (resp. ϕ_2) is logically equivalent to $(\phi_1' \wedge p) \vee (\phi_1'' \wedge \neg p)$ (resp. $(\phi_2' \wedge p) \vee (\phi_2'' \wedge \neg p)$). Prove that $\phi_1' \models \phi_2'$, $\phi_1'' \models \phi_2''$ and use the induction hypothesis. (10)