Introduction to Logic, Mid-semester exam

Chennai Mathematical Institute

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1. You are in a land inhabited by people who either always tell the truth or always tell falsehoods. You come to a fork in the road and you need to know which fork leads to the capital. There is a local resident there, but he has time only to reply to one yes-or-no question. What one question should you ask so as to learn which fork to take? Your answer should be justified by proper arguments. (10)

2. For the following derivations in Hilbert's proof system, explain how each step is obtained. Your explanation should be short and precise, like application of deduction theorem to a previous step, application of modus ponens to two previously derived formulas, instance of an axiom etc.

   a. 1. $\{A, \neg A\} \vdash \neg A$
      2. $\{A, \neg A\} \vdash A$
      3. $\{A, \neg A\} \vdash A \Rightarrow (\neg B \Rightarrow A)$
      4. $\{A, \neg A\} \vdash \neg A \Rightarrow (\neg B \Rightarrow \neg A)$
      5. $\{A, \neg A\} \vdash \neg B \Rightarrow A$
      6. $\{A, \neg A\} \vdash \neg B \Rightarrow \neg A$
      7. $\{A, \neg A\} \vdash (\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)$
      8. $\{A, \neg A\} \vdash (\neg B \Rightarrow A) \Rightarrow B$
      9. $\{A, \neg A\} \vdash B$
      10. $\{\neg A\} \vdash A \Rightarrow B$
      11. $\vdash \neg A \Rightarrow (A \Rightarrow B)$

   (2)

   b. 1. $\{A \Rightarrow (B \Rightarrow C), B\} \vdash A \Rightarrow (B \Rightarrow C)$
      2. $\{A \Rightarrow (B \Rightarrow C), B\} \vdash B$
      3. $\{A \Rightarrow (B \Rightarrow C), B\} \vdash B \Rightarrow (A \Rightarrow B)$
      4. $\{A \Rightarrow (B \Rightarrow C), B\} \vdash A \Rightarrow B$
      5. $\{A \Rightarrow (B \Rightarrow C), B\} \vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
      6. $\{A \Rightarrow (B \Rightarrow C), B\} \vdash (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$
      7. $\{A \Rightarrow (B \Rightarrow C), B\} \vdash (A \Rightarrow C)$
      8. $\{A \Rightarrow (B \Rightarrow C)\} \vdash B \Rightarrow (A \Rightarrow C)$ (2)

3. Prove $\vdash (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$ in the natural deduction system. (3)

4. Prove $\vdash \neg(\alpha \vee \beta) \Rightarrow \neg\alpha \wedge \neg\beta$ in the natural deduction system. (3)

5. Let ϕ be a propositional logic formula with vocabulary in V and let $v : V \to \{\top, \bot\}$ be a truth valuation. Let Γ_v be the set of formulas defined as $\{p \in V \mid v(p) = \top\} \cup \{\neg p \mid v(p) = \bot\}$. Recall that in class, we proved that if $v \models \phi$, then $\Gamma_v \vdash \phi$ and if $v \not\models \phi$, then $\Gamma_v \vdash \neg\phi$ in natural deduction system. The proof of this is by induction on structure of ϕ. Prove the induction step in the case where $\phi = \psi_1 \vee \psi_2$. (4)

6. Show that the ordering on natural numbers is definable in $(\mathbb{N}, +)$. (2)

7. Show that the ordering on integers is not definable in $(\mathbb{I}, +)$. (2)

8. Consider the first-order language consisting of two unary relation symbols A and B. For each of the following formulas, give an interpretation that will make the formula *false*.

   (a) $(\exists x A(x) \overset{T}{\Rightarrow} \exists x B(x)) \Rightarrow \forall x(A(x) \overset{F}{\Rightarrow} B(x))$ (2)

   (b) $(\forall x A(x) \Rightarrow \forall x B(x)) \Rightarrow \forall x(A(x) \Rightarrow B(x))$ (3)

9. Consider a first-order language consisting of one binary relation symbol P. Consider first-order structures built from directed graphs as follows: universe of the structure is the set of vertices of the graph. A pair of vertices (v_1, v_2) is in the relation P exactly when there is a directed path from v_1 to v_2. Let us call the set of such structures $\mathbb{M}$. Consider the formula $E(x, y)$ with two free variables x, y defined as $E(x, y) = P(x, y) \wedge \neg\exists z(P(x, z) \wedge P(z, y))$. Is it true that $E(v_1, v_2)$ is true in a structure from $\mathbb{M}$ exactly when there is a directed edge from v_1 to v_2? (7)

$A(x) = 1$

$B(x) = 2$

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T	F	F	T	T