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Mid Semester Examination, Aug–Nov 2023

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Date:	Subject: Mathematical logic
Course & Year: MSC CSI LOGIC 2023	Total No. of Pages:

36

	Points	Remarks
Part A		
Part B		
Total		

	Points	Remarks
B1		
B2		
B3		
B4		
B5		
B6		
Total		

B1. Write your solution to B1 below.

2 (a)

1 $\{A, \neg A\} \vdash \neg A$ — instance of a premise/axiom

2 $\{A, \neg A\} \vdash A$ — instance of a premise/axiom

3 $\{A, \neg A\} \vdash A \Rightarrow (\neg B \Rightarrow A)$ - Instance of Axiom (1) $(P = A, Q = \neg B)$

4 $\{A, \neg A\} \vdash \neg A \Rightarrow (\neg B \Rightarrow \neg A)$ - Instance of Ax 1 $(P = \neg A, Q = \neg B)$

5 $\{A, \neg A\} \vdash \neg B \Rightarrow A$ - Modus ponens on (2) and (3)

6 $\{A, \neg A\} \vdash \neg B \Rightarrow \neg A$ Modus Ponens on (1) and 4

7 $\{A, \neg A\} \vdash (\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)$ - Instance of Axiom (3)

8 $\{A, \neg A\} \vdash ((\neg B \Rightarrow A) \Rightarrow B)$ - Modus ponens on (6) and (7)

9 $\{A, \neg A\} \vdash B$ - Modus ponens on (5) and (8)

10 $\{\neg A\} \vdash A \Rightarrow B$ - Deduction theorem to (9)

11 $\vdash \neg A \Rightarrow (A \Rightarrow B)$ - Deduction theorem to (10)

2

~~$\neg B$~~,

~~$\neg A$~~

$\neg Q = \neg B$

$\neg P = \neg A$

3 rules are in terms of P, Q, R

1> $P \Rightarrow (Q \Rightarrow P)$ ax1

2> $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow ((P \Rightarrow Q) \Rightarrow (Q \Rightarrow R))$

2

3> $(\neg Q \Rightarrow \neg P) \Rightarrow ((\neg Q \Rightarrow P) \Rightarrow Q)$

Modus ponens $\dfrac{P,\ P \Rightarrow Q}{Q}$

Solution to B1 continued

2(b)

1. Instance of an axiom / a premise

2. Instance of an axiom / a premise

3. Instance of axiom ① $P = B$, $Q = A$

4. Modus ponens on ② and ③

5. Instance of axiom 2. $P = A$, $Q = B$, $R = C$

2

6. Modus ponens to ① and ⑤

7. Modus ponens to ④ and ⑥

8. Deduction theorem on 7

B2. Write your solution to B2 below.

① From the question I am assuming that.

The person who tells the falsehood will calculate everything correct in his mind but when answering me, he will lie.

So, My question will be,

Q. Are you a liar ?? Xor Is this the right road. ??

For the liar people, first question will always be true in his mind

So.

Right road	Are you a liar	his answer to the (Is this the right road)	Xor	Answer
T	T	T	F	T
F	T	F	T	F

If I ask the same question to the person who always tell the truth.

Right Road	Are you a liar	Is this the right road	Xor	ans
T	F	T	T	T
F	F	F	F	F

So irrespective of the person, I will find the correct path if I follow/obey them.

Also, Xor ($\wedge$; $\oplus$) can be written in term of $\wedge$ ~~and~~ $\vee$ and

So, $p \oplus q = (\neg p \wedge q) \vee (p \wedge \neg q)$

10

where p = Are you a liar ??

q = Is this the right road ??

So, the question can be formed in the way shown above

6. ~~$\mathcal{L}$~~ $L = (N, +)$

So, we will define a ~~formula~~ less s.t.

$\cancel{\phi(x, y)}$

$$\text{less}(x, y) = \exists z \, (\neg(z = z + z) \wedge (x + z = y))$$

~~if we~~, say, M is universe of Natural Number

$(+)^{I} = (+)$ (Normal meaning)

$(\text{less})^{I} = <$ (" ")

then $\boxed{I \models \text{less}(x, y)}$ ~~then~~ when $x < y$

2

So ordering is ~~possible~~ definable in $(N, +)$

also $\phi = \forall x \, \exists y \, \exists z \, (\neg(z \equiv z + z) \wedge (x + z \equiv y))$

So, $I \models \phi$ which also shows that order is definable.

B5. Write your solution to B5 below.

8\. In both the part questions we will have our universe as $\mathbb{N}$ $\Rightarrow$ $I : (\mathbb{N}, (A, B))$ as follows

$A = \{0\}$ (mean $A(x) = T$ for $x = 0$, $= F$ otherwise or $x \equiv 0$)

$B = \{1\}$ (mean $B(x) = T$ for $x = 1$, $= F$ otherwise or $x \equiv 1$)

Now,

(a) $\exists x A(x)$ is T $(x \equiv 0)$

$\exists x B(x)$ is T $(x \equiv 1)$

$A(x) \Rightarrow B(x)$ is false for $x \equiv 0$

So, $(\exists x A(x) \Rightarrow \exists x (B(x)) \Rightarrow \forall x (A(x) \Rightarrow B(x))$

T F

So, This interpretation $I \not\models$ The formula

(b) $\forall x A(x)$ is False $(\bot)$ as $A(x)$ only T for $x \equiv 0$

$\forall x B(x)$ is False $(\bot)$ as $B(x)$ only T for $x \equiv 1$

$A(x) \Rightarrow B(x)$ is false for $x \equiv 0$

F F

$\therefore (\forall x A(x) \Rightarrow \forall x B(x)) \Rightarrow \forall x (A(x) \rightarrow B(x))$

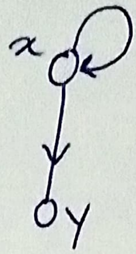
T F

So $I \not\models$ the formula

10

B6. Write your solution to B6 below.

(a) let us consider this case



Now, $E(x,y) = P(x,y) \wedge \neg \exists z(P(x,z) \wedge P(z,y))$

let's the statement be true. ($E(v_1, v_2)$ be true if there exists a directed path between v_1 and v_2)

Clearly, There exists a path from x to y.

So, $E(x,y)$ should be true according to the statement.

But, $E(x,y) = P(x,y) \wedge \neg \exists z(P(x,z) \wedge P(z,y))$

Now, from the above case,

$P(x,y)$ is true (as there exists a path between x and y)

also $\exists z(P(x,z) \wedge P(z,y))$ is true, as if we pick z as x. $P(x,x)$ is true due to the self-loop and $P(x,y)$ is true,

$\therefore \neg(\exists z(P(x,z) \wedge P(z,y)))$ is false

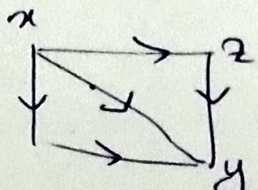
Solution to B6 continued.

So $E(x, y)$ is false

Hence contradiction.

So, the statement is not true

The statement is not true in general when the graph has self loops ~~bicycles~~ or we can follow two or more paths from a vertex v_1 to another vertex v_2.



this is also a case

For any part B solution continued in these blank pages, label it clearly AND note the page number of the continuation in the main part of the solution.

5. RTP

if $v \models \varphi_1 \vee \varphi_2$

then $\Gamma_v \vdash \varphi_1 \vee \varphi_2$

where φ_1 and φ_2 are two formulas with Vocabulary in $V \supseteq (Voc(\varphi_1) \cup Voc(\varphi_2))$

and $v : V \longrightarrow (T, \bot)$

we will use Induction on structure of ϕ:

Base case : p is an atomic proposition

if $v \models p$ then $p \in \Gamma_v$ from the definition of Γ_v

p is a negation of an atomic prop

if $v \models \neg p$.

~~then~~ $\neg p \in \Gamma_v$ from the definition of Γ_v

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Induction ~~step~~ hypothesis.

let $v \models \varphi_1 \vee \varphi_2$

then $\Gamma_v \vdash \varphi_1 \vee \varphi_2$

and ~~But~~ we know

if $v \models \varphi_1$ then $\Gamma_v \vdash \varphi_1$

and $v \models \varphi_2$ then $\Gamma_v \vdash \varphi_2$.

as φ_1, φ_2 are structurally smaller.

Induction step:

$v \models \varphi_1 \vee \varphi_2$

either of them will happen, so case of WLOG is redundant now

Then WLOG, (as v must satisfy atleast one of φ_1 and φ_2) from definition of or.

$v \models \varphi_1$

Therefore, $\Gamma_v \vdash \varphi_1$

So, $\Gamma_v \vdash (\varphi_1 \vee \varphi_2)$ $(\vee_{i_R})$

You also have to prove that if $v \not\models \varphi_1 \vee \varphi_2$, then $\Gamma_v \vdash \neg(\varphi_1 \vee \varphi_2)$

if $v \models \varphi_2$

Therefore $\Gamma_v \vdash \varphi_2$

$\Gamma_v \vdash (\varphi_1 \vee \varphi_2)$ $\vee_{i_L}$

X

So, proved

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(3)

$$\dfrac{}{\alpha \Rightarrow (\beta \Rightarrow \gamma), \alpha \Rightarrow \beta, \alpha \vdash \alpha \Rightarrow \beta}\ ax \qquad \dfrac{}{\alpha \Rightarrow (\beta \Rightarrow \gamma), \alpha \Rightarrow \beta, \alpha \vdash \alpha}\ ax$$

$$\rightarrow e$$

3

$$\dfrac{}{\alpha \Rightarrow (\beta \Rightarrow \gamma), (\alpha \Rightarrow \beta), \alpha \vdash \alpha}\ ax$$

diff C

$$\dfrac{}{\alpha \Rightarrow (\beta \Rightarrow \gamma), (\alpha \Rightarrow \beta), \alpha \vdash \alpha \Rightarrow (\beta \Rightarrow \gamma)}\ ax$$

$$\rightarrow e$$

$$\alpha \Rightarrow (\beta \Rightarrow \gamma), (\alpha \Rightarrow \beta), \alpha \vdash \beta \qquad \alpha \Rightarrow (\beta \Rightarrow \gamma), (\alpha \Rightarrow \beta), \alpha \vdash \beta \Rightarrow$$

$$\rightarrow e$$

$$\alpha \Rightarrow (\beta \Rightarrow \gamma), (\alpha \Rightarrow \beta), \alpha \vdash \gamma$$

$$\rightarrow i$$

$$\alpha \Rightarrow (\beta \Rightarrow \gamma), (\alpha \Rightarrow \beta) \vdash \alpha \Rightarrow \gamma.$$

$$\rightarrow i$$

$$\alpha \Rightarrow (\beta \Rightarrow \gamma) \vdash (\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma)$$

$$\rightarrow i$$

$$\vdash (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$$

4.

$$\neg(\alpha \vee \beta), \beta \vdash \beta \quad ax$$

$$\neg(\alpha \vee \beta), \beta \vdash \alpha \vee \beta \quad \vee_{iL}$$

$$\neg(\alpha \vee \beta), \beta \vdash \neg(\alpha \vee \beta)$$

3

$$\neg(\alpha \vee \beta), \alpha \vdash \alpha \quad ax$$

$$\neg(\alpha \vee \beta), \alpha \vdash \alpha \vee \beta \quad \vee_{ir}$$

$$\neg(\alpha \vee \beta), \alpha \vdash \neg(\alpha \vee \beta) \quad ax$$

$$\neg(\alpha \vee \beta), \alpha \vdash \bot \quad \neg e$$

$$\neg(\alpha \vee \beta), \beta \vdash \bot$$

$$\neg(\alpha \vee \beta) \vdash \neg\alpha \quad \neg i$$

$$\neg(\alpha \vee \beta) \vdash \neg\beta \quad \neg i$$

$$\neg(\alpha \vee \beta) \vdash (\neg\alpha \wedge \neg\beta) \quad \wedge i$$

$$\vdash \neg(\alpha \vee \beta) \Rightarrow (\neg\alpha \wedge \neg\beta) \quad \rightarrow i$$

(7) we will define an automorphism, ie homomorphism from $(I, +)$ to itself

Now, A, B be the structures defined to be the $(I, +)$

Now $h : |A| \to |B|$

$h(x) = -x$

we have to show that, $\vec{(+)}$ is preserved

$h((+)(a, b)) = h(a+b)$

$= -(a+b)$

$h((+)(a, b)) = +(h(a), h(b))$

$= (-a) + (-b)$

$= -(a+b)$

2

$\therefore$ $(+)$ is preserved.

So h is indeed a homomorphism

Now, let's say, we have defined the '<' relation in A and B' such that..

$< (x, y)$ is true if $(x < y)$
(ordering)

let it be true in A,

So, $< (h(x), h(y))$ should be true.
as relation is preserved in homomorphism

But, $< (-x, -y)$ is not true

as $\boxed{\begin{array}{l} x < y \\ \Rightarrow -x > -y \end{array}}$.

So, if (x, y) resides in $<$

$h(x), h(y)$ does not reside in $<$.

Hence ordering is not definable in $(I, +)$